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Higher order spreading models

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Abstract. We introduce higher order spreading models associated to a Banach space X. Their definition is based on $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$ with $\mathcal{F}$ a regular thin family and on plegma families. We show that the higher order spreading models of a Banach space X form an increasing transfinite hierarchy $(\mathcal{SM}_\xi(X))_{\xi<\omega_1}$. Each $\mathcal{SM}_\xi(X)$ contains all spreading models generated by $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$ with order of $\mathcal{F}$ equal to ξ. We also study the fundamental properties of this hierarchy.

1. Introduction. Spreading models were invented by A. Brunel and L. Sucheston [7] in the middle 70's and since then have a constant presence in the evolution of Banach space theory. Recall that a sequence $(e_n)_n$ in a seminormed space $(E,\|\cdot\|_*)$ is called a *spreading model* of the space X if there exists a sequence $(x_n)_n$ in X which is *Schreier almost isometric* to $(e_n)_n$, that is, for some null sequence $(\delta_n)_n$ of positive reals we have

$$\Big|\Big\|\sum_{i=1}^{k} a_i x_{n_i}\Big\| - \Big\|\sum_{i=1}^{k} a_i e_i\Big\|_*\Big| < \delta_k \tag{1.1}$$

for every $k\le n_1<\cdots<n_k$ and $(a_i)_{i=1}^k\in[-1,1]^k$. We also say that the sequence $(x_n)_n$ which satisfies (1.1) *generates* $(e_n)_n$ *as a spreading model.* By an iterated use of Ramsey's theorem [18], Brunel and Sucheston proved that every bounded sequence in a Banach space X has a subsequence generating a spreading model.

It is easy to see that any sequence $(e_n)_n$ satisfying (1.1) is spreading ([1]). The importance of spreading models comes from the fact that they connect in an asymptotic manner the structure of an arbitrary Banach space

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([1]) A sequence $(e_n)_n$ in a seminormed space $(E,\|\cdot\|_*)$ is called *spreading* if for every $n\in\mathbb{N}$, $k_1<\cdots<k_n$ in $\mathbb{N}$ and $a_1,\dots,a_n\in\mathbb{R}$ we have $\|\sum_{j=1}^n a_j e_j\|_*=\|\sum_{j=1}^n a_j e_{k_j}\|_*$.

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X to the corresponding one of spaces generated by spreading sequences. The definition of spreading model resembles the finite representability ([2]) of the space generated by the sequence $(e_n)_n$ into the space $(X, \|\cdot\|)$. However there exists a significant difference between the two concepts. Indeed in the realm of finite representability there are two classical achievements: Dvoretsky's theorem [9] asserting that ℓ^2 is finitely representable in every Banach space X and Krivine's theorem [13] asserting that for every linearly independent sequence $(x_n)_n$ in X there exists a $1 \leq p \leq \infty$ such that ℓ^p is block finitely representable in the subspace generated by $(x_n)_n$. On the other hand, E. Odell and Th. Schlumprecht [16] have shown that there exists a reflexive space X admitting no ℓ^p as a spreading model. Thus the spreading models of a space X lie strictly between the finitely representable spaces in X and the spaces that are isomorphic to a subspace of X.

The spreading models associated to a Banach space X can be considered as a cloud of Banach spaces, including many members with regular structure, surrounding the space X and offering information concerning the local structure of X in an asymptotic manner. Our aim is to enlarge that cloud and to fill in the gap between spreading models and the spaces which are finitely representable in X. More precisely we extend the Brunel–Sucheston concept of a spreading model and we show that under the new definition the spreading models associated to a Banach space X form a whole hierarchy of classes of spaces indexed by the countable ordinals. The first class of this hierarchy is the classical spreading models. The initial step of this extension has already been done in [4] where the class of k-spreading models was defined for every positive integer k. The transfinite extension introduced in the present paper requires analogous ingredients that we are about to describe.

The first one is $\mathcal{F}$-*sequences*, that is, sequences of the form $(x_s)_{s\in\mathcal{F}}$ where the index set $\mathcal{F}$ is a regular thin family of finite subsets of $\mathbb{N}$ (see Definition 2.7). Typical examples of such families are the k-element subsets of $\mathbb{N}$ and also the maximal elements of the ξth Schreier family $\mathcal{S}_\xi$ (see [2]). A subsequence of $(x_s)_{s\in\mathcal{F}}$ is a restriction of the $\mathcal{F}$-sequence to an infinite subset of $\mathbb{N}$, i.e. it is of the form $(x_s)_{s\in\mathcal{F}\restriction M}$ where $\mathcal{F}\restriction M = \mathcal{F} \cap [M]^{<\infty}$. Among other things we study the convergence of $\mathcal{F}$-sequences in a topological space $(X, \mathcal{T})$. In this setting we show that when the closure of $(x_s)_{s\in\mathcal{F}}$ in $(X,\mathcal{T})$ is a compact metrizable space then we can always restrict to an infinite subset M of $\mathbb{N}$ where the subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ is *subordinated*, that is, if $\widehat{\mathcal{F}} = \{t \in [\mathbb{N}]^{<\infty} : \exists s \in \mathcal{F}$ such that t is an initial segment of $s\}$ then there

([2]) A Banach space Y is *finitely representable* in X if for every finite-dimensional subspace F of Y and every $\varepsilon > 0$ there exists a bounded linear injection $T : F \to Y$ such that $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$.

exists a continuous map $\varphi : \widehat{\mathcal{F}}\restriction M \to X$ with $\varphi(s) = x_s$ for every $s \in \mathcal{F}\restriction M$ (see Definition 5.8 and Theorem 5.10).

The second ingredient is the notion of *plegma families* which extends the corresponding notion in [4]. Roughly speaking a plegma family is a sequence $(s_1, \dots, s_l)$ of nonempty finite subsets of $\mathbb{N}$ where the first elements of $s_1, \dots, s_l$ are in increasing order and they lie before their second elements which are also in increasing order and so on (see Definition 3.1). Here plegma families do not necessarily consists of sets of equal size.

$\mathcal{F}$-sequences and plegma families are the key components for the definition of higher order spreading models which goes as follows. Given an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in a Banach space X and a sequence (e_n) in a seminormed space $(E, \|\cdot\|_*)$ we will say that $(x_s)_{s\in\mathcal{F}}$ *generates* $(e_n)_n$ *as an* $\mathcal{F}$*-spreading model* if for some null sequence $(\delta_n)_n$ of positive reals we have

$$\left| \Big\| \sum_{i=1}^{k} a_i x_{s_i} \Big\| - \Big\| \sum_{i=1}^{k} a_i e_i \Big\|_* \right| < \delta_k \tag{1.2}$$

for every $(a_i)_{i=1}^k \in [-1,1]^k$ and every plegma family $(s_i)_{i=1}^k$ in $\mathcal{F}$ with $k \leq \min s_1$. Note that the $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$ generate a higher order spreading model just as the ordinary sequences $(x_n)_n$ do in the classical definition. Moreover, since the family of all k-element subsets of $\mathbb{N}$ is a regular thin family, the above definition extends the classical definition of the spreading model as well as the one of k-spreading models given in [4].

Brunel–Sucheston's theorem [6] extends to bounded $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$ in a Banach space X. Namely, every bounded $\mathcal{F}$-sequence in X contains a subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ generating an $\mathcal{F}$-spreading model. The proof is based on the fact that plegma families with elements in a regular thin family have strong Ramsey properties. It is notable that the concept of $\mathcal{F}$-spreading model is independent of the particular family $\mathcal{F}$ and actually depends only on the order (3) of the family $\mathcal{F}$. Namely, if $(e_n)_n$ is an $\mathcal{F}$-spreading model then it is also a $\mathcal{G}$-spreading model for every regular thin family $\mathcal{G}$ with $o(\mathcal{G}) \geq o(\mathcal{F})$. This fact allows us to classify all the $\mathcal{F}$-spreading models of a Banach space X as an increasing transfinite hierarchy of the form $(\mathcal{SM}_\xi(X))_{\xi<\omega_1}$. Let us point out that the ξ-spreading models of X have a weaker asymptotic relation to the space X as ξ increases to ω_1.

The infinite graphs with vertices from a regular thin family and edges the plegma pairs are the key for the proof of the above results. Specifically, it is shown that if $\mathcal{G}$ and $\mathcal{F}$ are two regular thin families with $o(\mathcal{G}) \geq o(\mathcal{F})$ then there exist an infinite subset M of $\mathbb{N}$ and a plegma preserving map

(3) The *order* of $\mathcal{F}$, denoted by $o(\mathcal{F})$, is a countable ordinal which measures the complexity of $\mathcal{F}$ (see Section 2 for the precise definition). For example the family of k-element subsets of $\mathbb{N}$ has order k, while the ξth Schreier family has order $o(\mathcal{S}_\xi) = \omega^\xi$.

$\varphi : \mathcal{G}\upharpoonright M \to \mathcal{F}$ (that is, $(\varphi(s_1), \varphi(s_2))$ is a plegma pair in $\mathcal{F}$ whenever (s_1, s_2) is plegma pair in $\mathcal{G}\upharpoonright M$). Moreover, it is also shown that such an embedding is forbidden if we wish to go from families of lower order to families of higher order. More precisely, if $o(\mathcal{F}) < o(\mathcal{G})$ then for every $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$ there exists $L \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is a plegma pair in $\mathcal{G}$ (see Theorems 3.17 and 3.19).

The paper is organized as follows. In Section 2 we review some basic facts concerning families of finite subsets of $\mathbb{N}$ and we define regular thin families. In Section 3 we study plegma families and their properties. In Section 4 we introduce the definition of higher order spreading models. In Section 5 we deal with $\mathcal{F}$-sequences $(x_s)_{s\in\mathcal{F}}$ in a general topological space. Finally, in Section 6 we study $\mathcal{F}$-sequences which generate several classes of spreading sequences as spreading models. In this last section we show that several well known results concerning the classical spreading models remain valid in the higher order setting. For instance, we show that a subordinated, seminormalized and weakly null $\mathcal{F}$-sequence generates an unconditional spreading model.

The present paper is an updated version of the first part of [3]. The second part which deals with certain examples will be presented elsewhere.

1.1. Preliminary notation and definitions. By $\mathbb{N} = \{1, 2, \ldots\}$ we denote the set of all positive integers. Throughout the paper we shall identify strictly increasing sequences in $\mathbb{N}$ with their corresponding range i.e. we view every strictly increasing sequence in $\mathbb{N}$ as a subset of $\mathbb{N}$ and conversely every subset of $\mathbb{N}$ as the sequence resulting from the increasing ordering of its elements. We will use capital letters $L, M, N, \ldots$ to denote infinite subsets and lower case letters $s, t, u, \ldots$ to denote finite subsets of $\mathbb{N}$. For every infinite subset L of $\mathbb{N}$, $[L]^{<\infty}$ (resp. $[L]^\infty$) stands for the set of all finite (resp. infinite) subsets of L. For an $L = \{l_1 < l_2 < \cdots\} \in [\mathbb{N}]^\infty$ and a positive integer $k \in \mathbb{N}$, we set $L(k) = l_k$. Similarly, for a finite subset $s = \{n_1 < \cdots < n_m\}$ of $\mathbb{N}$ and for $1 \le k \le m$ we set $s(k) = n_k$. For an $L = \{l_1 < l_2 < \cdots\} \in [N]^\infty$ and a finite subset $s = \{n_1 < \cdots < n_m\}$ (resp. for an infinite subset $N = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$) we set $L(s) = \{l_{n_1}, \ldots, l_{n_m}\} = \{L(s(1)), \ldots, L(s(m))\}$ (resp. $L(N) = \{l_{n_1}, l_{n_2}, \ldots\} = \{L(N(1)), L(N(2)), \ldots\}$).

For $s \in [\mathbb{N}]^{<\infty}$ by $|s|$ we denote the cardinality of s. For $L \in [\mathbb{N}]^\infty$ and $m \in \mathbb{N}$ we denote by $[L]^m$ the set of all $s \in [L]^{<\infty}$ with $|s| = m$. Also for every nonempty $s \in [\mathbb{N}]^{<\infty}$ and $1 \le k \le |s|$ we set $s|k = \{s(1), \ldots, s(k)\}$ and $s|0 = \emptyset$. Moreover, for $s, t \in [\mathbb{N}]^{<\infty}$, we write $t \sqsubseteq s$ (resp. $t \sqsubset s$) to denote that t is an initial (resp. *proper* initial) segment of s. Also, for $s, t \in [\mathbb{N}]^{<\infty}$ we write $t < s$ if either at least one of them is the empty set, or $\max t < \min s$.

Concerning Banach space theory, although our notation is standard, we present for completeness some basic concepts that we will need. Let X be a Banach space. We say that a sequence $(x_n)_n$ in X is *bounded* (resp. *seminormalized*) if there exists $M > 0$ (resp. $M_1, M_2 > 0$) such that $\|x_n\| \leq M$ (resp. $M_1 \leq \|x_n\| \leq M_2$) for all $n \in \mathbb{N}$. The sequence $(x_n)_n$ is called *Schauder basic* if there exists a constant $C \geq 1$ such that

$$\Big\| \sum_{n=1}^{k} a_n x_n \Big\| \leq C \Big\| \sum_{n=1}^{m} a_n x_n \Big\| \tag{1.3}$$

for every $k \leq m$ in $\mathbb{N}$ and $a_1, \ldots, a_m \in \mathbb{R}$. Finally, we say that $(x_n)_n$ is C-*unconditional* if for every $m \in \mathbb{N}$, $F \subseteq \{1, \ldots, m\}$ and $a_1, \ldots, a_m \in \mathbb{R}$,

$$\Big\| \sum_{n\in F} a_n x_n \Big\| \leq C \Big\| \sum_{n=1}^{m} a_n x_n \Big\|. \tag{1.4}$$

2. Regular thin families. In this section we define regular thin families of finite subsets of $\mathbb{N}$ and we study their basic properties. The definition is based on two well known concepts, namely that of regular families traced back to [2] and that of thin families defined in [15] and extensively studied in [17] and [14].

2.1. On families of finite subsets of $\mathbb{N}$. We start with a review of the basic concepts concerning families of finite subsets of $\mathbb{N}$. For a more detailed exposition the reader is referred to [5].

2.1.1. *Ramsey properties of families of finite subsets of* $\mathbb{N}$*.* For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^{\infty}$, we set

$$\mathcal{F}{\restriction}L = \{s \in \mathcal{F} : s \subseteq L\} = \mathcal{F} \cap [L]^{<\infty}. \tag{2.1}$$

Recall some terminology from [12]. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $M \in [\mathbb{N}]^{\infty}$. We say that $\mathcal{F}$ is *large* in M if for every $L \in [M]^{\infty}$, $\mathcal{F}{\restriction}L$ is nonempty. We say that $\mathcal{F}$ is *very large* in M if for every $L \in [M]^{\infty}$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq L$. The following is a restatement (see [12]) of a well known theorem of F. Galvin and K. Prikry [10].

THEOREM 2.1. *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $M \in [\mathbb{N}]^{\infty}$. If $\mathcal{F}$ is large in M then there exists $L \in [M]^{\infty}$ such that $\mathcal{F}$ is very large in L.*

2.1.2. *The order of a family of finite subsets of* $\mathbb{N}$*.* Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a nonempty family of finite subsets of $\mathbb{N}$. The *order* of $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ is defined as follows (see also [17]). First, we assign to $\mathcal{F}$ its ($\sqsubseteq$-)*closure*, i.e. the set

$$\widehat{\mathcal{F}} = \{t \in [\mathbb{N}]^{<\infty} : \exists s \in \mathcal{F} \text{ with } t \sqsubseteq s\}, \tag{2.2}$$

which is a tree under the initial segment ordering. If $\widehat{\mathcal{F}}$ is *ill-founded* (i.e. there exists an infinite sequence $(s_n)_n$ in $\widehat{\mathcal{F}}$ such that $s_n \sqsubset s_{n+1}$) then we set

$o(\mathcal{F}) = \omega_1$. Otherwise for every maximal element s of $\widehat{\mathcal{F}}$ we set $o_{\widehat{\mathcal{F}}}(s) = 0$ and recursively for every s in $\widehat{\mathcal{F}}$ we define

$$o_{\widehat{\mathcal{F}}}(s) = \sup\{o_{\widehat{\mathcal{F}}}(t) + 1 : t \in \widehat{\mathcal{F}} \text{ and } s \sqsubset t\}. \tag{2.3}$$

The order of $\mathcal{F}$, denoted by $o(\mathcal{F})$, is defined to be the ordinal $o_{\widehat{\mathcal{F}}}(\emptyset)$. For instance $o(\{\emptyset\}) = 0$ and $o([\mathbb{N}]^k) = k$ for every $k \in \mathbb{N}$.

For every $n \in \mathbb{N}$, we define

$$\mathcal{F}_{(n)} = \{s \in [\mathbb{N}]^{<\infty} : n < s \text{ and } \{n\} \cup s \in \mathcal{F}\}, \tag{2.4}$$

where $n < s$ means that either $s = \emptyset$ or $n < \min s$. It is easy to see that for every nonempty family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ we have

$$o(\mathcal{F}) = \sup\{o(\mathcal{F}_{(n)}) + 1 : n \in \mathbb{N}\}. \tag{2.5}$$

2.1.3. *Regular families.* A family $\mathcal{R} \subseteq [\mathbb{N}]^{<\infty}$ is said to be *hereditary* if for every $s \in \mathcal{F}$ and $t \subseteq s$ we have $t \in \mathcal{F}$, and *spreading* if for any $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_k$ with $n_1 \leq m_1, \ldots, n_k \leq m_k$ we have $\{m_1, \ldots, m_k\} \in \mathcal{R}$ whenever $\{n_1, \ldots, n_k\} \in \mathcal{R}$. Also, $\mathcal{R}$ is called *compact* if the set $\{\chi_s \in \{0,1\}^{\mathbb{N}} : s \in \mathcal{R}\}$ of characteristic functions of the members of $\mathcal{R}$ is a closed subset of $\{0,1\}^{\mathbb{N}}$ under the product topology.

A family $\mathcal{R}$ of finite subsets of $\mathbb{N}$ will be called *regular* if it is compact, hereditary and spreading. Notice that for every regular family $\mathcal{R}$, $\widehat{\mathcal{R}} = \mathcal{R}$ and $\mathcal{R}_{(n)}$ is also regular for every $n \in \mathbb{N}$. Moreover, using (2.5), by induction on the order of $\mathcal{R}$ we easily get the following.

Proposition 2.2. *Let $\mathcal{R}$ be a regular family. Then $o(\mathcal{R}\restriction L) = o(\mathcal{R})$ for every $L \in [\mathbb{N}]^{\infty}$.*

Exploiting the method of [17] we obtain the next result.

Proposition 2.3. *For every $\xi < \omega_1$ there exists a regular family $\mathcal{R}_\xi$ with $o(\mathcal{R}_\xi) = \xi$.*

Proof. For $\xi = 0$ we set $\mathcal{R}_0 = \{\emptyset\}$. We proceed by induction on $\xi < \omega_1$. Assume that for some $\xi < \omega_1$ and for each $\zeta < \xi$ we have defined a regular family $\mathcal{R}_\zeta$ with $o(\mathcal{R}_\zeta) = \zeta$. If ξ is a successor ordinal, i.e. $\xi = \zeta + 1$, then we set

$$\mathcal{R}_\xi = \{\{n\} \cup s : n \in \mathbb{N},\ s \in \mathcal{R}_\zeta \text{ and } n < s\}.$$

If ξ is a limit ordinal, then we choose a strictly increasing sequence $(\zeta_n)_n$ such that $\zeta_n \to \xi$ and we set

$$\mathcal{R}_\xi = \bigcup_n \Big\{s \in \mathcal{R}_{\zeta_n} : \min s \geq n\Big\} = \bigcup_n \mathcal{R}_{\zeta_n}\restriction[n, \infty).$$

It is easy to check that $\mathcal{R}_\xi$ is regular with $o(\mathcal{R}_\xi) = \xi$ for all $\xi < \omega_1$. ∎

We will need some combinatorial properties of regular families. To this end we give the following definition.

For every $\mathcal{R} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^\infty$, let

$$L(\mathcal{R}) = \{L(s) : s \in \mathcal{R}\}. \tag{2.6}$$

Notice that $o(\mathcal{R}) = o(L(\mathcal{R}))$ and if $\mathcal{R}$ is compact (or hereditary) then so is $L(\mathcal{R})$. It is also easily verified that if $\mathcal{R}$ is spreading then $L_1(\mathcal{R}) \subseteq L_2(\mathcal{R})$ for every $L_1 \subseteq L_2$ in $[\mathbb{N}]^\infty$, and more generally,

$$L_1(\mathcal{R}_{(k)}) \subseteq L_2(\mathcal{R}_{(k)}) \tag{2.7}$$

for every $k \in \mathbb{N}$ and $L_1, L_2 \in [\mathbb{N}]^\infty$ satisfying $\{L_1(j) : j > k\} \subseteq \{L_2(j) : j > k\}$ (where $L(\mathcal{R}_{(k)}) = \{L(s) : s \in \mathcal{R}_{(k)}\}$).

PROPOSITION 2.4. *Let $\mathcal{R}, \mathcal{S}$ be regular families of finite subsets of $\mathbb{N}$ with $o(\mathcal{R}) \leq o(\mathcal{S})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $L(\mathcal{R}) \subseteq \mathcal{S}$.*

Proof. If $o(\mathcal{R}) = 0$, i.e. $\mathcal{R} = \{\emptyset\}$, then the conclusion trivially holds. Suppose that for some $\xi < \omega_1$ the proposition is true for any regular families $\mathcal{R}', \mathcal{S}' \subseteq [\mathbb{N}]^{<\infty}$ such that $o(\mathcal{R}') < \xi$ and $o(\mathcal{R}') \leq o(\mathcal{S}')$. Let $\mathcal{R}, \mathcal{S}$ be regular with $o(\mathcal{R}) = \xi$ and let $M \in [\mathbb{N}]^\infty$. By (2.5) we have $o(\mathcal{R}_{(1)}) < o(\mathcal{R})$. Hence $o(\mathcal{R}_{(1)}) < o(\mathcal{S})$ and so there is some $l_1 \in \mathbb{N}$ such that $o(\mathcal{R}_{(1)}) \leq o(\mathcal{S}_{(l_1)})$. Since $\mathcal{S}$ is spreading we have $o(\mathcal{S}_{(l_1)}) \leq o(\mathcal{S}_{(n)})$ for all $n \geq l_1$ and therefore we may suppose that $l_1 \in M$. Since $\mathcal{R}_{(1)}$ and $\mathcal{S}_{(l_1)}$ are regular families, by our inductive hypothesis there is $L_1 \in [M]^\infty$ such that $L_1(\mathcal{R}_{(1)}) \subseteq \mathcal{S}_{(l_1)}$.

Proceeding in the same way we construct a strictly increasing sequence $(l_j)_j$ in M and a decreasing sequence $M = L_0 \supset L_1 \supset \cdots$ of infinite subsets of M such that (i) $l_{j+1} \in L_j$, (ii) $l_{j+1} > L_j(j)$, and (iii) $L_j(\mathcal{R}_{(j)}) \subseteq \mathcal{S}_{(l_j)}$, for all $j \geq 1$.

We set $L = \{l_j\}_j$ and we claim that $L(\mathcal{R}) \subseteq \mathcal{S}$. Indeed, by the above construction we see that for every $k \in \mathbb{N}$, $\{L(j)\}_{j>k} \subseteq \{L_k(j)\}_{j>k}$. Therefore by (2.7) and (iii) above, we get

$$L(\mathcal{R}_{(k)}) \subseteq L_k(\mathcal{R}_{(k)}) \subseteq \mathcal{S}_{(l_k)}. \tag{2.8}$$

It is easy to see that $L(\mathcal{R}_{(k)}) = L(\mathcal{R})_{(l_k)}$ and so (2.8) shows $L(\mathcal{R})_{(l_k)} \subseteq \mathcal{S}_{(l_k)}$. Since this holds for every $k \in \mathbb{N}$, we conclude that $L(\mathcal{R}) \subseteq \mathcal{S}$. ∎

The next corollary is an immediate consequence.

COROLLARY 2.5. *Let $\mathcal{R}, \mathcal{S}$ be regular families of finite subsets of $\mathbb{N}$ with $o(\mathcal{R}) = o(\mathcal{S})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $L(\mathcal{R}) \subseteq \mathcal{S}$ and $L(\mathcal{S}) \subseteq \mathcal{R}$.*

2.1.4. *Thin families.* A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is called *thin* if there do not exist s, t in $\mathcal{F}$ such that s is a proper initial segment of t. The following result is due to C. St. J. A. Nash-Williams [15]. Since it plays a crucial role in what follows, for the sake of completeness we present its proof.

PROPOSITION 2.6. *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a thin family. Then for every finite partition $\mathcal{F} = \bigcup_{i=1}^{k} \mathcal{F}_i$ ($k \geq 2$) of $\mathcal{F}$ and every $M \in [\mathbb{N}]^\infty$ there exist $L \in [M]^\infty$ and $1 \leq i_0 \leq k$ such that $\mathcal{F}\upharpoonright L \subseteq \mathcal{F}_{i_0}$.*

Proof. It suffices to show the result for $k = 2$ since the general case follows easily by induction. So let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and $M \in [\mathbb{N}]^\infty$. Then either there is $L \in [M]^\infty$ such that $\mathcal{F}_1\upharpoonright L = \emptyset$ or $\mathcal{F}_1$ is large in M. In the first case it is clear that $\mathcal{F}\upharpoonright L \subseteq \mathcal{F}_2$. In the second case by Theorem 2.1 there is $L \in [M]^\infty$ such that $\mathcal{F}_1$ is very large in L. We claim that $\mathcal{F}\upharpoonright L \subseteq \mathcal{F}_1$. Indeed, let $s \in \mathcal{F}\upharpoonright L$. We choose $N \in [L]^\infty$ such that $s \sqsubseteq N$ and let $t \sqsubseteq N$ be such that $t \in \mathcal{F}_1$. Then s, t are $\sqsubseteq$-comparable members of $\mathcal{F}$ and since $\mathcal{F}$ is thin, $s = t \in \mathcal{F}_1$. Therefore $\mathcal{F}\upharpoonright L \subseteq \mathcal{F}_1$. ■

2.2. Regular thin families. We are now ready to introduce the main concept of this section.

DEFINITION 2.7. A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ will be called *regular thin* if (a) $\mathcal{F}$ is thin, and (b) the $\sqsubseteq$-closure $\widehat{\mathcal{F}}$ of $\mathcal{F}$ is a regular family.

The next lemma allows us to construct regular thin families from regular ones. We will use the following notation. For a family $\mathcal{R} \subseteq [\mathbb{N}]^{<\infty}$ we set

$$\mathcal{M}(\mathcal{R}) = \{s \in \mathcal{R} : s \text{ is } \sqsubseteq\text{-maximal in } \mathcal{R}\}. \tag{2.9}$$

Notice that a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ is thin if and only if $\mathcal{F} = \mathcal{M}(\widehat{\mathcal{F}})$.

LEMMA 2.8. *Let $\mathcal{R}$ be a regular family. Then the family $\mathcal{M}(\mathcal{R})$ is thin and $\widehat{\mathcal{M}(\mathcal{R})} = \mathcal{R}$. Therefore $\mathcal{M}(\mathcal{R})$ is regular thin with $o(\mathcal{M}(\mathcal{R})) = o(\mathcal{R})$.*

Proof. Since $\mathcal{M}(\mathcal{R}) \subseteq \mathcal{R}$ and $\mathcal{R}$ is hereditary, we have $\widehat{\mathcal{M}(\mathcal{R})} \subseteq \mathcal{R}$. To show that $\mathcal{R} \subseteq \widehat{\mathcal{M}(\mathcal{R})}$ notice that for every $s \in \mathcal{R}$ there exists a $t \in \mathcal{M}(\mathcal{R})$ such that $s \sqsubseteq t$, otherwise $\mathcal{R}$ would not be compact. Hence $\widehat{\mathcal{M}(\mathcal{R})} = \mathcal{R}$ and clearly $\mathcal{M}(\mathcal{R})$ is thin. Thus $\mathcal{M}(\mathcal{R})$ is regular thin. Finally, by the definition of the order, we have $o(\mathcal{M}(\mathcal{R})) = o(\widehat{\mathcal{M}(\mathcal{R})})$, hence $o(\mathcal{M}(\mathcal{R})) = o(\mathcal{R})$. ■

COROLLARY 2.9. *For every $\xi < \omega_1$ there is a regular thin family $\mathcal{F}_\xi$ with $o(\mathcal{F}_\xi) = \xi$.*

Proof. Let $\xi < \omega_1$ and $\mathcal{R}_\xi$ be a regular family with $o(\mathcal{R}_\xi) = \xi$. Then $\mathcal{F}_\xi = \mathcal{M}(\mathcal{R}_\xi)$ is as desired. ■

COROLLARY 2.10. *The map which sends $\mathcal{F}$ to $\widehat{\mathcal{F}}$ is a bijection between the set of all regular thin families and the set of all regular ones. Moreover, the inverse map sends each regular family $\mathcal{R}$ to $\mathcal{M}(\mathcal{R})$.*

Proof. By the definition of regular thin families, the map $\mathcal{F} \to \widehat{\mathcal{F}}$ sends each regular thin family to a regular one. By Lemma 2.8 we deduce that the map is 1-1, onto and the inverse map sends each regular family $\mathcal{R}$ to $\mathcal{M}(\mathcal{R})$. ∎

REMARK 1. If $\mathcal{F}$ is a regular thin family with $o(\mathcal{F}) = k < \omega$, then it is easy to see that there exists n_0 such that $\mathcal{F}\restriction[n_0, \infty) = \{s \in [\mathbb{N}]^k : \min s \geq n_0\}$. Therefore, for each $k < \omega$, the family $[\mathbb{N}]^k$ is essentially the unique regular thin family of order k. However this does not remain valid for regular thin families of order $\xi \geq \omega$. For instance, for every unbounded increasing map $f : \mathbb{N} \to \mathbb{N}$ the family $\mathcal{F} = \{s \in [\mathbb{N}]^{<\infty} : |s| = f(\min s)\}$ is a regular thin family of order ω.

LEMMA 2.11. *Let $\mathcal{R}$ be a regular family and $L \in [\mathbb{N}]^\infty$. Then $\mathcal{M}(\mathcal{R})\restriction L = \mathcal{M}(\mathcal{R}\restriction L)$, and setting $\mathcal{M} = \mathcal{M}(\mathcal{R})$, we have $\widehat{\mathcal{M}\restriction L} = \mathcal{R}\restriction L$ and $o(\mathcal{M}\restriction L) = o(\mathcal{R})$.*

Proof. It is easy to see that $\mathcal{M}(\mathcal{R})\restriction L \subseteq \mathcal{M}(\mathcal{R}\restriction L)$. To show the converse inclusion let $s \in \mathcal{M}(\mathcal{R}\restriction L)$ and assume that $s \notin \mathcal{M}(\mathcal{R})$. Since $\mathcal{R}\restriction L \subseteq \mathcal{R}$, $s \in \mathcal{R}$ and therefore there exists some $t \in \mathcal{M}(\mathcal{R})$ with $s \sqsubset t$. Since $\mathcal{R}$ is spreading this implies there exists $t' \in \mathcal{R}\restriction L$ with $s \sqsubset t'$. Thus $s \notin \mathcal{M}(\mathcal{R}\restriction L)$, a contradiction. Therefore $s \in \mathcal{M}(\mathcal{R})$. Since $s \in [L]^{<\infty}$, we see that $s \in \mathcal{M}(\mathcal{R})\restriction L$. Therefore $\mathcal{M}(\mathcal{R}\restriction L) = \mathcal{M}(\mathcal{R})\restriction L$.

Since $\mathcal{M}\restriction L = \mathcal{M}(\mathcal{R})\restriction L \subseteq \mathcal{R}\restriction L$ and $\mathcal{R}\restriction L$ is hereditary, we have $\widehat{\mathcal{M}\restriction L} \subseteq \mathcal{R}\restriction L$. Conversely, let $s \in \mathcal{R}\restriction L$. Since $\mathcal{R}\restriction L$ is compact there is $t \in \mathcal{M}(\mathcal{R}\restriction L) = \mathcal{M}(\mathcal{R})\restriction L$ with $s \sqsubseteq t$. Hence $s \in \widehat{\mathcal{M}\restriction L}$ and $\widehat{\mathcal{M}\restriction L} = \mathcal{R}\restriction L$.

Finally, $o(\mathcal{M}\restriction L) = o(\widehat{\mathcal{M}\restriction L}) = o(\mathcal{R}\restriction L) = o(\mathcal{R})$, where the last equality follows by Proposition 2.2. ∎

COROLLARY 2.12. *Let $\mathcal{F}$ be a regular thin family and $L \in [\mathbb{N}]^\infty$. Then*

$$\mathcal{F}\restriction L = \mathcal{M}(\widehat{\mathcal{F}}\restriction L), \qquad \widehat{\mathcal{F}\restriction L} = \widehat{\mathcal{F}}\restriction L, \qquad o(\mathcal{F}\restriction L) = o(\mathcal{F}).$$

Proof. Since $\mathcal{F}$ is thin we have $\mathcal{F} = \mathcal{M}(\widehat{\mathcal{F}})$. Setting $\mathcal{R} = \widehat{\mathcal{F}}$ in Lemma 2.11 yields the result. ∎

COROLLARY 2.13. *Let $\mathcal{F}$ be a regular thin family. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F}\restriction L$ is very large in L.*

Proof. If $\mathcal{F}$ is regular thin then since $\widehat{\mathcal{F}}$ is spreading, $\widehat{\mathcal{F}}\restriction N$ is nonempty for every $N \in [\mathbb{N}]^\infty$. Since $\widehat{\mathcal{F}}\restriction N = \widehat{\mathcal{F}\restriction N}$, we see that $\mathcal{F}\restriction N$ is nonempty too, i.e. $\mathcal{F}$ is large in $\mathbb{N}$. Therefore, by Theorem 2.1, for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F}\restriction L$ is very large in L. ∎

DEFINITION 2.14. For two families $\mathcal{F}, \mathcal{G}$ of finite subsets of $\mathbb{N}$, we write $\mathcal{F} \sqsubseteq \mathcal{G}$ (resp. $\mathcal{F} \sqsubset \mathcal{G}$) if every element in $\mathcal{F}$ has an extension (resp. proper extension) in $\mathcal{G}$ and every element in $\mathcal{G}$ has an initial (resp. proper initial) segment in $\mathcal{F}$.

The following is a consequence of a more general result from [11].

PROPOSITION 2.15. *Let $\mathcal{F},\mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$ be regular thin families with $o(\mathcal{F}) < o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that $\mathcal{F}\upharpoonright L \sqsubset \mathcal{G}\upharpoonright L$.*

Proof. By Corollary 2.13 we deduce that there exists $L_1 \in [M]^\infty$ such that both $\mathcal{F},\mathcal{G}$ are very large in L_1. So for every $L \in [L_1]^\infty$ and every $t \in \mathcal{G}\upharpoonright L$ there exists $s \in \mathcal{F}\upharpoonright L$ such that s, t are comparable.

Let $\mathcal{G}_1$ be the set of all elements of $\mathcal{G}$ which have a proper initial segment in $\mathcal{F}$ and $\mathcal{G}_2 = \mathcal{G}\setminus\mathcal{G}_1$. By Proposition 2.6 there exist $i_0 \in \{1, 2\}$ and $L \in [L_1]^\infty$ such that $\mathcal{G}\upharpoonright L \subseteq \mathcal{G}_{i_0}$. It suffices to show that $i_0 = 1$. Indeed, if $i_0 = 2$ then for every $t \in G\upharpoonright L$ there is $s \in \mathcal{F}$ such that $t \sqsubseteq s$. This in conjunction with Corollary 2.12 yields $o(\mathcal{G}) = o(\mathcal{G}\upharpoonright L) \le o(\mathcal{F})$, which is a contradiction. ∎

A similar but weaker result holds when $o(\mathcal{F}) = o(\mathcal{G})$.

PROPOSITION 2.16. *Let $\mathcal{F},\mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$ be regular thin families such that $o(\mathcal{F}) = o(\mathcal{G})$. Then there exists $L_0 \in [\mathbb{N}]^\infty$ such that for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [L_0(M)]^\infty$ such that $L_0(\mathcal{F})\upharpoonright L \sqsubseteq \mathcal{G}\upharpoonright L$.*

Proof. By Proposition 2.4 there exists $L_0 \in [\mathbb{N}]^\infty$ such that $L_0(\widehat{\mathcal{F}}) \subseteq \widehat{\mathcal{G}}$. Let $M \in [\mathbb{N}]^\infty$. Notice that $L_0(\mathcal{F})$ and $\mathcal{G}$ are large in $L_0(M)$. Hence by Theorem 2.1 there exists $L \in [L_0(M)]^\infty$ such that $L_0(\mathcal{F})$ and $\mathcal{G}$ are very large in N. Since $L_0(\widehat{\mathcal{F}}) \subseteq \widehat{\mathcal{G}}$, we conclude that $L_0(\mathcal{F})\upharpoonright L \sqsubseteq \mathcal{G}\upharpoonright L$. ∎

Technically the above two propositions are incorporated in one as follows.

COROLLARY 2.17. *Let $\mathcal{F},\mathcal{G} \subseteq [\mathbb{N}]^{<\infty}$ be regular thin families with $o(\mathcal{F}) \le o(\mathcal{G})$. Then there exists $L_0 \in [\mathbb{N}]^\infty$ such that for every $M \in [\mathbb{N}]^\infty$ there exists $L \in [L_0(M)]^\infty$ such that $L_0(\mathcal{F})\upharpoonright L \sqsubseteq \mathcal{G}\upharpoonright L$.*

Proof. If $o(\mathcal{F}) < o(\mathcal{G})$, we set $L_0 = \mathbb{N}$. Then $L_0(\mathcal{F}) = \mathcal{F}$ and $L_0(M) = M$ and the conclusion follows by Proposition 2.15. If $o(\mathcal{F}) = o(\mathcal{G})$ the result is immediate by Proposition 2.16. ∎

3. Plegma families. In this section we introduce the notion of plegma families initially defined in [4] for k-subsets of $\mathbb{N}$. Here we do not assume that all members of a plegma family are necessarily of the same cardinality.

3.1. Definition and basic properties. We begin by stating the definition of a plegma family.

DEFINITION 3.1. Let $l \in \mathbb{N}$ and $s_1, \ldots, s_l$ be nonempty finite subsets of $\mathbb{N}$. The l-tuple $(s_j)_{j=1}^l$ will be called a *plegma family* if the following are satisfied:

(i) For every $i, j \in \{1, \ldots, l\}$ and $k \in \mathbb{N}$ with $i < j$ and $k \le \min(|s_i|, |s_j|)$, we have $s_i(k) < s_j(k)$.

(ii) For every $i,j\in\{1,\dots,l\}$ and $k\in\mathbb{N}$ with $k\le\min(|s_i|,|s_j|-1)$, we have $s_i(k)<s_j(k+1)$.

For instance a pair $(\{n_1\},\{n_2\})$ of singletons is plegma iff $n_1<n_2$, and a pair of doubletons $(\{n_1,m_1\},\{n_2,m_2\})$ is plegma iff $n_1<n_2<m_1<m_2$. More generally for two nonempty $s,t\in[\mathbb{N}]^{<\infty}$ with $|s|\le|t|$ the pair (s,t) is a plegma pair iff $s(1)<t(1)<s(2)<t(2)<\dots<s(|s|)<t(|s|)$. Of course the situation is more involved when the size of a plegma family is large.

As we have mentioned in [4] plegma families are related and, as we will see, have similar Ramsey properties to, shift graphs on thin families of finite subsets of $\mathbb{N}$ (see [8]).

Below we gather together some stability properties of plegma families. We omit the proof as it is a direct application of the definition.

Proposition 3.2. *Let $(s_j)_{j=1}^l$ be a family of finite subsets of $\mathbb{N}$. Then the following are satisfied:*

(i) *If $(s_j)_{j=1}^l$ is a plegma family then so is $(s_{j_m})_{m=1}^k$ for every $1\le k\le l$ and $1\le j_1<\dots<j_k\le l$.*
(ii) *The family $(s_j)_{j=1}^l$ is a plegma family iff (s_{j_1},s_{j_2}) is a plegma pair for every $1\le j_1<j_2\le l$.*
(iii) *If $(s_j)_{j=1}^l$ is a plegma family then so is $(t_j)_{j=1}^l$ whenever $\emptyset\neq t_j\sqsubseteq s_j$ for $1\le j\le l$.*
(iv) *If $(s_j)_{j=1}^l$ is a plegma family then so is $(L(s_j))_{j=1}^l$ for every $L\in[\mathbb{N}]^\infty$.*

For every family $\mathcal{F}\subseteq[\mathbb{N}]^{<\infty}$ and $l\in\mathbb{N}$ we denote by $\mathrm{Plm}_l(\mathcal{F})$ the set of all $(s_j)_{j=1}^l$ such that $s_1,\dots,s_l\in\mathcal{F}$ and $(s_j)_{j=1}^l$ is a plegma family. We also set $\mathrm{Plm}(\mathcal{F})=\bigcup_{l=1}^\infty\mathrm{Plm}_l(\mathcal{F})$. Our main aim is to show that for every $l\in\mathbb{N}$, $\mathrm{Plm}_l(\mathcal{F})$ is a Ramsey family. To this end we need some preparatory lemmas.

Lemma 3.3. *Let $\mathcal{F}$ be a regular thin family and $l\in\mathbb{N}$. Then for every $(s_j)_{j=1}^l\in\mathrm{Plm}_l(\mathcal{F})$ we have $|s_1|\le\dots\le|s_l|$.*

Proof. By Proposition 3.2(ii) it suffices to show the conclusion for $l=2$. Assume on the contrary that there exists a plegma pair (s_1,s_2) in $\mathcal{F}$ with $|s_1|>|s_2|$. We pick $s\in[\mathbb{N}]^{<\infty}$ such that $|s|=|s_1|$, $s_2\sqsubset s$ and $s(|s_2|+1)>\max s_1$. By the definition of the plegma family, we see that for every $1\le k\le|s_2|$, $s_1(k)<s_2(k)=s(k)$. Hence, for every $1\le k\le|s_1|$, we have $s_1(k)\le s(k)$. By the spreading property of $\widehat{\mathcal{F}}$ we get $s\in\widehat{\mathcal{F}}$. But since s_2 is a proper initial segment of s we get $s_2\notin\mathcal{F}$, which is a contradiction. ∎

Lemma 3.4. *Let $\mathcal{F}$ be a thin family of finite subsets of $\mathbb{N}$ and $l\in\mathbb{N}$. Let $(s_j)_{j=1}^l,(t_j)_{j=1}^l\in\mathrm{Plm}_l(\mathcal{F})$ with $|s_1|\le\dots\le|s_l|$, $|t_1|\le\dots\le|t_l|$ and*

$\bigcup_{j=1}^{l} s_j \sqsubseteq \bigcup_{j=1}^{l} t_j$. Then $(s_j)_{j=1}^{l} = (t_j)_{j=1}^{l}$ and consequently $\bigcup_{j=1}^{l} s_j = \bigcup_{j=1}^{l} t_j$.

Proof. Suppose that for some $1 \leq m \leq l$ we have $(s_i)_{i<m} = (t_i)_{i<m}$. We will show that $s_m = t_m$. Let $s = \bigcup_{j=m}^{l} s_j$ and $t = \bigcup_{j=m}^{l} t_j$. Then by our assumptions $s \sqsubseteq t$. Moreover since $|s_m| \leq \cdots \leq |s_l|$ and $|t_m| \leq \cdots \leq |t_l|$, we easily conclude that $s_m(j) = s((j-1)(l-m+1)+1)$ for all $1 \leq j \leq |s_m|$ and similarly $t_m(j) = t((j-1)(l-m+1)+1)$ for all $1 \leq j \leq |t_m|$. Hence, as $s \sqsubseteq t$, we see that for all $1 \leq j \leq \min\{|t_m|, |s_m|\}$, $s_m(j) = t_m(j)$. Therefore s_m and t_m are $\sqsubseteq$-comparable. Since $\mathcal{F}$ is thin we have $s_m = t_m$. By induction on $1 \leq m \leq l$, we obtain $s_j = t_j$ for every $1 \leq j \leq l$. ■

By the above two lemmas we have the following.

Corollary 3.5. *Let $\mathcal{F}$ be a regular thin family of finite subsets of $\mathbb{N}$. Let $l \in \mathbb{N}$ and set*

$$\mathcal{U} = \Big\{ \bigcup_{i=1}^{l} s_i : (s_i)_{i=1}^{l} \in \mathrm{Plm}_l(\mathcal{F}) \Big\}.$$

Then $\mathcal{U}$ is a thin family.

It is easy to check that the family $\mathcal{U}$ defined above is actually regular thin. Since we will not make use of this fact, we omit its proof.

Theorem 3.6. *Let M be an infinite subset of $\mathbb{N}$, $l \in \mathbb{N}$ and $\mathcal{F}$ be a regular thin family. Then for every finite partition $\mathrm{Plm}_l(\mathcal{F}\restriction M) = \bigcup_{i=1}^{p} \mathcal{P}_i$, there exist $L \in [M]^\infty$ and $1 \leq i_0 \leq p$ such that $\mathrm{Plm}_l(\mathcal{F}\restriction L) \subseteq \mathcal{P}_{i_0}$.*

Proof. Let $\mathcal{U} = \{\bigcup_{j=1}^{l} s_j : (s_j)_{j=1}^{l} \in \mathrm{Plm}_l(\mathcal{F}\restriction M)\}$. By Corollary 3.5 we find that $\mathcal{U}$ is thin. Moreover, by Lemma 3.4 the map $\Phi : \mathrm{Plm}_l(\mathcal{F}\restriction M) \to \mathcal{U}$ sending each plegma l-tuple $(s_j)_{j=1}^{l}$ with $s_i \in \mathcal{F}\restriction M$ for $1 \leq i \leq l$ to its union $\bigcup_{j=1}^{l} s_j$ is a bijection. We set $\mathcal{U}_i = \Phi(\mathcal{P}_i)$ for $1 \leq j \leq p$. Then $\mathcal{U} = \bigcup_{i=1}^{p} \mathcal{U}_i$ and since $\mathcal{U}$ is thin, by Proposition 2.6, there exist j_0 and $L \in [M]^\infty$ such that $\mathcal{U}\restriction L \subseteq_{i_0} \mathcal{U}_{i_0}$ or equivalently $\mathrm{Plm}_l(\mathcal{F}\restriction L) \subseteq \mathcal{P}_{j_0}$. ■

3.2. Plegma paths. In this subsection we introduce the definition of plegma paths in finite subsets of $\mathbb{N}$ and we present some of their properties. Such paths will be used in the next subsection for the study of plegma preserving maps.

Definition 3.7. Let $k \in \mathbb{N}$ and $s_0, \ldots, s_k$ be nonempty finite subsets of $\mathbb{N}$. We will say that $(s_j)_{j=0}^{k}$ is a *plegma path* of length k tfrom s_0 to s_k if for every $0 \leq j \leq k-1$, the pair (s_j, s_{j+1}) is plegma. Similarly, a sequence $(s_j)_j$ of nonempty finite subsets of $\mathbb{N}$ will be called an *infinite plegma path* if for every $j \in \mathbb{N}$ the pair (s_j, s_{j+1}) is plegma.

The next simple lemma will prove very useful.

LEMMA 3.8. *Let $(s_0, \ldots, s_{k-1}, s)$ be a plegma path of length k from s_0 to s such that $s_0 < s$. Then*

$$k \geq \min\{|s_i| : 0 \leq i \leq k-1\}.$$

Proof. Suppose that $k < \min\{|s_i| : 0 \leq i \leq k-1\}$. Then $s(1) < s_{k-1}(2) < s_{k-2}(3) < \cdots < s_1(k) < s_0(k+1)$, which contradicts $s_0 < s$. ■

For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ a *plegma path in $\mathcal{F}$* is a (finite or infinite) plegma path which consists of elements of $\mathcal{F}$. It is easy to verify the existence of infinite plegma paths in $\mathcal{F}$ whenever $\mathcal{F}$ is very large in an infinite subset L of $\mathbb{N}$. In particular, let $s \in \mathcal{F}\restriction L$ have the following property: for every $j = 1, \ldots, |s|-1$ there exists $l \in L$ such that $s(j) < l < s(j+1)$. Then it is straightforward that there exists $s' \in \mathcal{F}\restriction L$ such that the pair (s, s') is plegma and moreover s' shares the same property with s. Based on this one can built an infinite plegma path in $\mathcal{F}$ of elements with the above property.

These remarks motivate the following definition. For every $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^{\infty}$, we set

$$\mathcal{F}\restriction\restriction L = \{s \in \mathcal{F}\restriction L : \forall 1 \leq j \leq |s|-1 \ \exists l \in L \text{ with } s(j) < l < s(j+1)\}. \tag{3.1}$$

The proof of the next lemma follows the lines of the one of Lemma 2.11.

LEMMA 3.9. *Let $\mathcal{F}$ be a regular thin family and $L \in [\mathbb{N}]^{\infty}$. Then $\widehat{\mathcal{F}\restriction\restriction L} = \widehat{F}\restriction\restriction L$, i.e. $s \in \mathcal{F}\restriction\restriction L$ iff s is $\sqsubseteq$-maximal in $\widehat{F}\restriction\restriction L$.*

We are now ready to present the main result of this subsection. In terms of graph theory it states that in the (directed) graph with vertices the elements of $\mathcal{F}\restriction\restriction L$ and edges the plegma pairs (s, t) in $\mathcal{F}\restriction\restriction L$, the distance between two vertices s_0 and s with $s_0 < s$ is equal to the cardinality of s_0.

THEOREM 3.10. *Let $\mathcal{F}$ be a regular thin family and $L \in [\mathbb{N}]^{\infty}$. Assume that $\mathcal{F}$ is very large in L. Then for every $s_0, s \in \mathcal{F}\restriction\restriction L$ with $s_0 < s$ there exists a plegma path $(s_0, \ldots, s_{k-1}, s)$ in $\mathcal{F}\restriction\restriction L$ of length $k = |s_0|$. Moreover $|s_0|$ is the minimal length of a plegma path in $\mathcal{F}\restriction\restriction L$ from s_0 to s.*

Proof. By Lemmas 3.8 and 3.3 every plegma path in $\mathcal{F}$ from s_0 to s is of length at least $|s_0|$. Therefore for $s_0 < s$ a plegma path of the form $(s_0, \ldots, s_{k-1}, s)$ with $s_0, \ldots, s_{k-1}, s \in \mathcal{F}$ and $k = |s_0|$ certainly is of minimal length.

We will actually prove a slightly more general result. Namely we will show that for every t in $\widehat{\mathcal{F}}\restriction\restriction L$ and $s \in \mathcal{F}\restriction\restriction L$ with $t < s$ there exists a plegma path of length $|t|$ from t to s such that all its elements except perhaps t belong to $\mathcal{F}\restriction\restriction L$.

For the proof we will use induction on the length of t. The case $|t| = 1$ is trivial, since for every $s \in [\mathbb{N}]^{<\infty}$ with $t < s$ the pair (t, s) is already a plegma path of length 1 from t to s. Suppose that for some $k \in \mathbb{N}$ the above holds for all t in $\widehat{\mathcal{F}}\restriction\restriction L$ with $|t| = k$.

Let $t \in \widehat{\mathcal{F}}\lceil\lceil L$ with $|t| = k+1$ and $s \in \mathcal{F}\lceil\lceil L$ with $t < s$. Then there exist $n_1 < n_2 < \cdots < n_{k+1}$ in $\mathbb{N}$ such that $n_j - n_{j-1} > 1$ for $2 \le j \le k$ and $t = \{L(n_j) : 1 \le j \le k+1\}$. We set $t_0 = \{L(n_j - 1) : 2 \le j \le k+1\}$. Since $n_j - 1 > n_{j-1}$ we see that t_0 is of equal cardinality and pointwise strictly greater than $t \setminus \{\max t\}$. Hence, since $\widehat{\mathcal{F}}$ is spreading, we have $t_0 \in \widehat{\mathcal{F}}\lceil\lceil L$ and moreover t_0 cannot be a $\sqsubseteq$-maximal element of $\widehat{\mathcal{F}}\lceil\lceil L$. By Lemma 3.9 we find that $\mathcal{F}\lceil\lceil L$ is the set of all $\sqsubseteq$-maximal elements of $\widehat{\mathcal{F}}\lceil\lceil L$. Therefore, we conclude that $t_0 \in \widehat{\mathcal{F}} \setminus \mathcal{F}$. Thus, since $|t_0| = k$, by the inductive hypothesis, there exists a plegma path $(t_0, s_1, \ldots, s_{k-1}, s)$ of length $k = |t_0|$ from t_0 to s with all $s_1, \ldots, s_{k-1}, s$ in $\mathcal{F}\lceil\lceil L$.

Let $l = |s_1|$. Since (t_0, s_1) is a plegma pair with $t_0 \in \widehat{F} \setminus \mathcal{F}$ and $s_1 \in \mathcal{F}$, arguing as in Lemma 3.3, we see that $l \ge k+1$. Moreover since $s_1 \in \mathcal{F}\lceil\lceil L$, there exist $m_1 < \cdots < m_l$ in $\mathbb{N}$ such that $m_j - m_{j-1} > 1$ and $s_1 = \{L(m_j) : 1 \le j \le l\}$. Notice that $n_2 \le m_1 < n_3 \le m_2 < \cdots < n_k \le m_{k-1} < m_{k+1} - 1$.

We set $w = t_0 \cup \{L(m_j - 1) : k+1 \le j \le l\}$ and let $L' \in [L]^\infty$ be such that w is an initial segment of L'. Notice that $|w| = l$. Since $\mathcal{F}$ is very large in L there exists $s_0 \in \mathcal{F}$ with s_0 an initial segment of L'. Using again the fact that $\widehat{\mathcal{F}}$ is spreading it is shown that $|t_0| < |s_0| \le l$ and therefore $t_0 \sqsubset s_0 \sqsubseteq w$.

It is easy to check that (t, s_0) and (s_0, s_1) are plegma pairs. Hence the sequence $(t, s_0, \ldots, s_{k-1}, s)$ is a plegma path of length $k+1$ from t to s with $s_0, \ldots, s_{k-1}, s \in \mathcal{F}\lceil\lceil L$. The proof of the inductive step as well as of the theorem is complete. ∎

We close this section by presenting an application of the above theorem. We start with the following definition.

Let X be a set, $M \in [\mathbb{N}]^\infty$, $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $\varphi : \mathcal{F} \to X$. We will say that φ is *hereditarily nonconstant in* M if for every $L \in [M]^\infty$ the restriction of φ on $\mathcal{F}\lceil L$ is nonconstant. In particular if $M = \mathbb{N}$ then we will simply say that φ is *hereditarily nonconstant.*

Lemma 3.11. *Let $\mathcal{F}$ be a regular thin family, X be a set and $\varphi : \mathcal{F} \to X$ be hereditarily nonconstant. Then for every $N \in [\mathbb{N}]^\infty$ there exists $L \in [N]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\lceil L$, $\varphi(s_1) \neq \varphi(s_2)$.*

Proof. By Theorem 3.6 there exists an $L \in [N]^\infty$ such that either $\varphi(s_1) \neq \varphi(s_2)$ for all plegma pairs (s_1, s_2) in $\mathcal{F}\lceil L$, or $\varphi(s_1) = \varphi(s_2)$ for all plegma pairs (s_1, s_2) in $\mathcal{F}\lceil L$. The second alternative is excluded. Indeed, suppose that $\varphi(s_1) = \varphi(s_2)$, for every plegma pair (s_1, s_2) in $\mathcal{F}\lceil L$. By Corollary 2.13 we may also assume that $\mathcal{F}\lceil L$ is very large in L. Let s_0 be the unique initial segment of $L_0 = \{L(2\rho) : \rho \in \mathbb{N}\}$ in $\mathcal{F}\lceil L$ and let $k = |s_0|$. We set $L'_0 = \{L(2\rho) : \rho \in \mathbb{N} \text{ and } \rho > k\}$. By Theorem 3.10 for every $s \in \mathcal{F}\lceil L'_0$ there exists a plegma path $(s_0, s_1, \ldots, s_{k-1}, s)$ of length k in $\mathcal{F}\lceil L$. Therefore for every $s \in \mathcal{F}\upharpoonright L_0'$ we have $\varphi(s) = \varphi(s_{k-1}) = \cdots = \varphi(s_1) = \varphi(s_0)$, which contradicts that φ is hereditarily nonconstant. ∎

PROPOSITION 3.12. *Let $\mathcal{F}$ be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F} \to \mathbb{N}$ be hereditarily nonconstant in M. Let also $g : \mathbb{N} \to \mathbb{N}$. Then there exists $N \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$, $\varphi(s_2) - \varphi(s_1) > g(n)$, where $\min s_2 = N(n)$.*

Proof. By Theorem 3.6 there exists $L \in [M]^\infty$ such that one of the following holds:

(i) For every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$, we have $\varphi(s_1) = \varphi(s_2)$.
(ii) For every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$, we have $\varphi(s_1) > \varphi(s_2)$.
(iii) For every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$, we have $\varphi(s_1) < \varphi(s_2)$.

Since φ is hereditarily nonconstant in M, by Lemma 3.11, case (i) is excluded. Similarly (ii) cannot occur since otherwise $(\varphi(s_n))_n$ would form a strictly decreasing sequence in $\mathbb{N}$ whenever $(s_n)_n$ is an infinite plegma path in $\mathcal{F}\upharpoonright L$. Therefore, (iii) holds. We choose $N \in [L]^\infty$ such that for every $n \geq 2$,

$$|\{l \in L : N(n-1) < l < N(n)\}| \geq \max_{j \leq n} g(j).$$

Let (s_1, s_2) be a plegma pair in $\mathcal{F}\upharpoonright N$ and let $n \in \mathbb{N}$ be such that $\min s_2 = N(n)$. Notice for every $1 \leq k \leq |s_1|$,

$$|\{l \in L : s_1(k) < l < s_2(k)\}| \geq g(n).$$

Similarly for every $|s_1| < k \leq |s_2|$,

$$|\{l \in L : s_2(k-1) < l < s_2(k)\}| \geq g(n).$$

The above shows that there exist $t_1, \ldots, t_{g(n)} \in \mathcal{F}\upharpoonright L$ such that the $(g(n)+2)$-tuple $(s_1, t_1, \ldots, t_{g(n)}, s_2)$ is plegma. Hence $\varphi(s_2) - \varphi(s_1) > g(n)$. ∎

COROLLARY 3.13. *Let $\mathcal{F}$ be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F} \to \mathbb{N}$ be hereditarily nonconstant in M. Then there exists $N \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$ we have $\varphi(s_2) - \varphi(s_1) > 1$.*

3.3. Plegma preserving maps. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $\varphi : \mathcal{F} \to [\mathbb{N}]^{<\infty}$. We will say that the map φ is *plegma preserving* if $(\varphi(s_1), \varphi(s_2))$ is a plegma pair whenever (s_1, s_2) is a plegma pair in $\mathcal{F}$.

LEMMA 3.14. *Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $\varphi : \mathcal{F} \to [\mathbb{N}]^{<\infty}$. If φ is plegma preserving then for every $l \in \mathbb{N}$ and $(s_j)_{j=1}^l \in \mathrm{Plm}(\mathcal{F})$, $(\varphi(s_j))_{j=1}^l$ is a plegma l-tuple.*

Proof. Let $l \in \mathbb{N}$ and $(s_j)_{j=1}^l$ be a plegma l-tuple in $\mathcal{F}$. Then for every $1 \leq j_1 < j_2 \leq l$ we deduce that (s_{j_1}, s_{j_2}) is plegma and thus $(\varphi(s_{j_1}), \varphi(s_{j_2}))$ is plegma. Hence, by Proposition 3.2(ii), $(\varphi(s_j))_{j=1}^l$ is a plegma l-tuple. ∎

Proposition 3.15. *Let $\mathcal{F}$ be a regular thin family and $\varphi : \mathcal{F} \to [\mathbb{N}]^{<\infty}$. Then for every $M \in [\mathbb{N}]^\infty$ there is $L \in [M]^\infty$ such that exactly one of the following holds:*

(i) *The restriction of φ to $\mathcal{F}\upharpoonright L$ is plegma preserving.*
(ii) *For every $(s_1, s_2) \in \mathrm{Plm}_2(\mathcal{F}\upharpoonright L)$ neither the pair $(\varphi(s_1), \varphi(s_2))$ nor the pair $(\varphi(s_2), \varphi(s_1))$ is plegma.*

Proof. Assume that there is $M \in [\mathbb{N}]^\infty$ such that for every $L \in [M]^\infty$ neither (i) nor (ii) holds true. Then by Theorem 3.6 there exists $L \in [M]^\infty$ such that for every $(s_1, s_2) \in \mathrm{Plm}_2(\mathcal{F}\upharpoonright N)$, $(\varphi(s_2), \varphi(s_1))$ is plegma. But this is impossible. Indeed, otherwise for an infinite plegma path $(s_n)_n$ in $\mathcal{F}\upharpoonright N$ the sequence $(\min s_n)_n$ would form a strictly decreasing infinite sequence in $\mathbb{N}$. ∎

For a family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and a plegma preserving map $\varphi : \mathcal{F} \to [\mathbb{N}]^{<\infty}$ we will say that φ is *normal* provided that $|\varphi(s_1)| \le |\varphi(s_2)|$ for every plegma pair (s_1, s_2) in $\mathcal{F}$ and $|\varphi(s)| \le |s|$ for every $s \in \mathcal{F}$.

Theorem 3.16. *Let $\mathcal{F}$ be a regular thin family, M be an infinite subset of $\mathbb{N}$ and let $\varphi : \mathcal{F}\upharpoonright M \to [\mathbb{N}]^{<\infty}$ be a plegma preserving map. Then there exists $L \in [M]^\infty$ such that the restriction of φ to $\mathcal{F}\upharpoonright L$ is a normal plegma preserving map.*

Proof. By Theorem 3.6 there exists $N \in [M]^\infty$ such that either (a) $|\varphi(s_1)| \le |\varphi(s_2)|$ for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$, or (b) $|\varphi(s_1)| > |\varphi(s_2)|$ for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$. Alternative (b) cannot occur since otherwise for an infinite plegma path $(s_n)_n$ in $\mathcal{F}\upharpoonright N$ the sequence $(|\varphi(s_n)|)_n$ would form a strictly decreasing sequence in $\mathbb{N}$. By Proposition 2.6 there exists $L \in [N]^\infty$ such that either (c) $|\varphi(s)| \le |s|$ for every $s \in \mathcal{F}\upharpoonright L$, or (d) $|\varphi(s)| > |s|$ for every $s \in \mathcal{F}\upharpoonright L$. We claim that (d) cannot hold true. Indeed, since φ on $\mathcal{F}\upharpoonright L$ is plegma preserving, using a plegma path of sufficiently large length, we may choose s_0, s in $\mathcal{F}\upharpoonright\upharpoonright L$ such that $\min s_0 < \min s$ and $\min \varphi(s_0) < \min \varphi(s)$. Let $k_0 = |s_0|$. Then by Proposition 3.10 there exists a plegma path $(s_i)_{i=0}^{k_0}$ in $\mathcal{F}\upharpoonright\upharpoonright L$ from s_0 to $s = s_{k_0}$ of length k_0. By Lemma 3.14, $(\varphi(s_i))_{i=0}^{k_0}$ is also a plegma path of length k_0 from $\varphi(s_0)$ to $\varphi(s_{k_0})$ and by Lemma 3.8 we have

$$\min\{|\varphi(s_i)| : 0 \le i \le k_0 - 1\} \le k_0. \tag{3.2}$$

Moreover by Lemma 3.4, we have $|s_0| \le |s_1| \le \cdots \le |s_{k_0}|$. Hence if (d) holds true then

$$\min\{|\varphi(s_i)| : 0 \le i \le k_0 - 1\} > \min\{|s_i| : 0 \le i \le k_0 - 1\} \ge k_0, \tag{3.3}$$

which contradicts (3.2).

Therefore we conclude that (a) and (c) hold true, i.e. the restriction of φ on $\mathcal{F}\upharpoonright L$ is a normal plegma preserving map. ∎

3.4. Plegma preserving maps between thin families. In this subsection we are concerned with the question of the existence of a plegma preserving map $\varphi : \mathcal{G} \to \mathcal{F}$ where $\mathcal{G}$ and $\mathcal{F}$ are regular thin families. We shall show that such maps exist only when $o(\mathcal{G}) \geq o(\mathcal{F})$. We start with a positive result.

THEOREM 3.17. *Let $\mathcal{F}, \mathcal{G}$ be regular thin families with $o(\mathcal{F}) \leq o(\mathcal{G})$. Then for every $M \in [\mathbb{N}]^\infty$ there exist $N \in [\mathbb{N}]^\infty$ and a plegma preserving map $\varphi : \mathcal{G}\restriction N \to \mathcal{F}\restriction M$. Moreover, for every $l \in \mathbb{N}$ and $t \in \mathcal{G}\restriction N$, if $\min t \geq N(l)$ then $\min \varphi(t) \geq M(l)$.*

Proof. Let $M \in [\mathbb{N}]^\infty$. By Corollary 2.17 there exists $L_0 \in [\mathbb{N}]^\infty$ and $N \in [L_0(M)]^\infty$ such that $L_0(\mathcal{F})\restriction N \sqsubseteq \mathcal{G}\restriction N$. Thus for every $t \in \mathcal{G}\restriction N$ there exists a unique $s_t \in \mathcal{F}$ such that $L_0(s_t) \sqsubseteq t$. Moreover, $L_0(s_t) \sqsubseteq t \subseteq N \subseteq L_0(M)$ and therefore $s_t \subseteq M$. We define $\varphi : \mathcal{G}\restriction N \to \mathcal{F}\restriction M$ by setting $\varphi(t) = s_t$. To see that φ is plegma preserving, let (t_1, t_2) be a plegma pair in $\mathcal{G}\restriction N$. Then $L_0(\varphi(t_i)) \sqsubseteq t_i$ for $i \in \{1, 2\}$, and therefore by Proposition 3.2(iv), $(L_0(\varphi(t_1)), L_0(\varphi(t_2)))$ and $(\varphi(t_1), \varphi(t_2))$ are also plegma pairs. Hence φ is plegma preserving.

Finally, let $l \in \mathbb{N}$ and $t \in \mathcal{G}\restriction N$ with $\min t \geq N(l)$. Since $L_0(\varphi(t)) \sqsubseteq t$, we have $\min L_0(\varphi(t)) = \min t$ and therefore

$$L_0(\min \varphi(t)) = \min L_0(\varphi(t)) = \min t \geq N(l) \geq L_0(M)(l) = L_0(M(l)).$$

Hence $\min \varphi(t) \geq M(l)$. ■

For the following we shall need the next definition. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^\infty$. We define

$$L^{-1}(\mathcal{F}) = \{t \in [\mathbb{N}]^{<\infty} : L(t) \in \mathcal{F}\}. \tag{3.4}$$

It is easy to see that for every family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ and $L \in [\mathbb{N}]^\infty$ the following are satisfied:

(a) If $\mathcal{F}$ is very large in L then the family $L^{-1}(\mathcal{F})$ is very large in $\mathbb{N}$.
(b) If $\mathcal{F}$ is regular thin then so is $L^{-1}(\mathcal{F})$.
(c) $o(L^{-1}(\mathcal{F})) = o(\mathcal{F}\restriction L)$. In particular if $\mathcal{F}$ is regular thin then $o(L^{-1}(\mathcal{F})) = o(\mathcal{F})$.

LEMMA 3.18. *Let $\mathcal{F}$ be a regular thin family, $L \in [\mathbb{N}]^\infty$ be such that $\mathcal{F}$ is very large in L. Let $\varphi : \mathcal{F}\restriction L \to [\mathbb{N}]^{<\infty}$ be a normal plegma preserving map. Define $\psi : L^{-1}(\mathcal{F}) \to [\mathbb{N}]^{<\infty}$ by $\psi(u) = \varphi(L(u))$ for every $u \in L^{-1}(\mathcal{F})$. Then ψ is a normal plegma preserving map which in addition has the following property: If $u \in L^{-1}(\mathcal{F})\restriction\restriction\mathbb{N}$ and $w = \psi(u)$ then $u(i) \leq w(i)$ for every $1 \leq i \leq |w|$.*

Proof. It is easy to check that ψ is a normal plegma preserving map. Therefore we pass to the proof of the property of ψ. First, by induction on $k = u(1)$, we shall show that $u(1) \le \psi(u)(1)$ for all $u \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$. Indeed, if $u(1) = 1$ then obviously $\psi(u)(1) \ge 1 = u(1)$. Suppose that for some $k \in \mathbb{N}$ and every $u \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$ with $u(1) = k$ we have $\psi_1(u)(1) \ge k$. Let $u \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$ with $u(1) = k+1$. Since $L^{-1}(\mathcal{F})$ is regular thin and very large in $\mathbb{N}$, we easily see that there exists a unique $u' \in L^{-1}(\mathcal{F})$ with $u' \sqsubseteq \{u(\rho) - 1 : 1 \le \rho \le |u|\}$. Notice that (u', u) is a plegma pair in $L^{-1}(\mathcal{F})$ and $u'(1) = k$. Since ψ is a normal plegma preserving map we infer that $(\psi(u'), \psi(u))$ is also a plegma pair. Hence $\psi(u)(1) > \psi(u')(1) \ge u'(1) = k$, that is, $\psi(u)(1) \ge k + 1 = u(1)$.

Suppose now that for some $i \in \mathbb{N}$ and every $u \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$ with $i \le |\psi(u)|$, $u(i) \le \psi(u)(i)$. Let $u \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$ with $i + 1 \le |\psi(u)|$. Since $L^{-1}(\mathcal{F})$ is very large in $\mathbb{N}$, there exists $u' \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$ such that $\{u(\rho) - 1 : 2 \le \rho \le |u|\} \sqsubseteq u'$. Observe that (u, u') is plegma pair in $L^{-1}(\mathcal{F})$, $|u| \le |u'|$ and $u(i+1) = u'(i) + 1$. Since ψ is normal plegma preserving, we see that $(\psi(u), \psi(u'))$ is also a plegma pair and in addition $i+1 \le |\psi(u)| \le |\psi(u')|$. Hence, $\psi(u)(i+1) > \psi(u')(i) \ge u'(i) = u(i+1)-1$, that is, $\psi(u)(i+1) \ge u(i+1)$. By induction on $i \in \mathbb{N}$ the proof is complete. ■

Theorem 3.19. *Let $\mathcal{F}, \mathcal{G}$ be regular thin families with $o(\mathcal{F}) < o(\mathcal{G})$ and let $M \in [\mathbb{N}]^\infty$. Then there is no plegma preserving map from $\mathcal{F}\upharpoonright M$ to $\mathcal{G}$. More precisely for every $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$ there exists $L \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is a plegma pair.*

Proof. Assume that there exist $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$ such that φ is plegma preserving. By Theorem 3.16 there exists $L \in [M]^\infty$ such that the restriction of φ on $\mathcal{F}\upharpoonright L$ is a normal plegma preserving map. By Corollary 2.13 we may also assume that $\mathcal{F}$ is very large in L. Since $o(L^{-1}(\mathcal{F})) = o(\mathcal{F}) < o(\mathcal{G})$, by Proposition 2.15 there exists $N \in [\mathbb{N}]^\infty$ such that $L^{-1}(\mathcal{F})\upharpoonright N \sqsubset \mathcal{G}\upharpoonright N$. We may assume that $N(i+1) - N(i) > 1$ and therefore $L^{-1}(\mathcal{F})\upharpoonright N \subseteq L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$. Let $\psi : L^{-1}(\mathcal{F}) \to [\mathbb{N}]^{<\infty}$ be defined by $\psi(u) = \varphi(L(u))$ for every $u \in L^{-1}(\mathcal{F})$.

Pick $u_0 \in L^{-1}(\mathcal{F})\upharpoonright N$ and set $w_0 = \psi(u_0)$. Since $L^{-1}(\mathcal{F})\upharpoonright N \sqsubset \mathcal{G}\upharpoonright N$, we have $u_0 \in \widehat{\mathcal{G}} \setminus \mathcal{G}$, and since φ takes values in $\mathcal{G}$, we have $w_0 \in \mathcal{G}$. We are now ready to derive a contradiction. Indeed, by Lemma 3.18, ψ is a normal plegma preserving map, which implies that $|w_0| \le |u_0|$. Moreover, since $L^{-1}(\mathcal{F})\upharpoonright N \subseteq L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$, we have $u_0 \in L^{-1}(\mathcal{F})\upharpoonright\upharpoonright\mathbb{N}$. Hence, again by Lemma 3.18, we get $u_0(i) \le w_0(i)$ for every $1 \le i \le |w_0|$. Summarizing we have $u_0 \in \widehat{\mathcal{G}} \setminus \mathcal{G}$, $|w_0| \le |u_0|$ and $u_0(i) \le w_0(i)$ for every $1 \le i \le |w_0|$. Since $\widehat{\mathcal{G}}$ is spreading we conclude that $w_0 \in \widehat{\mathcal{G}} \setminus \mathcal{G}$, which is impossible. Therefore there is no $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$ such that φ is plegma preserving. By Proposition 3.15, for every $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$

there is $L \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\restriction L$ neither $(\phi(s_1), \phi(s_2))$ nor $(\phi(s_2), \phi(s_1))$ is plegma. ■

4. The hierarchy of spreading models. In this section we define the class of ξ-spreading models of a Banach space X for every countable ordinal $1 \le \xi < \omega_1$. The definition is a transfinite extension of the corresponding one of finite order spreading models given in [4]. The basic ingredients of this extension are the concepts of $\mathcal{F}$-sequences in X, i.e. sequences of the form $(x_s)_{s\in\mathcal{F}}$ with $x_s \in X$ for every $s \in \mathcal{F}$, and plegma families with members in $\mathcal{F}$ where $\mathcal{F}$ is a regular thin family.

4.1. The $\mathcal{F}$-spreading models of a Banach space X. Let X be a Banach space and $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a regular thin family. By an *$\mathcal{F}$-sequence* in X we will mean a map $\varphi : \mathcal{F} \to X$. An $\mathcal{F}$-sequence in X will be usually denoted by $(x_s)_{s\in\mathcal{F}}$, where $x_s = \varphi(s)$ for all $s \in \mathcal{F}$. Also, for every $M \in [\mathbb{N}]^\infty$, the map $\varphi : \mathcal{F}\restriction M \to X$ will be called an *$\mathcal{F}$-subsequence* of $(x_s)_{s\in\mathcal{F}}$ and will be denoted by $(x_s)_{s\in\mathcal{F}\restriction M}$. An $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in X will be called *bounded* (resp. *seminormalized*) if there exists $C > 0$ (resp. $0 < c < C$) such that $\|x_s\| \le C$ (resp. $c \le \|x_s\| \le C$) for every $s \in \mathcal{F}$.

Lemma 4.1. *Let $(x_s)_{s\in\mathcal{F}}$ be a bounded $\mathcal{F}$-sequence in X. Let $k \in \mathbb{N}$, $N \in [\mathbb{N}]^\infty$ and $\delta > 0$. Then there exists $M \in [N]^\infty$ such that*

$$\left| \Big\| \sum_{j=1}^{k} a_j x_{t_j} \Big\| - \Big\| \sum_{j=1}^{k} a_j x_{s_j} \Big\| \right| \le \delta \tag{4.1}$$

for every $(t_j)_{j=1}^k, (s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction M)$ and $a_1, \ldots, a_k \in [-1, 1]$.

Proof. Let $(\mathbf{a}_n)_{n=1}^{n_0}$ be a $\delta/(3l)$-net of the unit ball of $(\mathbb{R}^k, \|\cdot\|_\infty)$. Setting $\mathbf{a}_n = (a_1^n, \ldots, a_k^n)$ for every $1 \le n \le n_0$, we inductively construct $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_{n_0}$ satisfying

$$\left| \Big\| \sum_{j=1}^{k} a_j^n x_{t_j} \Big\| - \Big\| \sum_{j=1}^{k} a_j^n x_{s_j} \Big\| \right| \le \delta/3 \tag{4.2}$$

for every $1 \le n \le n_0$ and every $(s_j)_{j=1}^k, (t_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction N_n)$.

The inductive step is as follows. Suppose that $N_0, \ldots, N_{n-1}$ have been constructed. Define $g_n : \mathrm{Plm}_k(\mathcal{F}\restriction N_{n-1}) \to [0, lC]$ by $g_n((s_j)_{j=1}^k) = \|\sum_{j=1}^k a_j^n x_{s_j}\|$. By dividing $[0, lC]$ into disjoint intervals of length $\delta/3$ and applying Theorem 3.6, there is $N_n \subseteq N_{n-1}$ such that

$$|g_n((t_j)_{j=1}^k) - g_n((s_j)_{j=1}^k)| \le \delta/3$$

for every $(t_j)_{j=1}^k, (s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction N_n)$.

We set $M = M(N_{n_0})$. By (4.2) and since $(\mathbf{a}_n)_{n=1}^{n_0}$ is a $\delta/3$-net of the unit ball of $(\mathbb{R}^k, \|\cdot\|_\infty)$, it is easy to see that L is as desired. ■

LEMMA 4.2. *Let $(x_s)_{s\in\mathcal{F}}$ be a bounded $\mathcal{F}$-sequence in X. Let $l \in \mathbb{N}$, $N \in [\mathbb{N}]^\infty$ and $\delta > 0$. Then there exists $M \in [N]^\infty$ such that*

$$\left|\Big\|\sum_{j=1}^{k} a_j x_{t_j}\Big\| - \Big\|\sum_{j=1}^{k} a_j x_{s_j}\Big\|\right| \leq \delta \tag{4.3}$$

for every $1 \leq k \leq l$, $(t_j)_{j=1}^k, (s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction M)$ and $a_1, \ldots, a_k \in [-1, 1]$.

Proof. This follows easily by an iterated use of Lemma 4.1. ∎

LEMMA 4.3. *Let $(x_s)_{s\in\mathcal{F}}$ be a bounded $\mathcal{F}$-sequence in X. Then for every sequence $(\delta_n)_n$ of positive real numbers and $N \in [\mathbb{N}]^\infty$ there exists $M \in [N]^\infty$ satisfying*

$$\left|\Big\|\sum_{j=1}^{k} a_j x_{t_j}\Big\| - \Big\|\sum_{j=1}^{k} a_j x_{s_j}\Big\|\right| \leq \delta_l \tag{4.4}$$

for every $1 \leq k \leq l$, $a_1, \ldots, a_k \in [-1, 1]$ and $(t_j)_{j=1}^k, (s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction M)$ such that $s_1(1), t_1(1) \geq M(l)$.

Proof. This is straightforward by Lemma 4.2 and a standard diagonalization. ∎

Hence, assuming in the above lemma that $(\delta_n)_n$ is a null sequence, we deduce that for every $l \in \mathbb{N}$ and every sequence $((s_j^n)_{j=1}^l)_n$ of plegma l-tuples in $\mathcal{F}\restriction M$ with $s_1^n(1) \to \infty$ the sequence $(\|\sum_{j=1}^l a_j x_{s_j^n}\|)_n$ is convergent and its limit is independent of the choice of $((s_j^n)_{j=1}^l)_n$. Actually, we may define a seminorm $\|\cdot\|_*$ on $c_{00}(\mathbb{N})$ for which the natural Hamel basis $(e_n)_n$ satisfies

$$\left|\Big\|\sum_{j=1}^{k} a_j x_{s_j}\Big\| - \Big\|\sum_{j=1}^{k} a_j e_j\Big\|_*\right| \leq \delta_l \tag{4.5}$$

for all $1 \leq k \leq l$, $(a_i)_{i=1}^k$ in $[-1, 1]$ and $(s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\restriction M)$ with $s_1(1) \geq M(l)$.

Let us notice here that there exist bounded $\mathcal{F}$-sequences in Banach spaces such that no seminorm resulting from Lemma 4.3 is a norm. For example this happens in the case where $(x_s)_{s\in\mathcal{F}}$ is constant. Moreover, even if $\|\cdot\|_*$ is a norm on $c_{00}(\mathbb{N})$, the sequence $(e_n)_n$ is not necessarily Schauder basic.

We are now ready to give the definition of $\mathcal{F}$-spreading models of a Banach space X.

DEFINITION 4.4. Let X be a Banach space, $\mathcal{F}$ be a regular thin family, $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X. Let $(E, \|\cdot\|_*)$ be an infinite-dimensional seminormed linear space with Hamel basis $(e_n)_n$. Also let $M \in [\mathbb{N}]^\infty$ and $(\delta_n)_n$ be a null sequence of positive real numbers.

We will say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ *generates* $(e_n)_n$ *as an* $\mathcal{F}$*-spreading model* (with respect to $(\delta_n)_n$) if for every $l \in \mathbb{N}$, $1 \le k \le l$, $(a_i)_{i=1}^k$ in $[-1,1]$ and $(s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\upharpoonright M)$ with $s_1(1) \ge M(l)$, we have

$$\Big|\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\| - \Big\|\sum_{j=1}^k a_j e_j\Big\|_*\Big| \le \delta_l. \tag{4.6}$$

We will also say that $(x_s)_{s\in\mathcal{F}}$ *admits* $(e_n)_n$ *as an* $\mathcal{F}$*-spreading model* if there exists $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model.

Finally, for a subset A of X, we will say that $(e_n)_n$ is an $\mathcal{F}$*-spreading model* of A if there exists an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in A which admits $(e_n)_n$ as an $\mathcal{F}$-spreading model.

The next remark is straightforward.

Remark 2. Let $M \in [\mathbb{N}]^\infty$ be such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. Then the following are satisfied:

(i) The sequence $(e_n)_n$ is spreading, i.e. for every $n \in \mathbb{N}$, $k_1 < \cdots < k_n$ in $\mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ we have $\|\sum_{j=1}^n a_j e_j\|_* = \|\sum_{j=1}^n a_j e_{k_j}\|_*$.

(ii) For every $M' \in [M]^\infty$, $(x_s)_{s\in\mathcal{F}\upharpoonright M'}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model.

(iii) For every null sequence $(\delta'_n)_n$ of positive reals there exists $M' \in [M]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M'}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model with respect to $(\delta'_n)_n$.

By Lemma 4.3 we get the following.

Theorem 4.5. *Let $\mathcal{F}$ be a regular thin family and X be a Banach space. Then every bounded $\mathcal{F}$-sequence in X admits an $\mathcal{F}$-spreading model. In particular for every bounded $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in X and every $N \in [\mathbb{N}]^\infty$ there exists $M \in [N]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates an $\mathcal{F}$-spreading model.*

4.2. Spreading models of order ξ. In this subsection we show that Definition 4.4 is independent of the particular regular thin family $\mathcal{F}$ and actually depends on the order of $\mathcal{F}$. More precisely we have the following.

Lemma 4.6. *Let $\mathcal{F}, \mathcal{G}$ be regular thin families with $o(\mathcal{F}) \le o(\mathcal{G})$. Let X be a Banach space and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X which admits an $\mathcal{F}$-spreading model $(e_n)_n$. Then there exists a $\mathcal{G}$-sequence $(w_t)_{t\in\mathcal{G}}$ such that $\{w_t : t \in \mathcal{G}\} \subseteq \{x_s : s \in \mathcal{F}\}$ and which admits $(e_n)_n$ as a $\mathcal{G}$-spreading model.*

Proof. Let $M \in [\mathbb{N}]^\infty$ and $(\delta_n) \searrow 0$ be such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model with respect to $(\delta_n)_n$. By Theorem 3.17 there exist $N \in [\mathbb{N}]^\infty$ and a plegma preserving map $\varphi : \mathcal{G}\upharpoonright N \to \mathcal{F}\upharpoonright M$ such that $\min\varphi(t) \ge M(l)$ for every $l \in \mathbb{N}$ and $t \in \mathcal{G}\upharpoonright N$ with $\min t \ge N(l)$. For every

$t \in \mathcal{G}\upharpoonright N$ let $w_t = x_{\varphi(t)}$ and for every $t \in \mathcal{G} \setminus (\mathcal{G}\upharpoonright N)$ let $w_t = x_{s_0}$ where s_0 is an arbitrary element of $\mathcal{F}$. We claim that $(w_t)_{t\in\mathcal{G}\upharpoonright N}$ generates $(e_n)_n$ as a $\mathcal{G}$-spreading model with respect to $(\delta_n)_n$.

Indeed, fix $l \in \mathbb{N}$, $1 \le k \le l$, $(a_j)_{j=1}^k$ in $[0,1]$ and $(t_j)_{j=1}^k \in \mathcal{G}\upharpoonright N$ with $t_1(1) \ge N(l)$. Let $s_j = \varphi(t_j)$ for all $1 \le j \le k$. Then $(s_j)_{j=1}^k \in \mathrm{Plm}_l(\mathcal{F}\upharpoonright M)$ and $s_1(1) \ge M(l)$. Therefore, since $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model with respect to $(\delta_n)_n$, we have

$$\Big|\Big\|\sum_{j=1}^k a_j w_{t_j}\Big\| - \Big\|\sum_{j=1}^k a_j e_j\Big\|_*\Big| = \Big|\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\| - \Big\|\sum_{j=1}^k a_j e_j\Big\|_*\Big| \le \delta_l$$

and the proof is complete. ■

COROLLARY 4.7. *Let X be a Banach space, $A \subseteq X$ and $\mathcal{F}, \mathcal{G}$ be regular thin families with $o(\mathcal{F}) = o(\mathcal{G})$. Then $(e_n)_n$ is an $\mathcal{F}$-spreading model of A iff $(e_n)_n$ is a $\mathcal{G}$-spreading model of A.*

The above permits us to give the following definition.

DEFINITION 4.8. Let A be a subset of a Banach space X and $\xi \ge 1$ be a countable ordinal. We will say that $(e_n)_n$ is a ξ-*spreading model* of A if there exists a regular thin family $\mathcal{F}$ with $o(\mathcal{F}) = \xi$ such that $(e_n)_n$ is an $\mathcal{F}$-spreading model of A. The set of all ξ-spreading models of A will be denoted by $\mathcal{SM}_\xi(A)$.

Notice that by Lemma 4.6 we have

$$\mathcal{SM}_\zeta(A) \subseteq \mathcal{SM}_\xi(A) \tag{4.7}$$

for every $1 \le \zeta < \xi < \omega_1$.

The following is an extension of Example 1 in [4]. It shows that for a given $\xi < \omega_1$ and a regular thin family $\mathcal{G}$ there exists a norm on $c_{00}(\mathcal{G})$ such that setting $A = \{e_s : s \in \mathcal{G}\}$ (where $(e_s)_{s\in\mathcal{F}}$ is the natural Hamel basis of $c_{00}(\mathcal{G})$), we have $\mathcal{SM}_\zeta(A) \subsetneq \mathcal{SM}_\xi(A)$ for every $\zeta < \xi$.

EXAMPLE 1. Let $1 \le \xi < \omega_1$, $\mathcal{G}$ be a regular thin family of order ξ and $(e_s)_{s\in\mathcal{G}}$ be the natural Hamel basis of $c_{00}(\mathcal{G})$. Let $(E, \|\cdot\|)$ be a Banach space with a normalized spreading and 1-unconditional basis $(e_n)_n$ which in addition is not equivalent to the usual basis of c_0. Let $X_\mathcal{G}$ be the completion of $c_{00}(\mathcal{G})$ under the norm $\|\cdot\|_\mathcal{G}$ defined by

$$\|x\|_\mathcal{G} = \sup\Big\{\Big\|\sum_{i=1}^l a_{t_i} e_i\Big\| : l \in \mathbb{N},\ (t_i)_{i=1}^l \in \mathrm{Plm}_l(\mathcal{G}) \text{ and } l \le t_1(1)\Big\} \tag{4.8}$$

for every $x = \sum_{t\in\mathcal{G}} a_t e_t \in c_{00}(\mathcal{G})$.

Let $A = \{e_t : t \in \mathcal{G}\}$. It is easy to see that $(e_t)_{t\in\mathcal{G}}$ generates $(e_n)_n$ as a $\mathcal{G}$-spreading model, i.e. $(e_n)_n$ belongs to $\mathcal{SM}_\xi(A)$. Let $\zeta < \xi$. We claim that for every $(e'_n)_n \in \mathcal{SM}_\zeta(A)$ either $(e'_n)_n$ is generated by a constant

$\mathcal{F}$-sequence with $o(\mathcal{F}) = \zeta$, or it is isometric to the usual basis of c_0. Thus $(e_n)_n \notin \mathcal{SM}_\zeta(A)$.

Indeed, let $(e'_n)_n \in \mathcal{SM}_\zeta(A)$. Then there exists a regular thin family $\mathcal{F}$ of order ζ, an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in A and $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e'_n)_n$ as an $\mathcal{F}$-spreading model. Since $\{x_s\}_{s\in\mathcal{F}} \subseteq A$, we may define $\varphi : \mathcal{F}\upharpoonright M \to \mathcal{G}$ by choosing for each $s \in \mathcal{F}\upharpoonright M$ an element $\varphi(s) \in \mathcal{G}$ satisfying $e_{\varphi(s)} = x_s$.

Assume now that $(e'_n)_n \in \mathcal{SM}_\zeta(A)$ is not generated by a constant $\mathcal{F}$-sequence with $o(\mathcal{F}) = \zeta$. By Remark 2(ii) we deduce that φ is hereditarily nonconstant and by Lemma 3.11 there exists $N \in [M]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$, $\varphi(s_1) \neq \varphi(s_2)$. Moreover since $o(\mathcal{F}) < o(\mathcal{G})$ by Theorem 3.19 we find that there exists $L \in [N]^\infty$ such that for every plegma pair (s_1, s_2) in $\mathcal{G}\upharpoonright L$ neither $(\varphi(s_1), \varphi(s_2))$ nor $(\varphi(s_2), \varphi(s_1))$ is a plegma pair. Therefore, by Proposition 3.2(i), for every $1 \le k \le l$, $(s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\upharpoonright L)$ and $(t_i)_{i=1}^l \in \mathrm{Plm}_l(\mathcal{G})$, we must have

$$|\{j \in \{1,\dots,k\} : \varphi(s_j) \in \{t_i : 1 \le i \le l\}\}| \le 1. \tag{4.9}$$

Hence, for every $k \in \mathbb{N}$, $a_1,\dots,a_k \in \mathbb{R}$ and $(s_j)_{j=1}^k \in \mathrm{Plm}_k(\mathcal{F}\upharpoonright L)$, we have

$$\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\|_{\mathcal{G}} = \Big\|\sum_{j=1}^k a_j e_{\varphi(s_j)}\Big\|_{\mathcal{G}} \overset{(4.8),(4.9)}{=} \max_{1\le j\le k} |a_j|, \tag{4.10}$$

i.e. $(e'_n)_n$ is isometric to the usual basis of c_0.

A natural question arising from the above is the following.

Question. *Let X be a separable Banach space. Is it true that there exists a countable ordinal ξ such that $\mathcal{SM}_\zeta(X) = \mathcal{SM}_\xi(X)$ for every $\zeta > \xi$?*

The above question can also be stated in an isomorphic version, i.e. whether every sequence in $\mathcal{SM}_\zeta(X)$ is equivalent to some sequence in $\mathcal{SM}_\xi(X)$ and vice versa.

Remark 3. In a forthcoming paper we will provide examples establishing the hierarchy of higher order spreading models and also illustrating the boundaries of the theory. Specifically we will show the following:

(1) For every countable limit ordinal ξ there exist a Banach space X such that $\mathcal{SM}_\xi(X)$ properly includes $\bigcup_{\zeta<\xi} \mathcal{SM}_\zeta(X)$ up to equivalence.
(2) There exist a Banach space X such that, for every $\xi < \omega_1$ and every $(e_n)_n \in \mathcal{SM}_\xi(X)$, the space E generated by $(e_n)_n$ does not contain any isomorphic copy of c_0 or ℓ^p for $1 \le p < \infty$.

The above results require a deeper study of the structure of $\mathcal{F}$-sequences generating ℓ_1-spreading models (see also [4] for the finite order case).

5. $\mathcal{F}$-sequences in topological spaces. Let $(X,\mathcal{T})$ be a topological space and $\mathcal{F}$ be a regular thin family. As we have already defined in the previous section, an $\mathcal{F}$-sequence in X is any map of the form $\varphi:\mathcal{F}\to X$ and generally an $\mathcal{F}$-subsequence in X is any map of the form $\varphi:\mathcal{F}\restriction M\to X$. In this section we will study the topological properties of $\mathcal{F}$-sequences. The particular case where $\mathcal{F}=[\mathbb{N}]^k$, $k\in\mathbb{N}$, has been studied in [4].

5.1. Convergence of $\mathcal{F}$-sequences. We introduce the following natural definition of convergence of $\mathcal{F}$-sequences.

DEFINITION 5.1. Let $(X,\mathcal{T})$ be a topological space, $\mathcal{F}$ a regular thin family, $M\in[\mathbb{N}]^\infty$, $x_0\in X$ and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in X. We will say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ *converges to* x_0 if for every $U\in\mathcal{T}$ with $x_0\in U$ there exists $m\in\mathbb{N}$ such that for every $s\in\mathcal{F}\restriction M$ with $\min s\geq M(m)$ we have $x_s\in U$.

It is immediate that if an $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ in a topological space X is convergent to some x_0, then every further $\mathcal{F}$-subsequence is also convergent to x_0. Also notice that if $o(\mathcal{F})\geq 2$ then the convergence of $(x_s)_{s\in\mathcal{F}\restriction M}$ does not in general imply that $\{x_s : s\in\mathcal{F}\restriction M\}$ is a relatively compact subset of X. For instance, let $(x_s)_{s\in[\mathbb{N}]^2}$ be the $[\mathbb{N}]^2$-sequence in c_0 defined by $x_s=\sum_{i=s(1)}^{s(2)}e_i$, where $(e_i)_i$ is the usual basis of c_0. By Definition 5.1 the $[\mathbb{N}]^2$-sequence $(x_s)_{s\in[\mathbb{N}]^2}$ weakly converges to zero but $\overline{\{x_s : s\in[\mathbb{N}]^2\}}^w=\{x_s : s\in[\mathbb{N}]^2\}\cup\{0\}$, which is not a weakly compact subset of c_0.

PROPOSITION 5.2. *Let $(X,\mathcal{T}_X)$, $(Y,\mathcal{T}_Y)$ be two topological spaces and $f:Y\to X$ be a continuous map. Let $\mathcal{F}$ be a regular thin family, $M\in[\mathbb{N}]^\infty$ and $(y_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in Y. Suppose that the $\mathcal{F}$-subsequence $(y_s)_{s\in\mathcal{F}\restriction M}$ is convergent to some $y\in Y$. Then the $\mathcal{F}$-subsequence $(f(y_s))_{s\in\mathcal{F}\restriction M}$ is convergent to $f(y)$.*

Proof. Let $U_X\in\mathcal{T}_X$, with $f(y)\in U_X$. By the continuity of f there exists $U_Y\in\mathcal{T}_Y$ such that $y\in U_Y$ and $f[U_Y]\subseteq U_X$. Since $(y_s)_{s\in\mathcal{F}\restriction M}$ is convergent to y, there exists $m\in\mathbb{N}$ such that for every $s\in\mathcal{F}\restriction M$ with $\min s\geq M(m)$ we have $y_s\in U_Y$ and therefore $f(y_s)\in f[U_Y]\subseteq U_X$. ∎

For the rest of this section we shall restrict to $\mathcal{F}$-sequences in metric spaces.

DEFINITION 5.3. Let (X,ρ) be a metric space, $\mathcal{F}$ a regular thin family, $M\in[\mathbb{N}]^\infty$ and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in (X,ρ). We will say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ is *Cauchy* if for every $\varepsilon>0$ there exists $m\in\mathbb{N}$ such that for every $s_1,s_2\in\mathcal{F}\restriction M$ with $\min s_1,\min s_2\geq M(m)$, we have $\rho(x_{s_1},x_{s_2})<\varepsilon$.

PROPOSITION 5.4. *Let $M \in [\mathbb{N}]^\infty$, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a complete metric space (X,ρ). Then the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is Cauchy if and only if $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is convergent.*

Proof. If the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is convergent, then it is straightforward that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is Cauchy. Concerning the converse we have the following. Suppose that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is Cauchy. Let $(s_n)_n$ be a sequence in $\mathcal{F}\upharpoonright M$ such that $\min s_n \to \infty$. It is immediate that $(x_{s_n})_n$ forms a Cauchy sequence in X. Since (X,ρ) is complete, there exists $x \in X$ such that the sequence $(x_{s_n})_n$ converges to x. We will show that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ converges to x. Indeed, let $\varepsilon > 0$. Since $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is Cauchy, there exists $k_0 \in \mathbb{N}$ such that for every $t_1, t_2 \in \mathcal{F}\upharpoonright M$ with $\min t_1, \min t_2 \geq M(k_0)$ we have $\rho(x_{t_1}, x_{t_2}) < \varepsilon/2$. Since $(x_{s_n})_n$ converges to x and $\min s_n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\min s_{n_0} \geq M(k_0)$ and $\rho(x, x_{s_{n_0}}) < \varepsilon/2$. Hence for every $s \in \mathcal{F}\upharpoonright M$ such that $\min s \geq M(k_0)$, we have $\rho(x, x_s) \leq \rho(x, x_{s_{n_0}}) + \rho(x_{s_{n_0}}, x_s) < \varepsilon$ and the proof is complete. ■

LEMMA 5.5. *Let $M \in [\mathbb{N}]^\infty$, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in a metric space (X,ρ). Suppose that for every $\varepsilon > 0$ and $L \in [M]^\infty$ there exists a plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$ such that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$. Then the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ has a further Cauchy subsequence.*

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive reals such that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Using Theorem 3.6, we inductively construct a decreasing sequence $(L_n)_n$ in $[M]^\infty$ such that for every $n \in \mathbb{N}$ and for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L_n$ we have $\rho(x_{s_1}, x_{s_2}) < \varepsilon_n$. Let L' be a diagonalization of $(L_n)_n$, i.e. $L'(n) \in L_n$ for all $n \in \mathbb{N}$, and $L = \{L'(2n) : n \in \mathbb{N}\}$.

We claim that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is Cauchy. Indeed, let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty \varepsilon_n < \varepsilon/2$. Let s_0 be the unique initial segment of $\{L(n) : n \geq n_0\}$ in $\mathcal{F}$. If $\max s_0 = L(k)$ then we set $k_0 = k+1$. Then for every $s_1, s_2 \in \mathcal{F}\upharpoonright L$ with $\min s_1, \min s_2 \geq L(k_0)$, by Theorem 3.10 there exist plegma paths $(s_j^1)_{j=1}^{|s_0|}, (s_j^2)_{j=1}^{|s_0|}$ in $\mathcal{F}\upharpoonright L'$ from s_0 to s_1, s_2 respectively. Then for $i = 1,2$ we have

$$\rho(x_{s_0}, x_{s_i}) \leq \sum_{j=0}^{|s_0|-1} \rho(x_{s_j^i}, x_{s_{j+1}^i}) < \sum_{j=0}^{|s_0|-1} \varepsilon_{n_0+j} < \varepsilon/2,$$

which implies that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$. ■

DEFINITION 5.6. Let $\varepsilon > 0$, $L \in [\mathbb{N}]^\infty$, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in a metric space X. We will say that the subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is *plegma ε-separated* if for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$, $\rho(x_{s_1}, x_{s_2}) > \varepsilon$.

The following proposition is actually a restatement of Lemma 5.5.

Proposition 5.7. *Let $M \in [\mathbb{N}]^\infty$, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in a metric space X. Then the following are equivalent:*

(i) *The $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ has no further Cauchy subsequence.*
(ii) *For every $N \in [M]^\infty$ there exist $\varepsilon > 0$ and $L \in [N]^\infty$ such that the subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is plegma ε-separated.*

Proof. (i)⇒(ii). Assume that (ii) is not true. Then there is $N \in [M]^\infty$ such that for every $\varepsilon > 0$ and $L \in [N]^\infty$ there exists a plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright L$ such that $\rho(x_{s_1}, x_{s_2}) < \varepsilon$. By Lemma 5.5 the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright N}$ has a further Cauchy subsequence. Since $N \subseteq M$ this means that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ has a further Cauchy subsequence, which is a contradiction.

(ii)⇒(i). Suppose (i) does not hold. Then there exists $N \in [M]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright N}$ is Cauchy. Let $\varepsilon > 0$ and $L \in [N]^\infty$. Then $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is also Cauchy and therefore $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is not plegma ε-separated, a contradiction. ∎

5.2. Subordinated $\mathcal{F}$-sequences. By identifying every subset of $\mathbb{N}$ with its characteristic function, a thin family $\mathcal{F}$ becomes a discrete subspace of $\{0,1\}^{\mathbb{N}}$ (under the usual product topology) with $\widehat{\mathcal{F}}$ being its closure. This in particular implies that every $\phi : \mathcal{F} \to (X, \mathcal{T})$ is automatically continuous. In this subsection we show that for every regular thin family $\mathcal{F}$, any $M \in [\mathbb{N}]^\infty$ and $\varphi : \mathcal{F}\upharpoonright M \to (X, \mathcal{T})$ such that the closure of $\varphi(\mathcal{F}\upharpoonright M)$ is a compact metrizable subspace of X, there exist $L \in [M]^\infty$ and a continuous extension $\widehat{\varphi} : \widehat{\mathcal{F}}\upharpoonright L \to (X, \mathcal{T})$. We start with the following definition.

Definition 5.8. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X. We say that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is subordinated (with respect to $(X, \mathcal{T})$) if there exists a continuous map $\widehat{\varphi} : \widehat{\mathcal{F}}\upharpoonright M \to (X, \mathcal{T})$ with $\widehat{\varphi}(s) = x_s$ for every $s \in \mathcal{F}\upharpoonright M$.

Assume that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is subordinated. Then since $\mathcal{F}$ is dense in $\widehat{\mathcal{F}}$, there exists a unique continuous map $\widehat{\varphi} : \widehat{\mathcal{F}}\upharpoonright M \to (X, \mathcal{T})$ witnessing this. Moreover, for the same reason we have $\overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}} = \widehat{\varphi}(\widehat{\mathcal{F}}\upharpoonright M)$, where $\overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$ is the $\mathcal{T}$-closure of $\{x_s : s \in \mathcal{F}\upharpoonright M\}$ in X. Therefore $\overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$ is a countable compact metrizable subspace of $(X, \mathcal{T})$ with Cantor–Bendixson index at most $o(\mathcal{F})+1$. Another property of subordinated $\mathcal{F}$-sequences is stated in the next proposition.

Proposition 5.9. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X. Let $M \in [\mathbb{N}]^\infty$ be such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is subordinated. Then $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is a convergent $\mathcal{F}$-subsequence in X. In particular, if $\widehat{\varphi} : \widehat{\mathcal{F}}\upharpoonright M \to (X, \mathcal{T})$ is the continuous map witnessing the fact that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is subordinated then $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is convergent to $\widehat{\varphi}(\emptyset)$.*

Proof. Via the identity map we may consider the family $\mathcal{F}$ as an $\mathcal{F}$-sequence in the metric space $Y = \widehat{\mathcal{F}}$, i.e. let $(y_s)_{s\in\mathcal{F}}$ be the $\mathcal{F}$-sequence in

Y with $y_s = s$ for every $s \in \mathcal{F}$. As already noticed, $(y_s)_{s\in\mathcal{F}}$ converges to the empty set. Since $\widehat{\varphi} : \widehat{\mathcal{F}}\upharpoonright M \to (X, \mathcal{T})$ is continuous, by Proposition 5.2 we find that $(\widehat{\varphi}(y_s))_{s\in\mathcal{F}\upharpoonright M}$ converges to $\widehat{\varphi}(\emptyset)$. Since $\widehat{\varphi}(y_s) = \widehat{\varphi}(s) = x_s$ for every $s \in \mathcal{F}\upharpoonright M$, this means that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is convergent to $\widehat{\varphi}(\emptyset)$. ■

THEOREM 5.10. *Let $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a topological space $(X, \mathcal{T})$. Then for every $M \in [\mathbb{N}]^\infty$ such that $\overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$ is a compact metrizable subspace of $(X, \mathcal{T})$ there exists $L \in [M]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated.*

Proof. We will use induction on the order of the regular thin family $\mathcal{F}$. If $o(\mathcal{F}) = 0$ (i.e. the family $\mathcal{F}$ is the singleton $\mathcal{F} = \{\emptyset\}$) the result trivially holds. Let $\xi < \omega_1$ and assume that the theorem is true when $o(\mathcal{F}) < \xi$.

We fix a regular thin family $\mathcal{F}$ with $o(\mathcal{F}) = \xi$, an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in a topological space $(X, \mathcal{T})$ and $M \in [\mathbb{N}]^\infty$ such that $\overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$ is a compact metrizable subspace of $(X, \mathcal{T})$. By passing to an infinite subset of M if necessary, we may also suppose that $\mathcal{F}$ is very large in M. Let ρ be a compatible metric for the subspace $X_0 = \overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$. We shall construct (a) a strictly increasing sequence $(m_n)_n$ in M, (b) a decreasing sequence $M = M_0 \supseteq M_1 \supseteq \cdots$ of infinite subsets of M, (c) a sequence $\widehat{\varphi}_n$ of maps with $\widehat{\varphi}_n : \widehat{\mathcal{F}_{(m_n)}}\upharpoonright M_n \to X$, and (d) a decreasing sequence of closed balls $(B_n)_n$ in X_0 such that for every $n \in \mathbb{N}$ the following are satisfied:

(i) $m_n = \min M_{n-1}$ and $M_n \subseteq M_{n-1} \setminus \{m_n\}$,
(ii) $\operatorname{diam} B_n < 1/n$,
(iii) the map $\widehat{\varphi}_n$ is continuous,
(iv) $\widehat{\varphi}_n(u) = x_{\{m_n\}\cup u}$ for every $u \in \mathcal{F}_{(m_n)}\upharpoonright M_n$, and
(v) $\{\widehat{\varphi}_n(u) : u \in \widehat{\mathcal{F}_{(m_n)}}\upharpoonright M_n\} \subseteq B_n$.

We shall present the general inductive step of the above construction so let us assume that the construction has been carried out up to some $n \in \mathbb{N}$. We set $m_{n+1} = \min M_n$. Since $\mathcal{F}$ is very large in M we see that $\mathcal{G} = \mathcal{F}_{(m_{n+1})} = \{u \in [\mathbb{N}]^{<\infty} : m_{n+1} < u \text{ and } \{m_{n+1}\} \cup u \in \mathcal{F}\}$ is a regular thin family. For each $u \in \mathcal{G}$ we set $y_u = x_{\{m_{n+1}\}\cup u}$ and we form the $\mathcal{G}$-sequence $(y_u)_{u\in\mathcal{G}}$. Let $M_n' = M_n \setminus \{m_{n+1}\}$. Since $Y = \{y_u : u \in \mathcal{G}\upharpoonright M_n'\} \subseteq \overline{\{x_s : s \in \mathcal{F}\upharpoonright M\}}$, the closure of Y in $(X, \mathcal{T})$ is also a compact metrizable subspace of $(X, \mathcal{T})$. Thus Y is a totally bounded metric space and therefore, by Theorem 2.1 and passing to an infinite subset of M_n' if necessary, we may also suppose that there exists a ball B_{n+1} of X_0 with $\operatorname{diam} B_{n+1} < (n+1)^{-1}$ and

$$\{y_u : u \in \mathcal{G}\upharpoonright M_n'\} \subseteq B_{n+1}. \tag{5.1}$$

Moreover, $o(\mathcal{G}) = o(\mathcal{F}_{(m_{n+1})}) < o(\mathcal{F}) = \xi$. Hence, by our inductive hypothesis, there exists an infinite subset M_{n+1} of $M_n' = M_n \setminus \{m_{n+1}\}$ such that the $\mathcal{G}$-subsequence $(y_u)_{u\in\mathcal{G}\upharpoonright M_{n+1}}$ is subordinated. Let $\widehat{\varphi}_{n+1} : \widehat{\mathcal{G}}\upharpoonright M_{n+1} \to X$ be the continuous map witnessing this fact. Then $\widehat{\varphi}_{n+1}(u) = y_u = x_{\{m_{n+1}\}\cup u}$ for every $u \in \mathcal{F}_{(m_{n+1})}\restriction M_{n+1}$, and by the continuity of $\widehat{\varphi}_{n+1}$ we have

(5.2)

$$\{\widehat{\varphi}_{n+1}(u) : u \in \widehat{\mathcal{F}_{(m_{n+1})}}\restriction M_{n+1}\} \subseteq \overline{\{y_u : u \in \mathcal{F}_{(m_{n+1})}\restriction M_{n+1}\}} \overset{(5.1)}{\subseteq} B_{n+1},$$

which completes the proof of the inductive step.

We set $M' = \{m_n : n \in \mathbb{N}\}$. Since $\lim \operatorname{diam} B_n = 0$ and X_0 is a compact metric space there exists a strictly increasing sequence $(k_n)_n$ and $x_0 \in X_0$ such that

$$\lim \operatorname{dist}(x_0, B_{k_n}) = 0. \tag{5.3}$$

We set $L = \{m_{k_n} : n \in \mathbb{N}\}$ and we define $\widehat{\varphi} : \widehat{\mathcal{F}}\restriction L \to X$ as follows. For $s = \emptyset$, we set $\widehat{\varphi}(\emptyset) = x_0$. Otherwise, if n is the unique positive integer such that $m_{k_n} = \min s$ we set $\widehat{\varphi}(s) = \widehat{\varphi}_{k_n}(s \setminus \{\min s\}) = x_t$. It is easy to check that $\widehat{\varphi}$ is well defined. To see that $\widehat{\varphi}$ is continuous let $(s_n)_n$ be a sequence in $\widehat{\mathcal{F}}\restriction L$ and $s \in \widehat{\mathcal{F}}\restriction L$ such that $s_n \to s$. If $s = \emptyset$ then $\min s_n \to \infty$, thus using (v) and (5.3) we obtain $\widehat{\varphi}(s_n) \to x_0 = \widehat{\varphi}(\emptyset)$. Otherwise, let $m_{k_{n_0}} = \min s$. Then $\min s_n = \min s = m_{k_{n_0}}$ for all but finitely many n. Therefore, $\widehat{\varphi}(s_n) = \widehat{\varphi}_{k_{n_0}}(s_n)$, for all but finitely many n, and since $\widehat{\varphi}_{k_{n_0}}$ is continuous, $\widehat{\varphi}(s_n) \to \widehat{\varphi}(s)$. ■

6. $\mathcal{F}$-sequences generating spreading models. Let $(x_n)_n$ be a sequence in a Banach space X generating a spreading model $(e_n)_n$. It is well known (see [6], [7]) that if $(x_n)_n$ is norm convergent then the seminorm in the space generated by the sequence $(e_n)_n$ is not a norm. On the other hand, if $(x_n)_n$ is weakly null and seminormalized then $(e_n)_n$ is 1-unconditional ([7]). In this section we show that analogues of the above results remain true in the higher order setting of ξ-spreading models.

To make the presentation more clear and self-contained we start with a short review of the basic properties of spreading sequences. We divide the spreading sequences into four disjoint categories. The first category consists of those spreading sequences which we call *trivial*. A spreading sequence $(e_n)_n$ in a seminormed space $(E, \|\cdot\|_*)$ is trivial if the restriction of the seminorm $\|\cdot\|_*$ to the linear subspace generated by $(e_n)_n$ is not a norm (see Definition 6.1). The nontrivial spreading sequences are divided into three classes, namely the *singular*, the *unconditional* and the *conditional* ones. The singular ones are the nontrivial spreading sequences which are not Schauder basic, while the conditional ones are the Schauder basic spreading sequences which are not unconditional (see Definition 6.3). It is shown that every singular sequence $(e_n)_n$ admits a *natural decomposition* as $e_n = e'_n + e$ where e is the weak limit of $(e_n)_n$ and $(e'_n)_n$ is a 1-unconditional and Cesàro summable to the zero spreading sequence (Proposition 6.5).

In §6.2 we provide characterizations for $\mathcal{F}$-sequences generating a trivial spreading model (Theorem 6.6). Among other things it is shown that an $\mathcal{F}$-sequence generates a trivial spreading model if and only if it contains a further norm Cauchy subsequence. We also give a sufficient condition for an $\mathcal{F}$-sequence to generate a Schauder basic spreading model (see Theorem 6.9).

In §6.3 we proceed to an analysis of the spreading models generated by subordinated $\mathcal{F}$-sequences. Specifically, we show that seminormalized weakly null subordinated $\mathcal{F}$-sequences generate 1-unconditional spreading models (Theorem 6.11). Moreover, we present a classification of the spreading models generated by non-weakly null subordinated $\mathcal{F}$-sequences. In this case we show that the generated spreading model is either equivalent to the usual basis of ℓ^1, or singular (Theorem 6.14). These results imply that a nontrivial spreading model generated by a weakly relatively compact $\mathcal{F}$-sequence is either singular or unconditional (Corollary 6.17).

Finally, in §6.4 we study the $\mathcal{F}$-sequences which generate singular spreading models. In particular we show that the aforementioned natural decomposition of the singular spreading model is also reflected back to its generating $\mathcal{F}$-sequence (see Theorem 6.19 and Corollary 6.21).

6.1. Spreading sequences. Let $(E, \|\cdot\|_*)$ be a seminormed linear space. A sequence $(e_n)_n$ in E is called *spreading* if

$$\Big\|\sum_{j=1}^{n} a_j e_j\Big\|_* = \Big\|\sum_{j=1}^{n} a_j e_{k_j}\Big\|_*$$

for every $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ and $k_1 < \cdots < k_n$ in $\mathbb{N}$. As already mentioned, every spreading model of any order of a Banach space is a spreading sequence. In this subsection we shall briefly recall some well known results on spreading sequences that we shall later use (for a more detailed exposition see [1], [5], [6], [7]). Towards a classification of spreading sequences, we start with the following definition.

Definition 6.1. Let $(E, \|\cdot\|_*)$ be a seminormed linear space and $(e_n)_n$ be a spreading sequence in E. We will say that $(e_n)_n$ is *trivial* if there exist $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{R}$, not all zero, such that $\|\sum_{j=1}^k a_i e_i\|_* = 0$.

Concerning the trivial sequences, we have the following elementary lemma.

Lemma 6.2. *Let $(E, \|\cdot\|_*)$ be a seminormed linear space and $(e_n)_n$ be a spreading sequence in E. Then the sequence $(e_n)_n$ is trivial if and only if for every $n, m \in \mathbb{N}$ we have $\|e_n - e_m\|_* = 0$.*

Proof. Let $(e_n)_n$ be a trivial sequence. Pick $k \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$, not all zero, such that $\|\sum_{j=1}^k a_j e_j\|_* = 0$. Since $(e_n)_n$ is spreading we may suppose that $a_j \neq 0$ for all $1 \leq j \leq n$. Moreover notice that

$$\Big\|\sum_{j=1}^{k-1} a_j e_j + a_k e_k\Big\|_* = \Big\|\sum_{j=1}^{k-1} a_j e_j + a_k e_{k+1}\Big\|_* = 0. \tag{6.1}$$

Hence,

$$\|e_k - e_{k+1}\|_* \le \frac{1}{|a_k|}\Big(\Big\|\sum_{j=1}^{k-1} a_j e_j + a_k e_k\Big\|_* + \Big\|\sum_{j=1}^{k-1} a_j e_j + a_n e_{k+1}\Big\|_*\Big) = 0. \tag{6.2}$$

Since $(e_n)_n$ is spreading we get $\|e_n - e_m\|_* = 0$ for every $n, m \in \mathbb{N}$. The converse implication is straightforward. ■

Let us observe that if a sequence $(e_n)_n$ is nontrivial then the restriction of the seminorm $\|\cdot\|_*$ to the linear subspace generated by $(e_n)_n$ is actually a norm. Thus every nontrivial sequence can always be considered as a sequence in a Banach space. Following [4] we consider the classification of nontrivial sequences described by the next definition.

Definition 6.3. We classify all nontrivial spreading sequences into the following three categories:

(1) *singular* spreading sequences, i.e. those nontrivial spreading sequences which are not Schauder basic,
(2) *unconditional* spreading sequences, and
(3) *conditional Schauder basic* spreading sequences, i.e. spreading sequences which are Schauder basic but not unconditional.

Proposition 6.4. *Let $(e_n)_n$ be a nontrivial spreading sequence.*

(i) *If $(e_n)_n$ is weakly null then it is 1-unconditional.*
(ii) *If $(e_n)_n$ is unconditional then either it is equivalent to the usual basis of ℓ^1 or it is norm Cesàro summable to 0 (i.e. $\lim \big\|\frac{1}{n}\sum_{i=1}^n e_i\big\| = 0$).*

Proof. (i) See [1].

(ii) Since $(e_n)_n$ is unconditional there exists $C>0$ such that $\|\sum_{i=1}^n \varepsilon_i a_i e_i\| \le C\|\sum_{i=1}^n a_i e_i\|$ for any $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$. Also since it is spreading and nontrivial there exists $M > 0$ such that $\|e_n\| = M$ for all $n \in \mathbb{N}$. Suppose that $(e_n)_n$ is not Cesàro summable to zero. Then there exist $\theta > 0$ and a strictly increasing sequence of natural numbers $(p_n)_n$ such that $\big\|\frac{1}{p_n}\sum_{i=1}^{p_n} e_i\big\| > \theta$ for all $i \in \{1, \dots, p_n\}$. Hence for every $n \in \mathbb{N}$ there exists x_n^* with $\|x_n^*\| = 1$ such that $x_n^*\big(\frac{1}{p_n}\sum_{i=1}^{p_n} e_i\big) > \theta$. For every $n \in \mathbb{N}$, we set $I_n = \{1, \dots, p_n\}$ and let $A_n = \{i \in I_n : x_n^*(e_i) > \theta/2\}$. Then

$$\theta < x_n^*\Big(\frac{1}{p_n}\sum_{i\in I_n} e_i\Big) = \frac{1}{p_n} x_n^*\Big(\sum_{i\in A_n} e_i\Big) + \frac{1}{p_n} x_n^*\Big(\sum_{i\in I_n\setminus A_n} e_i\Big) \le \frac{1}{p_n}|A_n|C + \frac{\theta}{2}.$$

Hence $|A_n| \ge \frac{\theta}{2C} p_n$, which gives $\lim_{n\to\infty} |A_n| = \infty$. We are now ready to show that $(e_n)_n$ is equivalent to the usual basis of ℓ_1. Indeed, let $n \in \mathbb{N}$,

$a_1, \dots, a_n \in \mathbb{R}$ and choose $n_0 \in \mathbb{N}$ such that $|A_{n_0}| \geq n$. Then

$$M \sum_{i=1}^n |a_i| \geq \Big\| \sum_{i=1}^n a_i e_i \Big\| \geq \frac{1}{C} \Big\| \sum_{i=1}^n |a_i| e_i \Big\| = \frac{1}{C} \Big\| \sum_{i=1}^n |a_i| e_{A_{n_0}(i)} \Big\|$$
$$\geq \frac{1}{C} \cdot x_n^* \Big(\sum_{i=1}^n |a_i| e_{A_{n_0}(i)} \Big) \geq \frac{\theta}{2C} \sum_{i=1}^n |a_i|. \ \blacksquare$$

Proposition 6.5. *Let $(e_n)_n$ be a singular sequence and let E be the space generated by $(e_n)_n$. Then there is $e \in E \setminus \{0\}$ such that $(e_n)_n$ is weakly convergent to e. Moreover if $e'_n = e_n - e$ then $(e'_n)_n$ is spreading,* 1-*unconditional and Cesàro summable to zero.*

Proof. Since $(e_n)_n$ is equivalent to all its subsequences and it is not Schauder basic, no subsequence of $(e_n)_n$ is Schauder basic. In particular, a subsequence of $(e_n)_n$ cannot be nontrivial weak-Cauchy or weakly null. Hence, by Rosenthal's ℓ^1-theorem [19], $(e_n)_n$ is weakly convergent to a nonzero element $e \in E$.

Let $e'_n = e_n - e$. To show that $(e'_n)_n$ is spreading, let $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $k_1 < \dots < k_n$ in $\mathbb{N}$. If $\sum_{i=1}^n \lambda_i = 0$, then

$$\Big\| \sum_{i=1}^n \lambda_i e'_i \Big\| = \Big\| \sum_{i=1}^n \lambda_i e_i \Big\| = \Big\| \sum_{i=1}^n \lambda_i e_{k_i} \Big\| = \Big\| \sum_{i=1}^n \lambda_i e'_{k_i} \Big\|. \tag{6.3}$$

Generally let $\sum_{i=1}^n \lambda_i = \lambda$. Since $(e'_n)_n$ is weakly null we may choose a convex block subsequence $(w_m)_m$ of $(e'_n)_n$ which norm converges to zero. Let $m_0 \in \mathbb{N}$ be such that $k_n < \operatorname{supp}(w_m)$ for all $m \geq m_0$. Then by (6.3),

$$\Big\| \sum_{i=1}^n \lambda_i e'_i - \lambda w_m \Big\| = \Big\| \sum_{i=1}^n \lambda_i e'_{k_i} - \lambda w_m \Big\| \tag{6.4}$$

for all $m \geq m_0$. Hence, by taking limits, we get

$$\Big\| \sum_{i=1}^n \lambda_i e'_i \Big\| = \Big\| \sum_{i=1}^n \lambda_i e'_{k_i} \Big\|,$$

that is, the sequence $(e'_n)_n$ is spreading. Moreover, since

$$\|e_n - e_m\| = \|e'_n - e'_m\|,$$

by Lemma 6.2 we deduce that $(e'_n)_n$ is nontrivial. Finally, since $(e'_n)_n$ is weakly null, by Proposition 6.4 it is also 1-unconditional and norm Cesàro summable to zero. $\blacksquare$

The above decomposition $e_n = e'_n + e$ of a singular spreading sequence $(e_n)_n$ will be called the *natural decomposition* of $(e_n)_n$.

6.2. $\mathcal{F}$-sequences generating nonsingular spreading models. We start with a characterization of those $\mathcal{F}$-sequences in a Banach space X which generate a trivial spreading model.

THEOREM 6.6. *Let X be a Banach space, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X and $M\in[\mathbb{N}]^\infty$. Let $(E,\|\cdot\|_*)$ be an infinite-dimensional seminormed linear space with Hamel basis $(e_n)_n$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. Then the following are equivalent:*

(i) *The sequence $(e_n)_n$ is trivial.*
(ii) *For every $\varepsilon>0$ and every $L\in[M]^\infty$, there exists a plegma pair (s_1,s_2) in $\mathcal{F}\upharpoonright L$ such that $\|x_{s_1}-x_{s_2}\|<\varepsilon$.*
(iii) *The $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ contains a further norm Cauchy subsequence.*
(iv) *There exists $x\in X$ such that every subsequence of $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ contains a further subsequence convergent to x.*

Proof. (i)⇒(ii). Let $\varepsilon>0$ and $L\in[M]^\infty$. Since the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ also generates $(e_n)_n$ as an $\mathcal{F}$-spreading model (see Remark 2), there exists $n_0\in\mathbb{N}$ such that for every plegma pair (s_1,s_2) in $\mathcal{F}\upharpoonright L$ with $\min s_1\geq L(n_0)$,

$$\big|\|x_{s_1}-x_{s_2}\|-\|e_1-e_2\|_*\big|<\varepsilon. \tag{6.5}$$

Let (s_1,s_2) be such a plegma pair. Since $(e_n)_n$ is trivial we have $\|e_1-e_2\|_*=0$ and therefore by (6.5) we obtain $\|x_{s_1}-x_{s_2}\|<\varepsilon$.

(ii)⇒(iii). This follows by Lemma 5.5.

(iii)⇒(i). Using the fact that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ contains a further norm Cauchy subsequence, we easily construct a sequence $((s_1^n,s_2^n))_n$ of plegma pairs in $\mathcal{F}\upharpoonright M$ such that $s_1^n(1)\to\infty$ and $\|x_{s_1^n}-x_{s_2^n}\|<1/n$. Then

$$\|e_1-e_2\|_*=\lim_{n\to\infty}\|x_{s_1^n}-x_{s_2^n}\|=0 \tag{6.6}$$

and therefore the sequence $(e_n)_n$ is trivial.

(iv)⇒(iii). This is straightforward.

(i)⇒(iv). Since every subsequence of $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model we see that, for every $L\in[M]^\infty$, $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates a trivial spreading model. By the implication (i)⇒(iii) and Proposition 5.4, every subsequence of $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ contains a further convergent subsequence. It remains to show that all the convergent subsequences of $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ have a common limit.

To this end, let $L_1,L_2\in[M]^\infty$, $x_1,x_2\in X$ be such that $(x_s)_{s\in\mathcal{F}\upharpoonright L_i}$ converges to x_i for $i\in\{1,2\}$ and let $\varepsilon>0$. Hence there exists $n_0\in\mathbb{N}$ such that for every $s\in\mathcal{F}\upharpoonright L_1$ and $t\in\mathcal{F}\upharpoonright L_2$ with $\min s\geq L_1(n_0)$ and

$\min t \geq L_2(n_0)$ we have

$$(6.7) \qquad \|x_1 - x_s\|, \|x_2 - x_t\| < \varepsilon/3.$$

Since $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates the trivial sequence $(e_n)_n$ as an $\mathcal{F}$-spreading model, we may also assume that

$$(6.8) \qquad \|x_{s_1} - x_{s_2}\| = \big|\|x_{s_1} - x_{s_2}\| - \|e_1 - e_2\|_*\big| < \varepsilon/3$$

for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright M$ with $\min s_1 \geq M(n_0)$.

It is easy to see that we can choose $s_1 \in \mathcal{F}\upharpoonright L_1$ with $\min s_1 \geq L_1(n_0)$ and $s_2 \in \mathcal{F}\upharpoonright L_2$ with $\min s_2 \geq L_2(n_0)$ such that (s_1, s_2) is a plegma pair. Then by (6.7) and (6.8) we have

$$(6.9) \qquad \|x_1 - x_2\| \leq \|x_1 - x_{s_1}\| + \|x_{s_1} - x_{s_2}\| + \|x_2 - x_{s_2}\| < \varepsilon.$$

Since (6.9) holds for every $\varepsilon > 0$ we get $x_1 = x_2$. ∎

We proceed to present a sufficient condition for an $\mathcal{F}$-sequence to generate a Schauder basic spreading model. We need the next definition.

Definition 6.7. Let A be a countable seminormalized subset of a Banach space X. We say that A admits a *Skipped Schauder Decomposition* (*SSD*) if there exist $C \geq 1$ and a pairwise disjoint sequence $(A_k)_k$ of finite subsets of A such that the following are satisfied:

(i) $\bigcup_{k=1}^{\infty} A_k = A$.
(ii) For every $N \in [\mathbb{N}]^\infty$ not containing two successive integers, and for every sequence $(x_k)_{k\in N}$ with $x_k \in A_k$ for all $n \in N$, $(x_k)_{k\in N}$ is a Schauder basic sequence with constant C.

The following proposition is well known but for the sake of completeness we outline its proof.

Proposition 6.8. *Let $(x_n)_n$ be a seminormalized weakly null sequence in a Banach space X. Then for every $\varepsilon > 0$ the set $A = \{x_n : n \in \mathbb{N}\}$ admits a SSD with constant $C = 1 + \varepsilon$.*

Proof. We may assume that X has a Schauder basis $(e_n)_n$ with basis constant $K = 1$ (for example we may assume that $X = C[0,1]$). By induction and using the sliding hump argument, we define (1) a partition $(F_n)_n$ of $\mathbb{N}$ into finite pairwise disjoint sets, and (2) a sequence $(y_n)_n$ of finitely supported vectors in X such that:

(i) For every $k \in \mathbb{N}$ and $n \in F_k$, we have $\|x_n - y_n\| < \varepsilon/2^k$, and
(ii) for every $k_2 > k_1$ with $k_2 - k_1 > 1$, $n_1 \in F_{k_1}$ and $n_2 \in F_{k_2}$, we have $\max \operatorname{supp}(y_{n_1}) < \min \operatorname{supp}(y_{n_2})$.

Setting $A_k = \{x_n : n \in F_k\}$, $k \in \mathbb{N}$, it is easy to check that $(A_k)_k$ satisfies conditions (i) and (ii) of Definition 6.7. ∎

THEOREM 6.9. *Let A be a subset of a Banach space X. If A admits a SSD with constant C, then every nontrivial spreading model of any order of A is Schauder basic with constant C.*

Proof. Let $1 \leq \xi$ be a countable ordinal and $(e_n)_n$ be a nontrivial spreading model of order ξ of A. Let $\mathcal{F}$ be a regular thin family with $o(\mathcal{F}) = \xi$, $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in A and $M \in [\mathbb{N}]^\infty$ be such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. Let $(A_k)_k$ be a partition of A satisfying condition (ii) of Definition 6.7. Finally, let $\varphi : \mathcal{F}\upharpoonright M \to \mathbb{N}$ be defined by $\varphi(s) = k$ if $x_s \in F_k$.

Observe that φ is hereditarily nonconstant in M. Indeed, otherwise there exist $L \in [M]^\infty$ and $k_0 \in \mathbb{N}$ such that $x_s \in F_{k_0}$ for $s \in \mathcal{F}\upharpoonright L$ and $s \in \mathcal{F}\upharpoonright L$. Since F_{k_0} is finite, by Proposition 2.6 there exists $N \in [L]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright N}$ is constant. By part (ii) of Remark 2, the $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}\upharpoonright N}$ also generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. But then, since $(x_s)_{s\in\mathcal{F}\upharpoonright N}$ is constant, the sequence $(e_n)_n$ should be trivial, which is a contradiction. Hence φ is hereditarily nonconstant in M and therefore by Corollary 3.13 there exists $N \in [M]^\infty$ such that $\varphi(s_2) - \varphi(s_1) > 1$ for every plegma pair (s_1, s_2) in $\mathcal{F}\upharpoonright N$. By the SSD property of A we deduce that for every $1 \leq m < l \in \mathbb{N}$,

$$\Big\|\sum_{j=1}^{m} a_j x_{s_j}\Big\| \leq C\Big\|\sum_{j=1}^{l} a_j x_{s_j}\Big\| \tag{6.10}$$

for every plegma l-tuple $(s_j)_{j=1}^l$ in $\mathcal{F}\upharpoonright N$ and $a_1, \ldots, a_l \in \mathbb{R}$. This easily implies that $(e_n)_n$ is a Schauder basic sequence with constant C. ∎

6.3. Spreading models generated by subordinated $\mathcal{F}$-sequences. Let $(x_n)_n$ be a weakly convergent sequence in a Banach space X which is not norm Cauchy and assume that it generates a spreading model $(e_n)_n$. It is well known (see [6], [7]) that $(e_n)_n$ is either an unconditional or a singular spreading sequence. In [4] we extended this fact to subordinated k-sequences. Here we will show that similar results also hold true for $\mathcal{F}$-sequences where $\mathcal{F}$ is a regular thin family.

6.3.1. *Unconditional spreading models.* Let $n \in \mathbb{N}$ and for every $1 \leq i \leq n$ let $F_i \subseteq [\mathbb{N}]^{<\infty}$. We will say that $(F_i)_{i=1}^n$ is *completely plegma connected* if for every choice of $s_i \in F_i$, the n-tuple $(s_i)_{i=1}^n$ is a plegma family. Also, for a subset A of a Banach space X, $\operatorname{conv} A$ denotes the convex hull of A.

LEMMA 6.10. *Let X be a Banach space, $n \in \mathbb{N}$, $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be regular thin families, and $L \in [\mathbb{N}]^\infty$. Assume that for every $i = 1, \ldots, n$ there exists a continuous map $\widehat{\varphi}_i : \widehat{\mathcal{F}_i}\upharpoonright L \to (X, w)$. Then for every $\varepsilon > 0$ there exists* *a completely plegma connected family* $(F_i)_{i=1}^n$ *such that* $F_i \subseteq [\mathcal{F}_i \upharpoonright L]^{<\infty}$ *and* $\mathrm{dist}(\widehat{\varphi}_i(\emptyset), \mathrm{conv}\, \widehat{\varphi}_i(F_i)) < \varepsilon$ *for every* $i = 1, \dots, n$.

Proof. We will use induction on $o((\mathcal{F}_i)_{i=1}^l) := \max\{o(\mathcal{F}_i) : 1 \leq i \leq n\}$. If $o((\mathcal{F}_i)_{i=1}^l) = 0$, i.e. $\mathcal{F}_i = \{\emptyset\}$ for all $1 \leq i \leq n$, the result follows trivially. Let $1 \leq \xi < \omega_1$ and suppose that the lemma holds true if $o((\mathcal{F}_i)_{i=1}^n) < \xi$. Let $n \in \mathbb{N}$, $L \in [\mathbb{N}]^\infty$ and $\mathcal{F}_1, \dots, \mathcal{F}_n$ be regular thin families with $o((\mathcal{F}_i)_{i=1}^l) = \xi$ and assume that there exists a continuous map $\widehat{\varphi}_i : \widehat{\mathcal{F}_i} \upharpoonright L \to (X, w)$ for every $1 \leq i \leq n$

Fix $i \in \{1, \dots, n\}$. We may suppose that $\mathcal{F}_i$ is very large in L and therefore every singleton $\{l\}$ with $l \in L$ belongs to $\widehat{\mathcal{F}_i}$. By the continuity of $\widehat{\varphi}_i$, we find that $w\text{-}\lim_{l \in L} \widehat{\varphi}_i(\{l\}) = \widehat{\varphi}_i(\emptyset)$. By Mazur's theorem, we may choose a finite subset $\Lambda_i = \{l_1^i < \dots < l_{m_i}^i\}$ of L such that

$$\mathrm{dist}(\widehat{\varphi}_i(\emptyset), \mathrm{conv}\, \widehat{\varphi}_i(\Lambda_i)) < \varepsilon/2 \tag{6.11}$$

for every $1 \leq i \leq n$. We may also assume that

$$\Lambda_1 < \dots < \Lambda_n. \tag{6.12}$$

Let $\Lambda = \bigcup_{i=1}^n \Lambda_i$ and let $M = \{l \in L : l > \max \Lambda\}$. Fix $1 \leq i \leq n$ and $1 \leq j \leq m_i$. We set $\mathcal{G}_j^i = (\mathcal{F}_i)_{(l_j^i)} = \{s \in [\mathbb{N}]^{<\infty} : l_j^i < s \text{ and } \{l_j^i\} \cup s \in \mathcal{F}_i\}$ and let $\widehat{\varphi}_j^i : \widehat{\mathcal{G}_j^i} \upharpoonright M \to (X, w)$ be defined by $\widehat{\varphi}_j^i(s) = \widehat{\varphi}_i(\{l_j^i\} \cup s)$. Notice that $\widehat{\varphi}_j^i$ is a continuous map and since $o(\mathcal{G}_j^i) < o(\mathcal{F}_i)$, it follows that $o(((\mathcal{G}_j^i)_{j=1}^{m_i})_{i=1}^n) < o((\mathcal{F}_i)_{i=1}^n) = \xi$. Therefore using our inductive assumption we may choose a completely plegma connected family $((G_j^i)_{j=1}^{m_i})_{i=1}^n$ such that $G_j^i \subseteq [\mathcal{G}_j^i \upharpoonright M]^{<\infty}$ and

$$\mathrm{dist}(\widehat{\varphi}_j^i(\emptyset), \mathrm{conv}\, \widehat{\varphi}_j^i(G_j^i)) < \varepsilon/2 \tag{6.13}$$

for every $1 \leq i \leq n$ and $1 \leq j \leq m_i$.

For every $1 \leq i \leq n$ and $1 \leq j \leq m_i$ we set $F_j^i = \big\{\{l_j^i\} \cup s : s \in G_j^i\big\}$. By (6.12) and the choice of $((G_j^i)_{j=1}^{m_i})_{i=1}^n$, we easily see that $((F_j^i)_{j=1}^{m_i})_{i=1}^n$ is completely plegma connected. Moreover, observe that $\widehat{\varphi}_j^i(\emptyset) = \widehat{\varphi}_i(\{l_j^i\})$ and $\widehat{\varphi}_j^i(G_j^i) = \widehat{\varphi}_i(F_j^i)$. Hence, (6.13) translates to

$$\mathrm{dist}(\widehat{\varphi}_i(\{l_j^i\}), \mathrm{conv}\, \widehat{\varphi}_i(F_j^i)) < \varepsilon/2 \tag{6.14}$$

for every $1 \leq i \leq n$ and $1 \leq j \leq m_i$.

For every $1 \leq i \leq n$, let $F_i = \bigcup_{j=1}^{m_i} F_j^i$. Clearly $(F_i)_{i=1}^n$ is a completely plegma connected family with $F_i \subseteq [\mathcal{F}_i \upharpoonright L]^{<\infty}$ for every $i = 1, \dots, n$. Finally, fix $1 \leq i \leq n$. By (6.14) we have $\mathrm{dist}(x, \mathrm{conv}\, \widehat{\varphi}_i(F_i)) < \varepsilon/2$ for every $x \in \mathrm{conv}\, \widehat{\varphi}_i(\Lambda_i)$, and therefore $\mathrm{dist}(\widehat{\varphi}_i(\emptyset), \mathrm{conv}\, \widehat{\varphi}_i(F_i)) < \varepsilon$ by (6.11). ∎

We are ready to obtain the following generalization of a well known result about classical spreading models stating that every spreading model (of order 1) generated by a seminormalized weakly null sequence is 1-unconditional.

THEOREM 6.11. *Let X be a Banach space, $\mathcal{F}$ be a regular thin family and $L \in [\mathbb{N}]^\infty$. Let $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ be an $\mathcal{F}$-subsequence in X generating an $\mathcal{F}$-spreading model $(e_n)_n$. Also assume that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is seminormalized, subordinated* (*with respect to the weak topology of X*) *and weakly null. Then $(e_n)_n$ is a* 1*-unconditional spreading sequence.*

Proof. We first show that $(e_n)_n$ is nontrivial. Indeed, otherwise by Theorem 6.6 there exist $M \in [L]^\infty$ and $x_0 \in X$ such that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is norm convergent to x_0. Since $M \subseteq L$, $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is also weakly null and therefore $x_0 = 0$. But this is a contradiction since $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is seminormalized.

We proceed to show that $(e_n)_n$ is 1-unconditional. Fix $n \in \mathbb{N}$, $1 \le p \le n$ and $a_1, \dots, a_n \in [-1, 1]$. It suffices to show that for every $\varepsilon > 0$ we have

$$\Big\| \sum_{\substack{i=1\\ i\neq p}}^{n} a_i e_i \Big\|_* < \Big\| \sum_{i=1}^{n} a_i e_i \Big\|_* + \varepsilon. \tag{6.15}$$

Indeed, fix $\varepsilon > 0$. Since $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model, by passing to a final segment of L if necessary we may assume that

(6.16)
$$\bigg| \Big\| \sum_{\substack{i=1\\ i\neq p}}^{n} a_i x_{s_i} \Big\| - \Big\| \sum_{\substack{i=1\\ i\neq p}}^{n} a_i e_i \Big\|_* \bigg| < \frac{\varepsilon}{3} \quad \text{and} \quad \bigg| \Big\| \sum_{i=1}^{n} a_i x_{s_i} \Big\| - \Big\| \sum_{i=1}^{n} a_i e_i \Big\|_* \bigg| < \frac{\varepsilon}{3}$$

for every plegma n-tuple $(s_i)_{i=1}^n$ in $\mathcal{F}\upharpoonright L$. Since $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated with respect to the weak topology, there exists a continuous map $\widehat{\varphi} : \mathcal{F}\upharpoonright L \to (X, w)$ such that $\widehat{\varphi}(s) = x_s$ for every $s \in \mathcal{F}\upharpoonright L$. Since $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is weakly convergent to $\widehat{\varphi}(\emptyset)$ we have $\widehat{\varphi}(\emptyset) = 0$. Therefore by Lemma 6.10 (for $\mathcal{F}_i = \mathcal{F}$ and $\widehat{\varphi}_i = \widehat{\varphi}$ for all $i = 1, \dots, n$), there exist a completely plegma connected family $(F_i)_{i=1}^n$ and a sequence $(x_i)_{i=1}^n$ in X such that $F_i \subseteq [\mathcal{F}\upharpoonright L]^{<\infty}$, $x_i \in \operatorname{conv} \widehat{\varphi}(F_i)$ and $\|x_i\| < \varepsilon/3$ for every $1 \le i \le n$. Let $(\mu_s)_{s\in F_p}$ be a sequence in $[0, 1]$ such that $\sum_{s\in F_p} \mu_s = 1$ and $x_p = \sum_{s\in F_p} \mu_s \widehat{\varphi}(t)$ and for each $i \neq p$ choose $s_i \in F_i$. By the above, for every $s \in F_p$ the n-tuple $(s_1, \dots, s_{p-1}, s, s_{p+1}, \dots, s_n)$ is a plegma family and $\|x_p\| = \|\sum_{s\in F_p} \mu_s x_s\| < \varepsilon/3$.

Therefore by (6.16) we have

$$\Big\|\sum_{\substack{i=1\\ i\neq p}}^n a_ie_i\Big\|_* \le \Big\|\sum_{\substack{i=1\\ i\neq p}}^n a_ix_{s_i}\Big\| + \frac{\varepsilon}{3} \le \Big\|\sum_{\substack{i=1\\ i\neq p}}^n a_ix_{s_i} + a_p\sum_{s\in F_p}\mu_s x_s\Big\| + |a_p|\frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\le \sum_{s\in F_p}\mu_s\Big\|\sum_{\substack{i=1\\ i\neq p}}^n a_ix_{s_i} + a_px_s\Big\| + \frac{2\varepsilon}{3}$$

$$\le \sum_{s\in F_p}\mu_s\Big(\Big\|\sum_{i=1}^n a_ie_i\Big\|_* + \frac{\varepsilon}{3}\Big) + \frac{2\varepsilon}{3} = \Big\|\sum_{i=1}^n a_ie_i\Big\|_* + \varepsilon$$

and the proof is complete. ∎

6.3.2. *Singular or isomorphic to ℓ_1 spreading models.* We proceed to show an analogue of Theorem 6.11 for subordinated $\mathcal{F}$-sequences which are not weakly null. We will need the following lemma.

LEMMA 6.12. *Let $(e_n)_n$ and $(\widetilde{e}_n)_n$ be two nontrivial spreading sequences which are both Cesàro summable to zero. Suppose that for every $n\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_n$ with $\sum_{i=1}^n\lambda_i=0$ we have*

$$\Big\|\sum_{i=1}^n \lambda_ie_i\Big\|_* = \Big\|\sum_{i=1}^n \lambda_i\widetilde{e}_i\Big\|_{**}. \tag{6.17}$$

Then the map $e_n\to\widetilde{e}_n$ extends to a linear isometry from $\langle(e_n)_n\rangle$ onto $\langle(\widetilde{e}_n)_n\rangle$.

Proof. Let $n\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_n\in\mathbb{R}$. Since $(e_n)_n$ (resp. $(\widetilde{e}_n)_n$) is Cesàro summable to zero, we have $\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^m e_{n+j}=0$ (resp. $\lim_m\frac{1}{m}\sum_{j=1}^m\widetilde{e}_{n+j}=0$). Let $\lambda=\sum_{i=1}^n\lambda_i$. Then $\sum_{i=1}^n\lambda_i-\sum_{j=1}^m\lambda/m=0$ and therefore,

$$\Big\|\sum_{i=1}^n\lambda_ie_i\Big\|_* = \lim_{m\to\infty}\Big\|\sum_{i=1}^n\lambda_ie_i - \frac{\lambda}{m}\sum_{j=1}^m e_{n+j}\Big\|_*$$

$$\overset{(6.17)}{=} \lim_{m\to\infty}\Big\|\sum_{i=1}^n\lambda_i\widetilde{e}_i - \frac{\lambda}{m}\sum_{j=1}^m \widetilde{e}_{n+j}\Big\|_{**} = \Big\|\sum_{i=1}^n\lambda_i\widetilde{e}_i\Big\|_{**}. \ ∎$$

The next lemma is from [4]. We reproduce it for the sake of completeness.

LEMMA 6.13. *Let X be a Banach space, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ be an $\mathcal{F}$-subsequence in X. Let $x_0\in X$ and set $x'_s=x_s-x_0$ for all $s\in\mathcal{F}\upharpoonright L$. Assume that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ and $(x'_s)_{s\in\mathcal{F}\upharpoonright L}$ generate $\mathcal{F}$-spreading models $(e_n)_n$ and $(\widetilde{e}_n)_n$ respectively. Then:*

(i) $\|\sum_{i=1}^n a_ie_i\| = \|\sum_{i=1}^n a_i\widetilde{e}_i\|$ *for every $n\in\mathbb{N}$ and $a_1,\dots,a_n\in\mathbb{R}$ with* $\sum_{i=1}^n a_i=0$.

(ii) *The sequence* $(e_n)_n$ *is trivial if and only if* $(\widetilde{e}_n)_n$ *is trivial.*

(iii) *The sequence* $(e_n)_n$ *is equivalent to the usual basis of* ℓ^1 *if and only if* $(\widetilde{e}_n)_n$ *is equivalent to the usual basis of* ℓ^1.

Proof. (i) Notice that for every $n \in \mathbb{N}$, $s_1, \ldots, s_n$ in $\mathcal{F}\restriction L$ and $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$, we have $\sum_{i=1}^n a_i x_{s_i} = \sum_{i=1}^n a_i x'_{s_i}$. Since $(e_n)_n$ and $(\widetilde{e}_n)_n$ are generated by $(x_s)_{s\in\mathcal{F}\restriction L}$ and $(x_s)_{s\in\mathcal{F}\restriction L}$ the result follows.

(ii) This follows by part (i) and the definition of a trivial sequence.

(iii) We fix $\varepsilon > 0$. If $(\widetilde{e}_n)_n$ is not equivalent to the usual basis of ℓ^1 then there exist $n \in \mathbb{N}$ and $a'_1, \ldots, a'_n \in \mathbb{R}$ such that $\sum_{i=1}^n |a'_i| = 1$ and $\|\sum_{i=1}^n a'_i \widetilde{e}_i\| < \varepsilon$. Setting $a_i = a'_i/2$ and $a_{n+i} = -a'_i/2$ for all $1 \le i \le n$, we have $\sum_{i=1}^{2n} a_i = 0$ and therefore $\|\sum_{i=1}^{2n} a_i e_i\| = \|\sum_{i=1}^{2n} a_i \widetilde{e}_i\| < \varepsilon$. Since $\sum_{i=1}^{2n} |a_i| = 1$, $(e_n)_n$ is also not equivalent to the usual basis of ℓ^1. ∎

THEOREM 6.14. *Let* X *be a Banach space,* $\mathcal{F}$ *be a regular thin family and* $L \in [\mathbb{N}]^\infty$. *Let* $(x_s)_{s\in\mathcal{F}\restriction L}$ *be an* $\mathcal{F}$*-subsequence in* X *generating a nontrivial* $\mathcal{F}$*-spreading model* $(e_n)_n$. *Also assume that* $(x_s)_{s\in\mathcal{F}\restriction L}$ *is subordinated and let* x_0 *be the weak limit of* $(x_s)_{s\in\mathcal{F}\restriction L}$. *Finally, let* $x'_s = x_s - x_0$ *for every* $s \in \mathcal{F}\restriction L$. *If* $x_0 \neq 0$ *then exactly one of the following holds:*

(i) *The sequence* $(e_n)_n$ *as well as every spreading model of* $(x'_s)_{s\in\mathcal{F}\restriction L}$ *is equivalent to the usual basis of* ℓ^1.

(ii) *The sequence* $(e_n)_n$ *is singular, and if* $e_n = e'_n + e$ *is its natural decomposition then* $\|e\| = \|x_0\|$ *and* $(e'_n)_n$ *is the unique* (*up to isometry*) $\mathcal{F}$*-spreading model of* $(x'_s)_{s\in\mathcal{F}\restriction L}$.

Proof. Let $(\widetilde{e}_n)_n$ be an $\mathcal{F}$-spreading model of $(x'_s)_{s\in\mathcal{F}\restriction L}$. By passing to an infinite subset of L if necessary we may assume that $(x'_s)_{s\in\mathcal{F}\restriction L}$ generates $(\widetilde{e}_n)_n$ as an $\mathcal{F}$-spreading model.

If $(e_n)_n$ is equivalent to the usual basis of ℓ^1 then by Lemma 6.13 the same holds for $(\widetilde{e}_n)_n$ and hence (i) is satisfied. Otherwise, again by Lemma 6.13, $(\widetilde{e}_n)_n$ is also nontrivial and not equivalent to the ℓ^1-basis. Let us denote by $\|\cdot\|_*$ (resp. $\|\cdot\|_{**}$) the norm of the space generated by $(e_n)_n$ (resp. $(\widetilde{e}_n)_n$). Since $(\widetilde{e}_n)_n$ is nontrivial, we have $\|\widetilde{e}_n\|_{**} > 0$ and therefore (by passing to a final segment of L if necessary) we may assume that $(x'_s)_{s\in\mathcal{F}\restriction L}$ is seminormalized. It is also easy to see that $(x'_s)_{s\in\mathcal{F}\restriction M}$ is subordinated and weakly null. Therefore, by Theorem 6.11, $(\widetilde{e}_n)_n$ is 1-unconditional. Moreover, since $(\widetilde{e}_n)_n$ is not equivalent to the usual basis of ℓ^1, by Proposition 6.4(ii), we conclude that $(\widetilde{e}_n)_n$ is norm Cesàro summable to zero. Hence, by Lemma 6.13(i), we have

$$(6.18)\quad \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^n e_j - \frac{1}{n}\sum_{j=n+1}^{2n} e_j\Big\|_* = \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^n \widetilde{e}_j - \frac{1}{n}\sum_{j=n+1}^{2n} \widetilde{e}_j\Big\|_{**} = 0.$$

For every $n \in \mathbb{N}$ choose $(s_j^n)_{j=1}^n \in \mathrm{Plm}_n(\mathcal{F}\restriction L)$ such that $\min s_1^n \ge L(n)$.

Since $x_s - x_0 = x'_s$ for every $s \in \mathcal{F}\upharpoonright L$, we have

$$\text{(6.19)}\quad \lim_{n\to\infty}\Big\|x_0 - \frac{1}{n}\sum_{j=1}^{n} x_{s_j^n}\Big\| = \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^{n} x'_{s_j^n}\Big\| = \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^{n} \widetilde{e}_n\Big\|_{**} = 0.$$

Therefore

$$\text{(6.20)}\quad \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^{n} e_j\Big\|_{*} = \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{j=1}^{n} x_{s_j^n}\Big\| = \|x_0\| > 0.$$

By (6.18) and (6.20), we deduce that $(e_n)_n$ is not Schauder basic, i.e. it is singular. Let $e_n = e'_n + e$ be the natural decomposition of $(e_n)_n$. By (6.20) and the fact that $(e'_n)_n$ is Cesàro summable to zero, we have $\|e\| = \|x_0\|$.

To complete the proof it remains to show that $(\widetilde{e}_n)_n$ and $(e'_n)_n$ are isometrically equivalent. By Lemma 6.12 it suffices to show that

$$\text{(6.21)}\quad \Big\|\sum_{i=1}^{n} \lambda_i e'_i\Big\|_{*} = \Big\|\sum_{i=1}^{n} \lambda_i \widetilde{e}_i\Big\|_{**}$$

for every $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_1 = 0$. Indeed, fix $n \in \mathbb{N}$ and $\lambda_1, \ldots \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^{n} \lambda_i = 0$. For each $k \in \mathbb{N}$ choose $(s_j^k)_{j=1}^n \in \mathrm{Plm}_n(\mathcal{F}\upharpoonright L)$ such that $\lim_{k\to\infty} \min s_1^k = +\infty$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^{n} \lambda_i e'_i\Big\|_{*} &= \Big\|\sum_{i=1}^{n} \lambda_i e_i\Big\|_{*} = \lim_{k\to\infty}\Big\|\sum_{i=1}^{n} \lambda_i x_{s_i^k}\Big\| \\ &= \lim_{k\to\infty}\Big\|\sum_{i=1}^{n} \lambda_i x'_{s_i^k}\Big\| = \Big\|\sum_{i=1}^{n} \lambda_i \widetilde{e}_i\Big\|_{**}\end{aligned}$$

and the proof is complete. ∎

6.3.3. *Weakly relatively compact $\mathcal{F}$-sequences.* Let X be a Banach space and $\xi < \omega_1$. By $\mathcal{SM}_\xi^{\mathrm{wrc}}(X)$ we will denote the set of all spreading sequences $(e_n)_n$ such that there exists a weakly relatively compact subset W of X which admits $(e_n)_n$ as a ξ-spreading model. We also set

$$\mathcal{SM}^{\mathrm{wrc}}(X) = \bigcup_{\xi<\omega_1} \mathcal{SM}_\xi^{\mathrm{wrc}}(X).$$

Hence $(e_n)_n \in \mathcal{SM}_\xi^{\mathrm{wrc}}(X)$ if and only if there exists an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ such that $\overline{\{x_s : s \in \mathcal{F}\}}^w$ is a weakly compact subset of X and for some $L \in [\mathbb{N}]^\infty$, $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. The $\mathcal{F}$-sequences with weakly relatively compact range will be called *weakly relatively compact* ("wrc" for short). The following proposition says that every wrc $\mathcal{F}$-sequence always contains a subordinated subsequence.

PROPOSITION 6.15. *Let X be a Banach space, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be a weakly relatively compact $\mathcal{F}$-sequence in X. Then for every*

$M \in [\mathbb{N}]^\infty$ there exists $L \in [M]^\infty$ such that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated with respect to the weak topology.

Proof. Let $M \in [\mathbb{N}]^\infty$. Since the weak topology on every separable weakly compact subset of a Banach space is metrizable, we see that $\overline{\{x_s : s \in \mathcal{F}\}}^w$ is compact metrizable. By Theorem 5.10 the result follows. ∎

Proposition 6.16. *Let X be a Banach space, $\xi < \omega_1$ and let $(e_n)_n \in \mathcal{SM}_\xi^{\mathrm{wrc}}(X)$. Then for every regular thin family $\mathcal{G}$ with $o(\mathcal{G}) \geq \xi$ there exist a weakly relatively compact $\mathcal{G}$-sequence $(w_t)_{t\in\mathcal{G}}$ in X and $L \in [\mathbb{N}]^\infty$ such that $(w_t)_{t\in\mathcal{G}\upharpoonright L}$ is subordinated with respect to the weak topology and generates $(e_n)_n$ as a $\mathcal{G}$-spreading model. Consequently, $\mathcal{SM}_\zeta^{\mathrm{wrc}}(X) \subseteq \mathcal{SM}_\xi^{\mathrm{wrc}}(X)$ for any $1 \leq \zeta < \xi < \omega_1$.*

Proof. Since $(e_n)_n \in \mathcal{SM}_\xi^{\mathrm{wrc}}(X)$ there exists a weakly relatively compact subset A of X such that A admits $(e_n)_n$ as a ξ-spreading model. Hence there exists a regular thin family $\mathcal{F}$ of order ξ, an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in A and $M \in [\mathbb{N}]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. By Lemma 4.6 there exist a $\mathcal{G}$-sequence $(w_t)_{t\in\mathcal{G}}$ and $N \in [\mathbb{N}]^\infty$ such that $(w_t)_{t\in\mathcal{G}\upharpoonright N}$ generates $(e_n)_n$ as a $\mathcal{G}$-spreading model and moreover $\{w_t : t \in \mathcal{G}\} \subseteq \{x_s : s \in \mathcal{F}\} \subseteq A$. Hence $(w_t)_{t\in\mathcal{G}}$ is a weakly relatively compact $\mathcal{G}$-sequence. By Proposition 6.15 there exists $L \in [N]^\infty$ such that $(w_t)_{t\in\mathcal{G}\upharpoonright L}$ is subordinated with respect to the weak topology. Clearly $(w_t)_{t\in\mathcal{G}\upharpoonright L}$ also generates $(e_n)_n$ as a $\mathcal{G}$-spreading model and the proof is complete. ∎

Proposition 6.16 implies that every $(e_n)_n$ in $\mathcal{SM}^{\mathrm{wrc}}(X)$ is generated by a subordinated $\mathcal{F}$-subsequence. Hence, by Theorems 6.11 and 6.14 we obtain the following.

Corollary 6.17. *Let X be a Banach space, $\mathcal{F}$ be a regular thin family and $(x_s)_{s\in\mathcal{F}}$ be a weakly relatively compact $\mathcal{F}$-sequence. Let $(e_n)_n$ be a spreading sequence and assume that $(x_s)_{s\in\mathcal{F}}$ admits $(e_n)_n$ as an $\mathcal{F}$-spreading model. Then exactly one of the following holds:*

(i) *The sequence $(e_n)_n$ is trivial.*
(ii) *The sequence $(e_n)_n$ is singular. In this case there exist $L \in [\mathbb{N}]^\infty$ and $x_0 \in X$ such that if $e_n = e'_n + e$ is the natural decomposition of $(e_n)_n$ then the $\mathcal{F}$-subsequence $(x'_s)_{s\in\mathcal{F}\upharpoonright L}$, defined by $x'_s = x_s - x_0$ for all $s \in \mathcal{F}\upharpoonright L$, generates the sequence $(e'_n)_n$ as an $\mathcal{F}$-spreading model and $\|x_0\| = \|e\|$.*
(iii) *The sequence $(e_n)_n$ is Schauder basic. In this case $(e_n)_n$ is unconditional.*

6.4. $\mathcal{F}$-sequences generating singular spreading models. Let X be a Banach space and $(x_n)_n$ be a sequence in X which generates a singular spreading model $(e_n)_n$ and let $e_n = e'_n + e$ be the natural decomposition of

$(e_n)_n$. It can be shown that there exists $x \in X \setminus \{0\}$ such that $\|x\| = \|e\|$, and setting $x'_n = x_n - x$, $(e'_n)_n$ is the unique spreading model of $(x'_n)_n$. In the following we will present an extension of this fact to $\mathcal{F}$-sequences in a Banach space X. We start with the next lemma.

LEMMA 6.18. *Let $\mathcal{F}$ be a regular thin family, $M \in [\mathbb{N}]^\infty$ and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a Banach space X such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates a singular $\mathcal{F}$-spreading model $(e_n)_n$. Then there exists $L \in [M]^\infty$ with the following property: For every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that*

$$\left\| \frac{1}{n} \sum_{j=1}^{n} x_{s_j} - \frac{1}{m} \sum_{j=1}^{m} x_{t_j} \right\| < \varepsilon, \tag{6.22}$$

for every $n, m \geq m_0$ and $(s_j)_{j=1}^n, (t_j)_{j=1}^m \in \mathrm{Plm}(\mathcal{F}\upharpoonright L)$ with $s_1(1) \geq L(n)$ and $t_1(1) \geq L(m)$.

Proof. First, we notice that a weaker version of the lemma holds true, that is, for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $n, m \geq k_0$ and every $(s_j)_{j=1}^{n+m} \in \mathrm{Plm}(\mathcal{F}\upharpoonright M)$ with $s_1(1) \geq M(n+m)$, we have

$$\left\| \frac{1}{n} \sum_{j=1}^{n} x_{s_j} - \frac{1}{m} \sum_{j=1}^{m} x_{s_{n+j}} \right\| < \varepsilon. \tag{6.23}$$

Indeed, let $\varepsilon > 0$. Since $(e_n)_n$ is singular, it is weakly convergent to some e and moreover, setting $e'_n = e_n - e$, the sequence $(e'_n)_n$ is Cesàro summable to zero. Hence we may choose $n_0 \in \mathbb{N}$ such that $\left\| \frac{1}{n} \sum_{i=1}^n e'_i \right\|_* < \varepsilon/4$ for all $n \geq n_0$. Therefore, for every $n, m \geq n_0$, we have

$$\left\| \frac{1}{n} \sum_{i=1}^{n} e_i - \frac{1}{m} \sum_{i=1}^{m} e_{n+i} \right\|_* = \left\| \frac{1}{n} \sum_{i=1}^{n} e'_i - \frac{1}{m} \sum_{i=1}^{m} e'_{n+i} \right\|_* < \varepsilon/2 \tag{6.24}$$

Since $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model we can find $k_0 \geq n_0$ such that for every $n, m \geq k_0$ and every $(s_j)_{j=1}^{n+m} \in \mathrm{Plm}(\mathcal{F}\upharpoonright M)$ with $s_1(1) \geq M(n+m)$ inequality (6.23) is satisfied.

Let $(\varepsilon_k)_k$ be a sequence of positive real numbers such that $\sum_k \varepsilon_k < \infty$. By the above we can choose an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that for every $k \in \mathbb{N}$, $n, m \geq n_k$ and $(s_j)_{j=1}^{n+m} \in \mathrm{Plm}(\mathcal{F}\upharpoonright M)$ with $s_1(1) \geq M(n+m)$,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} x_{s_j} - \frac{1}{m} \sum_{j=1}^{m} x_{s_{n+j}} \right\| < \varepsilon_k. \tag{6.25}$$

We may also assume that $\mathcal{F}$ is very large in M, and $2n_k < n_{k+1}$ for every $k \in \mathbb{N}$.

We set $L = \{M(2n_k + n_{k+1}) : k \in \mathbb{N}\}$; we shall show that L satisfies the conclusion of the lemma. To this end we shall use an appropriate map sending each $s \in \mathcal{F}\upharpoonright L$ to a plegma family in $\mathcal{F}\upharpoonright M$. First, for every $s \in \mathcal{F}\upharpoonright L$

and $p = 1, \dots, |s|$, let $k(s(p))$ be the unique positive integer k satisfying $s(p) = L(k) = M(2n_k + n_{k+1})$. We define $\Phi : \mathcal{F}\restriction L \to \mathrm{Plm}(\mathcal{F}\restriction M)$ as follows. To every $s \in \mathcal{F}\restriction L$, we assign the $n_{k(s(1))}$-tuple $\Phi(s) = (v_j^s)_{j=1}^{n_{k(s(1))}}$, where v_j^s is the unique element of $\mathcal{F}\restriction M$ satisfying

$$v_j^s \sqsubseteq \{M(2n_{k(s(p))} + n_{k(s(p))+1} - n_{k(s(1))} + j) : p = 1, \dots, |s|\}. \tag{6.26}$$

The existence of v_j^s, $j = 1, \dots, n_{k(s(1))}$, follows easily from the fact that $\mathcal{F}$ is regular thin and very large in M.

Below we state some useful properties of Φ. Their verification is straightforward.

- (P1) For every $s \in \mathcal{F}\restriction L$, $\Phi(s) \in \mathrm{Plm}_{n_k}(\mathcal{F}\restriction M)$, $v_1^s(1) > M(n_k + n_{k+1})$ and $v_{n_k}^s = s$, where $k = k(s(1))$.
- (P2) For every $(s_1, s_2) \in \mathrm{Plm}_2(\mathcal{F}\restriction L)$, the concatenation $\Phi(s_1)^\frown\Phi(s_2)$ belongs to $\mathrm{Plm}(\mathcal{F}\restriction M)$.

We are now ready to prove that L is actually the desired set. Fix a positive integer k and denote by s the unique element of $\mathcal{F}\restriction L$ such that $s \sqsubseteq \{L(i) : i \geq k\}$. Notice that $s(1) = L(k) = M(2n_k + n_{k+1})$ and therefore $k(s(1)) = k$. Also let $m_k = \max\{n_k, k + |s| + 1\}$. We claim that

$$\Big\| \frac{1}{n_k} \sum_{j=1}^{n_k} x_{v_j^s} - \frac{1}{m} \sum_{j=1}^{m} x_{t_j} \Big\| < \sum_{l=k}^{k+|s|} \varepsilon_l \tag{6.27}$$

for every $m \geq m_k$ and $(t_j)_{j=1}^m \in \mathcal{F}\restriction L$ with $t_1(1) \geq L(m)$.

Indeed, let $m \geq m_k$ and $(t_j)_{j=1}^m \in \mathcal{F}\restriction L$ with $t_1(1) \geq L(m)$. Notice that $\max s = L(k + |s| - 1) < L(m) \leq t_1(1) = \min t_1$. Hence, by Theorem 3.10 there exists a plegma path $(w_l)_{l=0}^{l_0}$ in $\mathcal{F}\restriction L$ from $w_0 = s$ to $w_{l_0} = t_1$ of length $l_0 = |s|$. Notice that $k(w_l(1)) \geq k + l$, which implies that $n_{k(w_l(1))} \geq n_{k+l}$ and therefore $(v_1^{w_l}, \dots, v_{n_{k+l}}^{w_l})$ is a subfamily of $\Phi(w_l)$. Thus, by properties (P1) and (P2) above, $(v_1^{w_l}, \dots, v_{n_{k+l}}^{w_l}, v_1^{w_{l+1}}, \dots, v_{n_{k+l+1}}^{w_{l+1}})$ is a plegma family in $\mathcal{F}\restriction L$ of length $n_{k+l} + n_{k+l+1}$ with $v_1^{w_l}(1) > M(n_{k(w_l(1))} + n_{k(w_l(1)+1)}) > M(n_{k+l} + n_{k+l+1})$. Hence by (6.25) we get

$$\Big\| \frac{1}{n_{k+l}} \sum_{j=1}^{n_{k+l}} x_{v_j^{w_l}} - \frac{1}{n_{k+l+1}} \sum_{j=1}^{n_{k+l+1}} x_{v_j^{w_{l+1}}} \Big\| < \varepsilon_{k+l} \tag{6.28}$$

for every $l = 0, \dots, l_0 - 1$. Thus,

$$\Big\| \frac{1}{n_k} \sum_{j=1}^{n_k} x_{v_j^s} - \frac{1}{n_{k+|s|}} \sum_{j=1}^{n_{k+|s|}} x_{v_j^{t_1}} \Big\| < \sum_{l=k}^{k+|s|-1} \varepsilon_l. \tag{6.29}$$

Similarly, since $m > k + |s| = k + |l_0|$ we know that $n_m > n_{k+|s|}$. Also since $t_1(1) \geq L(m) = M(2n_m + n_{m+1})$ we have $k(t_1(1)) \geq m$. Hence $n_{k(t_1(1))} \geq n_m > n_{k+|s|}$, which implies that $(v_1^{t_1}, \dots, v_{n_{k+|s|}}^{t_1})$ is a proper subfamily of $\Phi(t_1)$. Therefore, $(v_1^{t_1},\dots,v_{n_{k+|s|}}^{t_1},t_1,\dots,t_m)$ is a plegma family in $\mathcal{F}\upharpoonright L$. Moreover $t_1(1)\geq M(2n_m+n_{m+1})\geq M(n_{k+|s|}+m)$ and so, again by (6.25), we have

$$\Big\|\frac{1}{n_{k+|s|}}\sum_{j=1}^{n_{k+|s|}} x_{v_j^{t_1}}-\frac{1}{m}\sum_{j=1}^{m}x_{t_j}\Big\|<\varepsilon_{k+|s|}. \tag{6.30}$$

Now (6.27) follows by (6.29) and (6.30).

Finally, by (6.27) and a triangle inequality we obtain

$$\Big\|\frac{1}{n}\sum_{j=1}^{n}x_{s_j}-\frac{1}{m}\sum_{j=1}^{m}x_{t_j}\Big\|<2\sum_{l=k}^{k+|s|}\varepsilon_l \tag{6.31}$$

for every $k\in\mathbb{N}$, $n,m\geq m_k$ and $(s_j)_{j=1}^n,(t_j)_{j=1}^m\in\mathrm{Plm}(\mathcal{F}\upharpoonright L)$ with $s_1(1)\geq L(n)$ and $t_1(1)\geq L(m)$. Since $\sum_k\varepsilon_k<\infty$ the proof is complete. ∎

Theorem 6.19. *Let $\mathcal{F}$ be a regular thin family, $M\in[\mathbb{N}]^\infty$ and $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a Banach space X such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates a singular $\mathcal{F}$-spreading model $(e_n)_n$. Let $e_n=e'_n+e$ be the natural decomposition of $(e_n)_n$. Then there exist $x\in X$ with $\|x\|=\|e\|_*$ and $L\in[M]^\infty$ such that setting $x'_s=x_s-x$ the $\mathcal{F}$-subsequence $(x'_s)_{s\in\mathcal{F}\upharpoonright L}$ admits $(e'_n)_n$ as a unique (up to isometry) $\mathcal{F}$-spreading model.*

Proof. We start by determining the element $x\in X$. Let $L\in[M]^\infty$ satisfy Lemma 6.18. For every $k\in\mathbb{N}$ we set

$$A_k=\Big\{\frac{1}{n}\sum_{i=1}^{n}x_{s_i}:(s_i)_{i=1}^n\in\mathrm{Plm}(\mathcal{F}\upharpoonright L)\text{ and }s_1(1)\geq n\geq k\Big\}.$$

Clearly the sequence $(A_k)_k$ is decreasing and, by Lemma 6.18, $\mathrm{diam}(A_k)\to 0$. Therefore there exists a unique $x\in X$ such that $\bigcap_{k=1}^\infty\overline{A_k}=\{x\}$.

We proceed to show that $\|e\|=\|x\|$. Notice that by the choice of x, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,

$$\Big\|\frac{1}{n}\sum_{j=1}^{n}x_{s_j}-x\Big\|<\varepsilon. \tag{6.32}$$

For each $n\in\mathbb{N}$ we pick $(s_i^n)_{i=1}^n\in\mathrm{Plm}(\mathcal{F}\upharpoonright L)$, with $s_i^n(1)\geq L(n)$. By (6.32),

$$\lim_n\Big\|\frac{1}{n}\sum_{i=1}^{n}x_{s_i^n}-x\Big\|=0. \tag{6.33}$$

Also, since $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model,

$$\lim_n\Big|\Big\|\frac{1}{n}\sum_{i=1}^{n}x_{s_i^n}\Big\|-\Big\|\frac{1}{n}\sum_{i=1}^{n}e_i\Big\|_*\Big|=0. \tag{6.34}$$

Moreover, since $(e'_n)_n$ is Cesàro summable to zero,

$$\lim_n \left\| \frac{\sum_{i=1}^n e_i}{n} - e \right\|_* = 0. \tag{6.35}$$

Hence,

$$\|e\|_* \overset{(6.35)}{=} \lim_n \left\| \frac{1}{n} \sum_{i=1}^n e_i \right\|_* \overset{(6.34)}{=} \lim_n \left\| \frac{1}{n} \sum_{i=1}^n x_{s_i^n} \right\| \overset{(6.33)}{=} \|x\|.$$

We now proceed to show that $(e'_n)_n$ is the unique $\mathcal{F}$-spreading model of $(x'_s)_{s\in\mathcal{F}\restriction L}$, where $x'_s = x_s - x$, $s \in \mathcal{F}\restriction L$. Let $N \in [L]^\infty$ be such that $(x'_s)_{s\in\mathcal{F}\restriction N}$ generates an $\mathcal{F}$-spreading model $(\widetilde{e}_n)_n$. We will show that $(\widetilde{e}_n)_n$ is isometric to $(e'_n)_n$. Since

$$\frac{1}{n} \sum_{j=1}^n x'_{s_j} = \frac{1}{n} \sum_{j=1}^n x_{s_j} - x \tag{6.36}$$

for every $n \in \mathbb{N}$, by (6.32) we conclude that $(\widetilde{e}_n)_n$ is Cesàro summable to zero. Hence by Lemma 6.12 it suffices to show that

$$\Big\| \sum_{i=1}^n \lambda_i e'_i \Big\|_* = \Big\| \sum_{i=1}^n \lambda_i \widetilde{e}_i \Big\|_{**} \tag{6.37}$$

for every $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_1 = 0$. Indeed, let $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$. Also let $((s_j^k)_{j=1}^n)_k$ be a sequence in $\mathrm{Plm}_n(\mathcal{F}\restriction L)$ such that $\lim_{k\to\infty} s_1^k(1) = \infty$. Then

$$\begin{aligned} \Big\| \sum_{i=1}^n \lambda_i e'_i \Big\|_* &= \Big\| \sum_{i=1}^n \lambda_i e_i \Big\|_* = \lim_{k\to\infty} \Big\| \sum_{i=1}^n \lambda_i x_{s_i^k} \Big\| \\ &= \lim_{k\to\infty} \Big\| \sum_{i=1}^n \lambda_i x'_{s_i^k} \Big\| = \Big\| \sum_{i=1}^n \lambda_i \widetilde{e}_i \Big\|_{**} \end{aligned}$$

and the proof is complete. ∎

We close by a strengthening of Theorem 6.19 for Banach spaces with separable dual. We will need the following lemma.

Lemma 6.20. *Let $\mathcal{F}$ be a regular thin family and $(y_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a Banach space X. Let $L \in [\mathbb{N}]^\infty$ and suppose that for every $\varepsilon > 0$ and $N \in [L]^\infty$ there exist $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k > 0$ and $(s_j)_{j=1}^k \in \mathrm{Plm}(\mathcal{F}\restriction N)$ such that $\sum_{j=1}^k \lambda_j = 1$ and $\|\sum_{j=1}^k \lambda_j y_{s_j}\| < \varepsilon$. Then for every $x^* \in X^*$, $\varepsilon > 0$ and $N \in [L]^\infty$ there exists $M \in [N]^\infty$ such that $|x^*(y_s)| < \varepsilon$ for every $s \in \mathcal{F}\restriction M$.*

Proof. Let $x^* \in X^*$, $\varepsilon > 0$ and $N \in [L]^\infty$. By Proposition 2.6 there exists $M \in [N]^\infty$ such that exactly one of the following holds: (a) $|x^*(y_s)| < \varepsilon$ for every $s \in \mathcal{F}\restriction M$, or (b) $x^*(y_s) \geq \varepsilon$ for every $s \in \mathcal{F}\restriction M$, or (c) $x^*(y_s) \leq -\varepsilon$

for every $s \in \mathcal{F}\restriction M$. It suffices to show that cases (b) and (c) cannot occur. Indeed, suppose (b) holds true (the proof for case (c) is similar). By our assumption there exist $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k > 0$ and $(s_j)_{j=1}^k \in \mathrm{Plm}(\mathcal{F}\restriction M)$ such that $\sum_{j=1}^k \lambda_j = 1$ and $\|\sum_{j=1}^k \lambda_j y_{s_j}\| < \varepsilon$. But then

$$\varepsilon > \Big\|\sum_{j=1}^{k} \lambda_j y_{s_j}\Big\| \geq x^*\Big(\sum_{j=1}^{k} \lambda_j y_{s_j}\Big) = \sum_{j=1}^{k} \lambda_j x^*(y_{s_j}) \geq \varepsilon, \tag{6.38}$$

which is a contradiction. ∎

Corollary 6.21. *Let X be a Banach space with separable dual. Let $\mathcal{F}$ be a regular thin family, $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in X and $M \in [\mathbb{N}]^\infty$ be such that $(x_s)_{s\in\mathcal{F}\restriction M}$ generates a singular $\mathcal{F}$-spreading model $(e_n)_n$. Let $e_n = e'_n + e$ be the natural decomposition of $(e_n)_n$. Then there exist $x \in X$ with $\|x\| = \|e\|_*$ and $N \in [M]^\infty$ such that setting $x'_s = x_s - x$, $(x'_s)_{s\in\mathcal{F}\restriction N}$ is weakly null and admits $(e'_n)_n$ as a unique (up to isometry) $\mathcal{F}$-spreading model.*

Proof. By Theorem 6.19, there exist $x \in X$ with $\|x\| = \|e\|_*$ and $L \in [M]^\infty$ such that $(x'_s)_{s\in\mathcal{F}\restriction L}$ admits $(e'_n)_n$ as a unique $\mathcal{F}$-spreading model. By applying Lemma 6.20 (for x_s in place of y_s) and a standard diagonalization for a countable dense subset of X^* we may choose $N \in [L]^\infty$ such that the $\mathcal{F}$-subsequence $(x'_s)_{s\in\mathcal{F}\restriction N}$ is in addition weakly null. ∎

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