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Normal restrictions of the noncofinal ideal on $P_\kappa(\lambda)$

by

Pierre Matet (Caen)

Abstract. We discuss the problem of whether there exists a restriction of the noncofinal ideal on $P_\kappa(\lambda)$ that is normal.

0. Introduction. *Let κ be a regular uncountable cardinal, and $\lambda > \kappa$ be a cardinal.*

$I_{\kappa,\lambda}$ (respectively, $NS_{\kappa,\lambda}$) denotes the noncofinal (respectively, nonstationary) ideal on $P_\kappa(\lambda)$. Johnson and Baumgartner (see [8]) showed that there may exist a stationary subset A of $P_\kappa(\lambda)$ such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. Shelah [20] later established that it is even possible to have $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ for some B. In fact the following holds:

Proposition 0.1 ([20], [14]). *The following are equivalent:*

- (i) $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ *for some B.*
- (ii) $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) = \lambda$.
- (iii) $\mathrm{cf}(\lambda) < \kappa$, *and* $\overline{\mathrm{cof}}(NS_{\kappa,\tau}) \leq \lambda$ *for every cardinal τ with $\kappa \leq \tau < \lambda$.*

So the problem of whether there is B with $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ is pretty much solved. This paper is concerned with the more general problem of the existence of A such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. The existence of such an A may seem like a local property, but actually it has consequences for the entire nonstationary ideal $NS_{\kappa,\lambda}$:

Proposition 0.2.

- (i) ([15]) *Suppose $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A. Then* $\mathrm{cof}(NS_{\kappa,\lambda}) = u(\kappa,\lambda)$.
- (ii) ([17]) *If* $\mathrm{cof}(NS_{\kappa,\lambda}) = u(\kappa,\lambda)$, *then $NS_{\kappa,\lambda}$ is nowhere precipitous.*

If SSH holds and $\mathrm{cof}(NS_{\kappa,\lambda}) = u(\kappa,\lambda)$, then clearly $\mathrm{cf}(\lambda) < \kappa$. Hence the following holds:

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Proposition 0.3. *Assuming* GCH, *the following are equivalent:*

(i) $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ *for some* A.
(ii) $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ *for some* B.
(iii) $\mathrm{cf}(\lambda) < \kappa$.

In case SSH fails, the picture may be quite different, and we will see that "λ is regular and $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A" and "$\kappa \le \mathrm{cf}(\lambda) < \lambda$ and $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A" are both consistent relative to a large cardinal.

The following was already known:

Proposition 0.4 ([14]). *Let* $\theta < \kappa$ *be a cardinal for which there exists a* $[\lambda]^{<\theta}$*-normal ideal on* $P_\kappa(\lambda)$*, and* J *be an ideal on* $P_\kappa(\lambda)$ *such that* $J \subseteq NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$ *and* $\overline{\mathrm{cof}}(J) \le \lambda^{<\theta}$. *Then* $J|A = I_{\kappa,\lambda}|A$ *for some* $A \in (NS_{\kappa,\lambda}^{[\lambda]^{<\theta}})^*$.

This result raises two issues that we will address in this paper. Suppose that $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) > \lambda$ and we want to apply Proposition 0.4 to get an A such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. Then (a) we will have to use a θ that is uncountable, so we will need that $\nu^{\aleph_0} < \kappa$ for every cardinal $\nu < \kappa$, and (b) our A will be everywhere of uncountable cofinality since for $\theta > \omega$, the set of all $a \in P_\kappa(\lambda)$ such that $\mathrm{cf}(\sup(a \cap \eta)) = \omega$ for some limit ordinal η with $\kappa \le \eta \le \lambda$ and $\mathrm{cf}(\eta) \ge \omega_1$ lies in $NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$.

Now suppose to be definite that $\mathrm{cf}(\lambda) < \kappa$ and $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) = \lambda^+$ (which can be arranged by adding λ^+ Cohen subsets of κ to V, assuming that V satisfies GCH). Note that by our assumptions $\mathrm{cof}(NS_{\kappa,\lambda}) = u(\kappa,\lambda)$. We will show that if $(\mathrm{cf}(\lambda))^+ < \kappa$ and the principle $\mathcal{A}_{\kappa,\lambda}((\mathrm{cf}(\lambda))^+, \lambda^+)$ holds, then there is A such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$.

It is not clear how large this A is going to be, but this approach has the advantage that there are many pairs (κ,λ) for which the principle holds (e.g. all pairs (κ,λ) with $\omega_4 \le \kappa < \omega_\omega$ and $\lambda = \omega_\omega$).

A second principle, $\mathcal{B}_{\kappa,\lambda}(\kappa, \lambda^+)$, will imply the existence of A with $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ such that $\{\sup(a \cap \kappa) : a \in A\} \in NS_\kappa^*$. The question of the strength of $\mathcal{B}_{\kappa,\lambda}(\kappa, \lambda^+)$ is given special attention in the paper.

A third principle, $\mathcal{C}_{\kappa,\lambda}(\kappa, \lambda^+)$, will give A with $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ such that $\{a \cap \nu : a \in A\} \in NS_{\kappa,\nu}^*$ for every cardinal ν with $\kappa \le \nu < \lambda$.

Finally, in the case when $\mathrm{cf}(\lambda) \ne \omega$, a fourth principle, $\mathcal{D}_{\kappa,\lambda}^J((\mathrm{cf}(\lambda))^+, \lambda^+)$, where J denotes the noncofinal ideal on $\mathrm{cf}(\lambda)$, will yield an A with $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ that is large in the sense that it lies in the filter dual to the game ideal $NG_{\kappa,\lambda}$.

All four principles follow from the Almost Disjoint Sets principle ADS_λ, and so they will hold unless there are inner models with (fairly) large cardinals.

For a simple situation where our results apply, suppose that $V = L$ and $\mathrm{cf}(\lambda) < \kappa$, and consider the generic extension $(V^{\mathbb{Q}})^{\mathbb{P}}$, where $\mathbb{Q}$ adds λ^+ Cohen subsets of κ and $\mathbb{P}$ adds κ Cohen reals. We will see that in M, (a) for any uncountable cardinal $\theta < \kappa$, there is no $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$, and (b) there is no B such that $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$, but (c) $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A such that $\{a \cap \nu : a \in A\} \in NS^*_{\kappa,\nu}$ for every regular cardinal ν with $\kappa \le \nu < \lambda$.

The paper is organized as follows. Section 1 reviews basic material concerning $P_\kappa(\lambda)$ and its ideals. In Section 2 we generalize Proposition 0.4. Section 3 is concerned with the principle $\mathcal{A}_{\kappa,\lambda}(\tau, \pi)$. In Section 4 we deal with the special case when there exists a $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$. Sections 5–8 are respectively devoted to $\mathcal{B}_{\kappa,\lambda}(\tau, \pi)$, $\mathcal{C}_{\kappa,\lambda}(\tau, \pi)$, $\mathcal{D}^J_{\kappa,\lambda}(\nu^+, \pi)$ and ADS_λ. Finally in Section 9 we investigate the situation obtained by adding λ^+ Cohen subsets of κ to L.

1. Basic material. For a set A and a cardinal ρ, set $P_\rho(A) = \{a \subseteq A : |a| < \rho\}$.

NS_κ denotes the nonstationary ideal on κ.

For a regular infinite cardinal $\tau < \kappa$, E^κ_τ denotes the set of all $\delta < \kappa$ with $\mathrm{cf}(\delta) = \tau$.

Let $\mu \ge \kappa$ be a cardinal. $I_{\kappa,\mu}$ denotes the set of all $A \subseteq P_\kappa(\mu)$ such that $\{a \in A : b \subseteq a\} = \emptyset$ for some $b \in P_\kappa(\mu)$. By an *ideal* on $P_\kappa(\mu)$ we mean a collection J of subsets of $P_\kappa(\mu)$ such that (i) $I_{\kappa,\mu} \subseteq J$ and $P_\kappa(\mu) \notin J$, (ii) $P(A) \subseteq J$ for all $A \in J$, and (iii) $\bigcup x \in J$ for every $x \in P_\kappa(J)$.

Let J be an ideal on $P_\kappa(\mu)$. Set $J^+ = \{A \subseteq P_\kappa(\mu) : A \notin J\}$ and $J^* = \{P_\kappa(\mu) \setminus A : A \in J\}$. Put $J|A = \{B \subseteq P_\kappa(\mu) : B \cap A \in J\}$ for every $A \in J^+$.

$\mathrm{cof}(J)$ (respectively, $\overline{\mathrm{cof}}(J)$) denotes the least cardinality of any $X \subseteq J$ with the property that for any $A \in J$, there is x in $P_2(X)$ (respectively, $P_\kappa(X)$) with $A \subseteq \bigcup x$.

Given two infinite cardinals σ and ρ, J is *(σ, ρ)-regular* if there is $C_\alpha \in J^*$ for $\alpha < \rho$ such that $\bigcap_{\alpha \in z} C_\alpha = \emptyset$ for any $z \subseteq \rho$ with $|z| = \sigma$.

Given a cardinal $\pi \ge \kappa$ and $f : P_\kappa(\mu) \to P_\kappa(\pi)$, set $f(J) = \{X \subseteq P_\kappa(\pi) : f^{-1}(X) \in J\}$.

Given $\delta \le \mu$ and a cardinal $\theta \le \kappa$, J is *$[\delta]^{<\theta}$-normal* if for any $A \in J^+$, and any $f : A \to P_\theta(\delta)$ with the property that $f(a) \in P_{|a \cap \theta|}(a)$ for all $a \in A$, there is $B \in J^+ \cap P(A)$ such that f is constant on B.

LEMMA 1.1 ([15]).

(i) *Suppose that $\theta < \kappa$. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\mu)$ if and only if $|P_\theta(\nu)| < \kappa$ for every cardinal $\nu < \kappa \cap (\delta + 1)$.*

(ii) *Suppose that κ is a limit cardinal and $\delta \geq \theta = \kappa$. Then there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\mu)$ if and only if κ is Mahlo.*

If there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\mu)$, then $NS_{\kappa,\mu}^{[\delta]^{<\theta}}$ denotes the smallest such ideal.

Lemma 1.2 ([15]). *Suppose that σ is a cardinal with $\kappa \leq \sigma < \mu$, and $\kappa \leq \delta \leq \sigma$. Then $NS_{\kappa,\sigma}^{[\delta]^{<\theta}} = p(NS_{\kappa,\mu}^{[\delta]^{<\theta}})$, where $p : P_\kappa(\mu) \to P_\kappa(\sigma)$ is defined by $p(x) = x \cap \sigma$.*

For $f : P_\theta(\delta) \to P_\kappa(\mu)$, $C_f^{\kappa,\mu}$ denotes the set of all $a \in P_\kappa(\mu)$ such that $a \cap \theta \neq \emptyset$, and $f(e) \subseteq a$ for every $e \in P_{|a\cap\theta|}(a \cap \delta)$.

Lemma 1.3 ([15]). *Suppose that $\kappa \leq \delta$ and $2 \leq \theta$, and let $B \subseteq P_\kappa(\mu)$. Then $B \in NS_{\kappa,\mu}^{[\delta]^{<\theta}}$ if and only if $B \cap C_f^{\kappa,\mu} = \emptyset$ for some $f : P_{3\cup\theta}(\delta) \to P_\kappa(\mu)$.*

Lemma 1.4 ([14]). *Suppose that $\kappa \leq \delta$, $2 \leq \theta$ and J is $[\delta]^{<\theta}$-normal. Then either $\mathrm{cf}(\overline{\mathrm{cof}}(J)) < \kappa$, or $\mathrm{cf}(\overline{\mathrm{cof}}(J)) > |\delta|^{<\overline{\theta}}$, where $\overline{\theta} = \theta$ if $\theta < \kappa$, or $\theta = \kappa$ and κ is a limit cardinal, and $\overline{\theta} = \nu$ if $\theta = \kappa = \nu^+$.*

It is simple to see that if J is $[\delta]^{<2}$-normal, then it is $[\delta]^{<\omega}$-normal.

Set $NS_{\kappa,\mu}^\delta = NS_{\kappa,\mu}^{[\delta]^{<2}}$.

J is *normal* if it is $[\mu]^{<2}$-normal. We put $NS_{\kappa,\mu} = NS_{\kappa,\mu}^\mu$.

Given four cardinals π, σ, τ and χ with $\pi \geq \sigma \geq \tau \geq \omega$ and $\tau \geq \chi \geq 2$, $\mathrm{cov}(\pi, \sigma, \tau, \chi)$ denotes the least cardinality of any $A \subseteq P_\sigma(\pi)$ with the property that for any $b \in P_\tau(\pi)$, there is $z \in P_\chi(A)$ with $b \subseteq \bigcup z$.

In case $\sigma = \tau$ and $\chi = 2$, we let $\mathrm{cov}(\pi, \sigma, \tau, \chi) = u(\sigma, \pi)$.

Shelah's Strong Hypothesis (SSH) asserts that given two uncountable cardinals ν and χ with $\mathrm{cf}(\nu) = \nu \leq \chi$, $u(\nu, \chi)$ equals χ if $\mathrm{cf}(\chi) \geq \nu$, and χ^+ otherwise.

Lemma 1.5 ([10]). *Given a cardinal σ with $\kappa \leq \sigma < \mu$, the following are equivalent:*

(i) $NS_{\kappa,\mu}^\sigma | A = I_{\kappa,\mu} | A$ *for some* $A \in NS_{\kappa,\mu}^*$.
(ii) $\overline{\mathrm{cof}}(NS_{\kappa,\sigma}) \leq \mu = \mathrm{cov}(\mu, \sigma^+, \sigma^+, \kappa)$.

$\overline{\partial}_\kappa$ *denotes the smallest cardinality of any $F \subseteq {}^\kappa\kappa$ with the property that for any $g \in {}^\kappa\kappa$, there is $z \in P_\kappa(F)$ such that $g(\alpha) < \bigcup_{f\in z} f(\alpha)$ for every $\alpha \in \kappa$.*

Lemma 1.6 ([16]). $\overline{\partial}_\kappa = \overline{\mathrm{cof}}(NS_{\kappa,\kappa})$.

For $B \subseteq P_\kappa(\mu)$, the two-player game $H_{\kappa,\mu}(B)$ is defined as follows. The game lasts ω moves, with player I making the first move. I and II alternately pick members of $P_\kappa(\mu)$, thus building a sequence $\langle a_n : n < \omega \rangle$. II wins the game whenever $\bigcup_{n<\omega} a_n \in B$.

$NG_{\kappa,\mu}$ denotes the collection of all $A \subseteq P_\kappa(\mu)$ such that player II has a winning strategy in the game $H_{\kappa,\mu}(P_\kappa(\mu) \setminus A)$.

LEMMA 1.7 ([11]).

(i) *$NG_{\kappa,\mu}$ is a normal ideal on $P_\kappa(\mu)$.*
(ii) *There is $A \in NG^*_{\kappa,\mu}$ such that $\mathrm{cf}(\sup(a \cap \eta)) = \omega$ whenever $a \in A$ and η is a limit ordinal with $\kappa \leq \eta \leq \mu$ and $\mathrm{cf}(\eta) \geq \kappa$.*
(iii) *Let σ be a cardinal with $\kappa \leq \sigma < \mu$. Then $NG_{\kappa,\sigma} = p(NG_{\kappa,\mu})$, where $p : P_\kappa(\mu) \to P_\kappa(\sigma)$ is defined by $p(x) = x \cap \sigma$.*

κ is *mildly μ-ineffable* if given $t_a : a \to 2$ for $a \in P_\kappa(\mu)$, there is $g : \mu \to 2$ with the property that for any $a \in P_\kappa(\mu)$, there is $b \in P_\kappa(\mu)$ such that $a \subseteq b$ and $g{\restriction}a = t_b{\restriction}a$.

LEMMA 1.8 ([22]). *Suppose that κ is mildly μ-ineffable and $\mathrm{cf}(\mu) \geq \kappa$. Then $\mu^{<\kappa} = \mu$.*

Suppose that

- σ is a cardinal with $\mathrm{cf}(\mu) \leq \sigma < \mu$.
- $\langle \mu_i : i < \sigma \rangle$ is a one-to-one sequence of regular cardinals less than μ such that $\sigma < \mu_0$ and $\sup(\{\mu_i : i < \sigma\}) = \mu$.
- I is a proper ideal on σ such that for any cardinal $\chi < \mu$, $\{i \in \sigma : \mu_i \leq \chi\} \in I$.
- π is a cardinal greater than μ.
- $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ is an increasing, cofinal sequence in $(\prod_{i<\sigma} \mu_i, <_I)$, where $g <_I h$ whenever $\{i < \sigma : g(i) < h(i)\} \in I^*$.

Then $\vec{f}$ is a *scale of length π for μ*.

Let $\delta < \pi$ be an infinite limit ordinal. Then δ is a *good* (respectively, *remarkably good*) *point* for $\vec{f}$ if we may find a cofinal (respectively, closed unbounded) subset $X \subseteq \delta$, and $Z_\xi \in I$ for $\xi \in X$, such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \sigma \setminus (Z_\beta \cup Z_\xi)$. Further, δ is a *better point* for $\vec{f}$ if we may find a closed unbounded subset X of δ, and $Z_\xi \in I$ for $\xi \in X$, such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \sigma \setminus Z_\xi$. Finally, δ is a *very good point* for $\vec{f}$ if there is a closed unbounded subset X of δ, and $Z \in I$, such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \sigma \setminus Z$.

Note that

$$\text{very good} \Rightarrow \text{better} \Rightarrow \text{remarkably good} \Rightarrow \text{good}.$$

It is easy to see that points of small cofinality are better:

LEMMA 1.9 ([3]). *Let $\delta < \pi$ be an infinite limit ordinal such that I is $\mathrm{cf}(\delta)$-complete. Then δ is a better point for $\vec{f}$.*

Proof. Select a closed unbounded subset X of δ with $\mathrm{o.t.}(X) = \mathrm{cf}(\delta)$. For $\beta < \xi$ in X pick $Z_{\beta\xi} \in I$ so that $f_\beta(i) < f_\xi(i)$ whenever $i \in \mu \setminus Z_{\beta\xi}$.

Now given $\xi \in X$, put $Z_\xi = \bigcup_{\beta \in X \cap \xi} Z_{\beta\xi}$. Then clearly $Z_\xi \in I$. Moreover, $f_\beta(i) < f_\xi(i)$ whenever $\beta \in X \cap \xi$ and $i \in \sigma \setminus Z_\xi$. ■

It immediately follows that every infinite limit ordinal $\delta < \pi$ such that I is $(\mathrm{cf}(\delta))^+$-complete is a very good point for $\vec{f}$.

Let us also mention the following, which is readily checked.

Lemma 1.10. *Let $\delta < \pi$ be an infinite limit ordinal such that $\mathrm{cf}(\delta)$ is a weakly compact cardinal greater than σ. Then δ is a good point for $\vec{f}$.*

The scale $\vec{f}$ is *good* (respectively, *remarkably good*, *better*, *very good*) if there is a closed unbounded subset C of π with the property that every limit ordinal δ in C such that $\mathrm{cf}(\delta) < \mu$ and I is not $\mathrm{cf}(\delta)$-complete is a good (respectively, remarkably good, better, very good) point for $\vec{f}$.

We refer to other sources for the definitions of other notions of pcf theory. The definitions of $\mathrm{pp}(\mu)$, $\mathrm{pp}^+(\mu)$ and $\mathrm{pp}_{\Gamma(\kappa,\omega_1)}(\mu)$ can be found in [19, pp. 39 and 41]. See [3, Definitions 2.3, 3.8 and 6.3] for the definition of the three principles $\square^*_\mu$, VGS_μ and AP_μ.

2. A sufficient condition for $K|A = I_{\kappa,\lambda}|A$. *Throughout the remainder of the paper τ will denote an infinite cardinal less than or equal to κ, and π a cardinal greater than λ.*

Definition. A (τ, λ, π)-*sequence* is a one-to-one sequence $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ of elements of $P_\tau(\lambda)$ with $y_\alpha = \{\alpha\}$ for every $\alpha < \lambda$.

Definition. For a (κ, λ, π)-sequence $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$, $A(\vec{y})$ denotes the set of all $x \in P_\kappa(\pi)$ such that $\{\alpha < \pi : y_\alpha \subseteq x\} \subseteq x$.

Let J be a normal ideal on $P_\kappa(\pi)$, and let $p : P_\kappa(\pi) \to P_\kappa(\lambda)$ be defined by $p(x) = x \cap \lambda$.

Lemma 2.1. *Let $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ be a (κ, λ, π)-sequence. Suppose that $C_\alpha \in (p(J))^*$ for $\alpha < \pi$. Then there is $D \in J^*$ such that $x \cap \lambda \in C_\alpha$ for every $x \in D \cap A(\vec{y})$, and every $\alpha < \pi$ such that $y_\alpha \subseteq x$.*

Proof. Let $D = \{x \in P_\kappa(\pi) : \forall \alpha \in x\ (x \cap \lambda \in C_\alpha)\}$. ■

Proposition 2.2. *Let $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ be a (κ, λ, π)-sequence. Suppose that $A(\vec{y}) \in J^+$, and $K \subseteq p(J)$ is an ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(K) \le \pi$. Then there is $D \in J^*$ such that $K|A = I_{\kappa,\lambda}|A$, where $A = p``(D \cap A(\vec{y}))$.*

Proof. Select $C_\alpha \in K^*$ for $\alpha < \pi$ so that for any $C \in K^*$, there is $z \in P_\kappa(\pi) \setminus \{\emptyset\}$ with $\bigcap_{\alpha \in z} C_\alpha \subseteq C$. By Lemma 2.1, there is $D \in J^*$ such that $x \cap \lambda \in C_\alpha$ whenever $x \in D \cap A(\vec{y})$ and $y_\alpha \subseteq x$. Now fix $B \in I^+_{\kappa,\lambda}$ with $B \subseteq \{x \cap \lambda : x \in D \cap A(\vec{y})\}$. Let us show that $B \in K^+$. Thus let $C \in K^*$. Pick $z \in P_\kappa(\pi) \setminus \{\emptyset\}$ with $\bigcap_{\alpha \in z} C_\alpha \subseteq C$, and set $b = \bigcup_{\alpha \in z} y_\alpha$. Then clearly $\{a \in B : b \subseteq a\} \subseteq C$. ■

Corollary 2.3. *Suppose that there exists a (κ,λ,π)-sequence $\vec{y}$ such that $A(\vec{y}) \in NS^+_{\kappa,\pi}$, and let $K \subseteq NS_{\kappa,\lambda}$ be an ideal on $P_\kappa(\lambda)$ with $\overline{\mathrm{cof}}(K) \leq \pi$. Then $K|A = I_{\kappa,\lambda}|A$ for some $A \in NS^+_{\kappa,\lambda}$.*

3. $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$. In this section we start our search for (κ,λ,π)-sequences $\vec{y}$ such that $A(\vec{y}) \in NS^+_{\kappa,\pi}$.

Definition. An $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$-*sequence* is a (τ,λ,π)-sequence $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ with $|\{\alpha < \pi : y_\alpha \subseteq a\}| < \kappa$ for every $a \in P_\kappa(\lambda)$.

The following is readily checked.

Proposition 3.1. *Let $\vec{y}$ be a (κ,λ,π)-sequence. Then $\vec{y}$ is an $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$-sequence whenever $A(\vec{y}) \in I^+_{\kappa,\pi}$.*

Definition. $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ asserts the existence of an $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$-sequence.

If $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ holds, then by a result of [12], $\pi \leq \mathrm{cov}(\lambda,\kappa,\tau,2)$. In particular, $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ implies that $\pi \leq u(\kappa,\lambda)$.

The following is immediate.

Proposition 3.2. *The following are equivalent:*

(i) *$\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ holds.*
(ii) *$I_{\kappa,\lambda}$ is (κ,π)-regular.*
(iii) *There is $B \in I^+_{\kappa,\lambda}$ such that $I_{\kappa,\lambda}|B$ is (κ,π)-regular.*

Proposition 3.3 ([13]). *Let $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ be an $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$-sequence. Then $\overline{\mathrm{cof}}(I_{\kappa,\pi}|A(\vec{y})) \leq \lambda$.*

Proof. Fix $c \in P_\kappa(\pi)$, and set $d = \bigcup_{\alpha\in c} y_\alpha$. Then

$$A(\vec{y}) \cap \{x \in P_\kappa(\pi) : d \subseteq x\} \subseteq \{z \in P_\kappa(\pi) : c \subseteq z\}. \blacksquare$$

Conversely, if $\overline{\mathrm{cof}}(J) \leq \lambda$ for some ideal J on $P_\kappa(\pi)$, then by [14, Proposition 5.7], $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ holds.

By Proposition 3.3, $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ implies that $\mathrm{cof}(I_{\kappa,\pi}) = \mathrm{cof}(I_{\kappa,\lambda})$. This can be generalized as follows.

Proposition 3.4 ([16]). *Suppose that $\vec{y}$ is an $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$-sequence. Then for some $D \in NS^*_{\kappa,\pi}$, there is an isomorphism f from $(P_\kappa(\lambda),\subset)$ onto $(D \cap A(\vec{y}),\subset)$ with the following property: for any $\delta \leq \lambda$, and any cardinal $\theta \leq \kappa$ for which there exists a $[\delta]^{<\theta}$-normal ideal on $P_\kappa(\pi)$, $f(NS^{[\delta]^{<\theta}}_{\kappa,\lambda}) = NS^{[\delta]^{<\theta}}_{\kappa,\pi}|(D \cap A(\vec{y}))$ (and hence $\overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\pi}|(D\cap A(\vec{y}))) \leq \overline{\mathrm{cof}}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda})$ and $\mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\pi}) = \mathrm{cof}(NS^{[\delta]^{<\theta}}_{\kappa,\lambda})$).*

Proposition 3.5. *Suppose that $\vec{y}$ is an $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$-sequence, where $\tau = \mathrm{cf}(\tau) < \kappa$. Then the following hold:*

(i) *For any regular cardinal* χ *with* $\tau \le \chi < \kappa$, *we have* $\{x \in A(\vec{y}) : \mathrm{cf}(\sup(x \cap \kappa)) = \chi\} \in NS^+_{\kappa,\pi}$.

(ii) *Let* $\theta \le \kappa$ *be an infinite cardinal such that there exists a* $[\pi]^{<\theta}$*-normal ideal on* $P_\kappa(\pi)$. *Then* $A(\vec{y}) \in (NS^{[\pi]^{<\theta}}_{\kappa,\pi})^+$.

Proof. By the proof of [14, Proposition 5.6(ii)]. ∎

Corollary 3.6. *Suppose that* $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ *holds, where* $\tau = \mathrm{cf}(\tau) < \kappa$, *and let* $K \subseteq NS_{\kappa,\lambda}$ *be an ideal on* $P_\kappa(\lambda)$ *with* $\overline{\mathrm{cof}}(K) \le \pi$. *Then there is* A *such that* (a) $\{a \in A : \mathrm{cf}(\sup(a \cap \kappa)) = \chi\} \in NS^+_{\kappa,\lambda}$ *for every regular cardinal* χ *with* $\tau \le \chi < \kappa$, *and* (b) $K|A = I_{\kappa,\lambda}|A$.

Proof. By Propositions 2.2 and 3.5. ∎

For example, suppose that in V, GCH holds, σ is a strong cardinal, and $\pi = \mathrm{cf}(\pi) > \sigma$. Then by work of Gitik and Magidor [6], there is a notion of forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, (a) all cardinals are preserved, (b) $\mathrm{cf}(\sigma) = \omega$, (c) $2^\chi = \chi^+$ for any infinite cardinal $\chi < \sigma$, and (c) $2^\sigma = \pi$ and in fact (as was kindly pointed out to the author by Moti Gitik) $2^\nu = \pi$ for any cardinal ν with $\sigma \le \nu < \pi$. Working in $V^{\mathbb{P}}$, suppose that $\kappa < \sigma \le \lambda < \pi$ and κ is not the successor of a cardinal of cofinality ω. Then by Proposition 0.4, there is $A \in (NS^{[\lambda]^{<\omega_1}}_{\kappa,\lambda})^*$ such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$.

Set $W = V^{\mathbb{P}}$. In W, let $\mathbb{Q}$ be the notion of forcing to add $\aleph_4$ Cohen reals. Then clearly in $W^{\mathbb{Q}}$, $2^{\aleph_j} = \aleph_4$ for every $j < 4$, and $(2^\nu)^{W^{\mathbb{Q}}} = (2^\nu)^W$ for every cardinal $\nu \ge \omega_4$. Working in $W^{\mathbb{Q}}$, suppose that $\kappa = \omega_4$ and $\sigma \le \lambda < \pi$. Proposition 0.4 no longer applies, since now there does not exist any $[\lambda]^{<\omega_1}$-normal ideal on $P_\kappa(\lambda)$. So we take another route. By a result of Shelah [7, p. 369], $\mathrm{pp}^+(\sigma) > \mathrm{cov}(\sigma,\sigma,\omega_1,2) = \pi$, so by [12, Proposition 4.6(i)], $\mathcal{A}_{\kappa,\lambda}(\omega_1,\pi)$ holds. Hence by Corollary 3.6, $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A.

Note that if $\mathrm{cf}(\lambda) \ge \kappa$ and $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$, then clearly $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}|A) \le \lambda$, and in fact $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}|A) < \lambda$ since by Lemma 1.4, $\mathrm{cf}(\overline{\mathrm{cof}}(NS_{\kappa,\lambda}|A)) < \kappa$.

Next we consider some situations when it can be deduced from $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ that $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ holds for some regular $\tau < \kappa$.

Proposition 3.7. *Suppose that* $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ *holds,* $\mathrm{cf}(\lambda) < \tau = \mathrm{cf}(\tau) < \kappa$, *and* $\mathrm{cov}(\lambda',\kappa,\kappa,\tau) \le \lambda$ *for every cardinal* λ' *with* $\kappa \le \lambda' < \lambda$. *Then* $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ *holds.*

Proof. The proof is an easy modification of that of [12, Corollary 2.13]. ∎

Proposition 3.8 ([12]). *Suppose that* $\mathcal{A}_{\kappa,\lambda}(\kappa,\pi)$ *holds,* κ *is a limit cardinal, and* $\mathrm{cf}(\pi) \ne \kappa$. *Then* $\mathcal{A}_{\kappa,\lambda}(\tau,\pi)$ *holds for some regular* $\tau < \kappa$.

LEMMA 3.9 (see [4, Theorem 7.12, p. 48]). *Let μ be an infinite cardinal. Then μ^ν assumes only finitely many values for ν with $2^\nu < \mu$.*

PROPOSITION 3.10. *Suppose that κ is inaccessible and $\lambda^{<\kappa} > \lambda$. Then $\mathcal{A}_{\kappa,\lambda}(\tau, \lambda^{<\kappa})$ holds for some regular $\tau < \kappa$.*

Proof. By Lemma 3.9, there is a regular infinite cardinal $\tau < \kappa$ such that $\lambda^{<\kappa} = \lambda^{<\tau}$. Then clearly $|P_\tau(\lambda) \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. ∎

PROPOSITION 3.11. *Suppose that $\mathcal{A}_{\kappa,\lambda}(\tau, \pi)$ holds, κ' is a regular cardinal with $\kappa < \kappa' < \lambda$, and $\mathrm{cov}(\nu, \kappa, \tau, 2) < \kappa'$ for every cardinal ν with $\kappa \le \nu < \kappa'$. Then $\mathcal{A}_{\kappa',\lambda}(\tau, \pi)$ holds.*

Proof. The proof is a straightforward modification of that of [14, Proposition 5.5]. ∎

PROPOSITION 3.12 ([12]). *Let ρ be the largest limit cardinal less than or equal to κ. Assume that $\mathrm{cf}(\lambda) < \kappa$ and one of the following conditions is satisfied:*

(a) $\rho = \kappa$.
(b) $\mathrm{cf}(\lambda) < \rho$ *and* $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\rho)$.
(c) $\mathrm{cf}(\lambda) = \mathrm{cf}(\rho) < \rho$ *and* $\min(\mathrm{pp}(\rho), \rho^{+3}) < \kappa$.
(d) $\mathrm{cf}(\lambda) \ge \rho$ *and* $\min(2^{\mathrm{cf}(\lambda)}, (\mathrm{cf}(\lambda))^{+3}) < \kappa$.
(e) *For some regular cardinal σ with* $\max(\rho, \mathrm{cf}(\lambda)) < \sigma \le \kappa$, *$\lambda$ carries a scale of length λ^+ for which almost all (in the sense of the nonstationary ideal) points with cofinality σ are good.*

Then $\mathcal{A}_{\kappa,\lambda}((\mathrm{cf}(\lambda))^+, \lambda^+)$ holds.

By a result of Todorcevic, it is consistent relative to a 2-huge cardinal that $\mathcal{A}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ fails (see [12, Propositions 3.18 and 3.19]).

Magidor (see [2, Theorem 17.1]) proved that under MM, there is no scale for ω_ω which is good at every point of cofinality ω_1.

QUESTION. Is it consistent relative to some large cardinal that "MM and $\mathcal{A}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ both hold"?

QUESTION. Is it consistent relative to some large cardinal that "$\overline{\mathrm{cof}}(NS_{\omega_1,\omega_\omega}) = \aleph_{\omega+1}$ but there is no A such that $NS_{\omega_1,\omega_\omega}|A = I_{\omega_1,\omega_\omega}|A$"?

Another problem which is worth mentioning is whether there is a converse to Corollary 3.6.

QUESTION. Suppose $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) > \lambda$ and $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. Does then $\mathcal{A}_{\kappa,\lambda}(\kappa, \overline{\mathrm{cof}}(NS_{\kappa,\lambda}))$ hold?

4. $NS_{\kappa,\lambda^{<\theta}}^{[\lambda^{<\theta}]^{<\theta}}$. Abe [1] proved that if κ is Mahlo and $\lambda^{<\kappa} > \lambda$, then we may find $B \in (NS_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}})^*$, and an isomorphism f from $(P_\kappa(\lambda), \subset)$

onto $(B, \subset)$ such that $f(NS_{\kappa,\lambda}^{[\lambda]^{<\kappa}}) = NS_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}}$. In this section we prove the corresponding result for $NS_{\kappa,\lambda^{<\theta}}^{[\lambda^{<\theta}]^{<\theta}}$ with $\theta < \kappa$.

Suppose $\theta < \kappa$ is a regular cardinal such that $\lambda^{<\theta} > \lambda$. Suppose further that there exists a $[\lambda]^{<\theta}$-normal ideal on $P_\kappa(\lambda)$.

Let $\vec{y} = \langle y_\alpha : \alpha < \lambda^{<\theta}\rangle$ be a one-to-one enumeration of the elements of $P_\theta(\lambda)$ such that $y_\alpha = \{\alpha\}$ for every $\alpha < \lambda$.

The following is immediate:

Proposition 4.1. $A(\vec{y}) \in (NS_{\kappa,\lambda^{<\theta}}^{[\lambda]^{<\theta}})^*$.

Proposition 4.2. *There is $D \in NS_{\kappa,\lambda^{<\theta}}^*$, and an isomorphism f from $(P_\kappa(\lambda), \subset)$ onto $(D \cap A(\vec{y}), \subset)$ such that*

$$f(NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}) = NS_{\kappa,\lambda^{<\theta}}^{[\lambda^{<\theta}]^{<\theta}} = NS_{\kappa,\lambda^{<\theta}}^{[\lambda]^{<\theta}} | (D \cap A(\vec{y})).$$

Proof. Set $\pi = \lambda^{<\theta}$ and $D = \{x \in P_\kappa(\pi) : \forall\alpha \in x\ (y_\alpha \subseteq x)\}$. It is immediate that $D \in NS_{\kappa,\pi}^*$. Define $f : P_\kappa(\lambda) \to P_\kappa(\pi)$ by setting $f(a) = \{\alpha < \pi : y_\alpha \subseteq a\}$. Note that $f(a) \cap \lambda = a$. Put $B = \mathrm{ran}(f)$. Then clearly $B = D \cap A(\vec{y})$, and moreover $B \in (NS_{\kappa,\pi}^{[\pi]^{<\theta}})^*$. It is simple to see that f is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(B, \subset)$. Furthermore, $f^{-1}(X) \in I_{\kappa,\lambda}$ for every $X \in I_{\kappa,\pi}$. Set $J = NS_{\kappa,\lambda}^{[\lambda]^{<\theta}}$. Then clearly $f(J)$ is an ideal on $P_\kappa(\pi)$. Note that $B \in (f(J))^*$.

Claim 1. *$f(J)$ is $[\pi]^{<\theta}$-normal.*

Proof of Claim 1. Fix $X \in (f(J))^+$ with $X \subseteq B \cap \{x \in P_\kappa(\pi) : \theta \subseteq x\}$, and $h : X \to P_\theta(\pi)$ with $h(x) \subseteq x$ for every $x \in X$. Define $k : f^{-1}(X) \to P_\theta(\lambda)$ by $k(a) = \bigcup_{\alpha \in h(f(a))} y_\alpha$. There are $A \in J^+ \cap P(f^{-1}(X))$ and $e \in P_\theta(\lambda)$ such that k takes the constant value e on A. Put $z = \{\alpha < \pi : y_\alpha \subseteq e\}$ and $T = f``A$. Then clearly $T \in (f(J))^+ \cap P(X)$, and moreover $h(x) \subseteq z$ for every $x \in T$. Since $|P_\theta(z)| < \kappa$, there must be $W \in (f(J))^+ \cap P(T)$ and $d \in P_\theta(z)$ such that $h(x) = d$ for all $x \in W$, which completes the proof of Claim 1.

Claim 2. *$f(J) \subseteq NS_{\kappa,\pi}^{[\lambda]^{<\theta}} | B$.*

Proof of Claim 2. Fix $Z \in f(J)$. Set $Q = Z \cap B \cap \{x \in P_\kappa(\pi) : \theta \subseteq x\}$. Since $f^{-1}(Q) \in J$, we may find $g : P_\theta(\lambda) \to P_\kappa(\lambda)$ such that $f^{-1}(Q) \cap C_g^{\kappa,\lambda} = \emptyset$. Then clearly $Q \cap C_g^{\kappa,\pi} = \emptyset$, and hence $Z \cap B \in NS_{\kappa,\pi}^{[\lambda]^{<\theta}}$. This completes the proof of Claim 2.

By Claims 1 and 2,

$$NS_{\kappa,\pi}^{[\pi]^{<\theta}} \subseteq f(J) \subseteq NS_{\kappa,\pi}^{[\lambda]^{<\theta}} | B \subseteq NS_{\kappa,\pi}^{[\pi]^{<\theta}},$$

so $f(J) = NS_{\kappa,\pi}^{[\pi]^{<\theta}} = NS_{\kappa,\pi}^{[\lambda]^{<\theta}} | B$. ■

5. $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$

Definition. A $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$-*sequence* is a (τ,λ,π)-sequence $\vec{y}=\langle y_\alpha : \alpha<\pi\rangle$ with the property that for each nonempty e in $P_{\kappa^+}(\pi)$, there is a $<\kappa$-to-one $g \in \prod_{\alpha\in e} y_\alpha$.

Definition. $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$ asserts the existence of a $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$-sequence.

Proposition 5.1. *Let $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ be a (τ,λ,π)-sequence. Then the following are equivalent:*

(i) *$\vec{y}$ is a $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$-sequence.*
(ii) *For any $D \in NS^*_{\kappa,\pi}$, there is $x_\beta \in D\cap A(\vec{y})$ for $\beta<\kappa$ such that*
  (1) *$x_\gamma \subset x_\beta$ and $\sup(x_\gamma\cap\kappa) < \sup(x_\beta\cap\kappa)$ for all $\gamma<\beta$, and*
  (2) *$x_\beta = \bigcup_{\gamma<\beta} x_\gamma$ in case β is a nonzero limit ordinal.*

Proof. (i)→(ii): By the proof of [14, Proposition 5.11].

(ii)→(i): Suppose that (ii) holds, and fix $e\subseteq\pi$ with $|e|=\kappa$. Let $\langle \alpha_i : i<\kappa\rangle$ be a one-to-one enumeration of the elements of e. Now let D be the set of all $x\in P_\kappa(\pi)$ such that (a) for any $i\in x\cap\kappa$, $y_{\alpha_i}\subseteq x$, (b) for any $i\in\kappa$ such that $\alpha_i\in x$, $i\in x\cap\kappa$, and (c) $x\cap\kappa$ is an infinite limit ordinal. Note that for any $x\in D\cap A(\vec{y})$, $x\cap\kappa = \{i\in\kappa : y_{\alpha_i}\subseteq x\}$. Since $D\in NS^*_{\kappa,\pi}$, we may find $x_\beta\in D\cap A(\vec{y})$ for $\beta<\kappa$ such that (1) $x_\gamma\subset x_\beta$ and $x_\gamma\cap\kappa < x_\beta\cap\kappa$ for all $\gamma<\beta$, and (2) $x_\beta = \bigcup_{\gamma<\beta} x_\gamma$ in case β is a nonzero limit ordinal. Define $k\in\prod_{i\in x_0\cap\kappa} y_{\alpha_i}$ by $k(i)=$ the least element of y_{α_i}, and $h_\beta\in\prod_{i\in(x_{\beta+1}\cap\kappa)\setminus x_\beta} y_{\alpha_i}$ for $\beta<\kappa$ by $h_\beta(i)=$ the least element of $y_{\alpha_i}\setminus x_\beta$. Set $h = k\cup\bigcup_{\beta<\kappa} h_\beta$. Then clearly, $h\in\prod_{i\in\kappa} y_{\alpha_i}$. Moreover, h is $<\kappa$-to-one. ∎

Proposition 5.2. *Let $\vec{y}$ be a $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$-sequence. Then the following hold:*

(i) *For any $D\in NS^*_{\kappa,\pi}$,*
$$\{\sup(x\cap\kappa) : x\in D\cap A(\vec{y})\}\in NS^*_\kappa.$$
(ii) *Let $\theta\le\kappa$ be an infinite cardinal such that there exists a $[\pi]^{<\theta}$-normal ideal on $P_\kappa(\pi)$. Then $A(\vec{y})\in (NS^{[\pi]^{<\theta}}_{\kappa,\pi})^+$.*

Proof. (i) By Proposition 5.1.

(ii) By the proof of Proposition 5.11 in [14]. ∎

Corollary 5.3. *Suppose that $\mathcal{B}_{\kappa,\lambda}(\kappa,\pi)$ holds, and let $K\subseteq NS_{\kappa,\lambda}$ be an ideal with $\overline{\mathrm{cof}}(K)\le\pi$. Then there is $A\in NS^+_{\kappa,\lambda}$ such that* (a) $\{\sup(a\cap\kappa) : a\in A\}\in NS^*_\kappa$, *and* (b) $K|A = I_{\kappa,\lambda}|A$.

Proof. By Propositions 2.2 and 5.2. ∎

Let us now show that we may find A as above with the additional property that $A\in NG^+_{\kappa,\lambda}$:

Proposition 5.4. *Let $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ be a $\mathcal{B}_{\kappa,\lambda}(\kappa,\pi)$-sequence. Then $A(\vec{y}) \in NG^+_{\kappa,\pi}$.*

Proof. Fix $X \in NG^*_{\kappa,\pi}$, and let σ be a winning strategy for II in $H_{\kappa,\pi}(X)$.

Given $\beta < \kappa$ and $n \in \omega \cap (\beta+1)$, let $K_{n\beta}$ denote the set of all increasing functions $k : n+1 \to \beta+1$. Now define $e_\beta \in P_\kappa(\pi)$ for $\beta < \kappa$ by

- $e_0 = \emptyset$.
- $e_\beta = \bigcup_{\gamma<\beta} e_\gamma$ if β is an infinite limit ordinal.
- $e_{\beta+1} = \beta \cup e_\beta \cup \sigma(\emptyset) \cup \{\alpha < \pi : y_\alpha \subseteq e_\beta\} \cup (\bigcup_{\alpha\in e_\beta} y_\alpha) \cup (\bigcup\{z_k : k \in \bigcup_{n\in\omega\cap(\beta+1)} K_{n\beta}\})$, where $z_k = \sigma(e_{k(0)}, \dots, e_{k(n)})$ if $n \in \omega \cap (\beta+1)$ and $k \in K_{n\beta}$.

Set $e = \bigcup_{\beta<\kappa} e_\beta$, and select a $<\kappa$-to-one $g \in \prod_{\alpha\in e} y_\alpha$. Let D be the set of all $\beta < \kappa$ such that $\bigcup\{g^{-1}(\{\xi\}) : \xi \in e_\eta \cap \mathrm{ran}(g)\} \subseteq e_\beta$ for every $\eta < \beta$. Then $D \in NS^*_\kappa$, since for any $\eta < \kappa$, $\bigcup\{g^{-1}(\{\xi\}) : \xi \in e_\eta \cap \mathrm{ran}(g)\} \in P_\kappa(e) = \bigcup_{\beta<\kappa} P(e_\beta)$. Pick $\beta \in D \cap E^\kappa_\omega$, and let $\langle\beta_i : i < \omega\rangle$ be an increasing sequence of ordinals cofinal in β. Put $x = \bigcup_{i<\omega} e_{\beta_i}$. Then $x \in X$, since for any $i < \omega$, $\sigma(e_{\beta_0}, \dots, e_{\beta_i}) \subseteq e_{\beta_i+1} \subseteq e_{\beta_{i+1}}$. Let us show that $x \in A(\vec{y})$. Thus let $\alpha < \pi$ be such that $y_\alpha \subseteq x$. Then obviously $y_\alpha \subseteq e_\gamma$ for some $\gamma < \kappa$, so $\alpha \in e$. There must be $\ell < \omega$ such that $g(\alpha) \in e_{\beta_\ell}$. Then $\alpha \in g^{-1}(\{g(\alpha)\}) \subseteq e_\beta = \bigcup_{i<\omega} e_{\beta_i}$, and consequently $\alpha \in x$. ■

We will now see that $\mathcal{B}_{\kappa,\lambda}(\kappa,\pi)$ follows from the existence of certain scales.

Suppose that μ is a cardinal with $\mathrm{cf}(\lambda) \le \mu < \kappa$, and $\langle\lambda_i : i < \mu\rangle$ is a one-to-one sequence of regular infinite cardinals less than λ with supremum λ. Suppose further that I is a proper ideal on μ such that for any cardinal $\sigma < \lambda$, $\{i < \mu : \lambda_i \le \sigma\} \in I$. Suppose finally that $\vec{f} = \langle f_\alpha : \alpha < \pi\rangle$ is a $<_I$-increasing, cofinal sequence of elements of $(\prod_{i<\mu} \lambda_i, <_I)$.

Note that if κ is mildly λ^+-ineffable, then by Lemma 1.8 the length of $\vec{f}$ (i.e. π) must be equal to λ^+.

Proposition 5.5. *Suppose that there is a closed unbounded subset C of π such that every δ in $C \cap E^\pi_\kappa$ is a remarkably good point for $\vec{f}$. Then $\mathcal{B}_{\kappa,\lambda}(\mu^+,\pi)$ holds.*

Proof. Pick a bijection $h : \mu \times \lambda \to \lambda$. For $\alpha \in C$, set $y^B_\alpha = \{h(i, f_\alpha(i)) : i \in \mu \setminus B\}$ for every $B \in I$, and put $y_\alpha = y^\emptyset_\alpha$. For $\eta < \pi$, $\Phi(\eta)$ asserts that for any order-type η subset z of C, and any $\varphi : z \to I$, there is a $<\kappa$-to-one function g in $\prod_{\alpha\in z} y^{\varphi(\alpha)}_\alpha$. Let us show by induction that $\Phi(\eta)$ holds for every $\eta < \kappa^+$. It is immediate that $\Phi(\eta)$ holds for every $\eta < \kappa$, and that $\Phi(\eta)$ implies $\Phi(\eta+1)$.

Next suppose that $\eta < \kappa^+$ is an infinite limit ordinal of cofinality less than κ with the property that $\Phi(\gamma)$ holds for every $\gamma < \eta$. Select an increasing continuous sequence $\langle \eta_\delta : \delta < \mathrm{cf}(\eta)\rangle$ of ordinals with supremum η. Let z be an order-type η subset of C, and let $\varphi : z \to I$. Let $\langle \zeta_\beta : \beta < \eta\rangle$ be the increasing enumeration of the elements of z. For $\delta < \mathrm{cf}(\eta)$, pick a $<\kappa$-to-one g_δ in $\prod_{\alpha\in z_\delta} y_\alpha^{\varphi(\alpha)}$, where $z_\delta = \{\alpha : \zeta_{\eta_\delta} \le \alpha < \zeta_{\eta_{\delta+1}}\}$. Then clearly, $\bigcup_{\delta<\mathrm{cf}(\eta)} g_\delta$ is a $<\kappa$-to-one function in $\prod_{\alpha\in z} y_\alpha^{\varphi(\alpha)}$.

Finally, suppose that η is a limit ordinal of cofinality κ such that $\Phi(\gamma)$ holds for every $\gamma < \eta$. Let v be an order-type η subset of C, and let $\psi : v \to I$. Put $\delta = \sup(v)$. Then there is a closed unbounded subset X of δ with o.t.$(X) = \kappa$, and $Z_\xi \in I$ for $\xi \in X$ such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \mu \setminus (Z_\beta \cup Z_\xi)$. We can assume that $0 \in X$. Let $\langle \xi_\sigma : \sigma < \kappa\rangle$ be the increasing enumeration of the elements of X. For $\sigma < \kappa$, set $v_\sigma = \{\alpha \in v : \xi_\sigma \le \alpha < \xi_{\sigma+1}\}$. Define $\psi_\sigma : v_\sigma \to I$ as follows. Given $\alpha \in v_\sigma$, pick $W \in I$ so that $f_{\xi_\sigma}(i) \le f_\alpha(i) < f_{\xi_{\sigma+1}}(i)$ for every $i \in \mu \setminus W$, and set

$$\psi_\sigma(\alpha) = Z_{\xi_\sigma} \cup Z_{\xi_{\sigma+1}} \cup W \cup \psi(\alpha).$$

There must be a $<\kappa$-to-one function g_σ in $\prod_{\alpha\in v_\sigma} y_\alpha^{\psi_\sigma(\alpha)}$. Set $g = \bigcup_{\sigma<\kappa} g_\sigma$. Note that $g \in \prod_{\alpha\in v} y_\alpha^{\psi(\alpha)}$. That g is $<\kappa$-to-one is easily derived from the following.

Claim. *Let $\alpha \in v_\sigma$ and $\beta \in v_\chi$, where $\sigma < \chi < \kappa$. Then $g(\alpha) \ne g(\beta)$.*

Proof of the Claim. Suppose otherwise. Then there is $i \in \mu \setminus (\psi_\sigma(\alpha) \cup \psi_\chi(\beta))$ such that $f_\alpha(i) = f_\beta(i)$. But clearly,

$$f_\alpha(i) < f_{\xi_{\sigma+1}}(i) \le f_{\xi_\chi}(i) \le f_\beta(i).$$

This contradiction completes the proof of the Claim. ∎

If κ is λ-Shelah, then by a result of [13], $\vec{f}$ cannot be good. We will now show that if κ is mildly λ^+-ineffable, then $\vec{f}$ cannot be remarkably good.

Lemma 5.6. *Let C be a closed unbounded subset of π, and ν be a cardinal with $0 < \nu < \kappa$. Suppose that for any regular infinite cardinal ρ with $\nu < \rho < \kappa$, and any $\delta \in C \cap E^\pi_\rho$, δ is a remarkably good point for $\vec{f}$. Then we may find $z_\beta \in P_{\mu^+}(\lambda)$ for $\beta < \pi$ with the property that for any $a \in P_\kappa(\pi) \setminus \{\emptyset\}$, there is a $\le\nu$-to-one g in $\prod_{\beta\in a} z_\beta$.*

Proof. Pick a bijection $h : \mu \times \lambda \to \lambda$. For $\beta \in C$, put $z_\beta = \{h(i, f_\beta(i)) : i < \mu\}$. For $a \in P_\kappa(C) \setminus \{\emptyset\}$ and $k : a \to \mu$, define $\psi^a_k : a \to \lambda$ by $\psi^a_k(\beta) = h(k(\beta), f_\beta(k(\beta)))$. Now for $\eta \in \kappa\setminus\{0\}$, let $\Phi(\eta)$ assert that for any order-type η subset a of C, there is $F_a : a \to I$ with the property that ψ^a_k is $\le\nu$-to-one for every $k \in \prod_{\beta\in a}(\mu \setminus F_a(\beta))$.

Claim. *$\Phi(\eta)$ holds for every $\eta \in \kappa \setminus \{0\}$.*

Proof of the Claim. We proceed by induction. Obviously, $\Phi(\eta)$ holds whenever $0 < \eta < \nu^+$. It is also immediate that for any $\eta \in \kappa \setminus \{0\}$, $\Phi(\eta)$ implies $\Phi(\eta+1)$. Now let $\eta \in \kappa \setminus \nu^+$ be a limit ordinal such that $\Phi(\zeta)$ holds for every $\zeta \in \eta \setminus \{0\}$. Fix $a \subseteq C$ with o.t.$(a) = \eta$.

First suppose $\mathrm{cf}(\eta) \leq \nu$. Set $a = \bigcup_{j<\mathrm{cf}(\eta)} a_j$, where $0 <$ o.t.$(a_j) < \eta$ for each $j < \mathrm{cf}(\eta)$, and $a_\ell \cap a_j = \emptyset$ whenever $\ell < j < \mathrm{cf}(\eta)$. Now put $F_a = \bigcup_{j<\mathrm{cf}(\eta)} F_{a_j}$.

Next suppose that $\mathrm{cf}(\eta) > \nu$. Set $\delta = \sup(a)$. Since δ is a remarkably good point for $\vec{f}$, we may find a closed unbounded subset X of δ with o.t.$(X) = \mathrm{cf}(\eta)$, and $Z_\xi \in I$ for $\xi \in X$ such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \mu \setminus (Z_\beta \cup Z_\xi)$. Let $\langle x_j : j < \mathrm{cf}(\eta)\rangle$ be the increasing enumeration of X. For $j < \mathrm{cf}(\eta)$, set $v_j = \{\beta \in a : x_j \leq \beta < x_{j+1}\}$. For $j < \mathrm{cf}(\eta)$ and $\beta \in v_j$, select $w_\beta \in I$ so that $f_{x_j}(i) \leq f_\beta(i) < f_{x_{j+1}}(i)$ whenever $i \in \mu \setminus w_\beta$. We now define F_a as follows. Given $j < \mathrm{cf}(\eta)$ and $\beta \in v_j$, we let $F_a(\beta) = F_{v_j}(\beta) \cup w_\beta \cup Z_{x_j} \cup Z_{x_{j+1}}$. Note that if $\gamma \in v_\ell$ and $\beta \in v_j$, where $\ell < j < \mathrm{cf}(\eta)$, then $f_\gamma(i) < f_\beta(i)$ for every $i \in \mu \setminus (F_a(\gamma) \cup F_a(\beta))$. This completes the proof of the claim.

Now fix $a \in P_\kappa(C) \setminus \{\emptyset\}$. Let $k \in \prod_{\beta\in a}(\mu \setminus F_a(\beta))$. Then clearly, $\psi_k^a \in \prod_{\beta\in a} z_\beta$. Moreover, ψ_k^a is $\leq\nu$-to-one. ∎

Lemma 5.7. *Let ν be a cardinal with $0 < \nu < \kappa$, and $z_\beta \in P_\kappa(\lambda)$ for $\beta < \lambda^+$ be such that for any $a \in P_\kappa(\lambda^+) \setminus \{\emptyset\}$, there is a $\leq\nu$-to-one g_a in $\prod_{\beta\in a} z_\beta$. Then κ is not mildly λ^+-ineffable.*

Proof. Suppose otherwise. Pick a bijection $h : \lambda^+ \times \kappa \to \lambda^+$. For $\beta < \lambda^+$, let $\langle \zeta^\beta(j) : j < |z_\beta|\rangle$ be a one-to-one enumeration of z_β. For $a \in P_\kappa(\lambda^+) \setminus \{\emptyset\}$, define $\ell_a \in \prod_{\beta\in a} |z_\beta|$ by $g_a(\beta) = \zeta^\beta(\ell_a(\beta))$, and $f_a : a \to 2$ by $f_a(\xi) = 1$ if and only if we may find $\beta \in a$ and $j < |z_\beta|$ such that $\xi = h(\beta, j)$ and $\ell_a(\beta) = j$. There must be $F : \lambda^+ \to 2$ with the property that for any $e \in P_\kappa(\lambda^+) \setminus \{\emptyset\}$,

$$\{a \in P_\kappa(\lambda^+) : e \subseteq a \text{ and } f_a \restriction e = F \restriction e\} \in I^+_{\kappa,\lambda^+}.$$

For $\beta < \lambda^+$, put $e_\beta = \{h(\beta, j) : j < |z_\beta|\} \cup \{\beta\}$ and pick $a_\beta \in P_\kappa(\lambda^+)$ so that $e_\beta \subseteq a_\beta$ and $f_{a_\beta} \restriction e_\beta = F \restriction e_\beta$. Now define $G \in \prod_{\beta<\lambda^+} z_\beta$ by $G(\beta) = \zeta^\beta(\ell_{a_\beta}(\beta))$.

Suppose toward a contradiction that we may find $\gamma < \lambda$ and $d \subseteq \lambda^+$ with $|d| = \nu^+$ such that $d \subseteq G^{-1}(\{\gamma\})$. Set $e = \bigcup_{\beta\in d} e_\beta$ and select $a \in P_\kappa(\lambda^+)$ so that $e \subseteq a$ and $f_a \restriction e = F \restriction e$. Then for each $\beta \in d$, $\ell_{a_\beta}(\beta) = \ell_a(\beta)$ since $f_{a_\beta} \restriction e_\beta = f_a \restriction e_\beta$, and consequently $g_a(\beta) = G(\beta) = \gamma$. Hence $|g_a^{-1}(\{\gamma\})| > \nu$, which yields the desired contradiction.

Thus G is a $\leq\nu$-to-one function from λ^+ to λ, a contradiction. ∎

Proposition 5.8. *Suppose that κ is mildly λ^+-ineffable. Then the set of all $\delta \in E_\rho^{\lambda^+}$ such that δ is not a remarkably good point for $\vec{f}$ is stationary in λ^+ for cofinally many regular infinite cardinals $\rho < \kappa$.*

Proof. By Lemmas 5.6 and 5.7. ∎

We now concentrate on the case when $\mathrm{cf}(\lambda) < \kappa = \omega_1$.

Definition. Given a cardinal $\nu \geq \omega_1$, $\mathrm{Refl}^*(P_{\omega_1}(\nu))$ means that for any stationary subset S of $P_{\omega_1}(\nu)$, there is a size $\aleph_1$ subset Y of ν such that $\mathrm{cf}(\mathrm{o.t.}(Y)) = \omega_1 \subseteq Y$, and $S \cap P_{\omega_1}(Y)$ is stationary in $P_{\omega_1}(Y)$.

Proposition 5.9. *Suppose that $\kappa = \omega_1$, $\mathrm{cf}(\lambda) = \omega$, and $\mathcal{B}_{\kappa,\lambda}(\kappa, \lambda^+)$ holds. Then* $\mathrm{Refl}^*(P_{\omega_1}(\lambda^+))$ *fails.*

Proof. Suppose otherwise. Fix a $\mathcal{B}_{\kappa,\lambda}(\kappa, \lambda^+)$-sequence $\langle y_\alpha : \alpha < \lambda^+\rangle$. Set $S = \{x \in P_{\omega_1}(\lambda^+) : y_{\sup(x)} \subseteq x\}$. It is not difficult to see that $S \in NS^+_{\omega_1,\lambda^+}$. Hence we may find a size $\aleph_1$ subset Y of λ^+ such that $\mathrm{cf}(\mathrm{o.t.}(Y)) = \omega_1 \subseteq Y$ and $S \cap P_{\omega_1}(Y)$ is stationary in $P_{\omega_1}(Y)$. Pick a $<\omega_1$-to-one function $g \in \prod_{\alpha\in Y} y_\alpha$, and define $h : S \cap P_{\omega_1}(Y) \to Y$ by $h(x) = g(\sup(x))$. We may find $T \subseteq S \cap P_{\omega_1}(Y)$ and $\gamma \in Y$ such that T is stationary in $P_{\omega_1}(Y)$ and h takes the constant value γ on T. Then clearly $\sup(x) \in g^{-1}(\{\gamma\})$ for every $x \in T$. Since $|g^{-1}(\{\gamma\})| \leq \aleph_0$, there must be $\delta \in Y$ such that $g^{-1}(\{\gamma\}) \subseteq \delta$, a contradiction. ∎

Foreman, Magidor and Shelah [5] established that (a) under MM, $\mathrm{Refl}^*(P_{\omega_1}(\rho))$ holds for every regular cardinal $\rho \geq \omega_2$, and (b) if ν is a supercompact cardinal, then in $V^{\mathrm{Coll}(\omega_1,<\nu)}$, $\mathrm{Refl}^*(P_{\omega_1}(\rho))$ holds for every regular cardinal $\rho \geq \omega_2$. Thus it is consistent relative to a supercompact cardinal that "$\mathcal{B}_{\omega_1,\sigma}(\omega_1, \sigma^+)$ fails for every singular cardinal σ of cofinality ω".

By a result of Magidor [9] (see also [3, Remark 6.3]), it is consistent relative to infinitely many supercompact cardinals that "$\mathrm{AP}_{\omega_\omega}$ and $\mathrm{Refl}^*(P_{\omega_1}(\omega_{\omega+1}))$ both hold". Hence it is consistent (relative to the assumption above) that "there is a good scale for ω_ω (and in fact every scale for ω_ω is good), so that $\mathcal{A}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ holds, but $\mathcal{B}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ fails". (Note that according to [21, Claim 6.9. 6)a)], if $E^{\omega_{\omega+1}}_{\omega_1} \in I[\omega_{\omega+1}]$, then $\mathcal{B}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ holds and in fact there is an $(\omega_1, \omega_\omega, \omega_{\omega+1})$-sequence $\vec{y} = \langle y_\alpha : \alpha < \omega_{\omega+1}\rangle$ with the property that for each nonempty e in $P_{\omega_2}(\omega_{\omega+1})$, there is a one-to-one g in $\prod_{\alpha\in e} y_\alpha$. This contradicts the consistency of "$\mathrm{AP}_{\omega_\omega}$ holds but $\mathcal{B}_{\omega_1,\omega_\omega}(\omega_1, \omega_{\omega+1})$ fails".)

On the other hand, Gitik and Sharon [7] proved that it is consistent relative to a supercompact cardinal that "λ is a strong limit cardinal of cofinality ω + $2^\lambda > \lambda^+$+ AP_λ fails (and in fact, as observed by Cummings and Foreman, there is a scale for λ that is not good) + VGS_λ + λ carries a very good scale of length λ^{++} (and hence $\mathcal{B}_{\omega_1,\lambda}(\omega_1, \lambda^{++})$ holds)".

6. $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$

Definition. A $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$-*sequence* is a (τ,λ,π)-sequence $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ with the property that for each nonempty e in $P_\lambda(\pi)$, there is a $<\kappa$-to-one $g \in \prod_{\alpha\in e} y_\alpha$.

Note that every $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$-sequence is a $\mathcal{B}_{\kappa,\lambda}(\tau,\pi)$-sequence.

The following is readily checked.

Proposition 6.1. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and* $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ *is a* $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$*-sequence. Then for any nonempty* e *in* $P_{\lambda^+}(\pi)$*, there is a* $<\kappa$*-to-one* $g \in \prod_{\alpha\in e} y_\alpha$.

Definition. $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$ asserts the existence of a $\mathcal{C}_{\kappa,\lambda}(\tau,\pi)$-sequence.

Proposition 6.2. *Suppose that* $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ *is a* $\mathcal{C}_{\kappa,\lambda}(\kappa,\pi)$*-sequence,* $D \in NS^*_{\kappa,\pi}$*, and* ν *is a regular cardinal with* $\kappa \le \nu < \lambda$*. Then* $\{x\cap\nu : x \in D\cap A(\vec{y})\} \in NS^*_{\kappa,\nu}$.

Proof. Fix $S \in NS^+_{\kappa,\nu}$. Then $T = \{x \in P_\kappa(\pi) : x\cap\nu \in S\}$ lies in $NS^+_{\kappa,\pi}$. Pick $F : P_\omega(\pi) \to P_\kappa(\pi)$ with $C_F^{\kappa,\pi} \subseteq D$. Define $e_\beta \in P_{\nu^+}(\pi)$ for $\beta < \nu$ by

- $e_0 = \nu$.
- $e_{\beta+1} = e_\beta \cup \{\alpha < \pi : y_\alpha \subseteq e_\beta\} \cup \bigcup F``P_\omega(e_\beta)$.
- $e_\beta = \bigcup_{\gamma<\beta} e_\gamma$ in case β is an infinite limit ordinal.

Put $e = \bigcup_{\beta<\nu} e_\beta$. Note that $\nu \subseteq e$, $|e| = \nu$, $\{\alpha < \pi : y_\alpha \subseteq e\} \subseteq e$ and $F``P_\omega(e) \subseteq P(e)$. Select a $<\kappa$-to-one $h \in \prod_{\alpha\in e} y_\alpha$, and let H be the set of all $z \in P_\kappa(\pi)$ such that $h^{-1}(\{\xi\}) \subseteq z$ for every $\xi \in z\cap\mathrm{ran}(h)$. Clearly $H \in (NS^\lambda_{\kappa,\pi})^*$, so we may find z such that $z \in H\cap T\cap C_F^{\kappa,\pi}$. It is easy to see that $z\cap e \in C_F^{\kappa,\pi}\cap A(\vec{y})$. Moreover, $(z\cap e)\cap\nu \in S$. ∎

Corollary 6.3. *Suppose that* $\mathcal{C}_{\kappa,\lambda}(\kappa,\pi)$ *holds and* $\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) \le \pi$*. Then* $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ *for some* A *with the property that* $\{a\cap\nu : a \in A\} \in NS^*_{\kappa,\nu}$ *for every regular cardinal* ν *with* $\kappa \le \nu < \lambda$.

Proof. By Propositions 2.2, 5.2 and 6.2. ∎

Let us now discuss the validity of $\mathcal{C}_{\kappa,\lambda}(\kappa,\pi)$. First, the positive side:

Proposition 6.4. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and* $u(\lambda^+,\pi) < \mathrm{cov}(\lambda,\lambda,\kappa,2)$*. Then* $\mathcal{C}_{\kappa,\lambda}(\kappa,\pi)$ *holds, and in fact we may find a* (κ,λ,π)*-sequence* $\vec{y} = \langle y_\alpha : \alpha < \pi\rangle$ *with the property that for any nonempty* e *in* $P_{\lambda^+}(\pi)$*, there is a* $<(\mathrm{cf}(\lambda))^+$*-to-one* $g \in \prod_{\alpha\in e} y_\alpha$.

Proof. By the proof of [14, Proposition 6.2]. ∎

Proposition 6.5. *Suppose that* $\pi = \lambda^+$ *and there is a closed unbounded subset* C *of* π *such that for any regular cardinal* θ *with* $\kappa \le \theta < \lambda$*, and any* $\delta \in C\cap E^\pi_\theta$*,* δ *is a remarkably good point for* $\vec{f}$*. Then* $\mathcal{C}_{\kappa,\lambda}(\kappa,\pi)$ *holds, and* *in fact we may find a (μ^+,λ,π)-sequence $\vec{y}=\langle y_\alpha:\alpha<\pi\rangle$ with the property that for any nonempty e in $P_\pi(\pi)$, there is a $<\kappa$-to-one g in $\prod_{\alpha\in e} y_\alpha$.*

Proof. Proceed as in the proof of Proposition 5.5, showing this time that $\Phi(\eta)$ holds for every $\eta<\pi$. ∎

Now for the negative side:

Proposition 6.6. *Suppose that there is a mildly λ^+-ineffable cardinal κ' with $\kappa<\kappa'<\lambda$. Then $\mathcal{C}_{\kappa,\lambda}(\kappa,\lambda^+)$ does not hold.*

Proof. By Lemma 5.7. ∎

7. $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$. Fix a bijection $j_\lambda:\kappa\times\lambda\to\lambda$.

Definition. Let $\nu<\kappa$ be an infinite cardinal, and J be a proper ideal on ν. A $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$*-sequence* is a (ν^+,λ,π)-sequence $\vec{y}=\langle y_\alpha:\alpha<\pi\rangle$ with the following property: there is $h_\alpha:\nu\to\lambda$ for $\lambda\le\alpha<\pi$ such that (a) for any nonempty $e\in P_{\kappa^+}(\pi\setminus\lambda)$, there is $g:e\to J$ such that $h_\alpha(i)\neq h_\beta(i)$ whenever $\alpha<\beta$ are in e and $i\in\nu\setminus(g(\alpha)\cup g(\beta))$, and (b) for any $\alpha\in\pi\setminus\lambda$, $y_\alpha=\{j_\lambda(i,h_\alpha(i)):i<\nu\}$.

It is easy to see that every $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$-sequence is a $\mathcal{B}_{\kappa,\lambda}(\nu^+,\pi)$-sequence.

Definition. $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$ asserts the existence of a $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$-sequence.

Proposition 7.1. *Suppose that $\vec{y}=\langle y_\alpha:\alpha<\pi\rangle$ is a $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$-sequence, where J is ω_1-complete. Then $A(\vec{y})\in NG^*_{\kappa,\pi}$.*

Proof. Let $\langle h_\alpha:\lambda\le\alpha<\pi\rangle$ be as in the definition above. Define $k:(\pi\setminus\lambda)\times P_\kappa(\lambda)\to P(\nu)$ by $k(\alpha,a)=\{i\in\nu:j_\lambda(i,f_\alpha(i))\in a\}$, and $\varphi:P_\kappa(\lambda)\to P(\pi\setminus\lambda)$ by $\varphi(a)=\{\alpha\in\pi\setminus\lambda:k(\alpha,a)\in J^+\}$.

Claim. *$|\varphi(a)|<\kappa$ for all $a\in P_\kappa(\lambda)$.*

Proof of the Claim. Suppose otherwise, and fix $a\in P_\kappa(\lambda)$ with $|\varphi(a)|\ge\kappa$. Pick $e\subseteq\varphi(a)$ with $|e|=\kappa$. There must be $g:e\to J$ such that $h_\alpha(i)\neq h_\beta(i)$ whenever α and β are two distinct elements of e and $i\in\nu\setminus(g(\alpha)\cup g(\beta))$. Pick $q\in\prod_{\alpha\in e}(k(\alpha,a)\setminus g(\alpha))$, and define $\psi:e\to a$ by $\psi(\alpha)=j_\lambda(q(\alpha),h_\alpha(q(\alpha)))$. Note that if $\alpha,\beta\in e$ are such that $\psi(\alpha)=\psi(\beta)$, then for $i=q(\alpha)=q(\beta)$, $i\in\nu\setminus(g(\alpha)\cup g(\beta))$ and $h_\alpha(i)=h_\beta(i)$, and therefore $\alpha=\beta$. Thus ψ is one-to-one. This contradiction completes the proof of the Claim.

We need to find a winning strategy σ for II in $H_{\kappa,\lambda}(A(\vec{y}))$. Consider a run of the game where I's successive moves are $s_0,s_1,\ldots$. We let $\varphi(s_0)=t_0\cup\varphi(t_0\cap\lambda)$, where $t_0=s_0$, and $\varphi(s_0,\ldots,s_{n+1})=t_{n+1}\cup\varphi(t_{n+1}\cap\lambda)$, where $t_{n+1}=s_{n+1}\cup\sigma(s_0,\ldots,s_n)$. Let us check that $x\in A(\vec{y})$, where $x=\bigcup_{n<\omega}(s_n\cup\sigma(s_0,\ldots,s_n))=\bigcup_{n<\omega}t_n$. Thus fix $\alpha\in\pi\setminus\lambda$ with $y_\alpha\subseteq x$. Then clearly $\nu=\bigcup_{n<\omega}k(\alpha,t_n\cap\lambda)$, so we may find $m<\omega$ such that $k(\alpha,t_m\cap\lambda)\in J^+$. Then $\alpha\in\varphi(t_m\cap\lambda)\subseteq\varphi(s_0,\ldots,s_m)$, and hence $\alpha\in x$. ∎

Corollary 7.2. *Suppose that* $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$ *holds, where* J *is* ω_1*-complete, and let* $K \subseteq NS_{\kappa,\lambda}$ *be an ideal on* $P_\kappa(\lambda)$ *with* $\overline{\mathrm{cof}}(K) \leq \pi$. *Then there is* $A \in NG^*_{\kappa,\lambda}$ *such that* $K|A = I_{\kappa,\lambda}|A$.

Proof. By Lemma 1.7 and Propositions 2.2 and 7.1. ■

Let us next consider some situations when $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$ holds.

Proposition 7.3 ([19, Claim 1.5 A and Remark 1.5 B (4), p. 51]). *Suppose that* $\mathrm{cf}(\lambda) < \lambda < \pi < \mathrm{pp}_{\Gamma(\kappa,\omega_1)}(\lambda)$. *Then for some infinite cardinal* $\nu < \kappa$, *and some* ω_1*-complete, proper ideal* J *on* ν, $\mathcal{D}^J_{\kappa,\lambda}(\nu^+,\pi)$ *holds.*

Proposition 7.4. *Let* C *be a closed unbounded subset of* π *such that for any regular infinite cardinal* $\theta \leq \kappa$, *and any* $\delta \in C \cap E^\pi_\theta$, δ *is a remarkably good point for* $\vec{f}$. *Then for any nonempty* $e \in P_{\kappa^+}(C)$, *there is* $g : e \to I$ *such that* $f_\alpha(i) < f_\beta(i)$ *whenever* $\alpha < \beta$ *are in* e *and* $i \in \mu \setminus (g(\alpha) \cup g(\beta))$ (*and hence* $\mathcal{D}^I_{\kappa,\lambda}(\mu^+,\pi)$ *holds*).

Proof. For $\eta \in \pi$, let $\Phi(\eta)$ assert that for any order-type η subset z of C, there is $F_z : z \to I$ with the property that $f_\gamma(i) < f_\beta(i)$ whenever $\gamma < \beta$ are in z and $i \in \mu \setminus (F_z(\gamma) \cup F_z(\beta))$. Let us show by induction that $\Phi(\eta)$ holds for every $\eta < \kappa^+$. Obviously, $\Phi(0)$ holds. Now assuming $\Phi(\eta)$, let us prove that $\Phi(\eta+1)$ holds. Thus let $z \subseteq C$ with $\mathrm{o.t.}(z) = \eta + 1$. Set $z = t \cup \{\alpha\}$, where $\mathrm{o.t.}(t) = \eta$. For $\gamma \in t$, pick $w_\gamma \in I$ so that $f_\gamma(i) < f_\alpha(i)$ whenever $i \in \mu \setminus w_\gamma$. We define $F_z : z \to I$ by $F_z(\alpha) = \emptyset$, and $F_z(\gamma) = F_t(\gamma) \cup w_\gamma$ for each $\gamma \in t$.

Finally, suppose that η is an infinite limit ordinal such that $\Phi(\theta)$ holds for every $\theta < \eta$. Fix $z \subseteq C$ with $\mathrm{o.t.}(z) = \eta$. Put $\delta = \sup(z)$. Since δ is a remarkably good point for $\vec{f}$, we may find a closed unbounded subset X of δ with $\mathrm{o.t.}(X) = \mathrm{cf}(\eta)$, and $Z_\xi \in X$ for $\xi \in X$ such that $f_\beta(i) < f_\xi(i)$ whenever $\beta < \xi$ are in X and $i \in \mu \setminus (Z_\beta \cup Z_\xi)$. Let $\langle x_j : j < \mathrm{cf}(\eta)\rangle$ be the increasing enumeration of X. For $j < \mathrm{cf}(\eta)$, set $v_j = \{\alpha \in z : x_j \leq \alpha < x_{j+1}\}$. For $j < \mathrm{cf}(\eta)$ and $\zeta \in v_j$, select $w_\zeta \in I$ so that $f_{x_j}(i) \leq f_\zeta(i) < f_{x_{j+1}}(i)$ whenever $i \in \mu \setminus w_\zeta$. We now define $F_z : z \to I$ as follows. Given $j < \mathrm{cf}(\eta)$ and $\zeta \in v_j$, we let

$$F_z(\zeta) = F_{v_j}(\zeta) \cup w_\zeta \cup Z_{x_j} \cup Z_{x_{j+1}}. \ ■$$

8. ADS_λ

Definition. ADS_λ asserts the existence of a sequence $\langle z_\beta : \beta < \lambda^+\rangle$ such that (i) for any $\beta < \lambda^+$, z_β is an order-type $\mathrm{cf}(\lambda)$, cofinal subset of λ, and (ii) for any $\delta < \lambda^+$, there is a $g : \delta \to \lambda$ with the property that $(z_\beta \setminus g(\beta)) \cap (z_\gamma \setminus g(\gamma)) = \emptyset$ whenever $\beta < \gamma < \delta$.

The principle ADS_λ was introduced by Shelah [18, p. 440], who observed that it automatically holds in case λ is regular (as witnessed by any sequence $\langle z_\beta : \beta < \lambda^+\rangle$ of almost disjoint subsets of λ of size λ).

Suppose that $\mathrm{cf}(\lambda) < \kappa$ and ADS_λ holds. Then clearly there exists a $\mathcal{C}_{\kappa,\lambda}((\mathrm{cf}(\lambda))^+, \lambda^+)$-sequence which is also a $\mathcal{D}^J_{\kappa,\lambda}((\mathrm{cf}(\lambda))^+, \lambda^+)$-sequence, where $J =$ the noncofinal ideal on $\mathrm{cf}(\lambda)$.

LEMMA 8.1. *Let C be a closed unbounded subset of π. Suppose that $\pi = \lambda^+$, and for any regular infinite cardinal $\theta < \lambda$, and any $\delta \in C \cap E^\pi_\theta$, δ is a remarkably good point for $\vec{f}$. Then for any $\beta < \pi$, there is $g : C \cap \beta \to I$ with the property that $f_\gamma(i) < f_\delta(i)$ whenever $\gamma < \delta$ are in $C \cap \beta$ and $i \in \mu \setminus (g(\gamma) \cup g(\delta))$.*

Proof. Modify the proof of Proposition 7.4 so as to show that $\Phi(\eta)$ holds for every $\eta < \pi$. ■

PROPOSITION 8.2. *Suppose that $\mu = \mathrm{cf}(\lambda), I =$ the noncofinal ideal on μ, $\pi = \lambda^+$ and $\vec{f}$ is remarkably good. Then ADS_λ holds.*

Proof. Select a bijection $h : \lambda \times \mu \to \lambda$. For $\beta < \pi$, set $t_\beta = \{h(f_\beta(i), i) : i < \mu\}$.

CLAIM 1. $\{\beta \in \pi : \sup(t_\beta) < \lambda\} \in NS_\pi$.

Proof of Claim 1. Suppose otherwise. Then we may find a cardinal $\chi < \lambda$ and a stationary subset T of π such that $\sup(t_\beta) \le \chi$ for every $\beta \in T$. Set $q = \{i \in \mu : \lambda_i \le \chi\}$. Then clearly, $q \in I$. Moreover, $|\{\alpha \in \lambda_i : h(\alpha, i) \le \chi\}| < \lambda_i$ for $i \in \mu \setminus q$. So we may find $k \in \prod_{i<\mu} \lambda_i$ such that $\{\alpha \in \lambda_i : h(\alpha, i) \le \chi\} \subseteq k(i)$ for all $i \in \mu \setminus q$. Then clearly $f_\beta(i) < k(i)$ whenever $\beta \in T$ and $i \in \mu \setminus q$. Hence, $f_\beta <_I k$ for all $\beta \in T$. This contradiction completes the proof of Claim 1.

Let C be a closed unbounded subset of π such that each limit ordinal δ in C is a remarkably good point for $\vec{f}$. By Claim 1 we may find a closed unbounded subset D of C with the property that $\sup(t_\beta) = \lambda$ for any $\beta \in D$. For $\beta \in D$, pick $w_\beta \subseteq t_\beta$ so that $\mathrm{o.t.}(w_\beta) = \mu$ and $\sup(w_\beta) = \lambda$. Let $\langle d_\gamma : \gamma < \pi\rangle$ be the increasing enumeration of D. For $\gamma < \pi$, put $z_\gamma = w_{d_\gamma}$ and $r_\gamma = \{i < \mu : h(f_{d_\gamma}(i), i) \in z_\gamma\}$.

Now fix ξ with $0 < \xi < \pi$. By Lemma 8.1, there is $\ell : \xi \to \mu$ with the property that $f_{d_\gamma}(i) < f_{d_\eta}(i)$ whenever $\gamma < \eta < \xi$ and $i \in \mu \setminus (\ell(\gamma) \cup \ell(\eta))$. Define $g : \pi \to \lambda$ so that for any $\gamma < \pi$, $\{h(f_{d_\gamma}(i), i) : i \in r_\gamma \cap \ell(\gamma)\} \subseteq g(\gamma)$.

CLAIM 2. Suppose that $\gamma < \eta < \xi$. Then $(z_\gamma \setminus g(\gamma)) \cap (z_\eta \setminus g(\eta)) = \emptyset$.

Proof of Claim 2. Suppose otherwise. Then there must be i in $(r_\gamma \setminus \ell(\gamma)) \cap (r_\eta \setminus \ell(\eta))$ such that $h(f_{d_\gamma}(i), i) = h(f_{d_\eta}(i), i)$. But for this i, $f_{d_\gamma}(i) < f_{d_\eta}(i)$. This contradiction completes the proof of Claim 2.

Thus $\langle z_\gamma : \gamma < \pi\rangle$ witnesses that ADS_λ holds. ■

Suppose that $\mathrm{cf}(\lambda) < \kappa$. Cummings, Foreman and Magidor [3] proved that if $\square^*_\lambda$ or VGS_λ holds, then there is a better scale for λ (and hence ADS_λ

holds). On the other hand, it is known [10] that ADS_λ fails in case there is a $\mathrm{cf}(\lambda)$-saturated ideal on $P_\kappa(\lambda)$.

9. Cohen forcing. If in V, $2^{<\kappa} = \kappa$ and $\mathbb{P}$ is the notion of forcing that adds σ Cohen subsets of κ, where σ is a cardinal greater than $\lambda^{<\kappa}$, then by [14, Corollary 8.4], in $V^{\mathbb{P}}$, $NS^\kappa_{\kappa,\lambda}|B = I_{\kappa,\lambda}|B$ for no $B \in (NS^\kappa_{\kappa,\lambda})^+$ (and hence $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for no $A \in NS^+_{\kappa,\lambda}$). Now suppose that $V = L$ and $\mathrm{cf}(\lambda) < \kappa$ holds in V. We will show that if $\lambda^{<\kappa}$ Cohen subsets of κ are added to V, then in the generic extension, $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A, but $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ for no B.

Lemma 9.1. *Suppose that in V, $2^{<\kappa} = \kappa$, F is a function from $\lambda \times \kappa$ to κ, and $\mathbb{P}$ is the notion of forcing that adds a Cohen subset of κ. Then in $V^{\mathbb{P}}$, there exists $g : \kappa \to \kappa$ such that for any $a \in P_\kappa(\lambda)$, there is $\alpha \in \kappa$ with $g(\alpha) > \bigcup_{\delta\in a} F(\delta, \alpha)$.*

Proof. $\mathbb{P}$ can be identified with the set $\bigcup_{\beta<\kappa}^{(\beta\times\beta)} 2$. For $p \in \mathbb{P}$, let $\beta_p \in \kappa$ be such that $\mathrm{dom}(p) = \beta_p \times \beta_p$. For $a \in P_\kappa(\lambda)$, let D_a be, in V, the set of all $p \in \mathbb{P}$ such that (i) for any $\alpha \in \beta_p$, there is $\gamma \in \beta_p$ with $p(\alpha, \gamma) = 1$, and (ii) there is $\alpha \in \beta_p$ such that $\bigcup_{\delta\in a} F(\delta, \alpha) < \xi$, where $\xi =$ the least $\gamma \in \beta_p$ with $p(\alpha, \gamma) = 1$.

Now suppose that G is $\mathbb{P}$-generic over V. Then clearly $G \cap D_a \neq \emptyset$ for all $a \in P_\kappa(\lambda)$. In $V[G]$, define $g : \kappa \to \kappa$ by $g(\alpha) =$ the least $\gamma < \kappa$ such that $p(\alpha, \gamma) = 1$ for some $p \in G$. It is easy to see that for any $a \in P_\kappa(\lambda)$, there is $\alpha < \kappa$ with $g(\alpha) > \bigcup_{\delta\in a} F(\delta, \alpha)$. ■

Proposition 9.2. *Suppose that V satisfies* GCH *and in V,* $\mathrm{cf}(\lambda) < \kappa$ *and* ADS_λ *holds. In V, let $\mathbb{Q}$ be the notion of forcing to add λ^+ Cohen subsets of κ. Then in $V^{\mathbb{Q}}$,* (a) *$NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some $A \in NS^+_{\kappa,\lambda}$, and* (b) *$NS^\kappa_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for no $C \in NS^*_{\kappa,\lambda}$ (and hence $NS_{\kappa,\lambda} = I_{\kappa,\lambda}|B$ for no $B \in NS^*_{\kappa,\lambda}$).*

Proof. $\mathbb{Q}$ can be identified with the set of all functions q such that $\mathrm{dom}(q) \in P_\kappa(\lambda^+ \times \kappa)$ and $\mathrm{ran}(q) \subseteq 2$. Let G be $\mathbb{Q}$-generic over V. Any sequence $\langle z_\beta : \beta < \lambda^+\rangle$ witnessing that ADS_λ holds in V will witness that ADS_λ holds in $V[G]$. Hence in $V[G]$, ADS_λ holds and since

$$\overline{\mathrm{cof}}(NS_{\kappa,\lambda}) \leq \mathrm{cof}(NS_{\kappa,\lambda}) \leq 2^\lambda = \lambda^+,$$

by Corollary 6.3 there is $A \in NS^+_{\kappa,\lambda}$ such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$.

Claim. *In $V[G]$, $\overline{\partial}_\kappa \geq \lambda^+$.*

Proof of the Claim. Suppose otherwise. Then in $V[G]$ we may find $F : \lambda \times \kappa \to \kappa$ with the property that for any $g : \kappa \to \kappa$, there is $a \in P_\kappa(\lambda)$ such that $g(\alpha) \leq \bigcup_{\delta \in a} F(\delta, \alpha)$ for all $\alpha \in \kappa$. For $X \subseteq \lambda^+$, set $G_X = \{q \in G : \mathrm{dom}(q) \subseteq X \times \kappa\}$. There must be $\xi < \lambda^+$ with $F \in V[G_\xi]$. But then in $V[G_\xi][G_{\{\xi\}}]$, by Lemma 9.1 there is $g : \kappa \to \kappa$ such that for any $a \in P_\kappa(\lambda)$, we may find $\alpha \in \kappa$ with $g(\alpha) > \bigcup_{\delta \in a} F(\delta, \alpha)$. This contradiction completes the proof of the Claim.

It follows from the Claim that in $V[G]$, $\overline{\partial}_\kappa = \lambda^+$ and therefore by Lemmas 1.5 and 1.6, $NS^\kappa_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for no $C \in NS^*_{\kappa,\lambda}$. ∎

In the generic extension of Proposition 9.2, GCH holds below κ. We now show that it is consistent that "(a) and (b) both hold but $2^{\aleph_0}$ is large".

Proposition 9.3. *Suppose that V, κ, λ and $\mathbb{Q}$ are as in Proposition* 9.2. *In $V^{\mathbb{Q}}$, let ν be an infinite cardinal, and $\mathbb{P}$ be the notion of forcing that adjoins ν Cohen reals. Then in $(V^{\mathbb{Q}})^{\mathbb{P}}$,* (a) *$NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A, and* (b) *$NS^\kappa_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for no $C \in NS^*_{\kappa,\lambda}$.*

Proof. Set $M = V^{\mathbb{Q}}$ and $W = M^{\mathbb{P}}$. Since $\mathbb{P}$ is ω_1-cc, by a result of [15] $(\overline{\partial}_\kappa)^W \leq (\overline{\partial}_\kappa)^M$ and $(\overline{\mathrm{cof}}(NS_{\kappa,\lambda}))^W \leq (\overline{\mathrm{cof}}(NS_{\kappa,\lambda}))^M$. It is easy to see that ADS_λ still holds in W, so by Corollary 6.3, in W, $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$ for some A.

Claim. $(\overline{\partial}_\kappa)^W = \lambda^+$.

Proof of the Claim. We already saw that $(\overline{\partial}_\kappa)^W \leq \lambda^+$. Suppose toward a contradiction that $(\overline{\partial}_\kappa)^W \leq \lambda$. Then in W, there is $h : \lambda \times \kappa \to \kappa$ with the property that for any $g : \kappa \to \kappa$, there is $e \in P_\kappa(\lambda)$ such that $g(\xi) \leq \bigcup_{\alpha \in e} h(\alpha, \xi)$ for all $\xi < \kappa$. There must be $H : \lambda \times \kappa \to P_{\omega_1}(\kappa)$ in M such that for every $\alpha < \lambda$ and every $\xi < \kappa$, $h(\alpha, \xi) \in H(\alpha, \xi)$. In M, define $k : \lambda \times \kappa \to \kappa$ by $k(\alpha, \xi) = \bigcup H(\alpha, \xi)$. Now, let $g : \kappa \to \kappa$ in M. In W, there is $e \in P_\kappa(\lambda)$ such that $g(\xi) \leq \bigcup_{\alpha \in e} h(\alpha, \xi)$ for every $\xi < \kappa$. We may find $d \in P_\kappa(\lambda)$ in M with $e \subseteq d$. Then clearly in M, $g(\xi) \leq \bigcup_{\alpha \in d} k(\alpha, \xi)$ for all $\xi < \kappa$. Hence $(\overline{\partial}_\kappa)^M \leq (\overline{\partial}_\kappa)^W \leq \lambda$.

This contradiction completes the proof of the Claim.

We can now appeal to Lemma 1.5 and conclude that $NS^\kappa_{\kappa,\lambda}|C = I_{\kappa,\lambda}|C$ for no $C \in NS^*_{\kappa,\lambda}$. ∎

Returning now to Proposition 9.2, let us make the extra assumption that in V, $\lambda < \kappa^{+\omega_1}$. Then in $V^{\mathbb{Q}}$, by Proposition 5.4 and (the proof of) Proposition 9.2 we may find $A \in NG^+_{\kappa,\lambda}$ such that $NS_{\kappa,\lambda}|A = I_{\kappa,\lambda}|A$. Moreover by [11, Corollary 4.4 and Proposition 4.6], there is $D \in NG^*_{\kappa,\lambda}$ such that $NG_{\kappa,\lambda} = NS_{\kappa,\lambda}|D$. Hence $NG_{\kappa,\lambda}|T = NS_{\kappa,\lambda}|T = I_{\kappa,\lambda}|T$, where $T = A \cap D$.

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Pierre Matet
Université de Caen – CNRS
Laboratoire de Mathématiques
BP 5186
14032 Caen Cedex, France
E-mail: pierre.matet@unicaen.fr

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