Pure virtual braids homotopic to the identity braid

by

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Abstract. Two virtual link diagrams are homotopic if one may be transformed into the other by a sequence of virtual Reidemeister moves, classical Reidemeister moves, and self crossing changes. We recall the pure virtual braid group. We then describe the set of pure virtual braids that are homotopic to the identity braid.

1. Introduction. A *virtual link diagram* is a decorated immersion of n copies of S^1 with two types of crossings: classical and virtual. Classical crossings are indicated by over/under markings and virtual crossings are indicated by a solid encircled X. An example of a virtual link diagram is shown in Figure 1.
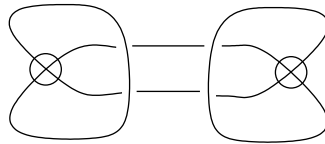

Fig. 1. Kishino's knot

Virtual link theory is a generalization of classical knot theory that was introduced by Louis H. Kauffman in 1996 (see [9]). Two virtual link diagrams are said to be *equivalent* if one may be transformed into the other by a sequence of classical Reidemeister moves (shown in Figure 2) and virtual Reidemeister moves (shown in Figure 3). Classical link diagrams contain no virtual crossings and form a subset of the virtual link diagrams.
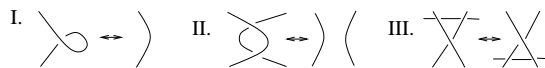

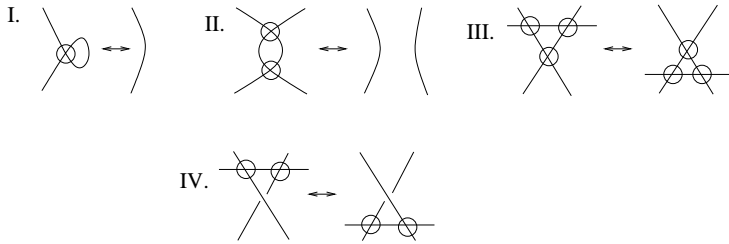Fig. 2. Classical Reidemeister moves

[225]

Fig. 3. Virtual Reidemeister moves

A *representation* of a virtual link diagram is a pair (F, L) where L is a link diagram on a closed, two-dimensional surface F taken up to Dehn twists and handle $(S^1 \times I)$ additions and cancellations. Reidemeister moves can be performed on the surface.

REMARK 1.1. Detailed descriptions of representations are given in [3] or [9]. Abstract surfaces (punctured representations) are described in [7].

We recall the following theorems:

THEOREM 1.1 (See [7] and [9]). *Classes of representations are in one-to-one correspondence with equivalence classes of virtual link diagrams.*

THEOREM 1.2 (Kuperberg [11]). *Representations of virtual link diagrams have a unique representative embedding class in the minimal genus surface that can support the diagram.*

Recalling [4], two virtual link diagrams are defined to be *homotopic* if one diagram may be transformed into the other by a sequence of virtual Reidemeister moves, classical Reidemeister moves, and self crossing changes. (By *self crossing change*, we mean changing the over/under markings at a crossing between two segments of the same link component.)

In this paper, we focus on virtual braids (see [8] and [10]) and pure virtual braids. A *n-strand virtual braid diagram* is a decorated immersion of n copies of $[0, 1]$ into the plane. Let $\{f_1, \ldots, f_n\}$ denote the n components. The sets of endpoints $\{f_1(i), \ldots, f_n(i)\}$, $i \in \{0, 1\}$, are contained in a line for each i. We refer to the set of points where $i = 0$ as the *upper endpoints* and the other set $(i = 1)$ as the *lower endpoints*, as shown in Figure 4. Two virtual braid diagrams are said to be *virtually homotopic* if one can be transformed into the other by a sequence of Reidemeister moves, virtual Reidemeister moves, and self crossing changes that leave the endpoints fixed. A virtual braid is *pure* if the braid is equivalent to a braid with no self crossings, that is, every crossing involves two distinct components. For a pure virtual braid, the order of the set of upper endpoints is the same as the order of the set of lower endpoints. We denote the set of n-strand pure virtual braids as VP_n
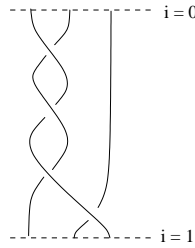
Fig. 4. Example of a braid diagram

and the set of n-strand pure classical braids as BP_n. We will discuss the group structure of VP_n and BP_n in the next section.

Two virtual braid diagrams are *virtually homotopic* if one diagram can be transformed into the other by a sequence of Reidemeister moves, virtual Reidemeister moves and self crossing changes. We will denote the set of homotopic n-strand pure virtual braids as $H(\mathrm{VP}_n)$ and the set of homotopic n-strand pure classical braids as $H(\mathrm{BP}_n)$ following the notation in [5].

A *representation* of an n-strand pure virtual braid is a pair (D, b) where D is a once punctured, two-dimensional oriented surface with an immersed braid b (where the boundary points of b are contained in the boundary of D) modulo Reidemeister moves, Dehn twists, and handle cancellations and additions.

REMARK 1.2. We can view the surface D as $I \times I$ with m attached handles ($m \geq 0$). Theorems 1.1 and 1.2 apply to representations of pure virtual braids. In a diagram of a representation with genus one, we will draw only the handle as shown in Figure 5. Elements of VP_n with genus one representations have a natural correspondence with elements of BP_{n+2} and BP_{n+1}.
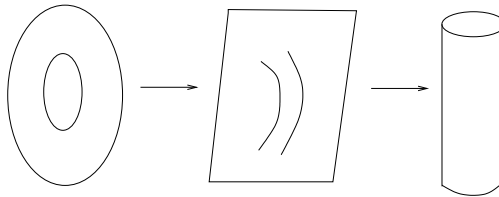

Fig. 5. Surfaces for genus one representations

Classical homotopy has been studied by Milnor [12], Goldsmith [5], and more recently Habegger and Lin [6]. Significant differences exist between the classical case and the virtual case. Strikingly, not all virtual knot diagrams are homotopic to the unknot. (This topic has been explored in [4] where Milnor's link groups and μ-invariants are applied to virtual link diagrams.)

The difference in the case of braids is illustrated by the fact that every classical braid with a fixed ordering on the endpoints can be homotoped into a braid with no self crossings. This is not true in the virtual case, as shown in Figure 6. In this paper, we determine which pure virtual braids are homotopic to the identity braid.
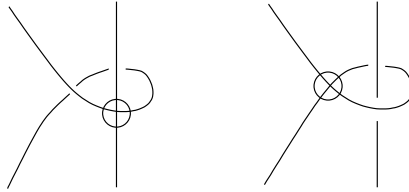

Fig. 6. Braids not homotopic to a pure braid

2. Pure braids. The structure of the n-strand pure virtual braid group is described in [2]. VP_n is generated by the set $\{\lambda_{ik}, \lambda_{ki} \mid i, k \in \{1, \ldots, n\}\}$. These generators are illustrated in Figure 7.
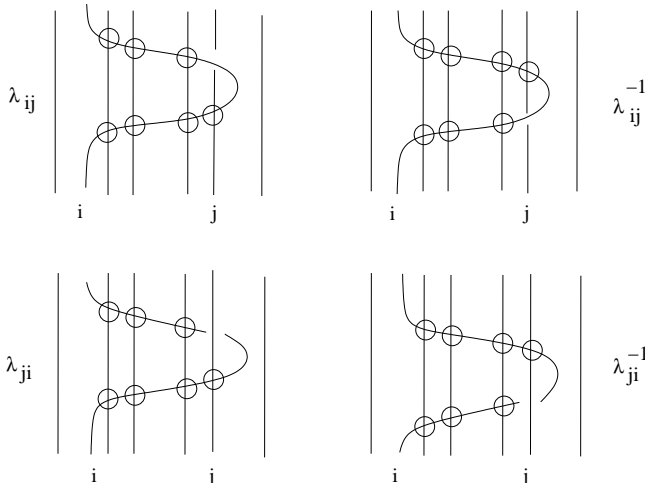
Fig. 7. Generators of the pure virtual braid group

The relations in VP_n are given in equations (1) and (2) below (cf. [2]). First,

$$(1) \qquad \lambda_{jk}\lambda_{in} = \lambda_{in}\lambda_{jk} \quad \text{for distinct } i, j, k, n.$$

Let $s(ij) = 1$ if $i < j$ and -1 otherwise. Then

$$(2) \qquad \lambda_{ki}^{s(ki)} \lambda_{kj}^{s(kj)} \lambda_{ij}^{s(ij)} = \lambda_{ij}^{s(ij)} \lambda_{kj}^{s(kj)} \lambda_{ki}^{s(ki)}.$$

REMARK 2.1. The naming convention for the generators differs slightly from that given in [2]. In this paper, the two indices indicate the strands involved. The classical crossing is in the upper tier of crossings. The first index number indicates the over crossing strand while the second indicates the under crossing strand.

The n-strand pure virtual braids have a group structure where multiplication is performed by concatenating braids as shown in Figure 8.
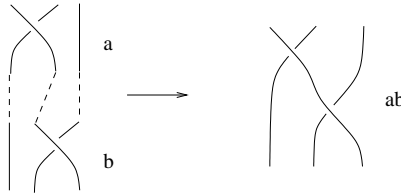

Fig. 8. Braid multiplication

The n-strand pure classical braids, denoted BP_n, form a subgroup of VP_n. The generators of BP_n are denoted as σ_{ij} with $1 \leq i < j \leq n$. Each σ_{ij} can be expressed as a product of the virtual generators:

$$\sigma_{ij} = (\lambda_{i,i+1}\lambda_{i,i+2}\ldots\lambda_{i,j-1})(\lambda_{ij}\lambda_{ji}^{-1})(\lambda_{i,j-1}^{-1}\ldots\lambda_{i,i+2}^{-1}\lambda_{i,i-1}^{-1}).$$

Recall the *commutator* of two elements:

$$[x, y] = xyx^{-1}y^{-1}.$$

Let F_i denote the subgroup of BP_n generated by $\{\sigma_{i,i+1}, \sigma_{i,i+2}, \ldots, \sigma_{in}\}$. The set of n-strand pure classical braids homotopic to the identity is the smallest normal subgroup generated by the commutators

$$(3) \qquad [\sigma_{ij}, g\sigma_{ij}g^{-1}] \quad \text{where } g \in F_i.$$

From [2], we can describe VP_n as a semidirect product and give a normal form for pure virtual braids. Let V_n denote the set of generators $\{\lambda_{in}^{\pm 1}, \lambda_{ni}^{\pm 1} \mid i \in 1, \ldots, n-1\}$. Then V_n^* denotes the smallest normal subgroup generated by V_n in VP. The subgroup V_n^* is normal in VP_n, and VP_n is the semidirect product $V_n^* \rtimes \mathrm{VP}_{n-1}$. That is, if $w_n \in V_n^*$ then

$$w_n = \prod_{j=1}^{k} g_j a_j g_j^{-1} \quad \text{where } a_j \in V_n,\, g_j \in \mathrm{VP}_{n-1}.$$

Based on the normal form, we define the *length* of w_n to be k. The braid w_n has *minimal homotopic length* if every strand contains at least one real crossing and the braid does not contain a subsequence such that its deletion produces a braid which homotopic to w_n but not virtually equivalent.

REMARK 2.2. If a braid has a strand with only virtual crossings then this braid is equivalent to a conjugate of a braid with only classical crossings on this strand. However, the minimal length of the braid (based on the normal form) increases as shown in Figure 9.
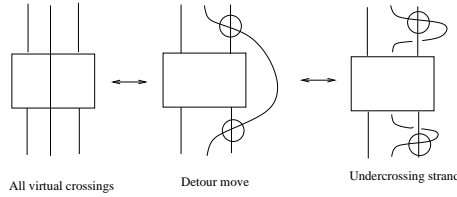

All virtual crossings   Detour move   Undercrossing strand

Fig. 9. Converting to a braid with minimal homotopic length

The following theorem describes a normal form for elements of VP_n.

THEOREM 2.1 (see [2]). *Let b be an element of VP_n. Then $b = w_2 w_3 \ldots w_n$ where $w_j \in V_j^*$.*

REMARK 2.3. There is a reduced form for g_j when describing elements of V_n^* based on the relations given earlier.

This result parallels Artin's theorem [1] about the normal form of pure classical braids. In the next section, we describe the set of pure virtual braids that are homotopic to the identity braid. We incorporate Goldsmith's methodology from the classical case [5] and representations of virtual braids [3].

3. Genus and homotopy. We prove that the set of n-strand pure braids homotopic to the identity braid is a normal subgroup of VP_n. We will denote this subgroup as $I(\mathrm{VP}_n)$. To describe it, we will first prove a sequence of lemmas about minimal genus and V_n^*. We then apply this result to a braid in normal form in the next section.

LEMMA 3.1. *Let $w \in \mathrm{VP}_2$. Then a minimal genus representation of w has genus less than or equal to one.*

Proof. The braid w contains two strands. Immerse the second strand in a surface with one handle so that the second strand follows the longitude of the handle as shown in Figure 10. ∎

COROLLARY 3.2. *The elements of VP_2 can be viewed as elements of BP_3 and there are non-classical elements of VP_2 homotopic to the identity braid.*

Proof. If w is an element of VP_2 then w has a genus one representation. This representation can be viewed as an element of BP_3. There are non-trivial braids in BP_3 that are homotopic to the identity braid. These braids
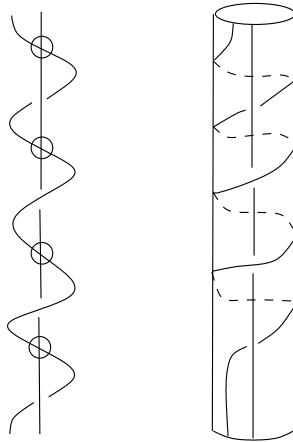
Fig. 10. A representation of an element of VP$_2$

in BP$_3$ correspond to representations of non-trivial elements of VP$_2$, which are then homotopic to the identity braid. ∎

We apply the following lemma in the cases where $n \geq 3$.

LEMMA 3.3. *Let h be a handle in a minimal genus representation of a pure virtual braid. Let $\{s_1, \ldots, s_k\}$ be the collection of strands such that the removal of s_i admits a cancellation curve. Then $k \leq 2$.*

Proof. Let $[m]$ denote the meridian and $[l]$ the longitude of handle h. Suppose that removing strand s_1 from the representation admits a cancellation curve α_1 on handle h. Then the strand s_1 intersects the cancellation curve α_1 transversely. By hypothesis, removing strand s_2 admits a cancellation curve α_2 on h. Note that s_2 intersects α_2 transversely.

Suppose that α_2 is homotopic to α_1; then s_1 also intersects both α_1 and α_2 transversely. As a result, α_2 cannot be a cancellation curve after the removal of s_2.

We observe that α_1 and α_2 are not homotopic but both pass through handle h. Assume without loss of generality that $\alpha_1 = [m]$ and $\alpha_2 = [l]$ (implying that s_1 parallels the longitude and s_2 parallels the meridian in the handle). Let s_3 be a third strand in the braid whose removal admits a canceling curve α_3 for h. The curve α_3 is homotopic to neither curve. But if α_3 passes through h then α_3 is homotopic to a curve that wraps around the meridian a times and the longitude b times. Hence, α_3 intersects at least one of the pair, α_1 or α_2. But then at least one of the pair, s_1 or s_2, intersects α_3 transversely. Hence, α_3 cannot be a cancellation curve. ∎

Lemma 3.3 can be applied to show that certain elements of V_n^* with $n \geq 4$ have representations with minimal genus less than or equal to one.

LEMMA 3.4. *Let $n \geq 4$. If w is a non-classical element of V_n^* with minimal homotopic length then the minimal genus of a representation of w is one.*

Proof. Let w be an element of V_n^* that is homotopic to the identity such that every strand contains at least one real crossing. Suppose that a minimal genus surface for w has m handles. If strand s_n is removed then each handle admits a canceling curve since removing s_n results in the identity braid. If strand s_i is removed then $m-1$ handles admit a canceling curve. As a result, if some handle admits only two canceling curves then all other curves admit three or more canceling curves since $n \geq 4$. Now, by Lemma 3.3 there is at most one handle and the representation has genus less than or equal to one. ∎

The remaining case occurs when $n = 3$. We will need the following lemma about linking number. Recall that the *sign* of a classical crossing, c, is determined by its relative orientation as shown in Figure 11.

+1   −1

Fig. 11. Sign of a classical crossing

Let V be the collection of crossings where strand a passes over strand b. Then

$$\mathrm{Link}(a,b) = \sum_{c \in V} \mathrm{sgn}(c).$$

PROPOSITION 3.5. *Let braid $b \in \mathrm{VP}_n$ be homotopic to the n-strand identity braid. If b contains n copies of the generator λ_{ij}, then b contains n copies of the generator λ_{ij}^{-1}.*

Proof. For the generator λ_{ij}, $\mathrm{Link}(i,j) = -1$ and $\mathrm{Link}(j,i) = 1$. Linking number is a homotopy invariant and in the identity braid $\mathrm{Link}(i,j) = 0$. Hence the generator λ_{ij} is paired with the generator λ_{ij}^{-1}. ∎

Let $\lambda_{p(ij)}$ represent either λ_{ij} or λ_{ji}. We use this notation in the following lemma.

LEMMA 3.6. *If w is a braid in V_3^* with minimal homotopic length that is homotopic to the identity then the minimal genus of a representation of w is less than or equal to one.*

Proof. Let w be a braid in V_3^* with minimal homotopic length that is homotopic to the identity braid. Suppose that a minimal genus representation of w is a surface with m handles. Consider the representation of w. Since

w is in normal form, the removal of the 3rd strand from the representation result in the 2-strand identity braid and m canceling curves on the surface.

The removal of the first strand results in the braid x, a 2-strand braid that is homotopic to the identity, and as a result, $m-1$ canceling curves on the surface. Similarly, the removal of the 2nd strand results in the braid y, a 2-strand braid homotopic to the identity and $m-1$ canceling curves on the surface.

Applying Lemma 3.3, we observe that the representation of w contained at most two handles. Note that if x is classical, then x is the identity braid and the representation of w has genus less than or equal to one. As a result, we will assume that both x and y are non-classical for the remainder of the proof.

Let $\widehat{x}$ denote the subsequence of w consisting of conjugates of $\lambda_{p(13)}^{\pm 1}$. The removal of the second strand from $\widehat{x}$ results in a 2-strand braid homotopic to the identity, while the removal of the 1st and 3rd strands results in the identity braid. Hence, a minimal genus representation of $\widehat{x}$ has genus one. (Note that if the genus is zero, then $\widehat{x}$ is a classical braid, which contradicts our assumption that x is a non-classical braid.) Let $\widehat{y}$ denote the subsequence of w consisting of conjugates of $\lambda_{p(23)}^{\pm 1}$. Following the argument given for $\widehat{x}$, a minimal genus representation of $\widehat{y}$ also has genus one.

Now, $w = x_1 y_1 x_2 y_2 \ldots x_n y_n$ where $\widehat{x} = x_1 x_2 \ldots x_n$ and $\widehat{y} = y_1 y_2 \ldots y_n$. Because minimal genus representations of $\widehat{x}$ and $\widehat{y}$ have genus one, we may assume that (after isotopy) minimal genus representations of each subbraid x_i and y_i occur on a tube (possibly) with handles as shown in Figure 12.
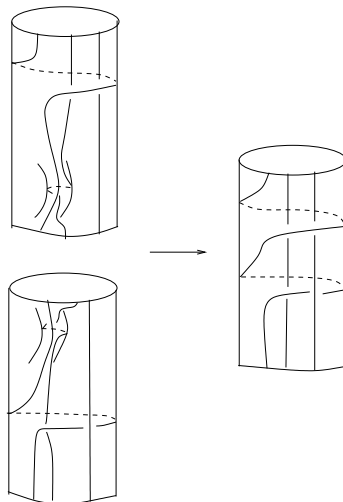

Fig. 12. Minimal genus surface

However, a representation of w has at most genus two. If handles occur in a representation of some x_i (or y_i) then all the handles are canceled in a representation of $\hat{x}$ (or $\hat{y}$). Without loss of generality, assume that the tube representation of some x_i contains at least one handle. (The handles can be selected so that a handle either contains strands 1 and 3 or strands 1 and 2.) Suppose that this handle contains strands 1 and 3; then the subbraid x_{i+1} must have a corresponding handle involving these strands.

In the braid w, the braid y_i occurs between x_i and x_{i+1}. As a result, the handle in x_i cancels with a handle occurring in either y_{i-1} or y_i. But $\hat{y}$ contains no virtual or classical crossings between 1 and 3. Hence, the handles cannot be canceled in w and the minimal genus of w is three, a contradiction.

Suppose that the handles in x_i that cannot be removed in w contain strands 1 and 2. That is, g_i^{-1} does not cancel with g_{i+1}. Now, the deletion of either strand 1 or strand 2 admits a canceling curve for each of these handles, but not the tube. The deletion of strand 3 admits a canceling curve for every handle. Again by applying Lemma 3.3, we observe that the representation of w has genus one. ∎

We have proved the following theorem:

THEOREM 3.7. *Let w_n be a minimal length, non-trivial element of V_n^* such that every strand contains a real crossing. If w_n is homotopic to the identity braid then a minimal genus representation of w_n has genus less than or equal to one.*

REMARK 3.1. If a braid contains a strand with only virtual crossings then some representation contains a handle with this strand immersed along the longitude.

LEMMA 3.8. *Let $b = w_2 w_3 \ldots w_n$ be an element of VP_n (where $w_j \in V_j^*$). Then b is homotopic to the n-strand identity braid if and only if each w_i is homotopic to the identity braid.*

Proof. If each w_i is an element of $I(\mathrm{VP}_n)$ then $b = w_1 w_2 \ldots w_n$ is an element of $I(\mathrm{VP}_n)$. Let w be an element of $I(\mathrm{VP}_n)$. Then there is a homotopy sequence $\{p_0, p_1, \ldots, p_m\}$, with $w = p_0$ and p_m equivalent to the identity braid, that transforms w into the identity. Remove strand n from each diagram and replace it with the identity strand. This reduces w to the braid b_{n-1} where $b_{n-1} = w_1 \ldots w_{n-1}$. This preserves the homotopy sequence so that b_{n-1} is homotopic to the identity strand. Removing strands, we observe that $b_i = w_1 \ldots w_i$ is homotopic to the identity braid. Hence, each w_i is homotopic to the identity braid. ∎

4. Braids homotopic to the identity braid. We show that the set of braids homotopic to the identity braid forms a normal subgroup of VP_n. Let

x be an element of VP_n with $x = (\lambda_{i-1,i}\lambda_{i-2,i}\ldots\lambda_{1i})(\lambda_{ni}^{-1})(\lambda_{n-1,i}^{-1}\ldots\lambda_{i+1,i}^{-1})$. Let g_a and g_b denote classical braids generated by the set

$$\{\sigma_{1i}, \sigma_{2i}\ldots\sigma_{i-1,i}\sigma_{i,i+1}\ldots\sigma_{in}\}.$$

PROPOSITION 4.1. *If an element of* VP_n *has the form*

$$[\sigma_{ij}, g_a x g_b \sigma_{ij} g_b^{-1} x^{-1} g_a^{-1}]$$

(*as shown below in Figure* 17 *where* g_a *and* g_b *are denoted as A and B*) *then the braid is homotopic to the identity braid.*

Proof. We show sample homotopy sequences for elements of VP_2 and VP_3 in Figures 13 and 14. ∎
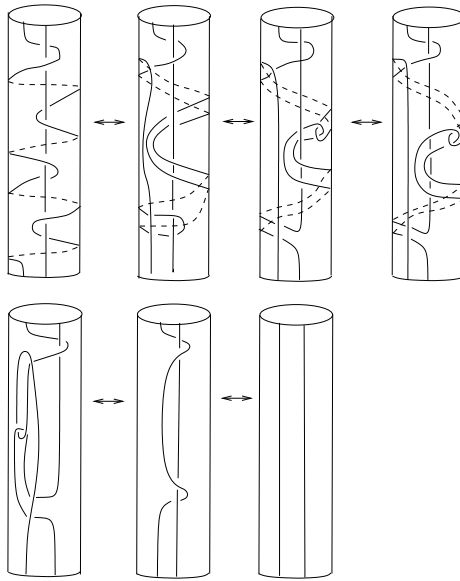
Fig. 13. A homotopy sequence in VP_2

REMARK 4.1. This braid can be expressed as the product of an element of V_n^* and VP_{n-1}.

PROPOSITION 4.2. *If* w *is an element of* V_n^* *that is homotopic to the identity with minimal homotopic length then some representation of* w *corresponds to an element of* BP_{n+1} *of the form* $[\sigma_{ij}, g\sigma_{ij}g^{-1}]$.

Proof. Let w be a braid in V_n^* with minimal homotopic length that is homotopic to the identity. By Theorem 3.7, the braid w has a representation with minimal genus one, a torus. View the torus as the complement of two linked curves to obtain an element of BP_{n+2}. After removing strand
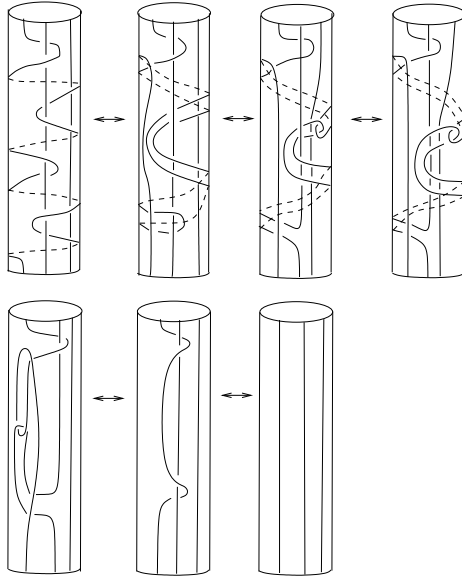
Fig. 14. A homotopy sequence in VP₃

$n + 2$, we obtain a classical braid in BP_{n+1} that is homotopic to the identity. The braid w, as an element of BP_{n+1}, can be written in the classical normal form, $b_1 b_2 \ldots b_{n+1}$, where b_i is an element of the group generated by $\{\sigma_{i,i+1}, \sigma_{i,i+2}, \ldots, \sigma_{i,n+1}\}$. In BP_{n+1}, a braid homotopic to the identity has the form $[\sigma_{ij}, g\sigma_{ij}g^{-1}]$.

In a representation of w the longitude of the torus corresponds to strand $n + 1$. As a result, consider the first b_i that includes strand $n + 1$.

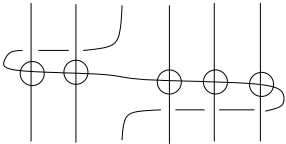Now, $\sigma_{i,n+1}$ either holds the place of σ_{ij} or is a term in g from equation (3). Let x denote the element of VP_n shown in Figure 15.
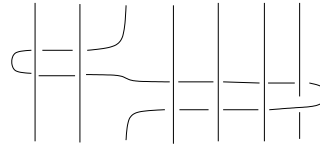
Fig. 15. Braid x in VP_n Fig. 16. Braid $\widehat{x}$ in BP_{n+1}

In VP_n,

$$x = (\lambda_{i-1,i}\lambda_{i-2,i}\ldots\lambda_{1i})(\lambda_{ni}^{-1}\lambda_{n-1,i}^{-1}\ldots\lambda_{i+1,i}^{-1}).$$

From a representation of x, we obtain $\widehat{x}$, the element of BP_{n+1} shown in Figure 16. In BP_{n+1},

$$\widehat{x} = (\sigma_{i,i+1}^{-1}\sigma_{i,i+2}^{-1}\ldots\sigma_{i,n}^{-1})\sigma_{i,n+1}^{-1}(\sigma_{in}\ldots\sigma_{i,i+1}).$$

If $\sigma_{i,n+1}$ corresponds to σ_{ij} then the homotopy sequence requires a crossing change on strand $n + 1$. Hence g must contain the term σ_{in+1} and the braid

b_i has the form

$$[\sigma_{ij}, q_a \widehat{x} q_b \sigma_{ij} q_b^{-1} \widehat{x}^{-1} q_a^{-1}]$$

where q_a and q_b are classical braids in the group generated by $\{\sigma_{i,i+1} \ldots \sigma_{in}\}$. This corresponds to the braid shown in Figure 17. In terms of VP_n this is a braid of the form

(4)
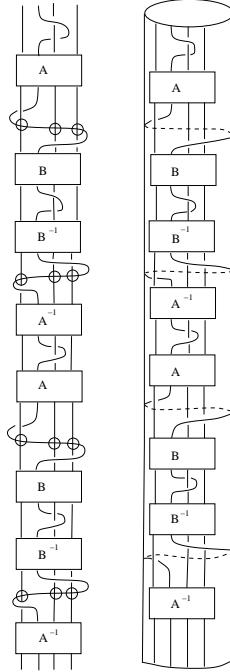$$[\sigma_{ij}, G_a \lambda_{ni}^{-1} G_b \sigma_{ij} G_b^{-1} \lambda_{ni} G_a^{-1}].$$

Fig. 17. A representation of w

(Note that G_a and G_b are the images of q_a and q_b with the appropriate part of $\widehat{x}$ in VP_n.) Let

(5)
$$\lambda = \lambda_{ni},$$

(6)
$$z = G_a^{-1} \sigma_{ij} G_a,$$

(7)
$$y = G_b \sigma_{ij} G_b.$$

Then the braid in equation (4) can be rewritten as

$$G_a(z\lambda^{-1} y \lambda z^{-1} \lambda^{-1} y^{-1} \lambda) G_a^{-1}.$$

This can be rewritten in normal form for VP_n:

$$G_a(zyz^{-1}y^{-1})(yzy^{-1}\lambda^{-1}yz^{-1}y^{-1})(yz\lambda z^{-1}y^{-1})(y\lambda^{-1}y^{-1})\lambda G_a^{-1}.$$

The braid $zyz^{-1}y^{-1}$ is an element of VP_{n-1} and homotopic to the identity. Then yzy^{-1} is homotopic to the braid z. Canceling terms, we obtain the

braid

$$(8) \qquad G_a z \lambda^{-1} y \lambda z^{-1} \lambda^{-1} y^{-1} \lambda G_a^{-1}.$$

Using equations (5)–(7) to rewrite, we obtain

$$(9) \qquad \sigma_{ij} G_a \lambda^{-1} G_b \sigma_{ij} G_b^{-1} \lambda G_a^{-1} \sigma_{ij} G_a \lambda^{-1} G_b \sigma_{ij} G_b^{-1} \lambda G_a^{-1}.$$

This is the same form as the original braid. ∎

REMARK 4.2. Note that in this form, strand i underpasses all $i-1$ previous strands. Any weaving on these strands can be described by multiplying w by the braid b_k with $k \leq i$. Note that if the original braid w over crosses the $i-1$ previous strands, then b_k with $k < i$ involves crossings on strand $n+1$. This contradicts the fact that b_i is the first strand to have crossings involving strand $n+1$.

We have proved the following theorem:

THEOREM 4.3. *The set of pure virtual braids homotopic to the identity is the smallest normal subgroup generated by elements of the form*

$$[\sigma_{ij}, g_a x g_b \sigma_{ij} g_b^{-1} x^{-1} g_a^{-1}]$$

where $x = (\lambda_{i-1,i} \lambda_{i-2,i} \ldots \lambda_{1i})(\lambda_{ni}^{-1} \lambda_{n-1,i}^{-1} \ldots \lambda_{i+1,i}^{-1})$ or x is the identity braid and g_a, g_b are classical braids generated by the set $\{\sigma_{1i}, \sigma_{2i}, \ldots \sigma_{i-1,i}, \sigma_{i,i+1} \ldots \sigma_{in}\}$.

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