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The consistency of $\mathfrak{b} = \kappa$ and $\mathfrak{s} = \kappa^+$

by

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Abstract. Using finite support iteration of ccc partial orders we provide a model of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for κ an arbitrary regular, uncountable cardinal.

1. Introduction. S. Shelah obtains the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$ using countable support iteration of a proper forcing notion which adds a real not split by the ground model reals and which satisfies the almost ${}^\omega\omega$-bounding property (see [9]). This paper will show that it is possible to find ccc suborders of Shelah's original order which behave very similarly to the larger order. Being ccc, they can be iterated with finite support. Under the assumption that the covering number of the meagre ideal is κ it will be shown that for any unbounded family $\mathcal{H} \subseteq {}^\omega\omega$ of size κ, such that every subfamily of size smaller than κ is dominated by an element of $\mathcal{H}$, there is a ccc forcing notion which preserves $\mathcal{H}$ unbounded and adds a real not split by the ground model reals. Thus under a suitable finite support iteration of length κ^+ of ccc forcing notions, the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$ for arbitrary regular κ will be established (Section 6). Using a different model Joerg Brendle obtains the consistency of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \kappa$ for arbitrary regular κ (see [5, Theorem 12.16] and [4]).

2. Preliminaries. Let f and g be functions in ${}^\omega\omega$. The function f is *dominated* by the function g if there is $n \in \omega$ such that $f \leq_n g$, i.e. $(\forall i \geq n)(f(i) \leq g(i))$. Then $<^* = \bigcup_{n\in\omega} \leq_n$ is called the *bounding relation* on ${}^\omega\omega$. A family of functions $\mathcal{F}$ in ${}^\omega\omega$ is *dominated* by the function g, denoted $\mathcal{F} <^* g$, if $f <^* g$ for every $f \in \mathcal{F}$. Also, $\mathcal{F}$ is *unbounded* (equiv. *not dominated*) if and only if there is no function g which dominates it. Then

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the *bounding number* $\mathfrak{b}$ is defined as the minimal size of an unbounded family. That is, $\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^\omega\omega$ and $\mathcal{B}$ is unbounded$\}$. A family S of infinite subsets of ω is *splitting* if for every $A \in [\omega]^\omega$ there is $B \in S$ such that $A \cap B$ and $A \cap B^c$ are infinite. Then the *splitting number* $\mathfrak{s}$ is defined as the minimal size of a splitting family. That is, $\mathfrak{s} = \min\{|S| : S \subseteq [\omega]^\omega$ and S is splitting$\}$. A family $\mathcal{H} \subseteq {}^\omega\omega$ is $<^*$*-directed* if every subfamily of size less than $|\mathcal{H}|$ is dominated by an element of $\mathcal{H}$.

3. Centred families of pure conditions. The notion of logarithmic measure is due to S. Shelah. In the presentation of logarithmic measures and their basic properties (Definitions 3.1, 3.4, 3.8, Lemmas 3.3, 3.5, 3.7) we follow [1].

Definition 3.1. Let $s \subseteq \omega$ and let $h : [s]^{<\omega} \to \omega$ where $[s]^{<\omega}$ is the family of finite subsets of s. Then h is a *logarithmic measure* if for all $A \in [s]^{<\omega}$ and all A_0, A_1 such that $A = A_0 \cup A_1$, $h(A_i) \geq h(A) - 1$ for $i = 0$ or $i = 1$ unless $h(A) = 0$. Whenever s is a finite set and h a logarithmic measure on s, the pair $x = (s, h)$ is called a *finite logarithmic measure*. The value $h(s) = \|x\|$ is called the *level* of x; the underlying set of integers s is denoted $\mathrm{int}(x)$.

Definition 3.2. Whenever h is a finite logarithmic measure on x and $e \subseteq x$ is such that $h(e) > 0$, we will say that e is *h-positive*.

Lemma 3.3. *If h is a logarithmic measure and $h(A_0 \cup \cdots \cup A_{n-1}) \geq l+1$ then $h(A_j) \geq l - j$ for some j, $0 \leq j \leq n-1$.*

Definition 3.4. Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family. Then P induces a logarithmic measure h on $[\omega]^{<\omega}$ defined inductively on $|s|$ for $s \in [\omega]^{<\omega}$ as follows:

(1) $h(e) \geq 0$ for every $e \in [\omega]^{<\omega}$;
(2) $h(e) > 0$ iff $e \in P$;
(3) for $l \geq 1$, $h(e) \geq l+1$ iff $|e| > 1$ and whenever $e_0, e_1 \subseteq e$ are such that $e = e_0 \cup e_1$, then $h(e_0) \geq l$ or $h(e_1) \geq l$.

Then $h(e) = l$ if l is maximal for which $h(e) \geq l$. The elements of P are called *positive sets* and h is said to be *induced by P*.

Lemma 3.5. *If h is a logarithmic measure induced by positive sets and $h(e) \geq l$, then $h(a) \geq l$ for every a such that $e \subseteq a$.*

Example 1 (Shelah, [10]). Let $P \subseteq [\omega]^{<\omega}$ be the family of sets containing at least two points and h the logarithmic measure induced by P. Then $h(x) = \min\{i : |x| \leq 2^i\}$ for all $x \in P$. This measure is called the *standard logarithmic measure*.

REMARK 3.6. From now on we assume that all logarithmic measures have the additional property that singletons are not positive sets.

LEMMA 3.7. *Let $P \subseteq [\omega]^{<\omega}$ be an upwards closed family and let h be the logarithmic measure induced by P. If for every $n \in \omega$ and every partition of ω into n sets $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that A_j contains a positive set, then for every $k \in \omega$, every $n \in \omega$ and every partition $\omega = A_0 \cup \cdots \cup A_{n-1}$ there is $j \in n$ such that A_j contains a set of h-measure greater than or equal to k.*

DEFINITION 3.8. Let Q be the set of all pairs (u, T) where $u \in [\omega]^{<\omega}$ and $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a sequence of finite logarithmic measures such that $\max u < \min s_0$, $\max s_i < \min s_{i+1}$ for all $i \in \omega$ and $\langle h_i(s_i) : i \in \omega \rangle$ is unbounded. If $u = \emptyset$ we say that $(\emptyset, T)$ is *a pure condition* and denote it by T. The underlying set $\bigcup\{s_i : i \in \omega\}$ of integers is denoted $\operatorname{int}(T)$. We say that (u_1, T_1) is *extended by* (u_2, T_2), where $T_l = \langle (s_i^l, h_i^l) : i \in \omega \rangle$ for $l = 1, 2$, and write $(u_2, T_2) \leq (u_1, T_1)$, if the following conditions hold:

(1) u_2 is an end-extension of u_1 and $u_2 \backslash u_1 \subseteq \operatorname{int}(T_1)$;
(2) $\operatorname{int}(T_2) \subseteq \operatorname{int}(T_1)$ and there is an infinite sequence $\langle B_i : i \in \omega \rangle$ of finite subsets of ω such that $\max u_2 < \min s_j^1$ for $j = \min B_0$, $\max(B_i) < \min(B_{i+1})$ and $s_i^2 \subseteq \bigcup\{s_j^1 : j \in B_i\}$;
(3) for every subset e of s_i^2 such that $h_i^2(e) > 0$ there is $j \in B_i$ such that $h_j^1(e \cap s_j^1) > 0$.

In the case of $u_1 = u_2$, (u_2, T_2) is called a *pure extension* of (u_1, T_1).

Whenever $T = \langle t_i : i \in \omega \rangle$ is a pure condition and $k \in \omega$, let $i_T(k) = \min\{i : k < \min \operatorname{int}(t_i)\}$ and $T \backslash k = T_{i_T(k)} = \langle t_i : i \geq i_T(k) \rangle$. For $u \in [\omega]^{<\omega}$ let $(u, T) = (u, T \backslash u) = (u, T_{i_T(\max u)})$. Note that if $R \leq T$ and $k \in \operatorname{int}(R)$, then $R \backslash k \leq T \backslash k$.

DEFINITION 3.9. If $\mathcal{F}$ is a family of pure conditions, then $Q(\mathcal{F})$ is the suborder of Q consisting of all $(u, T) \in Q$ such that $(\exists R \in \mathcal{F})(R \leq T)$.

Observe that if C is a centred family of pure conditions, then any two conditions in $Q(C)$ with equal stems have a common extension in $Q(C)$ and so $Q(C)$ is σ-centred. From now on by a *centred family* we mean a centred family of pure conditions. We also assume that all centred families are closed with respect to final segments, that is, if C is a centred family and $T \in C$ then $T \backslash v \in C$ for every $v \in [\omega]^{<\omega}$.

LEMMA 3.10. *Any two conditions of $Q(C)$ are compatible as conditions in $Q(C)$ if and only if they are compatible in Q.*

LEMMA 3.11. *Let $T = \langle t_i : i \in \omega \rangle$, where $t_i = (s_i, h_i)$, be a pure condition and $\omega = A_0 \cup \cdots \cup A_{n-1}$ a finite partition. Then there is $j \in n$ such that $\langle h_i(s_i \cap A_j) : i \in \omega \rangle$ is unbounded.*

Definition 3.12. Whenever $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a pure condition and $A \subseteq \omega$, let $T{\restriction}A = \langle (s_i \cap A, h_i{\restriction}\mathcal{P}(s_i \cap A)) : i \in \omega \rangle$.

If $T = \langle (s_i, h_i) : i \in \omega \rangle$ is a pure condition, $A \subseteq \omega$ and $\langle h_i(s_i \cap A) : i \in \omega \rangle$ is bounded, then T has no pure extension R with $\mathrm{int}(R) \subseteq A$. A pure condition T compatible with every element of a family $\mathcal{F}$ of pure conditions is said to be *compatible* with $\mathcal{F}$, denoted $T \not\perp \mathcal{F}$. If C' is a centred family such that $C \subseteq Q(C')$ then C' is said to *extend* C.

Lemma 3.13. *Let C be a centred family, T a pure condition compatible with C, and $\omega = A_0 \cup \cdots \cup A_{n-1}$ a finite partition. Then there is $j \in n$ such that $T{\restriction}A_j$ is a pure condition compatible with C.*

Proof. By Lemma 3.11, $I = \{j \in n : T{\restriction}A_j$ is a pure condition$\} \neq \emptyset$. Suppose that for every $j \in I$ there is $T_j \in C_j$ such that $T{\restriction}A_j$ and T_j are incompatible. However, I is finite, C is centred and so there exists $X \in C$ such that $(\forall j \in I)(X \leq T_j)$. By hypothesis X and T have a common extension $R \in Q$. By Lemma 3.11 again, there exists $i \in n$ such that $R{\restriction}A_i$ is a pure condition. However, $R{\restriction}A_i \leq T{\restriction}A_i$ and so $i \in I$. Also, $R{\restriction}A_i \leq R \leq X \leq T_i$ and so T_i and $T{\restriction}A_i$ are compatible, which is a contradiction. ■

Definition 3.14. Let Q_{fin} be the partial order of all sequences $\bar{r} = \langle r_0, \ldots, r_n \rangle$, $n \in \omega$, of finite logarithmic measures $r_i = (s_i, h_i)$ such that for all $i \in n$, $\max(s_i) < \min(s_{i+1})$ and $h_i(s_i) < h_{i+1}(s_{i+1})$ with extension relation being end-extension. The *level* of the sequence $\bar{r} = \langle r_0, \ldots, r_n \rangle$ is the level of r_n, denoted $\|\bar{r}\|$.

Definition 3.15. The sequence $\bar{r} \in Q_{\mathrm{fin}}$ *extends* the pure condition T if there is $R \leq T$ such that $\bar{r} \subseteq R$. The finite logarithmic measure r *extends* T if $\bar{r} = \langle r \rangle$ extends T.

Definition 3.16. Let $\tau = \langle T_n : n \in \omega \rangle$ be a sequence of pure conditions such that $(\forall n)(T_{n+1} \leq T_n)$. Then $\mathbb{P}_\tau$ is the suborder of Q_{fin} of all $\bar{r}$ such that $(\forall i \in |\bar{r}|)(r_i \leq T_{j_i})$ where $j_0 = 0$ and for $i \geq 1$, $j_i = \max \mathrm{int}(r_{i-1})$.

Lemma 3.17. *Let X be a pure condition compatible with τ and $n \in \omega$. Then $D_\tau(X, n) = \{\bar{r} \in \mathbb{P}_\tau : (\exists r_j \in \bar{r})(r_j \leq X$ and $\|r_j\| \geq n)\}$ is dense.*

Proof. Let $\bar{r} \in \mathbb{P}_\tau$ and $j = \max \mathrm{int}(\bar{r})$. Since $T_j \setminus \mathrm{int}(\bar{r})$ and X are compatible, there is a finite logarithmic measure r such that $\|r\| > \max\{\|\bar{r}\|, n\}$, which is their common extension. Then $\bar{r}^\frown\langle r \rangle$ is an extension of $\bar{r}$ which belongs to $D_\tau(X, n)$. ■

Corollary 3.18. *Let C be a centred family such that $(\forall X \in C)(X \not\perp \tau)$ and let G be a $\mathbb{P}_\tau$-generic filter. Then $R = \bigcup G = \langle r_i : i \in \omega \rangle$ is a pure condition of finite logarithmic measures of strictly increasing levels. In $V[G]$ there is a centred family C' such that $|C'| = |C|$ and $C \cup \tau \subseteq Q(C')$.*

Proof. For every $X \in C$ and $n \in \omega$ the set $D_\tau(X,n)$ is dense in $\mathbb{P}_\tau$ and so $G \cap D_\tau(X,n) \neq \emptyset$. Then $I_X = \langle i : r_i \leq X\rangle$ is infinite and so $R \wedge X := \langle r_i : i \in I_X\rangle$ is a pure condition which is a common extension of R and X. Furthermore, if $X \leq Y$ then $I_X \subseteq I_Y$, which implies $R \wedge X \leq R \wedge Y$ and so the family $\{R \wedge X\}_{X \in C}$ is centred. ∎

4. Preprocessed conditions. We use the fact that all reals have simple names of the form $\dot{f} = \bigcup\{\langle\langle i, j^i_p\rangle, p\rangle : p \in \mathcal{A}_i,\ i \in \omega,\ j^i_p \in \omega\}$, where for every $i \in \omega$, $\mathcal{A}_i = \mathcal{A}_i(\dot{f})$ is a maximal antichain of conditions deciding $\dot{f}(i)$.

Definition 4.1. Let C be a centred family and let $\dot{f}$ be a $Q(C)$-name for a real. Then $\dot{f}$ is a *good name* if for every centred family C' extending C, $\dot{f}$ is a $Q(C')$-name for a real.

Remark 4.2. If $\dot{f}$ is a $Q(C)$-name for a real and there is a centred family C' extending C such that $\dot{f}$ is not a $Q(C')$-name for a real, then there is a centred family C'' extending C which has the same cardinality as C and is such that $\dot{f}$ is not a $Q(C'')$-name for a real.

Definition 4.3. Let C be a centred family, $\dot{f}$ a good $Q(C)$-name for a real, and $i,k \in \omega$. A pure condition $T \in Q(C)$ such that $k < \min \operatorname{int}(T)$ is *preprocessed* for $\dot{f}(i), k, C$ (note that Abraham [1] uses the same terminology) if for every $v \subseteq k$ the following holds. If there is a centred family C' extending C such that $|C'| = |C|$, a pure condition $R \in Q(C')$ extending T and a condition $q \in \mathcal{A}_i(\dot{f})$ such that $(v,R) \leq q$, then there is $p \in \mathcal{A}_i(\dot{f})$ such that $(v,T) \leq p$.

Remark 4.4. Let C be a centred family, $\dot{f}$ a good $Q(C)$-name for a real, $i,k \in \omega$, and $T \in Q(C)$ a pure condition preprocessed for $\dot{f}(i),k,C$. Let C' be a centred family extending C with $|C'| = |C|$, and $T' \in Q(C')$ a pure extension of T. Then T' is preprocessed for $\dot{f}(i),k,C'$.

Corollary 4.5. *Let C be a centred family, $\dot{f}$ a good $Q(C)$-name for a real, $\tau = \langle T_n : n \in \omega\rangle \subseteq Q(C)$ a decreasing sequence of pure conditions such that for all n and $i \leq n$, T_n is preprocessed for $\dot{f}(i), n, C$, and let G be a $\mathbb{P}_\tau$-generic filter and $R = \bigcup G = \langle r_i : i \in \omega\rangle$. Then in $V[G]$ there is a centred family C' with $C \cup \{R\} \subseteq Q(C')$ and $|C'| = |C|$ such that for all $n \in \omega$ and $k \in \operatorname{int}(R_n)$, $R_n \backslash k$ is preprocessed for $\dot{f}(n), k, C'$, where $R_n = R \backslash \operatorname{int}(r_{n-1})$.*

Proof. Repeat the proof of Corollary 3.18 to obtain the family C'. Let $n \in \omega$, $k \in \operatorname{int}(R_n)$ and $i_{R_n}(k) = m$. Then $k \leq j_m = \max \operatorname{int}(r_{m-1})$. By definition T_{j_m} is preprocessed for $\dot{f}(n), j_m, C$ (note $n \leq m \leq j_m$). Since $R_n \backslash k = R_m \leq T_{j_m}$, $R_n \backslash k$ is preprocessed for $\dot{f}(n), k, C'$. ∎

5. Induced logarithmic measures. For completeness we state $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ (see [7]).

DEFINITION 5.1. $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ is the statement: for every countable partial order $\mathbb{P}$ and every family $\mathcal{D}$ with $|\mathcal{D}| < \kappa$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $(\forall D \in \mathcal{D})(G \cap D \neq \emptyset)$.

Let $\mathcal{M}$ be the ideal of meagre subsets of the real line. Recall that the *covering number* of $\mathcal{M}$, $\mathrm{cov}(\mathcal{M})$, is the minimal size of a family of meagre sets which covers the real line. For every regular uncountable cardinal κ, $\mathrm{cov}(\mathcal{M}) \geq \kappa$ if and only if $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ (see [3]).

LEMMA 5.2. *Let C be a centred family, $|C| < \mathrm{cov}(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, $n \in \omega$, and $T = \langle (s_i, h_i) : i \in \omega \rangle \in Q(C)$ be such that for all $k \in \mathrm{int}(T)$, $T \backslash k$ is preprocessed for $\dot{f}(n), k, C$. Let $v \in [\omega]^{<\omega}$. Then the logarithmic measure induced by the family $\mathcal{P}_v(T, \dot{f}(n))$ consisting of all $x \in [\mathrm{int}(T)]^{<\omega}$ such that $(\exists i \in \omega)(h_i(x \cap s_i) > 0)$ and $(\exists w \subseteq x)(\exists p \in \mathcal{A}_n(\dot{f}))((v \cup w, T\backslash x) \leq p)$ takes arbitrarily high values.*

Proof. To see that the induced measure takes arbitrarily high values consider an arbitrary finite partition $\omega = A_0 \cup \cdots \cup A_{M-1}$. By Lemma 3.13 there is $j \in M$ such that $T \restriction A_j$ is a pure condition compatible with C. As $|C| < \mathrm{cov}(\mathcal{M})$, by Corollary 3.18 there is a centred family C' extending C with $|C'| = |C|$ and a pure extension $R \in Q(C')$ of $T \restriction A_j$. Then $\dot{f}$ is a $Q(C')$-name for a real and so $\mathcal{A}_n(\dot{f})$ is a maximal antichain in $Q(C')$. Therefore there is a common extension $(v \cup w, R') \in Q(C')$ of (v, R) and some $q \in \mathcal{A}_n(\dot{f})$. Let $\bar{r}$ be a finite subsequence of R such that $w \subseteq x = \mathrm{int}(\bar{r})$. We can assume that $\|\bar{r}\| > 0$. However, $R \leq T$ and so there is $i \in \omega$ such that $h_i(x \cap s_i) > 0$. Since $R' \leq T$ and $T\backslash x$ is preprocessed for $\dot{f}(n), \max x, C$, there is $p \in \mathcal{A}_n(\dot{f})$ such that $(v \cup w, T\backslash x) \leq p$. ∎

COROLLARY 5.3. *Let C be a centred family, $|C| < \mathrm{cov}(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, $m, n \in \omega$, and let $T = \langle (s_i, h_i) : i \in \omega \rangle \in Q(C)$ be such that for all $k \in \mathrm{int}(T)$, $T\backslash k$ is preprocessed for $\dot{f}(n), k, C$. Then the logarithmic measure induced by the family $\mathcal{P}_m(T, \dot{f}(n))$ of all $x \in [\mathrm{int}(T)]^{<\omega}$ such that $(\exists i \in \omega)(h_i(s_i \cap x) > 0)$ and $(\forall v \subseteq m)(\exists w \subseteq x)(\exists p \in \mathcal{A}_n(\dot{f}))$ $((v \cup w, T\backslash x) \leq p)$ takes arbitrarily high values.*

Proof. Let $v_0, \ldots, v_{L-1}$ enumerate the subsets of m and let $\omega = A_0 \cup \cdots \cup A_{M-1}$ be a finite partition. By Lemma 3.13 there is $j \in M$ such that $T \restriction A_j$ is a pure condition compatible with C. Since $|C| < \mathrm{cov}(\mathcal{M})$, by Corollary 3.18 there is a centred family C' extending C with $|C'| = |C|$ and a pure extension $R \in Q(C')$ of $T \restriction A_j$. For every $k \in \mathrm{int}(R)$, $R\backslash k \leq T\backslash k$ and so $R\backslash k$ is preprocessed for $\dot{f}(n), k, C'$. Therefore by Lemma 5.2 for every $i \in L$ there is $x_i \in \mathcal{P}_{v_i}(R, \dot{f}(n))$. It will be shown that $x = \bigcup_{i \in L} x_i \in \mathcal{P}_m(T, \dot{f}(n))$. Let $v \subseteq m$. Then $v = v_i$ for some $i \in L$. Since $x_i \in \mathcal{P}_{v_i}(R, \dot{f}(n))$, there is $w_i \subseteq x_i$ and $q_i \in \mathcal{A}_n(\dot{f})$ such that $(v_i \cup w_i, R\backslash x_i) \leq q_i$, and so $(v_i \cup w_i, R\backslash x) \leq q_i$. However, $R \leq T$, C' extends C, $|C'| = |C|$ and $T\backslash x$ is preprocessed for $\dot{f}(n), \max x, C$. Hence for all $i \in L$ there is $p_i \in \mathcal{A}_n(\dot{f})$ such that $(v_i \cup w_i, T\backslash x) \leq p_i$. ∎

Until the end of the section let C be a centred family, $|C| < \mathrm{cov}(\mathcal{M})$, $\dot{f}$ a good $Q(C)$-name for a real, and $T = \langle t_i : i \in \omega\rangle \in Q(C)$ be a pure condition such that for all $n \in \omega$ and $k \in \mathrm{int}(T_n)$, $T\backslash k$ is preprocessed for $\dot{f}(n), k, C$, where $T_n = T\backslash \mathrm{int}(t_{n-1})$.

DEFINITION 5.4. Let $\mathbb{P}(C, T, \dot{f})$ be the suborder of Q_{fin} of all sequences $\bar{r} = \langle (x_i, g_i) : i \in l\rangle$ extending T, such that for all $i \in l$, all $v \subseteq \max x_{i-1}$ and all $s \subseteq x_i$ such that $g_i(s) > 0$, $(\exists w \subseteq s)(\exists p \in \mathcal{A}_i(\dot{f}))((v \cup w, T\backslash s) \leq p)$.

LEMMA 5.5. *Let $X \in Q(C)$ and $n \in \omega$. Then $D_{X,n}(C, T, \dot{f}) = \{\bar{r} \in \mathbb{P}(C, T, \dot{f}) : (\exists r_j \in \bar{r})(r_j \leq X$ and $\|r_j\| \geq n)\}$ is dense.*

Proof. Let $\bar{r} \in \mathbb{P}(C, T, \dot{f})$, $j = |\bar{r}|$, $m = \max \mathrm{int}(\bar{r})$. Let $Y \in C$ be a common extension of X and $T\backslash \mathrm{int}(\bar{r})$. For every $k \in \mathrm{int}(Y)$, $Y\backslash k \leq T_j\backslash k$ and so $Y\backslash k$ is preprocessed for $\dot{f}(j), k, C$. By Corollary 5.3 the logarithmic measure h induced by $\mathcal{P}_m(Y, \dot{f}(j))$ takes arbitrarily high values and so $(\exists x)(h(x) > \max\{\|\bar{r}\|, n\})$. Let $r = (x, h\restriction\mathcal{P}(x))$, $v \subseteq m$, and let $s \subseteq x$ be such that $h(s) > 0$. By definition of h there are $w \subseteq s$ and $q \in \mathcal{A}_j(\dot{f})$ such that $(v \cup w, Y\backslash s) \leq q$. But $T_j\backslash s$ is preprocessed for $\dot{f}(j), \max s, C$ and so there is $p \in \mathcal{A}_j(\dot{f})$ such that $(v \cup w, T\backslash s) \leq p$. ∎

COROLLARY 5.6. *Let G be a filter in $\mathbb{P}(C, T, \dot{f})$ meeting $D_{X,n}(C, T, \dot{f})$ for all $X \in C$, $n \in \omega$, and let $R = \bigcup G = \langle r_i : i \in \omega\rangle$. Then for all i and $v \subseteq i$ and every $s \subseteq \mathrm{int}(r_i)$ which is r_i-positive, $(\exists w \subseteq s)(\exists p \in \mathcal{A}_i(\dot{f}))((v \cup w, R) \leq p)$. Moreover, in $V[G]$ there is a centred family C' such that $C \cup \{R\} \subseteq Q(C')$ and $|C'| = |C|$.*

Proof. Let $i \in \omega$, $v \subseteq i$ and let $s \subseteq \mathrm{int}(r_i)$ be r_i-positive. Then by definition there are $w \subseteq s$ and $p \in \mathcal{A}_i(\dot{f})$ such that $(v \cup w, T\backslash s) \leq p$. However, $R \leq T$ and so $(v \cup w, R) = (v \cup w, R\backslash s) \leq p$. ∎

REMARK 5.7. If $X \notin Q(C)$, then the analogous $D_{X,n}(C, T, \dot{f})$ is not necessarily dense. In fact, the notion of a preprocessed condition is not defined for such X. Thus $\mathbb{P}(C, T, \dot{f})$ and $\mathbb{P}_\tau$ are distinct forcing notions.

6. Mimicking the almost bounding property

THEOREM 6.1. *Let κ be a regular uncountable cardinal, $\mathrm{cov}(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ an unbounded, $<^*$-directed family with $|\mathcal{H}| = \kappa$, C a centred family with $|C| < \kappa$, and let $\dot{f}$ be a good $Q(C)$-name for a real. Then there are a centred family C' extending C, $|C'| = |C|$ and $h \in \mathcal{H}$ such that for every centred family C'' extending C', $\Vdash_{Q(C'')}$ "$\check{h} \not<^* \dot{f}$".*

Proof. Let $T \in Q(C)$. There is a centred family C_0 extending C with $|C_0| = |C|$ and a sequence $\tau = \langle T_n : n \in \omega \rangle \subseteq Q(C_0)$ such that for all n, $T_n \leq T_{n-1}$ where $T_{-1} = T$ and, for all n and $i \leq n$, T_n is preprocessed for $\dot{f}(i), n, C_0$. By Corollary 4.5 and as $|C| < \mathrm{cov}(\mathcal{M})$, there is a centred family C_1 extending C with $|C_1| = |C|$ and a pure condition $T_1 \in Q(C_1)$ such that if $T_1 = \langle t_i^1 : i \in \omega \rangle$ then, for all $n \in \omega$ and $k \in \mathrm{int}(T_1) \backslash \mathrm{int}(t_{n-1}^1)$, $T_1 \backslash k$ is preprocessed for $\dot{f}(n), k, C_1$. Since $|C_1| < \mathrm{cov}(\mathcal{M})$ there is a filter $G \subseteq \mathbb{P}(C_1, T_1, \dot{f})$ meeting $D_{X,n}(C_1, T_1, \dot{f})$ for all $n \in \omega$ and $X \in C_1$. Then by Corollary 5.6 the pure condition $T_2 = \bigcup G = \langle r_i : i \in \omega \rangle$ extends T_1 and for all $i \in \omega$ and $v \subseteq i$, and each $s \subseteq \mathrm{int}(r_i)$ which is r_i-positive, $(\exists w \subseteq s)(\exists p \in \mathcal{A}_i(\dot{f}))((v \cup w, T_2) \leq p)$.

For all $i \in \omega$ let $g(i)$ be the maximal k such that there are $v \subseteq i$, $w \subseteq \mathrm{int}(r_i)$ and $p \in \mathcal{A}_i(\dot{f})$ such that $p \Vdash \dot{f}(i) = \check{k}$ and $(v \cup w, T_2) \leq p$. We can assume that g is nondecreasing. For all $X \in C_1$ let $J_X = \{i : r_i \leq X\}$ and let F_X be the following step function:

$$F_X(l) = g(J_X(i+1)) \quad \text{iff } l \in (J_X(i), J_X(i+1)],$$

where $J_X(m)$ is the mth element of J_X. Since $\mathcal{H}$ is unbounded, for all $X \in C_1$ there is $h_X \in \mathcal{H}$ such that $h_X \not\leq^* F_X$. However, $|C_1| < |\mathcal{H}|$ and so there exists $h \in \mathcal{H}$ such that $(\forall X \in C_1)(h_X \leq^* h)$. We can assume that h is nondecreasing. Note that $(\forall X \in C_1)(g \leq_0 F_X)$ and so $J = \{i \in \omega : g(i) < h(i)\}$ is infinite. Furthermore, $(\exists^\infty i \in J_X)(F_X(i) < h(i))$ and since $(\forall i \in J_X)(F_X(i) = g(i))$, the set $I_X = J_X \cap J$ is infinite. Let $R = \langle r_i : i \in J \rangle$ and for all $X \in C_1$ let $R \wedge X := \langle r_i : i \in I_X \rangle$. Then $C' = \{R \wedge X\}_{X \in C_1}$ is a centred family such that $C_1 \cup \{T\} \subseteq Q(C')$ and $|C| = |C'|$.

Let C'' be centred, $C' \subseteq Q(C'')$, $a \in [\omega]^{<\omega}$, $k_0 \in \omega$ and let $(b, R') \in Q(C'')$ be an extension of (a, R). There is $i \in J$ with $i > k_0$ such that $b \subseteq i$ and $s = \mathrm{int}(R') \cap \mathrm{int}(r_i)$ is r_i-positive. Then there are $w \subseteq s$ and $p \in \mathcal{A}_i(\dot{f})$ such that $(b \cup w, T_2) \leq p$. However, $R' \backslash w \leq T_2 \backslash w$. Therefore $(b \cup w, R') \leq (b, R')$ and $(b \cup w, R') \leq p$. Let $k \in \omega$ be such that $p \Vdash \dot{f}(i) = \check{k}$. Then $k \leq g(i)$ by definition of g, and $g(i) < h(i)$ since $i \in J$. Thus $(b \cup w, R') \Vdash_{Q(C'')}$ "$\dot{f}(i) = \check{k} \leq \check{g}(i) < \check{h}(i)$". ∎

LEMMA 6.2 (Main Lemma). *Let κ be a regular uncountable cardinal,* $\mathrm{cov}(\mathcal{M}) = \kappa$, $\mathcal{H} \subseteq {}^\omega\omega$ *an unbounded, $<^*$-directed family with $|\mathcal{H}| = \kappa$, and* $(\forall \lambda < \kappa)(2^\lambda \leq \kappa)$. *Then there is a centred family C with $|C| = \kappa$ such that* $(\mathcal{H}$ *is unbounded*$)^{V^{Q(C)}}$ *and $Q(C)$ adds a real not split by $V \cap [\omega]^\omega$.*

Proof. Let $\mathcal{N} = \{\dot{f}_\alpha\}_{\alpha < \kappa}$ enumerate all names for functions in ${}^\omega\omega$ for partial orders $Q(C')$ where C' is a centred family with $|C'| < \kappa$, and let $\mathcal{A} = \{A_{\alpha+1}\}_{\alpha < \kappa}$ enumerate $[\omega]^\omega \cap V$. The centred family C will be obtained by transfinite induction of length κ. Begin with an arbitrary pure condition T and $C_0 = \{T \backslash v : v \in [\omega]^{<\omega}\}$. If $\alpha = \beta + 1$ and we have defined the centred family C_β, let $\dot{g}_\alpha$ be the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\beta}$ which is a $Q(C_\beta)$-name for a real. If $\dot{g}_\alpha$ is a good $Q(C_\beta)$-name, by Theorem 6.1 there are a centred family C'_α extending C_β with $|C'_\alpha| = |C_\beta|$ and $h_\alpha \in \mathcal{H}$ such that for every centred family C'' extending C'_α, $\Vdash_{Q(C'')}$ "$\check{h}_\alpha \not<^* \dot{g}_\alpha$". If $\dot{g}_\alpha$ is not a good $Q(C_\beta)$-name, then by Remark 4.2 there is a centred family C'_α extending C_β with $|C'_\alpha| = |C_\beta|$ such that $\dot{g}_\alpha$ is not a $Q(C'_\alpha)$-name for a real. In either case, let $T' \in Q(C'_\alpha)$. Then by Lemma 3.13 there is $T_\alpha \le T'$ such that $\mathrm{int}(T_\alpha) \subseteq A_\alpha$ or $\mathrm{int}(T_\alpha) \subseteq A^{\mathrm{c}}_\alpha$ and $T_\alpha \not\perp C'_\alpha$. By Corollary 3.18 applied to the sequence of all final segments of T_α and $|C'_\alpha| < \mathrm{cov}(\mathcal{M})$ there is a centred family C_α such that $C'_\alpha \cup \{T_\alpha\} \subseteq Q(C_\alpha)$ and $|C_\alpha| = |C'_\alpha|$. If α is a limit let $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$. Then $|C_\alpha| < \kappa$ and $(\forall \beta < \alpha)(C_\beta \subseteq Q(C_\alpha))$. With this the inductive construction is complete. Let $C = \bigcup_{\alpha<\kappa} C_\alpha$. Then C is centred, $|C| = \kappa$ and $(\forall \alpha < \kappa)(C_\alpha \subseteq Q(C))$.

Let $\dot{f}$ be a $Q(C)$-name for a real and let $\alpha < \kappa$ be minimal such that $\dot{f}$ is a $Q(C_\alpha)$-name. Then $\dot{f}$ is a name in $\mathcal{N}$ and there is $\delta < \kappa$ (with $\alpha \le \delta$) such that $\dot{f}$ is the name with least index in $\mathcal{N}\backslash\{\dot{g}_{\gamma+1}\}_{\gamma<\delta}$ which is a $Q(C_\delta)$-name and so $\dot{f} = \dot{g}_{\delta+1}$. Note also that $\dot{f}$ is a good $Q(C_\delta)$-name. Then by the choice of $C'_{\delta+1}$, $\Vdash_{Q(C)}$ "$\check{h}_{\delta+1} \not<^* \dot{f}$". Let G be a $Q(C)$ generic filter and $\bigcup G = \bigcup\{u : (\exists T)((u,T) \in G)\}$. For every $\alpha \in \kappa$ the set $D_{\alpha+1} = \{(u,T) \in Q(C) : T \le T_{\alpha+1}\}$ is dense and so $\bigcup G \subseteq^* \mathrm{int}(T_{\alpha+1})$, which implies that $\bigcup G$ is almost contained in $A_{\alpha+1}$ or in $A^{\mathrm{c}}_{\alpha+1}$. ∎

The proof of Theorem 6.3 can be found in [8].

THEOREM 6.3. *Let $\mathcal{H} \subseteq {}^\omega\omega$ be an unbounded family such that $(\forall \mathcal{H}' \in [\mathcal{H}]^{\le\omega})$ $(\exists h \in \mathcal{H})(\mathcal{H}' \le^* h)$ and let $\langle \mathbb{P}_\gamma : \gamma \le \alpha\rangle$ be a finite support iteration of ccc forcing notions of length α with $\mathrm{cf}(\alpha) = \omega$ such that $(\forall \gamma < \alpha)$ $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}_\gamma}}$. Then $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}_\alpha}}$.*

The proof of Lemma 6.4 can be found in [2].

LEMMA 6.4. *Let κ be a regular uncountable cardinal, and $\mathcal{H} \subseteq {}^\omega\omega$ an unbounded, $<^*$-directed family with $|\mathcal{H}| = \kappa$. Then for every partial order $\mathbb{P}$ of size less than κ, $(\mathcal{H}$ is unbounded$)^{V^{\mathbb{P}}}$.*

Recall that if $\mathcal{A} \subseteq {}^\omega\omega$ is infinite then the Hechler forcing $\mathbb{H}(\mathcal{A})$ (see [7]) consists of all pairs (s, F) where $s \in \bigcup_{n\in\omega} {}^n\omega$ and $F \in [\mathcal{A}]^{<\omega}$, with extension relation $(s_1, F_1) \le (s_2, F_2)$ iff $s_2 \subseteq s_1$, $F_2 \subseteq F_1$ and for all $f \in F_2$ and $k \in \mathrm{dom}(s_1)\backslash\mathrm{dom}(s_2)$ we have $s_1(k) \ge f(k)$. Note that $\mathbb{H}(\mathcal{A})$ is σ-centred, adds a real dominating $\mathcal{A}$, and $|\mathbb{H}(\mathcal{A})| = |\mathcal{A}|$.

THEOREM 6.5 (GCH). *Let κ be a regular uncountable cardinal. Then there is a ccc generic extension in which $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$.*

Proof. Obtain a model V of $\mathfrak{b} = \mathfrak{c} = \kappa$ by adding κ Hechler reals (see [6]) and let $\mathcal{H} = V \cap {}^\omega\omega$. Inductively define a finite support iteration $\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+ \rangle$ of ccc forcing notions as follows. Suppose that for all $\beta < \alpha$, $\mathbb{P}_\beta$ has been defined so that in $V^{\mathbb{P}_\beta}$, $\mathcal{H}$ is unbounded, $<^*$-directed and $(\forall \lambda < \kappa)(2^\lambda \leq \kappa)$. If α is a limit, let $\mathbb{P}_\alpha$ be the finite support iteration of $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$. Then $\mathbb{P}_\alpha$ is ccc and by Theorem 6.3 the inductive hypothesis holds in $V^{\mathbb{P}_\alpha}$.

If $\alpha = \beta + 1$ and $\mathbb{P}_\beta$ has been defined, then let $V_\beta = V^{\mathbb{P}_\beta}$ and let $\mathbb{H}_1$ be the forcing notion for adding κ Cohen reals. Then in $V_\beta^{\mathbb{H}_1}$ the family $\mathcal{H}$ is unbounded, $<^*$-directed, $(\forall \lambda < \kappa)(2^\lambda \leq \kappa)$ and $\mathrm{cov}(\mathcal{M}) = \kappa$. Therefore in $V_\beta^{\mathbb{H}_1}$ the hypothesis of Lemma 6.2 holds and so there is a centred family C such that $Q(C)$ adds a real not split by $V_\beta^{\mathbb{H}_1} \cap [\omega]^\omega$ and preserves $\mathcal{H}$ unbounded. Let $\mathbb{H}_2$ be an $\mathbb{H}_1$-name for $Q(C)$ and in $V_\beta^{\mathbb{H}_1 * \mathbb{H}_2}$ let $\mathcal{A} \subseteq V_\beta \cap {}^\omega\omega$ be an unbounded family of cardinality less than κ. Let $\mathbb{H}_3$ be an $\mathbb{H}_1 * \mathbb{H}_2$ name for $\mathbb{H}(\mathcal{A})$. Then in $V_\beta^{(\mathbb{H}_1 * \mathbb{H}_2) * \mathbb{H}_3}$ the family $\mathcal{A}$ is dominated and since $|\mathbb{H}(\mathcal{A})| < \kappa$, $\mathcal{H}$ remains unbounded. Let $\dot{\mathbb{Q}}_\beta$ be a $\mathbb{P}_\beta$-name for $(\mathbb{H}_1 * \mathbb{H}_2) * \mathbb{H}_3$, and let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$.

Let $\mathbb{P} = \mathbb{P}_{\kappa^+}$. Let G be a $\mathbb{P}$-generic filter and let $\mathcal{A} \subseteq [\omega]^\omega \cap V[G]$, $|\mathcal{A}| < \kappa^+$. Then there exists $\alpha < \kappa^+$ such that $\mathcal{A} \subseteq V[G_\alpha]$ where $G_\alpha = G \cap \mathbb{P}_\alpha$. By the inductive construction of $\mathbb{P}$, in $V[G_{\alpha+1}]$ there is a real not split by $\mathcal{A}$. Therefore $V^{\mathbb{P}} \vDash \mathfrak{s} = \kappa^+$. By Theorem 6.3 and the construction of $\mathbb{P}$ the family $\mathcal{H}$ is unbounded in $V^{\mathbb{P}}$. Since every family of reals in $V^{\mathbb{P}}$ of size less than κ is obtained at some initial stage of the iteration, a suitable bookkeeping device can guarantee that any such family is bounded and so $V^{\mathbb{P}} \vDash \mathfrak{b} = \kappa$. ∎

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