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The entropy of algebraic actions of countable torsion-free abelian groups

by

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Abstract. This paper is concerned with the entropy of an action of a countable torsion-free abelian group G by continuous automorphisms of a compact abelian group X. A formula is obtained that expresses the entropy in terms of the Mahler measure of a greatest common divisor, complementing earlier work by Einsiedler, Lind, Schmidt and Ward. This leads to a uniform method for calculating entropy whenever G is free. In cases where these methods do not apply, a possible entropy formula is conjectured. The entropy of subactions is examined and, using a theorem of P. Samuel, it is shown that a mixing action of an infinitely generated group of finite rational rank cannot have a finitely generated subaction with finite non-zero entropy. Applications to the concept of entropy rank are also considered.

1. Introduction and statement of results. Let G be a countable torsion-free abelian group and X a compact metrizable abelian group with Borel σ-algebra $\mathcal{B}(X)$ and normalized Haar measure μ_X. If α is a G-action by μ_X-preserving transformations α^g of X, since G is amenable, we may consider the metric entropy $h(\alpha)$ with respect to μ_X. A detailed theory of metric entropy for amenable group actions can be found in the seminal article [20]. In this paper we will be interested in the case where α is an *algebraic G-action*, that is, each α^g is a continuous automorphism of X. Therefore, there is also a notion of topological entropy whose correspondence with metric entropy is summarized in [4].

In [15], Kitchens and Schmidt develop a framework for the study of algebraic G-actions; the resulting theory is the subject of Schmidt's monograph [22]. A fundamental observation is that there is a correspondence between these actions and modules over the group ring $\mathbb{Z}G$, facilitated by

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Pontryagin duality (see Section 2). Write α_M for the action corresponding to a $\mathbb{Z}G$-module M.

When $G = \mathbb{Z}^d$, $d < \infty$, $\mathbb{Z}G$ can be identified with a ring of Laurent polynomials and the formula of Lind, Schmidt and Ward [16] expresses $h(\alpha_M)$ in terms of Mahler measures of Laurent polynomials associated with M. For a Laurent polynomial $f = f(u_1^{\pm 1}, \ldots, u_n^{\pm 1})$ with integer coefficients, the *Mahler measure* of f is

$$\mathsf{m}(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| \, dt_1 \ldots dt_n.$$

For background on Mahler measure, consult [12]. If α_M corresponds to a $\mathbb{Z}G$-module $M = \mathbb{Z}G/(f)$, $f \in \mathbb{Z}G$, then $h(\alpha_M) = \mathsf{m}(f)$. Lind, Schmidt and Ward's full results cover all $\mathbb{Z}G$-modules M and generalize earlier formulae for single automorphisms due to Yuzvinskiĭ [25] and Lind and Ward [17].

More recently, Deninger and Schmidt [4], [5] have studied the entropy of actions of residually finite groups, relaxing the assumption that G is abelian. In [5], their main result is that if f is a unit in the convolution algebra $L^1(G)$, $\mathbb{Z}Gf \subset \mathbb{Z}G$ is a left ideal and $M = \mathbb{Z}G/\mathbb{Z}Gf$, then $h(\alpha_M)$ is given by the logarithm of a Fuglede–Kadison–Lück determinant ([4, Sec. 3] describes the relationship between these determinants and Mahler measure). In the setting of C^*-dynamical systems, the entropy of actions of infinitely generated torsion-free abelian groups has been considered by Golodets and Neshveyev [14]. The study of algebraic actions of such groups was initiated in [18] in relation to the dynamical property of expansiveness and continues here with the investigation of entropy.

When dealing with algebraic G-actions with G infinitely generated, a significant obstacle is that $\mathbb{Z}G$ is non-Noetherian; the importance of this condition and its dynamical interpretation is discussed in [22, Sec. 3]. When G is free, $G \cong \mathbb{Z}^d$ for some $d \le \infty$ (for a countable abelian group Γ, Γ^∞ is used to mean $\bigoplus_{\mathbb{N}} \Gamma$), so $\mathbb{Z}G$ is a unique factorization domain and the greatest common divisor of any non-trivial subset of $\mathbb{Z}G$ is well defined. For our purposes, this condition turns out to be more important than whether or not $\mathbb{Z}G$ is Noetherian. Also, whenever G is free, $\mathbb{Z}G$ can be viewed as a Laurent polynomial ring in countably many variables. By using the work of Lind, Schmidt and Ward as a foundation, the following result is obtained.

THEOREM 1.1. *Let G be a countable torsion-free abelian group and let $\mathfrak{a} \subset \mathbb{Z}G$ be a non-zero ideal. If there is a free group $G' \le G$ such that $\mathfrak{a} = (\mathfrak{a} \cap \mathbb{Z}G')\mathbb{Z}G$ then*

$$h(\alpha_{\mathbb{Z}G/\mathfrak{a}}) = \mathsf{m}(\gcd(\mathfrak{a} \cap \mathbb{Z}G')). \tag{1}$$

In particular, such a G' exists if G itself is free or if $\mathfrak{a}$ is finitely generated.

The use of greatest common divisors for calculating the entropy of algebraic actions of finitely generated G was initiated by Einsiedler [6] and developed by Einsiedler and Ward [9].

Using Theorem 1.1, a general entropy formula is established for the case $G = \mathbb{Z}^\infty$. For continuity with Lind, Schmidt and Ward's work, the result is phrased to include all free G. The fact that the $\mathbb{Z}^\infty$ case is so similar to that of $\mathbb{Z}^d$, $d < \infty$, is a little surprising, since this is certainly not true for all dynamical properties (see [19], for example). When G is free, for a $\mathbb{Z}G$-module M, let $\mathrm{Ass}^1(M)$ be the set of non-zero principal prime ideals in $\mathbb{Z}G$ that are annihilators of elements of M. Upon identifying $\mathbb{Z}G$ with a ring of Laurent polynomials, if $\mathfrak{p} \subset \mathbb{Z}G$ is a non-zero principal prime, $\mathsf{m}(\mathfrak{p})$ is defined in an obvious way.

Theorem 1.2. *Let $G = \mathbb{Z}^d$, $d \leq \infty$, and suppose M is a finitely generated $\mathbb{Z}G$-module. If M is a torsion module, then*

$$h(\alpha_M) = \sum_{\mathfrak{p} \in \mathrm{Ass}^1(M)} \mathsf{m}(\mathfrak{p}) \dim_{\mathbb{K}(\mathfrak{p})} M_{\mathfrak{p}}, \tag{2}$$

where $\mathbb{K}(\mathfrak{p})$ is the field of fractions of $\mathbb{Z}G/\mathfrak{p}$ and $M_{\mathfrak{p}}$ is the localization of M at $\mathfrak{p}$. If M is not a torsion module then $h(\alpha_M) = \infty$.

Standard results for algebraic G-actions allow one to extend Theorem 1.2 to the case where M is not finitely generated (see Remark 5.2). If neither Theorem 1.1 nor Theorem 1.2 can be applied, the situation is somewhat more complicated. In Section 5, a general formula for the entropy of actions arising from cyclic modules is proposed (Conjecture 5.4) and supporting evidence considered. The formula arises from a fundamental inequality (Theorem 5.3) that turns out to be useful for identifying actions with zero entropy.

In Section 6, the entropy of subactions is investigated. Some well-known examples of infinitely generated actions have finitely generated subactions with finite non-zero entropy (Example 6.1). However, the following theorem isolates a large class of algebraic actions for which this is not the case.

Theorem 1.3. *Suppose G is an infinitely generated torsion-free abelian group of finite rational rank and α is a mixing algebraic G-action on a compact abelian group X. Then α has no finitely generated subactions with finite non-zero entropy.*

The proof of Theorem 1.3 relies on an application of Samuel's theorem on finitely generated unit groups [21], reflecting the algebraic nature of this phenomenon. If the mixing assumption is dropped, the same conclusion cannot be reached (Example 6.6).

The concept of entropy rank was introduced by Boyle and Lind [2] in relation to the expansive subdynamics of $\mathbb{Z}^d$-actions, $d < \infty$, although the definition given by Einsiedler and Lind [7] extends more easily to our setting (Definition 6.8). Roughly speaking, entropy rank expresses the least rank such that all subactions of that rank have finite entropy. Applying results from Section 5, it is possible to characterize the relationship between entropy rank and algebraic dependence (Theorem 6.9), in line with the situation for finitely generated actions [8].

2. Preliminaries. Let G be a countable torsion-free abelian group. The *rational rank*, or simply the *rank*, of G is $\mathrm{rk}(G) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes G$. A *Følner sequence* (F_i) in G is a sequence of finite sets $F_i \subset G$ for which, given any $g \in G$,

$$|F_i \triangle g + F_i|/|F_i| \to 0$$

as $i \to \infty$. Note that Følner sequences can always be found in G. Suppose X is a compact metrizable abelian group with Borel σ-algebra $\mathcal{B}(X)$ and normalized Haar measure μ_X. If α is a G-action by μ_X-preserving transformations α^g of X, the *metric entropy* of α with respect to μ_X is the quantity

$$h(\alpha) = \sup_{\xi} h(\alpha, \xi),$$

where ξ runs over all finite measurable partitions of X and

$$h(\alpha, \xi) = \lim_{i\to\infty} \frac{1}{|F_i|} H(\xi^{F_i}),$$

$H(\cdot)$ being the entropy of a partition with respect to μ_X and $\xi^{F_i} = \bigvee_{g\in F_i} \alpha^g \xi$. It can be shown that the definition of entropy is independent of the choice of the Følner sequence. For any finite measurable partitions ξ, η of X,

$$h(\alpha, \xi) \le h(\alpha, \eta) + H(\xi \mid \eta),$$

so the Rokhlin inequality holds. If (ξ_n) is an increasing sequence of finite measurable partitions such that $\sigma(\bigvee_{n\ge1} \bigvee_{g\in G} \alpha^g \xi_n) = \mathcal{B}(X)$, then

$$h(\alpha) = \sup_{n\ge1} h(\alpha, \xi_n) = \lim_{n\to\infty} h(\alpha, \xi_n).$$

For further background and explanation of these statements, see [20] and [24].

Now suppose α is an algebraic action, that is, each α^g is a continuous automorphism of X. If Y is a closed α-invariant subgroup of X, the subscript notation α_Y, $\alpha_{X/Y}$ is also used for the induced actions on Y and X/Y respectively. Note that

$$h(\alpha_{X/Y}) \le h(\alpha_X) \quad \text{and} \quad h(\alpha_Y) \le h(\alpha_X).$$

If $Y_1 \supset Y_2 \supset \cdots$ is a sequence of closed α-invariant subgroups of X with $\bigcap_{n\ge1} Y_n = \{0\}$, then

$$h(\alpha) = \lim_{n\to\infty} h(\alpha_{X/Y_n}). \tag{3}$$

These results are straightforward generalizations of the situation for $\mathbb{Z}^d$-actions [22, Sec. 13]. Note also that Haar measure is a "dominant" measure for entropy in the sense of the variational principle (the proof of [22, Prop. 13.5] can be adapted to show this).

When α is an algebraic action, via Pontryagin duality, the discrete countable dual group $M = \widehat{X}$ becomes a module over the group ring $\mathbb{Z}G$ by setting

$$fa = \sum_{g\in G} c_g(f)\widehat{\alpha^g}(a),$$

where $a \in M$, $c_g(f) \in \mathbb{Z}$ and $f = \sum_{g\in G} c_g g$ has $c_g = 0$ for all but finitely many $g \in G$. Conversely, any countable $\mathbb{Z}G$-module M gives rise to an algebraic G-action α_M on the compact abelian group $X_M = \widehat{M}$ by setting α^g, $g \in G$, to be the automorphism dual to multiplication by g on M. For any module homomorphism $\phi : M \to N$, the continuous (and hence measurable) dual homomorphism $\widehat{\phi} : X_N \to X_M$ satisfies $\widehat{\phi}\cdot\alpha_N^g = \alpha_M^g \cdot \widehat{\phi}$ for all $g \in G$. For further details of the correspondence between algebraic G-actions and $\mathbb{Z}G$-modules, consult [22]. When it is necessary to emphasize the acting group G, write α_M^G. If $J \le G$, then M is naturally a $\mathbb{Z}J$-module and α_M^J denotes the induced subaction on X_M.

Any cyclic $\mathbb{Z}G$-module may be written in the form $M = \mathbb{Z}G/\mathfrak{a}$, where $\mathfrak{a} \subset \mathbb{Z}G$ is an ideal. Let $\mathcal{A}$ be a set of generators for $\mathfrak{a}$. Then X_M may be identified with the subgroup of $\mathbb{T}^G$ consisting of elements $x = (x_g)$ satisfying

$$\sum_{g\in G} c_g(f)x_{g+g'} = 0 \tag{4}$$

for all $g' \in G$ and $f \in \mathcal{A}$. Set

$$S(\mathcal{A}) = \{g \in G : c_g(f) \neq 0 \text{ for some } f \in \mathcal{A}\}. \tag{5}$$

According to the description of X_M just given, for each $g' \in G$, the automorphism $\alpha^{g'}$ identifies with the appropriate restriction of the G-*shift*; that is, for $x = (x_g) \in X_M$,

$$\alpha^{g'}(x)_g = x_{g+g'}.$$

3. Realization of actions and Følner sequences. Suppose G is infinitely generated. Let $\mathfrak{a} \subset \mathbb{Z}G$ be an ideal with generating set $\mathcal{A} \subset \mathbb{Z}G$, let $S(\mathcal{A})$ be given by (5) and set $M = \mathbb{Z}G/\mathfrak{a}$. Consider the following two cases:

(1) G is free and $\mathfrak{a}$ is finitely generated. Then G splits as $J \oplus K$, where $J \cong \mathbb{Z}^d$, $d < \infty$, J contains $S(\mathcal{A})$ and $K \cong \mathbb{Z}^\infty$.

(2) G is not free and there exists a free subgroup $G' \le G$ such that $\mathfrak{a} = (\mathfrak{a} \cap \mathbb{Z}G')\mathbb{Z}G$. In this case, without loss of generality, we may assume $\mathcal{A} \subset \mathbb{Z}G'$. By the axiom of choice, we can find $G' \le J \le G$ such that $J \cong \mathbb{Z}^d$, $d \le \infty$, and $K = G/J$ is a torsion group.

In case (2), identify J with $\mathbb{Z}^d$ and use the subsequent embedding of G into $\mathbb{Q} \otimes J = \mathbb{Q}^d$ to identify K with a subgroup of $(\mathbb{Q}/\mathbb{Z})^d$, choosing coset representatives for $\mathbb{Q}/\mathbb{Z}$ in the unit interval. This gives rise to a bijection $G \to J \times K$, with addition in G corresponding to

$$(a, q) + (b, q) = (a + b + \delta(q, r), q + r),$$

where $(a, q) = (a_n, q_n), (b, r) = (b_n, r_n) \in J \times K$, and

$$\delta(q, r)_n = \begin{cases} 0 & \text{if } q_n +_{\mathbb{Q}} r_n < 1, \\ 1 & \text{if } q_n +_{\mathbb{Q}} r_n \geq 1. \end{cases}$$

Since $S(\mathcal{A})$ lies in J, for any $(b, r) \in J \times K$, the left hand side of (4) becomes

$$\sum_{(a,q)\in J\times K} c_{(a,q)}(f)x_{(a,q)+(b,r)} = \sum_{a\in J} c_a(f)x_{(a,0)+(b,r)} = \sum_{a\in J} c_a(f)x_{(a+b,r)},$$

where $c_a = c_{(a,0)}$. Hence, X_M may be identified with the subgroup of $\mathbb{T}^{J\times K}$ consisting of the elements $x = (x_{(a,q)})$ satisfying

$$\sum_{a\in J} c_a(f)x_{(a+b,r)} = 0 \tag{6}$$

for all $(b, r) \in J \times K$ and $f \in \mathcal{A}$. For any $(b, r) \in J \times K$ and $x = x_{(a,q)} \in X_M$,

$$\alpha^{(b,r)}(x)_{(a,q)} = x_{(a+b+\delta(q,r),q+r)}. \tag{7}$$

In the simpler case (1), X_M identifies with the subgroup of $\mathbb{T}^{J\oplus K} = \mathbb{T}^{J\times K}$ consisting of the elements $x = (x_{(a,q)})$ satisfying (6) for all $(b, r) \in J \times K$. The G-action is given by (7) with $\delta \equiv 0$.

It will often be useful to consider the coordinate restriction homomorphism $\pi_J : \mathbb{T}^{J\times K} \to \mathbb{T}^J$ given by

$$\pi_J(x_{(a,q)}) = (y_a), \tag{8}$$

where $y_a = x_{(a,0)}$ for all $a \in J$. Let β be the J-shift on $\mathbb{T}^J$ and $W = \pi_J(X_M)$. Notice that X_M may also be regarded as W^K and its Haar measure as the product measure obtained from Haar measure on W. Furthermore,

$$\pi_J \cdot \alpha^{(b,0)} = \beta_W^b \cdot \pi_J$$

for all $b \in J$. So, $\alpha^J_{\mathbb{Z}J/\mathfrak{a}\cap\mathbb{Z}J}$ is algebraically isomorphic to β_W.

It is straightforward to obtain Følner sequences in G. However, we wish to use a specific form of Følner sequence to complement the description of the dynamical system (X_M, α_M) given above.

Lemma 3.1. *Let $d \leq \infty$. For $d = \infty$ set $s(i) = \lfloor \frac{1}{2} \log_2 i \rfloor$ and for $d < \infty$ set $s(i) = d$, $i \geq 1$. The sequence (F_i) defined by*

$$F_i = \{(a_n) \in \mathbb{Z}^d : -i \leq x_n \leq i \text{ when } n \leq s(i),\ x_n = 0 \text{ when } n > s(i)\}$$

is a Følner sequence in $\mathbb{Z}^d$*. If* G *is not free and* J *and* K *are as above, there is a sequence* $K_1 \le K_2 \le \cdots$ *of finite subgroups of* K *such that* $(E_i) = (F_i \times K_i)$ *corresponds to a Følner sequence in* G*.*

Proof. Let $b = (b_n) \in \mathbb{Z}^d$ and let $\|b\| = \max\{b_n\}$. For a group of the form $\varGamma^\infty$ and any $\gamma = (\gamma_n) \in \varGamma^\infty$, set $\ell(\gamma) = \max\{n : \gamma_n \ne 0\}$. Then

$$|F_i \triangle b + F_i| \ll (2i+1)^{s(i)} - (2i+1)^{s(i)-\ell(b)}(2i+1-\|b\|)^{\ell(b)} \ll |F_i| i^{-1},$$

so (F_i) is a Følner sequence. At this stage, $(s(i))$ could be any sequence of positive integers tending to infinity.

Let $K_1 \le K_2 \le \cdots$ be a sequence of finite subgroups of K such that $\bigcup_{i\ge 1} K_i = K$. Each K_i can be chosen so that $\ell(q) \le s(i)$ for all $q \in K_i$. Let $r \in K$ and suppose i is large enough so that F_i contains b and K_i contains r. Notice that

$$(b,r) + E_i = \{(a + b + \delta(p-r,r), p) : a \in F_i,\, p \in K_i\},$$

with $\delta(p-r,r)_n = 0$ for all $n > s(i)$. Set $\varLambda_i = \{\delta(p-r,r) : p \in K_i\}$. Then the sets $K_i(\lambda) = \{p \in K_i : \delta(p-r,r) = \lambda\}$, $\lambda \in \varLambda_i$, partition K_i and

$$E_i \triangle (b,r) + E_i = \bigcup_{\lambda \in \varLambda_i} (F_i \triangle b + \lambda + F_i) \times K_i(\lambda).$$

Therefore,

$$|E_i \triangle (b,r) + E_i| = \sum_{\lambda \in \varLambda_i} |K_i(\lambda)|\, |F_i \triangle b + \lambda + F_i|. \tag{9}$$

Furthermore,

$$\begin{aligned}|F_i \triangle b + \lambda + F_i| &\ll (2i+1)^{s(i)} - (2i)^{s(i)-\ell(b)}(2i - \|b\|)^{\ell(b)} \ll \sum_{j=0}^{s(i)-1} \binom{s(i)}{j} (2i)^j \\ &\le (2i)^{s(i)-1} \sum_{j=0}^{s(i)} \binom{s(i)}{j} = (2i)^{s(i)-1} 2^{s(i)}.\end{aligned}$$

Substituting into (9) gives

$$|E_i \triangle (b,r) + E_i| \ll (2i)^{s(i)-1} 2^{s(i)} \sum_{\lambda \in \varLambda_i} |K_i(\lambda)| \le (2i)^{s(i)-1} i^{1/2} |K_i|.$$

So, $|E_i \triangle (b,r) + E_i| / |E_i| \ll i^{-1/2}$. ∎

Finally, note that if G is free and $G' \le G$ has finite index, then

$$h(\alpha^{G'}) = |G/G'| h(\alpha^G). \tag{10}$$

To see this, first establish a bijection $G \to J \times K$, where $J = G'$ and $K = G/G'$. As in Lemma 3.1, it follows that $E_i = F_i \times K$ corresponds to a

Følner sequence in G. For any finite measurable partition ξ of X,

$$\frac{1}{|F_i|} H\Big(\bigvee_{b\in F_i} \alpha^{(b,0)} \bigvee_{r\in K} \alpha^{(0,r)}\xi\Big) = \frac{|K|}{|E_i|} H(\xi^{E_i}).$$

Hence,

$$h\Big(\alpha^{G'}, \bigvee_{r\in K} \alpha^{(0,r)}\xi\Big) = |K| h(\alpha^G, \xi).$$

From the same argument as for a single transformation, the result follows.

4. Entropy and the greatest common divisor. In this section, the proof of Theorem 1.1 is given. We begin by expressing Lind, Schmidt and Ward's formula in a form more suitable for our needs. The following result is a more straightforward version of [6, Lem. 4.5] and we supply an alternative proof.

LEMMA 4.1. *Suppose $J = \mathbb{Z}^d$, $d < \infty$, and let $\mathfrak{b} \subset \mathbb{Z}J$ be a non-zero ideal. Then*

$$h(\alpha_{\mathbb{Z}J/\mathfrak{b}}) = \mathsf{m}(\gcd(\mathfrak{b})).$$

Proof. Let $R = \mathbb{Z}J$ and note that this ring is a Noetherian unique factorization domain. Let $g = \gcd(\mathfrak{b})$. Since $\mathfrak{b} \subset (g)$, there is a surjective R-module homomorphism $R/\mathfrak{b} \to R/(g)$ and a corresponding dual inclusion $X_{R/(g)} \to X_{R/\mathfrak{b}}$ yielding

$$h(\alpha_{R/\mathfrak{b}}) \ge h(\alpha_{R/(g)}) = \mathsf{m}(g),$$

by [16, Th. 4.2].

If $g \ne 1$, we may write $g = g_1^{e_1} \cdots g_m^{e_m}$, where each g_i is prime and $e_i \ge 1$ for all $1 \le i \le m$. If $g = 1$, set $m = 0$. There is a list $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of associated primes of $R/\mathfrak{b}$ such that $\mathfrak{b}$ has an irredundant primary decomposition $\mathfrak{b} = \bigcap_{i=1}^n \mathfrak{b}_i$, where each $\mathfrak{b}_i$ is a $\mathfrak{p}_i$-primary ideal, $\mathfrak{p}_i = (g_i)$ for all $1 \le i \le m$, and the remaining primes are non-principal. For each $1 \le i \le m$,

$$\mathfrak{b}_i = \mathfrak{b}R_{\mathfrak{p}_i} \cap R = \{f \in R : sf \in \mathfrak{b} \text{ for some } s \in R \setminus \mathfrak{p}_i\}.$$

By the definition of the greatest common divisor, there exists $s \in R \setminus \mathfrak{p}_i$ such that $sg_i^{e_i} \in \mathfrak{b}_i$. Therefore, $g_i^{e_i} \in \mathfrak{b}_i$ and there is a surjective R-module homomorphism $R/(g_i^{e_i}) \to R/\mathfrak{b}_i$. Hence,

$$h(\alpha_{R/\mathfrak{b}_i}) \le h(\alpha_{R/(g_i^{e_i})}). \tag{11}$$

The R-module homomorphism $R \to N = \bigoplus_{i=1}^n R/\mathfrak{b}_i$ given by sending f to $(f + \mathfrak{b}_1) \oplus \cdots \oplus (f + \mathfrak{b}_n)$ has kernel $\mathfrak{b}$, so there is a dual projection $X_N \to X_{R/\mathfrak{b}}$. For $m < i \le n$, each of the modules $R/\mathfrak{b}_i$ has no principal associated primes, and therefore $h(\alpha_{R/\mathfrak{b}_i}) = 0$, by [16, Lem. 4.3 and Th. 4.2].

Thus, by (11),

$$h(\alpha_{R/\mathfrak{b}}) \le h(\alpha_N) = \sum_{1\le i\le n} h(\alpha_{R/\mathfrak{b}_i}) \le \sum_{1\le i\le m} \mathsf{m}(g_i^{e_i}) = \mathsf{m}(g). \ \blacksquare$$

Proof of Theorem 1.1. First note that if G is finitely generated, the result follows from Lemma 4.1, so let G be infinitely generated. Assume that either case (1) or case (2) of Section 3 holds. According to the hypotheses of the theorem, the only remaining possibility is that G is free and the ideal $\mathfrak{a}$ is infinitely generated; we return to this in a moment.

Let $M = \mathbb{Z}G/\mathfrak{a}$, $\alpha = \alpha_M$ and, as in Section 3, let X_M be regarded as the subgroup of $\mathbb{T}^{J\times K}$ defined by the relations (6). Let π_J be given by (8), set $W = \pi_J(X_M)$ and let β_W be the restriction of the J-shift to W. Write H_X and H_W for the entropy functions with respect to μ_X and μ_W.

If G is not free, let (E_i) be the Følner sequence defined in Lemma 3.1. If G is free, let (F_i) and (K_i) be Følner sequences for J and K respectively, defined by Lemma 3.1 for groups of the form $\mathbb{Z}^d$, $d \le \infty$. In this case, set $E_i = F_i \times K_i$, so that (E_i) corresponds to a Følner sequence in G.

Let η be any finite measurable partition of W. Even in the non-free case, for a set of the form $A = (A_q) \in W^K$ with $A_q = W$ for all $q \ne 0$, we have $\alpha^{(0,r)}A = (B_q)$, where $B_q = A_0$ for $q = r$ and $B_q = W$ for $q \ne r$. Hence,

$$H_X\Big(\bigvee_{r\in K_i} \alpha^{(0,r)}\pi_J^{-1}\eta\Big) = |K_i| H_W(\eta),$$

so

$$\frac{1}{|E_i|} H_X((\pi_J^{-1}\eta)^{E_i}) = \frac{1}{|E_i|} H_X\Big(\bigvee_{r\in K_i} \alpha^{(0,r)}\pi_J^{-1}\eta^{F_i}\Big) = \frac{1}{|F_i|} H_W(\eta^{F_i}).$$

Therefore, $h(\alpha, \pi_J^{-1}\eta) = h(\beta_W, \eta)$. This means that for all $i \ge 1$,

$$h(\alpha, \xi_i(\eta)) = h(\beta_W, \eta),$$

where $\xi_i(\eta) = \bigvee_{r\in K_i} \alpha^{(0,r)}\pi_J^{-1}\eta$. Thus,

$$\sup_{\eta,i}\{h(\alpha, \xi_i(\eta))\} = h(\beta_W),$$

where η runs over all finite measurable partitions of W and i runs over all natural numbers. Since the collection of partitions of the form $\xi_i(\eta)$ is dense in the space of all finite measurable partitions of X_M (under the Rokhlin metric), it follows that $h(\alpha) = h(\beta_W) = h(\alpha^J_{\mathbb{Z}J/\mathfrak{a}\cap\mathbb{Z}J})$.

When G is free and $\mathfrak{a}$ is finitely generated, (1) follows from Lemma 4.1. If G is free and $\mathfrak{a}$ is not finitely generated, proceed as follows. Let $g = \gcd(\mathfrak{a})$, so $\mathfrak{a} \subset (g)$. Since there is a finite set of elements in $\mathfrak{a}$ which determines g, there is a finitely generated ideal $\mathfrak{b} \subset \mathfrak{a}$ with $\gcd(\mathfrak{b}) = g$. Therefore, there is a chain of surjective $\mathbb{Z}G$-module homomorphisms

$$\frac{\mathbb{Z}G}{\mathfrak{b}} \to \frac{\mathbb{Z}G}{\mathfrak{a}} \to \frac{\mathbb{Z}G}{(g)}.$$

Using the result established for finitely generated ideals, we deduce that

$$\mathsf{m}(g) = h(\alpha_{\mathbb{Z}G/(g)}) \le h(\alpha_{\mathbb{Z}G/\mathfrak{a}}) \le h(\alpha_{\mathbb{Z}G/\mathfrak{b}}) = \mathsf{m}(g).$$

Finally, note that in the case where G is not free, it has been shown that

$$h(\alpha) = h(\alpha^J_{\mathbb{Z}J/\mathfrak{a}\cap\mathbb{Z}J}) = \mathsf{m}(\gcd(\mathfrak{a}\cap\mathbb{Z}J)).$$

When choosing J in Section 3, if $J \neq G'$, we can assume $J \cong G' \oplus G''$ where G'' is also free. Since the generators of $\mathfrak{a}$ lie in $\mathbb{Z}G'$, the result follows. ∎

5. Entropy for general modules. In this section, the calculation of entropy for actions arising from more general modules is pursued. Yuzvinskiĭ's entropy addition formula [25] is useful here. This was generalized to actions of $\mathbb{Z}^d$, $d < \infty$, by Lind, Schmidt and Ward in [16]. Schmidt's proof in [22] is more complete and can be adapted for actions of $\mathbb{Z}^\infty$. The necessary modifications are slightly technical and are consigned to the Appendix.

PROPOSITION 5.1 (Yuzvinskiĭ's formula for actions of $G = \mathbb{Z}^\infty$). *Let α be an algebraic $\mathbb{Z}^\infty$-action on a compact abelian group X. Then for any closed α-invariant subgroup Y of X,*

$$h(\alpha_X) = h(\alpha_Y) + h(\alpha_{X/Y}). \tag{12}$$

When $G = \mathbb{Z}^d$, $d < \infty$, the ring $\mathbb{Z}G$ is Noetherian and the technique of prime filtration, using associated primes, is very useful for decomposing algebraic G-actions [22]. For non-Noetherian rings there are several different notions of "associated prime" with a variety of uses [13], but there is no method of prime filtration in general. Nonetheless, Theorem 1.1 facilitates an algorithm similar to prime filtration, using principal primes, that can be used to prove Theorem 1.2.

Let $G = \mathbb{Z}^d$, $d \le \infty$, and recall that $\mathbb{Z}G$ is a unique factorization domain. For any $\mathbb{Z}G$-module M, let $\mathrm{Ass}^1(M)$ be the set of non-zero principal prime ideals in $\mathbb{Z}G$ that are annihilators of elements of M. Write $\mathrm{Ass}^1_{\mathbb{Z}G}(M)$ if emphasis of the underlying ring is required. Note that possibly $\mathrm{Ass}^1(M) = \emptyset$. Suppose $\mathfrak{a} \subset \mathbb{Z}G$ is a non-zero ideal. Then $\mathrm{Ass}^1(\mathbb{Z}G/\mathfrak{a})$ is the set of principal primes that contain $\mathfrak{a}$. Hence, $\mathrm{Ass}^1(\mathbb{Z}G/\mathfrak{a}) = \emptyset$ if and only if $\gcd(\mathfrak{a}) = 1$. If $L \subset M$ are $\mathbb{Z}G$-modules then

$$\mathrm{Ass}^1(M) = \mathrm{Ass}^1(L) \cup \mathrm{Ass}^1(M/L). \tag{13}$$

Proof of Theorem 1.2. Suppose that M is a torsion module. Since M is finitely generated, $\mathrm{Ass}^1(M)$ is finite. If $\mathrm{Ass}^1(M) = \emptyset$ then for all $x \in M$, $\mathrm{Ass}^1(\mathbb{Z}G/\mathrm{ann}(x)) = \emptyset$, so $h(\alpha_{\mathbb{Z}G/\mathrm{ann}(x)}) = 0$ for all $x \in M$. Since there is a surjective module homomorphism $\bigoplus_{x\in M}\mathbb{Z}G/\mathrm{ann}(x)\to M$, this means $h(\alpha_M)=0$ and (2) follows.

Now assume $\mathrm{Ass}^1(M)\neq\emptyset$ and proceed using the following algorithm. Set $M_0=\{0\}$. There exists $x\in M$ such that $\mathrm{ann}(x)=\mathfrak{p}_1$ for some $\mathfrak{p}_1\in\mathrm{Ass}^1(M)$. Let $M_1=\mathbb{Z}Gx$, consider M/M_1 and note that $\mathrm{Ass}^1(M/M_1)\subset\mathrm{Ass}^1(M)$. If $\mathrm{Ass}^1(M/M_1)=\emptyset$, terminate the algorithm. Otherwise, there exists $y\in M\setminus M_1$ such that $\mathrm{ann}(y+M_1)=\mathfrak{p}_2$ for some $\mathfrak{p}_2\in\mathrm{Ass}^1(M)$. Set $M_2=\mathbb{Z}Gy+M_1$. Note that $M_i/M_{i-1}\cong\mathbb{Z}G/\mathfrak{p}_i$, $i=1,2$. Continuing in this way, we obtain a chain of submodules

$$M_0\subset M_1\subset\cdots\tag{14}$$

of M, possibly terminating at M_n say, with $\mathrm{Ass}^1(M/M_n)=\emptyset$. If (14) does terminate in this way, a simple induction using (13) shows that every prime in $\mathrm{Ass}^1(M)$ must appear at least once in $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$. If the chain does not terminate, since $\mathrm{Ass}^1(M)$ is finite, at least one prime in $\mathrm{Ass}^1(M)$ must appear infinitely often in the sequence $(\mathfrak{p}_i)$.

We may localize (14) at any prime $\mathfrak{q}\in\mathrm{Ass}^1(M)$ to obtain a new module chain

$$(M_0)_{\mathfrak{q}}\subset(M_1)_{\mathfrak{q}}\subset\cdots$$

and

$$(M_i)_{\mathfrak{q}}/(M_{i-1})_{\mathfrak{q}}\cong(M_i/M_{i-1})_{\mathfrak{q}}\cong(\mathbb{Z}G/\mathfrak{p}_i)_{\mathfrak{q}}=\begin{cases}\mathbb{K}(\mathfrak{p}_i) & \text{if } \mathfrak{p}_i=\mathfrak{q},\\ \{0\} & \text{if } \mathfrak{p}_i\neq\mathfrak{q}.\end{cases}$$

If (14) terminates with $M_n\neq M$, note also that $(M/M_n)_{\mathfrak{q}}=\{0\}$ as M/M_n is a torsion module and $\mathrm{Ass}^1(M/M_n)=\emptyset$. Thus, each $\mathfrak{q}\in\mathrm{Ass}^1(M)$ appears with a well-defined multiplicity, namely $\dim_{\mathbb{K}(\mathfrak{q})}M_{\mathfrak{q}}$. Formula (2) follows by induction and Proposition 5.1.

Finally, if M is not a torsion module, then there is a submodule $L\subset M$ such that $L\cong\mathbb{Z}G$ and $h(\alpha_M)\geq h(\alpha_L)=\infty$. ∎

REMARK 5.2. Just as for $\mathbb{Z}^d$-actions, $d<\infty$, Theorem 1.2 and (3) combine to give a method for calculating the entropy of any $\mathbb{Z}^\infty$-action.

The following result is motivated by the subsequent conjecture; evidence for the latter is discussed at the end of the section.

THEOREM 5.3. *Let G be a countable torsion-free abelian group and let $\mathfrak{a}\subset\mathbb{Z}G$ be a non-zero ideal. Then there exists a chain $G_0\leq G_1\leq\cdots$ of free subgroups of G with $\bigcup_{n\geq0}G_n=G$ and G_n/G_0 finite such that*

$$h(\alpha_{\mathbb{Z}G/\mathfrak{a}})\leq\lim_{n\to\infty}\sum_{\mathfrak{p}\in P}\frac{\mathsf{m}(\mathfrak{p})\dim_{\mathbb{K}(\mathfrak{p})}(M_n)_{\mathfrak{p}}}{|G_n/G_0|},\tag{15}$$

where $M_n=\mathbb{Z}G_n/\mathfrak{a}\cap\mathbb{Z}G_n$ and $P=\mathrm{Ass}^1_{\mathbb{Z}G_0}(M_0)$.

Conjecture 5.4. *Equality holds in* (15).

Corollary 5.5. *If* $P = \emptyset$, *then* $h(\alpha_{\mathbb{Z}G/\mathfrak{a}}) = 0$.

A complete algebraic characterization of the property of zero entropy seems difficult, even if equality holds in (15). For further consideration of this issue, see Proposition 6.10 and Section 7.

Proof of Theorem 5.3. Let $G_0 \le G$ be such that G_0 is free and G/G_0 is a torsion group. Since $\mathbb{Z}G_0 \subset \mathbb{Z}G$ is an integral extension and $\mathfrak{a} \neq 0$, $\mathfrak{a} \cap \mathbb{Z}G_0 \neq 0$. Let $G_0 \le G_1 \le \cdots \le G$ be a sequence of subgroups such that G_n/G_0 is finite for all $n \ge 1$ (therefore, each G_n is free). Let $N = \mathbb{Z}G/\mathfrak{a}$ and for each $n \ge 0$, set $\mathfrak{a}_n = \mathfrak{a} \cap \mathbb{Z}G_n$, $M_n = \mathbb{Z}G_n/\mathfrak{a}_n$ and $N_n = \mathbb{Z}G/\mathfrak{a}_n\mathbb{Z}G$. There is a chain of surjective $\mathbb{Z}G$-module homomorphisms

$$N_0 \to N_1 \to N_2 \to \cdots \to N$$

and hence a corresponding dual chain of embeddings with the arrows reversed. It follows that $(h(\alpha^G_{N_n}))$ is decreasing and

$$h(\alpha^G_N) \le \lim_{n\to\infty} h(\alpha^G_{N_n}). \tag{16}$$

Let $n \ge 1$. The proof of Theorem 1.1 shows that

$$h(\alpha^G_{N_n}) = h(\alpha^{G_n}_{M_n}). \tag{17}$$

On the other hand, since G_0 has finite index in G_n, by (10),

$$h(\alpha^{G_0}_{M_n}) = |G_n/G_0|\, h(\alpha^{G_n}_{M_n}). \tag{18}$$

Combining (17) and (18) gives

$$h(\alpha^G_{N_n}) = \frac{1}{|G_n/G_0|}\, h(\alpha^{G_0}_{M_n}). \tag{19}$$

Let $P = \mathrm{Ass}^1_{\mathbb{Z}G_0}(M_0)$, so P is finite. If $\mathfrak{p} \in \mathrm{Ass}^1_{\mathbb{Z}G_0}(M_n)$, then $\mathfrak{p}$ is a principal prime containing $\mathfrak{a}_0$, so $\mathfrak{p} \in P$. On the other hand, since there is an injective $\mathbb{Z}G_0$-module homomorphism $M_0 \to M_n$, $P \subset \mathrm{Ass}^1_{\mathbb{Z}G_0}(M_n)$. Hence, $\mathrm{Ass}^1_{\mathbb{Z}G_0}(M_n) = P$ and it follows from Theorem 1.2 that

$$h(\alpha^{G_0}_{M_n}) = \sum_{\mathfrak{p}\in P} \mathsf{m}(\mathfrak{p}) \dim_{\mathbb{K}(\mathfrak{p})} (M_n)_{\mathfrak{p}}\,.$$

Combined with (19) and (16), this gives (15). ∎

From the proof of Theorem 5.3, it can be seen that equality holds in (15) if, given an algebraic G-action α on a compact abelian group X, a subgroup $Y \subset X$ and a decreasing sequence $Y_1 \supset Y_2 \supset \cdots$ of closed α-invariant subgroups of X whose intersection is Y, we have

$$h(\alpha_Y) = \lim_{n\to\infty} h(\alpha_{Y_n}).$$

This is true if $h(\alpha_{Y_1}) < \infty$ and Yuzvinskiĭ's formula (12) holds for the G-action in question, for (12) implies

$$h(\alpha_Y) + h(\alpha_{Y_1/Y}) = h(\alpha_{Y_1}) = h(\alpha_{Y_n}) + h(\alpha_{Y_1/Y_n}),$$

and (3) shows $h(\alpha_{Y_1/Y_n}) \to h(\alpha_{Y_1/Y})$ as $n \to \infty$. Unfortunately, all proofs of Yuzvinskiĭ's formula [22], [23], [25] use a scaling argument that requires the presence of finite index subgroups in the acting group G. In general, G may have a shortage of these or no finite index subgroups whatsoever.

6. Subactions and entropy rank. In this section, the entropy of subactions of infinitely generated algebraic actions is considered. This is partly motivated by the potential for obtaining additional information concerning the structure of zero entropy actions. The following example recalls some infinitely generated actions that have finitely generated subactions with finite non-zero entropy.

EXAMPLE 6.1 (Berend's group flows [1]). Berend refers to the following algebraic actions as *group flows*. Let G be a countable torsion-free abelian group and X a finite-dimensional compact connected abelian group, or more briefly, a *solenoid*. If α is a mixing algebraic G-action on X, then for each $g \in G$, we have $0 < h(\alpha^g) < \infty$. One of the simplest examples of an infinitely generated action of this kind is given by taking an infinite set $P = \{p_1, p_2, \dots\}$ of distinct rational primes and considering the collection of automorphisms $\{x \mapsto px\}_{p \in P}$ on the one-dimensional solenoid $X = \widehat{\mathbb{Q}}$. This generates an algebraic action of $G = \mathbb{Z}^\infty$. The corresponding $\mathbb{Z}G$-module is obtained by identifying $\mathbb{Z}G$ with the ring of Laurent polynomials $\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}, \dots]$ and defining a ring homomorphism $\mathbb{Z}G \to \mathbb{Q}$ via the evaluation map $f \mapsto f(p_1, p_2, \dots)$.

We now turn to the proof of Theorem 1.3. This requires the following result, which is sometimes considered to be a generalization of Dirichlet's unit theorem.

THEOREM 6.2 (Samuel [21]). *If A is a finitely generated reduced $\mathbb{Z}$-algebra then the group of units $A^\times$ is finitely generated.*

It is also helpful to have an algebraic characterization of mixing. This is provided by [22, Th. 1.6].

LEMMA 6.3. *Let G be a countable torsion-free abelian group and α_M an algebraic G-action. Then α_M is mixing if and only if given any non-zero $x \in M$ we have $(g-1)x \neq 0$ for all non-zero $g \in G$.*

The following results are also required.

LEMMA 6.4. *Let $G = \mathbb{Z}^d$, $d < \infty$, and let A be a $\mathbb{Z}G$-algebra which is a domain. Then $h(\alpha_F) = h(\alpha_A)$, where F is the field of fractions of A.*

Proof. For the non-trivial case, suppose A is a torsion $\mathbb{Z}G$-module. There is a unique non-zero prime ideal $\mathfrak{p} \subset \mathbb{Z}G$ such that $\mathrm{ann}(x) = \mathfrak{p}$ for all non-zero $x \in A$. Furthermore, $\mathfrak{p} \subsetneq \mathrm{ann}(y)$ for all non-zero $y \in K/A$. Therefore, $\mathrm{Ass}^1(K/A) = \emptyset$. The result now follows from Proposition 5.1 and Theorem 1.2. ∎

LEMMA 6.5. *Let $J = \mathbb{Z}^d$, $d < \infty$, and let $I \leq J$ be a non-trivial subgroup. If K is a Noetherian $\mathbb{Z}J$-module and $h(\alpha_K^I) > 0$, then $h(\alpha_{\mathbb{Z}J/\mathfrak{p}}^I) > 0$ for some $\mathfrak{p} \in \mathrm{Ass}_{\mathbb{Z}J}(K)$.*

Proof. For a contradiction, suppose $h(\alpha_{\mathbb{Z}J/\mathfrak{p}}^I) = 0$ for all associated primes $\mathfrak{p} \in \mathrm{Ass}_{\mathbb{Z}J}(K)$. Hence, if $\mathfrak{q} \supset \mathfrak{p}$ is an ideal of $\mathbb{Z}J$ then the surjective $\mathbb{Z}I$-module homomorphism $\mathbb{Z}J/\mathfrak{p} \to \mathbb{Z}J/\mathfrak{q}$ induces a dual inclusion. Therefore, $h(\alpha_{\mathbb{Z}J/\mathfrak{q}}^I) = 0$. Since K is a Noetherian $\mathbb{Z}J$-module, it is possible to take a prime filtration of K over $\mathbb{Z}J$. With the use of this filtration, an induction argument applying Proposition 5.1 now shows $h(\alpha_K^I) = 0$, giving a contradiction. ∎

Proof of Theorem 1.3. Let $M = \widehat{X}$ be the dual $\mathbb{Z}G$-module and let I be a finitely generated subgroup of G with $h(\alpha_M^I) > 0$. Our aim is to show that $h(\alpha_M^I) = \infty$.

Firstly, there is a surjective $\mathbb{Z}G$-module homomorphism $\bigoplus_{x\in M} \mathbb{Z}Gx \to M$. So, $h(\alpha_L^I) > 0$ for a cyclic submodule $L \subset M$ which may be identified with $\mathbb{Z}G/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathbb{Z}G$. Let J be a finitely generated subgroup of G containing I and having $\mathrm{rk}(J) = \mathrm{rk}(G)$. There is a surjective $\mathbb{Z}J$-module homomorphism $\bigoplus_{x\in L} \mathbb{Z}Jx \to L$, which implies $h(\alpha_K^I) > 0$ for some cyclic $\mathbb{Z}J$-submodule $K \subset L$. Using Lemma 6.5, we deduce that there exists $\mathfrak{p} \in \mathrm{Ass}_{\mathbb{Z}J}(K)$ with $h(\alpha_{\mathbb{Z}J/\mathfrak{p}}^I) > 0$. Thus,

$$h(\alpha_{\mathbb{K}(\mathfrak{p})}^I) > 0. \tag{20}$$

Because $\mathrm{rk}(J) = \mathrm{rk}(G)$, $\mathbb{Z}G$ is an integral $\mathbb{Z}J$-algebra. Furthermore, the prime ideal $\mathfrak{p}$ necessarily contains $\mathfrak{a} \cap \mathbb{Z}J$. Therefore, by [10, Prop 9.2], there is a prime ideal $\mathfrak{q} \subset \mathbb{Z}G$ containing $\mathfrak{a}$ such that

$$\mathfrak{q} \cap \mathbb{Z}J = \mathfrak{p}. \tag{21}$$

Since $\mathfrak{a} \subset \mathfrak{q}$, there is a surjective $\mathbb{Z}I$-module homomorphism $L \to \mathbb{Z}G/\mathfrak{q}$. It follows that

$$h(\alpha_M^I) \geq h(\alpha_L^I) \geq h(\alpha_{\mathbb{Z}G/\mathfrak{q}}^I). \tag{22}$$

Considering $\mathbb{Z}G/\mathfrak{q}$ as a $\mathbb{Z}J$-algebra, by (21), we see that there is an induced inclusion of fields

$$\mathbb{K}(\mathfrak{p}) \to \mathbb{K}(\mathfrak{q}), \tag{23}$$

and $\mathbb{K}(\mathfrak{q})$ is a $\mathbb{K}(\mathfrak{p})$-vector space. So,

$$\begin{aligned}
h(\alpha^I_{\mathbb{Z}G/\mathfrak{q}}) &= h(\alpha^I_{\mathbb{K}(\mathfrak{q})}) \qquad (24)\\
&= h(\alpha^I_{\mathbb{K}(\mathfrak{p})}) \dim_{\mathbb{K}(\mathfrak{p})}(\mathbb{K}(\mathfrak{q}))\\
&= h(\alpha^I_{\mathbb{Z}J/\mathfrak{p}}) \dim_{\mathbb{K}(\mathfrak{p})}(\mathbb{K}(\mathfrak{q})), \qquad (25)
\end{aligned}$$

where the equalities in (24) and (25) are obtained using Lemma 6.4.

We claim that $(\mathbb{Z}G/\mathfrak{q})^\times$ is infinitely generated. Suppose not. Then the natural map $\mathbb{Z}G \to \mathbb{Z}G/\mathfrak{q}$ is not injective on G. That is, there is a non-trivial element $g \in G$ such that $g-1 \in \mathfrak{q}$. Since $\mathrm{rk}(J) = \mathrm{rk}(G)$, it follows that there is a non-trivial $g' \in J$ such that $g'-1 \in \mathfrak{q}$. So, $g'-1 \in \mathfrak{p}$ by (21). Since $\mathfrak{p} \in \mathrm{Ass}_{\mathbb{Z}J}(K)$, there exists a non-zero $x \in L$ such that $(g'-1)x = 0$. By Lemma 6.3, this contradicts the mixing hypothesis. Therefore, $(\mathbb{Z}G/\mathfrak{q})^\times$ is infinitely generated.

Now, suppose (23) is a finite extension. Then [10, Th. 4.14] implies that the integral closure of $\mathbb{Z}J/\mathfrak{p}$ in $\mathbb{K}(\mathfrak{q})$ is a finitely generated $\mathbb{Z}J/\mathfrak{p}$-module and hence so is $\mathbb{Z}G/\mathfrak{q}$, as the integral closure contains $\mathbb{Z}G/\mathfrak{q}$. It follows that $\mathbb{Z}G/\mathfrak{q}$ is a finitely generated $\mathbb{Z}$-algebra, which by Theorem 6.2 means $\mathbb{Z}G/\mathfrak{q}$ has a finitely generated unit group. This contradiction shows that (23) is not finite.

Thus, (25) and (20) give $h(\alpha^I_{\mathbb{Z}G/\mathfrak{q}}) = \infty$ and hence $h(\alpha^I_M) = \infty$ by (22). ∎

The following example illustrates the need for the mixing hypothesis in Theorem 1.3.

Example 6.6. Let $G = \mathbb{Z} \times \mathbb{Q}$, let X be the shift space $\{0,1\}^{\mathbb{Z}}$ and denote the $\mathbb{Z}$-shift on X by β. For each $(a,q) \in G$, define an automorphism $\alpha^{(a,q)}$ of X by

$$\alpha^{(a,q)}(x) = \beta^a(x).$$

This produces an algebraic G-action which is not mixing as $\alpha^{(0,1)}$ is the identity. Moreover, $0 < h(\alpha^{(1,0)}) = \log 2 < \infty$.

If α is not an algebraic action, Theorem 1.3 also fails to hold.

Example 6.7. Suppose $\{\alpha^t : X \to X\}_{t\in\mathbb{R}}$ is a geodesic flow on a compact Riemannian manifold X of negative curvature (see [3, Ch. 7, Sec. 4.1]). Then the flow is mixing and $0 < h(\alpha^1) < \infty$. The maps α^t, $t \in \mathbb{Q}$, generate an action of $\mathbb{Q}$ and, by Abramov's formula [3, Ch. 3, Th. 2.1], for each non-zero $t \in \mathbb{Q}$,

$$0 < h(\alpha^t) < \infty.$$

For algebraic $\mathbb{Q}$-actions, it is tempting to try to use Abramov's formula to obtain Theorem 1.3. However, the required conclusion would only follow if the entropy of every element of the action was bounded below by some uniform positive constant. The existence of such a constant would give a solution to Lehmer's problem [12], which asks if zero is an isolated point in the range of Mahler measure; this problem has been open for over seventy years.

We now consider the concept of entropy rank, which generalizes easily from actions of $\mathbb{Z}^d$, $d < \infty$, to actions of more general torsion-free abelian groups G, as follows.

DEFINITION 6.8. An algebraic G-action α has *entropy rank* $k \leq \mathrm{rk}(G)$ if $h(\alpha^J) < \infty$ for all rank k subgroups J of G and k is minimal with this property. The entropy rank of α is denoted $\mathrm{entrk}(\alpha)$.

Example 6.1 describes some entropy rank one actions (see also [7]). The definition of entropy rank applies only to actions for which, given any $J \leq G$ with $\mathrm{rk}(J) = \mathrm{rk}(G)$, we have $h(\alpha_M^J) < \infty$. In some circumstances, this condition can be difficult to resolve; for example, see [7, Sec. 4] for a connection with Lehmer's problem.

The next result shows how entropy rank is closely related to algebraic dependence. In [7] and [8], the same idea is illustrated using Krull dimension. However, this perspective does not extend directly to our setting; the obvious extension of [8, Prop. 7.3] is violated by Example 6.1, with P as the set of all rational primes.

Let G be a countable torsion-free abelian group and $\mathfrak{a} \subset \mathbb{Z}G$ an ideal. A prime $\mathfrak{p} \subset \mathbb{Z}G$ is a *minimal prime* of $\mathfrak{a}$ if, given any prime $\mathfrak{q}$ with $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$, we have $\mathfrak{q} = \mathfrak{p}$. The set of all such $\mathfrak{p}$ is denoted $\mathrm{Min}(\mathfrak{a})$. Let $\mathfrak{p} \subset \mathbb{Z}G$ be a prime ideal and set

$$\tau(\mathfrak{p}) = \begin{cases} 1 + \mathrm{tr.deg}(\mathbb{K}(\mathfrak{p})|\mathbb{Q}) & \text{if } \mathrm{char}(\mathbb{K}(\mathfrak{p})) = 0, \\ \mathrm{tr.deg}(\mathbb{K}(\mathfrak{p})|\mathbb{F}_p) & \text{if } \mathrm{char}(\mathbb{K}(\mathfrak{p})) = p > 0. \end{cases}$$

THEOREM 6.9. *Let G be a countable torsion-free abelian group and α_M an algebraic G-action. Set*

$$s = \sup\{\tau(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(\mathrm{ann}(x)),\ x \in M\}.$$

If $s < \mathrm{rk}(G)$ then the entropy rank of α_M is defined. Moreover, whenever it is defined, $\mathrm{entrk}(\alpha_M) \in \{s, s+1\}$.

Proof. First, we show that if $s < \mathrm{rk}(G)$, then all subactions of rank at least $s+1$ have finite entropy. Let $J \leq G$ have $\mathrm{rk}(J) \geq s+1$ and for any ideal $\mathfrak{b} \subset \mathbb{Z}G$, let $\mathfrak{b}' = \mathfrak{b} \cap \mathbb{Z}J$. Our aim is to show that $h(\alpha_M^J) = 0$. Note that

$$h(\alpha^J_{\mathbb{Z}G/\mathrm{ann}(x)}) = 0 \text{ for all } x \in X \ \Rightarrow\ h(\alpha_M^J) = 0.$$

Let $x \in X$ and $\mathfrak{a} = \mathrm{ann}(x)$. Then

$$h(\alpha^J_{\mathbb{Z}J/\mathfrak{a}'}) = 0 \ \Rightarrow\ h(\alpha^J_{\mathbb{Z}G/\mathfrak{a}}) = 0.$$

Therefore, it suffices to calculate $h(\alpha^J_{\mathbb{Z}J/\mathfrak{a}'})$. Let $I \leq J$ be a free subgroup such that $\mathrm{rk}(I) = \mathrm{rk}(J)$, and for any ideal $\mathfrak{b} \subset \mathbb{Z}G$, set $\mathfrak{b}'' = \mathfrak{b} \cap \mathbb{Z}I$. Theorems 1.2 and 5.3 show that

$$h(\alpha^J_{\mathbb{Z}J/\mathfrak{a}'}) \leq h(\alpha^I_{\mathbb{Z}I/\mathfrak{a}''}), \tag{26}$$

which reduces the problem to showing $h(\alpha^I_{\mathbb{Z}I/\mathfrak{a}''}) = 0$.

If $\mathfrak{q} \in \mathrm{Ass}^1_{\mathbb{Z}I}(\mathbb{Z}I/\mathfrak{a}'')$, then $\sqrt{\mathfrak{a}''} \subset \mathfrak{q}$ and there is a surjective $\mathbb{Z}I$-module homomorphism $\mathbb{Z}I/\sqrt{\mathfrak{a}''} \to \mathbb{Z}I/\mathfrak{q}$. So, if $h(\alpha^I_{\mathbb{Z}I/\sqrt{\mathfrak{a}''}}) = 0$, then $h(\alpha^I_{\mathbb{Z}I/\mathfrak{q}}) = 0$ for all $\mathfrak{q} \in \mathrm{Ass}^1(\mathbb{Z}I/\mathfrak{a}'')$. Therefore,

$$h(\alpha^I_{\mathbb{Z}I/\sqrt{\mathfrak{a}''}}) = 0 \ \Rightarrow\ h(\alpha^I_{\mathbb{Z}I/\mathfrak{a}''}) = 0. \tag{27}$$

Thus, in view of (26) and (27), it remains to show that $h(\alpha^I_{\mathbb{Z}I/\sqrt{\mathfrak{a}''}}) = 0$.

Write $\mathrm{Min}(\mathbb{Z}G/\mathfrak{a}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots\}$. Since $\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots$, we have

$$\sqrt{\mathfrak{a}''} = (\sqrt{\mathfrak{a}})'' = \mathfrak{p}''_1 \cap \mathfrak{p}''_2 \cap \cdots . \tag{28}$$

By (28), it follows that the $\mathbb{Z}I$-module homomorphism from $\mathbb{Z}I$ to $N = \mathbb{Z}I/\mathfrak{p}''_1 \oplus \mathbb{Z}I/\mathfrak{p}''_2 \oplus \cdots$ given by $f \mapsto (f + \mathfrak{p}''_1, f + \mathfrak{p}''_2, \ldots)$ has kernel $\sqrt{\mathfrak{a}''}$, so

$$h(\alpha^I_{\mathbb{Z}I/\sqrt{\mathfrak{a}''}}) \leq h(\alpha^I_N).$$

Hence, the required result will follow by showing $h(\alpha^I_{\mathbb{Z}I/\mathfrak{p}''}) = 0$ for all $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \ldots$. To see this, note that there is a natural embedding of domains $\mathbb{Z}I/\mathfrak{p}'' \to \mathbb{Z}G/\mathfrak{p}$, so $\tau(\mathfrak{p}'') \leq \tau(\mathfrak{p}) \leq s$. Since $I \cong \mathbb{Z}^{\mathrm{rk}(J)}$, if $\mathfrak{p}''$ were principal, it would follow that $\tau(\mathfrak{p}'') = \mathrm{rk}(J) \geq s+1$, which is a contradiction. Hence, $\mathfrak{p}''$ is non-principal and $h(\alpha^I_{\mathbb{Z}I/\mathfrak{p}''}) = 0$.

Thus, if $s < \mathrm{rk}(G)$ then the entropy rank of α_M is defined, and whenever it is defined, $\mathrm{entrk}(\alpha_M) \leq s + 1$.

To complete the proof, it is necessary to show that, assuming it is defined, $\mathrm{entrk}(\alpha_M) \geq s$. Let $x \in M$, $\mathfrak{a} = \mathrm{ann}(x)$ and $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$. If $\tau(\mathfrak{p})$ is infinite, it is straightforward to find a subaction of α_M with infinite rank and infinite entropy. Hence, assume $\tau(\mathfrak{p})$ is finite and let j be the transcendence degree of $\mathbb{K}(\mathfrak{p})$ over its prime subfield $\mathbb{F}$. So, there exist $g_1, \ldots, g_j \in G$ such that their images in $\mathbb{Z}G/\mathfrak{p}$ are algebraically independent over $\mathbb{F}$. Let $I = \langle g_1, \ldots, g_{\tau(\mathfrak{p})-1}\rangle$, $J = \langle I, g_j\rangle$ and note that $\mathrm{rk}(I) = \tau(\mathfrak{p}) - 1$.

For any ideal $\mathfrak{b} \subset \mathbb{Z}G$, let $\mathfrak{b}' = \mathfrak{b} \cap \mathbb{Z}J$. There is an injective $\mathbb{Z}I$-module homomorphism $\mathbb{Z}J/\mathfrak{a}' \to M$, and a surjective $\mathbb{Z}I$-module homomorphism $\mathbb{Z}J/\mathfrak{a}' \to \mathbb{Z}J/\mathfrak{p}'$, so

$$h(\alpha^I_{\mathbb{Z}J/\mathfrak{p}'}) \leq h(\alpha^I_{\mathbb{Z}J/\mathfrak{a}'}) \leq h(\alpha^I_M). \tag{29}$$

Due to the choice of J and the natural embedding $\mathbb{K}(\mathfrak{p}') \to \mathbb{K}(\mathfrak{p})$, we have $\mathfrak{p}' = (\mathrm{char}(\mathbb{F}))$. So, $h(\alpha^I_{\mathbb{Z}J/\mathfrak{p}'}) = \infty$. Furthermore, by (29), $h(\alpha^I_M) = \infty$. Since x and $\mathfrak{p}$ were arbitrary, we conclude that $\mathrm{entrk}(\alpha_M) \geq s$. ■

Entropy rank may or may not be defined for zero entropy actions. For an example of a zero entropy action for which entropy rank is not defined, consider the following action of $G = \mathbb{Z}^\infty$. Let $\mathfrak{p} \subset \mathbb{Z}G$ be any finitely generated non-principal prime ideal with $\mathfrak{p} \cap \mathbb{Z} = 0$. Then $h(\alpha_{\mathbb{Z}G/\mathfrak{p}}) = 0$, but there exists $J \le G$ such that $\mathrm{rk}(J) = \mathrm{rk}(G)$ and $\mathfrak{p} \cap \mathbb{Z}J = 0$. So,

$$h(\alpha^J_{\mathbb{Z}J/\mathfrak{p}\cap\mathbb{Z}J}) = \infty \;\Rightarrow\; h(\alpha^J_{\mathbb{Z}G/\mathfrak{p}}) = \infty.$$

When G has finite rank, the following proposition gives a rich supply of zero entropy actions for which entropy rank is defined. It is not clear if the proposition identifies all zero entropy actions (see Section 7). For an ideal $\mathfrak{a} \subset \mathbb{Z}G$, let $\mathrm{Min}^1(\mathfrak{a})$ be the subset of $\mathrm{Min}(\mathfrak{a})$ consisting of height one primes (that is, the primes $\mathfrak{p}$ for which $(\mathbb{Z}G)_\mathfrak{p}$ has Krull dimension one).

PROPOSITION 6.10. *Let G be a countable torsion-free abelian group and α_M an algebraic G-action. Suppose that for every non-zero $x \in M$ and every $\mathfrak{p} \in \mathrm{Min}^1(\mathrm{ann}(x))$, there exists a non-zero $g \in G$ such that $g - 1 \in \mathfrak{p}$. Then $h(\alpha^J_M) = 0$ for every $J \le G$ such that G/J is a torsion group.*

Proof. As in the proof of Theorem 6.9, this may be seen by considering only free J. Let $x \in M$, $\mathfrak{a} = \mathrm{ann}(x)$ and consider $\mathbb{Z}G/\mathfrak{a}$ as a $\mathbb{Z}J$-algebra. Since $\mathbb{Z}G$ is integral over $\mathbb{Z}J$, it follows from [10, Prop. 9.2] that

$$\mathrm{Ass}^1_{\mathbb{Z}J}(\mathbb{Z}G/\mathfrak{a}) = \{\mathfrak{p} \cap \mathbb{Z}J : \mathfrak{p} \in \mathrm{Min}^1(\mathfrak{a})\}.$$

Hence, if $\mathrm{Min}^1(\mathfrak{a}) = \emptyset$, then $h(\alpha^J_{\mathbb{Z}G/\mathfrak{a}}) = 0$. If $\mathrm{Min}^1(\mathfrak{a}) \ne \emptyset$, a similar argument to that used in the proof of Theorem 1.3 shows that for each $\mathfrak{p} \in \mathrm{Ass}^1_{\mathbb{Z}J}(\mathbb{Z}G/\mathfrak{a})$, there exists a non-zero $g' \in G$ such that $g' - 1 \in \mathfrak{p}$. So, by [22, Th. 19.5], $\mathsf{m}(\mathfrak{p}) = 0$ and again $h(\alpha^J_{\mathbb{Z}G/\mathfrak{a}}) = 0$. Thus, $h(\alpha^J_M) = 0$. ∎

EXAMPLE 6.11. To see that the range $\{s, s+1\}$ cannot be improved in Theorem 6.9, consider the following two actions of $G = \mathbb{Z} \times \mathbb{Q}$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Define group homomorphisms $\phi'_1, \phi'_2 : G \to \overline{\mathbb{Q}}^\times$ by

$$\phi'_1(a, b/n) = 2^a \quad \text{and} \quad \phi'_2(a, b/n) = 2^a r_n^b,$$

where r_n is the positive real nth root of 3. Then ϕ'_1, ϕ'_2 extend in an obvious way to ring homomorphisms $\phi_1, \phi_2 : \mathbb{Z}G \to \overline{\mathbb{Q}}$. Denote their kernels by $\mathfrak{p}_1, \mathfrak{p}_2$ respectively. Note that

$$\tau(\mathfrak{p}_1) = 1 \quad \text{and} \quad \mathrm{entrk}(\alpha_{\mathbb{Z}G/\mathfrak{p}_1}) = 1,$$

because $X_{\mathbb{Z}G/\mathfrak{p}_1}$ is a solenoid. However,

$$\tau(\mathfrak{p}_2) = 1 \quad \text{and} \quad \mathrm{entrk}(\alpha_{\mathbb{Z}G/\mathfrak{p}_2}) = 2.$$

To verify the final statement, first note that by Proposition 6.10, all rank 2 subactions have zero entropy. On the other hand, Lemma 6.3 shows $\alpha_{\mathbb{Z}G/\mathfrak{p}_2}$ is mixing, and since $h(\alpha^{(1,0)}_{\mathbb{Z}G/\mathfrak{p}_2}) > 0$, it follows that $h(\alpha^{(1,0)}_{\mathbb{Z}G/\mathfrak{p}_2}) = \infty$, by Theorem 1.3.

7. Concluding remarks and problems. 1. For a $\mathbb{Z}^\infty$-action α_M arising from a cyclic module M, Theorem 1.1 shows that $h(\alpha_M)$ is also the entropy of some algebraic $\mathbb{Z}^d$-action, $d < \infty$. Is a similar statement true for all algebraic $\mathbb{Z}^\infty$-actions?

2. It is expected that Yuzvinskiĭ's formula (12) holds for all countable residually finite (not necessarily abelian) groups. If (12) holds for all countable torsion-free abelian groups G, not only is Conjecture 5.4 true, but together with (12) and (3), this gives a method for calculating the entropy of any algebraic G-action. Ward and Zhang's [24] extension of the Abramov–Rokhlin formula provides some hope that (12) might hold in the generality required, although it is interesting to note that Elek [11] has found situations where (12) fails in the non-amenable setting.

3. Is it possible that the sequence on the right hand side of (15) converges to zero without being eventually zero? Since each term in the sequence can also be expressed as a Mahler measure, an affirmative answer would require a solution to Lehmer's problem.

4. Let M be a countable $\mathbb{Z}G$-module. In light of the situation for $\mathbb{Z}^d$-actions, $d < \infty$, is the following statement true: $h(\alpha_M) = 0$ if and only if for every non-zero $x \in M$ and every $\mathfrak{p} \in \mathrm{Min}^1(\mathrm{ann}(x))$, there exists a non-zero $g \in G$ such that $g - 1 \in \mathfrak{p}$? Note that the "if" statement is dealt with by Proposition 6.10, and when $G = \mathbb{Z}^\infty$ the "only if" statement follows from Theorem 1.2.

5. If the above statement is true, then Proposition 6.10 shows that for algebraic actions of finite rank groups, the property of zero entropy is sufficient for entropy rank to be defined (this is obvious when G is finitely generated).

6. A complete algebraic characterization of the property of zero entropy holds potential for an explicit realization of the Pinsker algebra, as in [16, Th. 6.5].

Appendix. The following proof of Proposition 5.1 uses ideas from Schmidt's proof [22, Sec. 14] for $\mathbb{Z}^d$-actions, $d < \infty$, as a basis. However, there are some developments necessary for the infinite rank case that are not necessarily obvious.

Set $Z = X/Y$ and just as in [22, Sec. 14] take a Borel cross-section $s : Z \to X$, so that $\theta \cdot s$ is the identity on Z, where $\theta : X \to Z$ is the quotient map. For each $g \in G$, define $\tau^g : Z \to Y$ by

$$\tau^g(z) = \alpha_X^g(s(z)) - s(\alpha_Z^g(z)).$$

For every closed α_Y-invariant subgroup U of Y, define a measure preserving G-action $T_{(U)}$ on $(Z \times Y/U, \mu_Z \times \mu_{Y/U})$ by

$$T_{(U)}^g(z, y + U) = (\alpha_Z^g(z), \tau^g(z) + \alpha_{Y/U}^g(y + U)).$$

Define $\psi_{(U)} : X/U \to Z \times Y/U$ by
$$\psi_{(U)}(x+U) = (\theta(x)+U, (x+U) - s(\theta(x))).$$
Then $\psi_U \cdot \alpha^g_{X/U} = T^g_{(U)}\psi_{(U)}$ for all $g \in G$, and $h(T^{G'}_{(U)}) = h(\alpha^{G'}_{X/U})$ for any non-trivial $G' \leq G$.

For any finite measurable partition ξ of Y/U and $w \in Y/U$, let $w + \eta = \{w + B : B \in \eta\}$. For any finite set $F \subset G$ and $z \in Z$, let $\xi^F(z) = \bigvee_{g\in F} \tau^g(z) + \alpha^g_{Y/U}(\xi)$.

First assume Y can be described as the dual of a cyclic module $\mathbb{Z}G/\mathfrak{a}$. For a non-trivial case, suppose $h(\alpha_Y) < \infty$, so $\mathfrak{a} \neq 0$. Proceeding as in [22, Sec. 14], with the aid of [24], we find that (12) holds if Y is zero-dimensional, and more generally, for any non-trivial $G' \leq G$, we have

$$h(\alpha^{G'}_Z \times \alpha^{G'}_{Y/U}) \leq h(\alpha^{G'}_Z) + \sup_{n\geq 1} \lim_{i\to\infty} \frac{1}{|F_i|} \int_Z H_{Y/U}(\xi^{F_i}_n | \xi^{F_i}_n(z))\, d\mu_Z(z) + \sup_{n\geq 1} \lim_{i\to\infty} \frac{1}{|F_i|} \int_Z H_{Y/U}(\xi^{F_i}_n(z))\, d\mu_Z(z) \tag{30}$$

and

$$h(T^{G'}_{(U)}) \leq h(\alpha^{G'}_Z) + \sup_{n\geq 1} \lim_{i\to\infty} \frac{1}{|F_i|} \int_Z H_{Y/U}(\xi^{F_i}_n(z) | \xi^{F_i}_n)\, d\mu_Z(z) + \sup_{n\geq 1} \lim_{i\to\infty} \frac{1}{|F_i|} \int_Z H_{Y/U}(\xi^{F_i}_n)\, d\mu_Z(z), \tag{31}$$

where $H_{Y/U}$ is the entropy function with respect to $\mu_{Y/U}$, (F_i) is a Følner sequence in G' and (ξ_n) is a sequence of finite measurable partitions of Y/U satisfying $\sigma(\bigvee_{n\geq 1} \bigvee_{g\in G'} \gamma^g \xi_n) = \mathcal{B}(Y/U)$.

Since $\mathfrak{a} \neq 0$, we may write $G = J \oplus K$, where $J \cong \mathbb{Z}^d$, $d < \infty$, $K \cong \mathbb{Z}^\infty$ and $\mathfrak{a}\cap\mathbb{Z}J \neq 0$. Set $R = \mathbb{Z}J/\mathfrak{a}\cap\mathbb{Z}J$ and note that there is a ring isomorphism between $\mathbb{Z}G/\mathfrak{a}$ and $RK/\mathfrak{b}$, where $\mathfrak{b} \subset RK$ is an ideal with $\mathfrak{b} \cap R = 0$. This means Y identifies with a closed subgroup of W^K, where $W = \widehat{R}$. Let $\beta = \alpha^J_R$ and define a G-action γ on W^K by
$$\gamma^{(b,r)}(w)_q = \beta^b w_{q+r},$$
where $w = (w_q) \in W^K$, $(b,r) \in J \times K$. Under the same identification, α_Y becomes the appropriate restriction of the G-action γ.

Let $k \geq 1$. Since $h(\beta) < \infty$, the proof of [22, Th. 14.1] shows that there exists a subgroup $I_k \leq J$ of index k^d, a closed zero-dimensional β^{I_k}-invariant subgroup V_k of W and a sequence $(\eta_{k,n})$ of finite measurable partitions of $W_k = W/V_k$ that has the following properties:
$$\sigma\Big(\bigvee_{n\geq 1} \bigvee_{b\in I_k} \beta^b \eta_{k,n}\Big) = \mathcal{B}(W_k),$$

and for each $n \geq 1$, $w \in W_k$ and $A \in \eta_{n,k}$, the number of non-empty sets in $\{A \cap B : B \in w + \eta_{n,k}\}$ does not exceed $C_k = C^{dk^{d-1}}$, where C is a positive integer constant independent of k and n.

Let Y_k be the image of Y in W_k^K. So $Y_k \cong Y/U_k$, where U_k is the zero-dimensional group $Y \cap V_k^K$. Define $\phi_k : W_k^K \to W_k$ by $\phi_k((w_q)) = w_0$. Let $\xi_{k,n} = Y_k \cap \phi_k^{-1}\eta_{k,n}$ and $G_k = I_k \oplus K$. Then

$$\sigma\Big(\bigvee_{n\geq 1}\bigvee_{g\in G_k}\gamma^g\xi_{k,n}\Big) = \mathcal{B}(Y_k).$$

Furthermore, for each $n \geq 1$, $w \in Y_k$ and $A \in \xi_{k,n}$, the number of non-empty sets in $\{A \cap B : B \in w + \xi_{n,k}\}$ does not exceed C_k. Just as in [22, Sec. 14], it follows from (30) and (31) that

$$h(\alpha_Z^{G_k} \times \alpha_{Y_k}^{G_k}) - \log C_k \leq h(T_{(U_k)}) = h(\alpha_{X/U_k}^{G_k}) \leq h(\alpha_Z^{G_k} \times \alpha_{Y_k}^{G_k}) + \log C_k.$$

Since (12) holds when Y is zero-dimensional, adding $h(\alpha_{U_k}^{G_k})$ to both sides and applying (10) gives

$$h(\alpha_Z) + h(\alpha_Y) - \frac{\log C_k}{k^d} \leq h(\alpha_X) \leq h(\alpha_Z) + h(\alpha_Y) + \frac{\log C_k}{k^d}.$$

Since this holds for all $k \geq 1$, (12) follows when Y is the dual of a cyclic module.

For more general Y, consider the dual situation. Let $M = \widehat{X}$ be the dual $\mathbb{Z}G$-module and let $L \subset M$ be any submodule. Then there exists a chain of $\mathbb{Z}G$-modules

$$L = L_0 \subset L_1 \subset L_2 \subset \cdots \subset M$$

such that $\bigcup_{n\geq 1} L_n = M$ and L_n/L_{n-1} is isomorphic to a cyclic module for all $n \geq 1$. Hence, by the version of (12) just established and by induction, $h(\alpha_{L_n}) = h(\alpha_L) + h(\alpha_{L_n/L})$ for all $n \geq 1$. Applying (3) gives $h(\alpha_{L_n}) \to h(\alpha_M)$ and $h(\alpha_{L_n/L}) \to h(\alpha_{M/L})$ as $n \to \infty$. Thus, $h(\alpha_M) = h(\alpha_{M/L}) + h(\alpha_L)$, which is precisely the dual statement of (12).

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