On divisibility in definable groups

by

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Abstract. Let $M$ be an o-minimal expansion of a real closed field. It is known that a definably connected abelian group is divisible. We show that a definably compact definably connected group is divisible.

Let $M$ be an o-minimal expansion of a real closed field. A group is said to be definable if both the set and the graph of the group operation are definable in $M$. By results of Pillay in [10], a definable group can be equipped with a definable manifold topology making the group a topological group. Since topological groups are regular spaces, we can suppose that the manifold topology is induced by that of the ambient space (see Theorem 10.1.8 in [6]). In that setting, a definably compact group is a closed and bounded definable group. A definable group is definably connected provided it has no definable subgroups of finite index. A definably connected group which is abelian is also divisible, by Strebonski’s Theorem on the finiteness of torsion subgroups (see, e.g., the proof of Theorem 2.1 in [9]).

In this note, using the available literature on both definable groups and topological groups, we prove the following.

Theorem 1. Let $G$ be a definably compact definably connected definable group. Then $G$ is divisible.

In proving divisibility of the groups we are concerned with, the continuous definable maps $p_k : G \rightarrow G : a \mapsto a^k$ for $k > 0$ will play an important role (in both the Abelian definable case and the classical topological case).

First, we consider the o-minimal cohomology with coefficients in $\mathbb{Q}$, as defined in Section 3 of [9]. Recall that if $X$ is a definable set then $H^*(X; \mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space such that $H^m(X; \mathbb{Q}) = 0$ for $m > \dim X$, and moreover $H^0(X; \mathbb{Q}) \cong \mathbb{Q}$ provided $X$ is also definably connected. For an element $x \in H^m(X; \mathbb{Q})$, we say $x$ has degree $m$ and write $\deg x = m$. In [9],

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it is observed that \( H^*(X; \mathbb{Q}) \cong \text{Hom}_\mathbb{Z}(H_*(X), \mathbb{Q}) \), where \( H_*(X) \) is the o-minimal homology with coefficients in \( \mathbb{Z} \). It will be more convenient for our purposes to take o-minimal homology with coefficients in \( \mathbb{Q} \). In this case, we also get \( H^*(X; \mathbb{Q}) \cong \text{Hom}_\mathbb{Q}(H_*(X; \mathbb{Q}), \mathbb{Q}) \), because \( H^*(X; \mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q} \) and \( \text{Hom}_\mathbb{Q}(H_*(X) \otimes \mathbb{Q}, \mathbb{Q}) \cong \text{Hom}_\mathbb{Z}(H_*(X), \mathbb{Q}) \).

Notice that \( H^*(X; \mathbb{Q}) \) is a \( \mathbb{Q} \)-algebra with product defined as follows: let \( d: X \to X \times X : x \mapsto (x, x) \) be the diagonal map, identify \( H^*(X \times X; \mathbb{Q}) \) with \( H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) \) via the o-minimal K"unneth formula for cohomology, and let \( x \cdot y := d^*(x \otimes y) \) (see [9] for details).

Moreover, we have the following result.

**Lemma 2.** Let \( G \) be a nontrivial definably connected definably compact definable group. Then there is a unique integer \( r > 0 \) and elements \( y_1, \ldots, y_r \) in the o-minimal \( \mathbb{Q} \)-cohomology algebra of \( G \) such that

\[
\begin{align*}
(1) \quad & \deg y_i \text{ is odd } (i = 1, \ldots, r), \\
(2) \quad & H^*(G; \mathbb{Q}) \text{ is freely generated, as a } \mathbb{Q}\text{-vector space, by } 1 \in H^0(G; \mathbb{Q}) \text{ and the monomials } y_i(1) \cdots y_i(l) \text{ with } 1 \leq i(1) < \cdots < i(l) \leq r.
\end{align*}
\]

**Proof.** By Corollary 3.6 in [9], there is a unique \( r \geq 0 \) and \( y_1, \ldots, y_r \) satisfying the requirements. Now, since \( G \) is definably connected, by Theorem 5.2 in [3], the top o-minimal homology group \( H_n(G) \) is nontrivial, where \( n = \dim G > 0 \). Therefore, \( r > 0 \).

We write \( \text{len } x = l \) if \( x \) is a monomial of length \( l \), i.e., \( x = y_{i(1)} \cdots y_{i(l)} \) with \( 1 \leq i(1) < \cdots < i(l) \leq r \) (with the notation of the above lemma). In the following, we are going to consider the maps \( p_k, k > 0 \), mentioned above. The computations in [4], for such maps, apply to our o-minimal context and yield the following.

**Lemma 3.** Let \( G \) be a definably connected definable group. For each \( k > 0 \), consider the definable continuous map \( p_k: G \to G : a \mapsto a^k \) for \( a \in G \). Then the map \( p_k^*: H^*(G; \mathbb{Q}) \to H^*(G; \mathbb{Q}) \) sends each monomial \( x \) to \( k^{\text{len } x} x \).

**Proof.** See Lemma 5.2 in [9].}

Let \( X \) be a definable set of dimension \( n \) and let \( f: X \to X \) be a continuous definable map. The **Lefschetz number** of \( f \) is defined as follows:

\[
L(f) = \sum_{m=0}^{n} (-1)^m \text{trace}(f_*: H_m(X; \mathbb{Q}) \to H_m(X; \mathbb{Q})).
\]

(See [5] for the semialgebraic case, and compare with the definition of the Lefschetz number as an intersection number in the o-minimal differentiable case given in [2].)
Note that the matrix of \( f^* : H^m(X; \mathbb{Q}) \to H^m(X; \mathbb{Q}) \) is the transpose of the matrix of \( f_* : H_n(X; \mathbb{Q}) \to H_n(X; \mathbb{Q}) \) \((f^* = \text{Hom}_{\mathbb{Q}}(f_*))\). Hence, we also get \( L(f) = \sum_{m=0}^{n} (-1)^m \text{trace}(f^* : H^m(X; \mathbb{Q}) \to H^m(X; \mathbb{Q})) \).

The next step is to make use of the definable version of the Lefschetz fixed point theorem (see Theorem 1.1 in [8], and Proposition 2 in [5] for the semialgebraic case).

**Theorem 4.** Let \( X \) be a closed and bounded definably connected set. If \( f : X \to X \) is a continuous definable map and \( L(f) \neq 0 \), then \( f \) has a fixed point.

**Corollary 5.** Let \( G \) be a definably compact definable group. If \( f : G \to G \) is a continuous definable map and \( L(f) \neq 0 \), then there is an element \( b \in G \) such that \( f(b) = b \).

Finally, we shall follow Brown [4] to compute \( L(p_k) \) for each \( k \geq 2 \), and prove Theorem 1.

**Lemma 6.** Let \( G \) be a definably connected definably compact definable group. Then, for each \( k \geq 2 \), \( L(p_k) = (1 - k)^r \), where \( r \) is as in Lemma 2.

**Proof.** Let \( \{1, x_1, \ldots, x_s\} \) be a basis of the \( \mathbb{Q} \)-vector space \( H^*(G; \mathbb{Q}) \), where the \( x_i \)'s are monomials. By Lemma 3, \( p_k^*(x_i) = k^{\text{len}x_i}x_i \) \((i = 1, \ldots, s)\) and \( p_k^*(1) = 1 \). Then the matrix of \( p_k^* : H^m(G; \mathbb{Q}) \to H^m(G; \mathbb{Q}) \) is either \( 0 \) (if \( H^m(G; \mathbb{Q}) = 0 \)) or a diagonal matrix with entry \( k^{\text{len}x_i} \) corresponding to each \( x_i \) in \( H^m(G; \mathbb{Q}) \). Therefore, \( L(p_k) = \sum_{i=1}^{s} (-1)^{\text{deg}x_i}k^{\text{len}x_i} + 1 \). On the other hand, the \( x_i \)'s are monomials (products of the \( y_j \)'s of Lemma 2) and the \( y_j \)'s are of odd degree, so that \( \text{deg}x_i \equiv \text{len}x_i \pmod{2} \), and hence \( L(p_k) = \sum_{i=1}^{s} (-1)^{\text{len}x_i}k^{\text{len}x_i} + 1 \). Since there are \( \binom{s}{l} \) monomials of length \( l \), we get \( L(p_k) = \sum_{l=1}^{r} \binom{s}{l}(-1)^{l}k^l + 1 = (1 - k)^r \).

**Proof of Theorem 1.** Fix \( a \in G \) and \( k \geq 2 \). We shall prove the existence of an element \( b \in G \) such that \( b^k = a \). Let \( f : G \to G : c \mapsto c^{k+1}a^{-1} \) for \( c \in G \). Since \( G \) is definably connected, there is a definable path \( \gamma : [0, 1] \to G \) such that \( \gamma(0) = a^{-1} \) and \( \gamma(1) = e \), where \( e \) is the neutral element of \( G \). Let \( F : [0, 1] \times G \to G : (t, c) \mapsto F(t, c) : = e^{k+1}\gamma(t) \).

Clearly, \( F \) is a definable homotopy between the maps \( F(0, -) = f \) and \( F(1, -) = p_{k+1} \). Hence, the induced cohomology morphisms \( f^* \) and \( (p_{k+1})^* \) (both from \( H^*(G; \mathbb{Q}) \to H^*(G; \mathbb{Q}) \)) coincide. Therefore, \( L(f) = L(p_{k+1}) \), and by Lemma 6, \( L(p_{k+1}) = (-k)^r (\neq 0) \). By Corollary 5, there is an element \( b \) in \( G \) such that \( b^{k+1}a^{-1} = b \), as required.

After this note was written, two alternative proofs of Theorem 1 have been given in [7] and [1], both papers making reference to a preprint version of this note.
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References


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