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Covering maps over solenoids which are not covering homomorphisms

by

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Abstract. Let Y be a connected group and let $f : X \to Y$ be a covering map with the total space X being connected. We consider the following question: Is it possible to define a topological group structure on X in such a way that f becomes a homomorphism of topological groups. This holds in some particular cases: if Y is a pathwise connected and locally pathwise connected group or if f is a finite-sheeted covering map over a compact connected group Y. However, using shape-theoretic techniques and Fox's notion of an overlay map, we answer the question in the negative. We consider infinite-sheeted covering maps over solenoids, i.e. compact connected 1-dimensional abelian groups. First we show that an infinite-sheeted covering map $f : X \to \Sigma$ with a total space being connected over a solenoid Σ does not admit a topological group structure on X such that f becomes a homomorphism. Then, for an arbitrary solenoid Σ, we construct a connected space X and an infinite-sheeted covering map $f : X \to \Sigma$, which provides a negative answer to the question.

1. Introduction. In studying covering maps over topological groups a natural question arises: Is it always possible to define a topological group structure on a total space X in such a way that a covering map $f : X \to Y$ over a topological group Y becomes a homomorphism of topological groups? In some important cases this holds, e.g. if Y is a pathwise connected, locally pathwise connected group and X is a pathwise connected space ([13, Theorem 79]) or if f is a finite-sheeted covering map over a compact connected group Y and X is connected ([5, Theorem 1], [6, Theorem 1], [1, Lemma 2.9]). Moreover, the topological group structure on X is unique up to isomorphism of topological groups and in both cases covering homomorphisms $f : X \to Y$ and $f' : X' \to Y$ are equivalent as covering maps (via a homeomorphism) if and only if they are equivalent as covering homomorphisms (via a topological isomorphism) ([1, Corollary 2.6, Theorem 2.13]).

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In 1972, R. H. Fox, in attempt to extend the classical classification theorem of the covering space theory to arbitrary connected metric spaces, introduced the notion of an overlay map ([2], [3]). Every overlay map is a covering map. The converse holds in some particular cases: if Y is a connected locally connected paracompact space ([10, Lemma 4]) or if Y is a connected paracompact space and the number of sheets is finite ([11, Theorem 1]) (see also [3, Theorem 3] for metric case). Fox has given an example of a covering map over a metric continuum (the so-called razor clam shell), which is not an overlay map ([3, p. 86]). Apparently, in his example the number of sheets was infinite.

It turns out that the answer to our question is related to the notion of an overlay map. In the present paper we prove that a covering map $f : X \to Y$ over a compact connected group Y with a connected total space X admits a topological group structure on X such that f becomes a homomorphism if and only if f is an overlay map (Theorems 2.4 and 2.6 and Corollaries 2.5 and 2.7). We investigate infinite-sheeted covering maps over solenoids, i.e. compact connected 1-dimensional abelian groups. First we show that an infinite-sheeted covering map $f : X \to \Sigma$ with a connected total space over a solenoid Σ does not admit a topological group structure on X such that f becomes a homomorphism (Corollary 2.8). Then, for each solenoid Σ we construct a connected space X and an infinite-sheeted covering map $f : X \to \Sigma$, which provides the negative answer to the question (Theorem 3.1 and Corollary 3.2).

2. Overlays vs. covering homomorphisms. We start with a definition of an overlay map.

Let Y be a connected topological space, let $f : X \to Y$ be a continuous map and let S be a set of cardinality s. Let $\mathcal{B} = \{B\}$ be an open covering of Y and let $\mathcal{A} = \{A_B^\sigma : B \in \mathcal{B}, \sigma \in S\}$ be an open covering of X. We will say that $(\mathcal{A}, \mathcal{B})$ is an *s-sheeted covering pair* for $f : X \to Y$ provided the following three conditions are fulfilled:

(C1) $f^{-1}(B) = \bigcup_{\sigma\in S} A_B^\sigma$, $B \in \mathcal{B}$;
(C2) $A_B^\sigma \cap A_B^\tau = \emptyset$ for $\sigma, \tau \in S$, $\sigma \neq \tau$, and $B \in \mathcal{B}$;
(C3) $f|_{A_B^\sigma} : A_B^\sigma \to B$ is a homeomorphism for each $A_B^\sigma \in \mathcal{A}$.

Recall that a mapping $f : X \to Y$ is an s-sheeted *covering mapping* provided it admits an s-sheeted covering pair $(\mathcal{A}, \mathcal{B})$.

An s-sheeted covering pair $(\mathcal{A}, \mathcal{B})$ for $f : X \to Y$ is said to be an *s-sheeted overlay pair* for f provided $\mathcal{B}$ is a normal covering and the following additional condition is fulfilled:

(C4) If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then every $\sigma \in S$ admits a unique $\sigma' \in S$ such that $A_B^\sigma \cap A_{B'}^{\sigma'} \neq \emptyset$.

A mapping $f : X \to Y$ between topological spaces is said to be an *s-sheeted overlay mapping* provided it admits an s-sheeted overlay pair.

Definition 2.1. Let X and Y be topological groups and let $f : X \to Y$ be a map. We say that f is a *covering homomorphism* if f is a covering map and a homomorphism of topological groups as well.

Note that in the definition of a covering map we assume that the base space Y is connected in order to have all fibers of f of the same cardinality s. However, if $f : X \to Y$ is a homomorphism of topological groups, all fibers of f are of the same cardinality $s = \mathrm{card}(\ker f)$. So, in the definition of a covering homomorphism we omit the assumption of Y being connected.

Theorem 2.2. *Let X and Y be topological groups and let $f : X \to Y$ be a continuous epimorphism. If there exist an open neighborhood $A \subseteq X$ of the identity $e_X \in X$ and an open neighborhood $B \subseteq Y$ of the identity $e_Y \in Y$ such that $f|_A : A \to B$ is a homeomorphism, then f is an s-sheeted overlay map, where $s = \mathrm{card}(\ker f)$. In particular, every covering homomorphism $f : X \to Y$ is an overlay map.*

Proof. Let U be an open symmetric neighborhood of e_X such that $UU \subseteq A$, and let $V = f(U) \subseteq B$. Note that V is open in Y. We claim that $\{Ue : e \in \ker f\}$ evenly covers V.

First we show that $f^{-1}(V) = \bigcup_{e\in\ker f} Ue$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since $f|U : U \to V$ is a homeomorphism, there exists a unique $x' \in U$ such that $f(x') = f(x)$. Then $(x')^{-1}x \in \ker f$ and consequently $x \in \bigcup_{e\in\ker f} Ue$. Conversely, if $x \in \bigcup_{e\in\ker f} Ue$, then $f(x) \in f(U) = V$, i.e. $x \in f^{-1}(V)$.

Next we show that the subsets Ue, $e \in \ker f$, are pairwise disjoint. To see this, assume $x \in Ue \cap Ue'$, $e, e' \in \ker f$. Then there exist $u, u' \in U$ such that $x = ue = u'e'$. Since $f(u) = f(ue) = f(u'e') = f(u')$, it follows that $u = u'$. Hence, $e = e'$.

Since $f|U : U \to V$ is a homeomorphism, $f|_{Ue} : Ue \to V$, $e \in \ker f$, is a homeomorphism too. Put $\mathcal{B} = \{Vy : y \in Y\}$ and $\mathcal{A} = \{Ux : x \in f^{-1}(y), y \in Y\}$. Then $\mathcal{B}$ is an open covering of Y and $\mathcal{A}$ is an open covering of X. Since f is surjective, each fiber $f^{-1}(y)$ is non-empty and $f^{-1}(y) = (\ker f)x$, where $x \in f^{-1}(y)$ is arbitrary. We claim that $(\mathcal{A}, \mathcal{B})$ is an overlay pair for f. Obviously, $\{Ux : x \in f^{-1}(y)\}$ evenly covers Vy for each $y \in Y$. It remains to prove that, whenever $Vy \cap Vy' \neq \emptyset$, each Ux, $x \in f^{-1}(\{y\})$, intersects exactly one Ux', $x' \in f^{-1}(\{y'\})$. Assume Ux, $x \in f^{-1}(\{y\})$, intersects Ux' and Ux'', $x', x'' \in f^{-1}(y')$. Then there are $u_1, u_2, u_3, u_4 \in U$ such that $u_1x = u_2x'$ and $u_3x = u_4x''$. Since $u_2^{-1}u_1, u_4^{-1}u_3 \in A$ and $f(u_2^{-1}u_1) = f(x'x^{-1}) = f(x''x^{-1}) = f(u_4^{-1}u_3)$, it follows that $u_2^{-1}u_1 = u_4^{-1}u_3$ and consequently $x'x^{-1} = x''x^{-1}$. Hence, $x' = x''$ and Ux intersects exactly one Ux'. ∎

In what follows we consider covering maps $f : X \to Y$ over compact connected groups Y. At first we do not assume that the total space X is connected. However, since covering homomorphisms are overlay maps, the problem of connectedness of the total space is related to Fox's notion of indecomposability. Namely, Fox has noticed that for overlay maps, connectedness of the total space X has to be replaced by the *indecomposability* of the overlay map f, a property which he calls *vertical connectedness* of f.

We say that an overlay pair $(\mathcal{A}, \mathcal{B})$ for a map $f : X \to Y$ is *decomposable* provided there exist non-empty disjoint open sets X^1, X^2 whose union is X, and there exist non-empty disjoint subsets S^1, S^2 whose union is S. Moreover, the collections $\mathcal{A}^i = (A_B^{\sigma^i}, B \in \mathcal{B}, \sigma^i \in S^i)$, $i = 1, 2$, together with $\mathcal{B}$ form overlay pairs $(\mathcal{A}^i, \mathcal{B})$ for the mappings $f^i = f|_{X^i} : X^i \to Y$, $i = 1, 2$. We say that an overlay map $f : X \to Y$ is decomposable provided it admits a decomposable overlay pair $(\mathcal{A}, \mathcal{B})$.

Clearly, connectedness of the total space X always implies indecomposability of the overlay map $f : X \to Y$, but Fox exhibited an example of an indecomposable overlay map between metric spaces, where the total space is not connected ([2]). Again the number of sheets was infinite. In some important cases the idecomposability of an overlay map f implies the connectedness of the total space X. In particular, this holds if Y is a paracompact and locally connected space ([10, Lemma 5]) or if Y is paracompact and the number of sheets is finite ([11, Theorem 2]).

A simple consequence of Theorem 2.2 is the following corollary.

COROLLARY 2.3. *Let $f : X \to Y$ be a covering map over a connected group Y. If f is a covering homomorphism and the total space X is connected, then f is an indecomposable overlay map.*

If Y is a connected compact group, we are able to prove the converse of Corollary 2.3. To do this we need the notion of an ANR-*pull-back-expansion* of an s-sheeted overlay map $f : X \to Y$, denoted by $\boldsymbol{E}$. It consists of an ANR-resolution $\boldsymbol{q} = (q_\lambda : Y \to Y_\lambda,\ \lambda \in \Lambda) : Y \to \boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ (see [10, Section 5]) of a map $\boldsymbol{p} = (p_\lambda : X \to X_\lambda,\ \lambda \in \Lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and of a map $\boldsymbol{f} = (f_\lambda : X_\lambda \to Y_\lambda,\ \lambda \in \Lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ such that $\boldsymbol{f}\boldsymbol{p} = \boldsymbol{q}f$ and the following diagrams D_λ, $\lambda \in \Lambda$, and $D_{\lambda\lambda'}$, $\lambda \le \lambda'$, are pull-back diagrams:

$$\begin{array}{ccc} X_\lambda & \xleftarrow{p_\lambda} & X \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle f} \\ Y_\lambda & \xleftarrow{q_\lambda} & Y \end{array} \qquad \begin{array}{ccc} X_\lambda & \xleftarrow{p_{\lambda\lambda'}} & X_{\lambda'} \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle f_{\lambda'}} \\ Y_\lambda & \xleftarrow{q_{\lambda\lambda'}} & Y_{\lambda'} \end{array}$$

Furthermore, we require that the maps $f_\lambda : X_\lambda \to Y_\lambda$, $\lambda \in \Lambda$, be s-sheeted covering maps. If all maps in D_λ and $D_{\lambda\lambda'}$ are pointed maps, we speak of a *pointed* ANR-*pull-back-expansion* $\boldsymbol{E}_*$ of f.

We say that an s-sheeted overlay map $f : X \to Y$ is the *inverse limit* of an ANR-pull-back-expansion $\boldsymbol{E}$ if $X = \varprojlim \boldsymbol{X}$, $Y = \varprojlim \boldsymbol{Y}$ and $f = \varprojlim \boldsymbol{f}$.

Theorem 2.4. *Let Y be a compact connected group with the identity e and let $f : (X, x_0) \to (Y, e)$ be a pointed covering map. Then the following statements are equivalent.*

(i) *f is a pointed s-sheeted indecomposable overlay map.*

(ii) *If (Y, e) is the inverse limit of a pointed inverse system $((Y_\lambda, e_\lambda), q_{\lambda\lambda'}, \Lambda)$, where each Y_λ is a compact connected ANR, then f is the inverse limit of a pointed ANR-pull-back expansion $\boldsymbol{E}_*$ consisting of pointed s-sheeted covering maps $f_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, e_\lambda)$, $\lambda \geq \lambda_0$, with connected total space.*

(iii) *X is a connected space and there exists a multiplication $\cdot$ on X such that $(X, \cdot)$ is a topological group with the identity x_0 and f is an s-sheeted covering homomorphism.*

Proof. (i)⇒(ii). Let (Y, e) be the inverse limit of a pointed inverse system $((Y_\lambda, e_\lambda), q_{\lambda\lambda'}, \Lambda)$, where each Y_λ is a compact connected ANR. According to [9, Ch. I, §6.1, Theorem 1] $\boldsymbol{q} = (q_\lambda : Y \to Y_\lambda, \lambda \in \Lambda) : (Y, e) \to ((Y_\lambda, e_\lambda), q_{\lambda\lambda'}, \Lambda)$ is a pointed ANR-resolution of (Y, e). Take an s-sheeted indecomposable overlay pair $(\mathcal{A}, \mathcal{B})$ for f and apply Lemma 23 and Remark 8 of [10]. We get a pointed (enriched) pull-back expansion of f, where each $f_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, e_\lambda)$, $\lambda \geq \lambda_0$, is a pointed s-sheeted covering map with the connected total space X_λ.

$$\begin{array}{ccccccccc}
(X_{\lambda_0}, x_{\lambda_0}) & \xleftarrow{p_{\lambda_0\lambda}} & (X_\lambda, x_\lambda) & \xleftarrow{p_{\lambda\lambda'}} & (X_{\lambda'}, x_{\lambda'}) & \longleftarrow & \cdots & \longleftarrow & (X, x_0) \\
\Big\downarrow{\scriptstyle f_{\lambda_0}} & & \Big\downarrow{\scriptstyle f_\lambda} & & \Big\downarrow{\scriptstyle f_{\lambda'}} & & & & \Big\downarrow{\scriptstyle f} \\
(Y_{\lambda_0}, e_{\lambda_0}) & \xleftarrow{q_{\lambda_0\lambda}} & (Y_\lambda, e_\lambda) & \xleftarrow{q_{\lambda\lambda'}} & (Y_{\lambda'}, e_{\lambda'}) & \longleftarrow & \cdots & \longleftarrow & (Y, e)
\end{array}$$

Since $Y = \varprojlim(Y_\lambda, , q_{\lambda\lambda'}, \lambda \geq \lambda_0)$, by [10, Lemma 11] it follows that $X = \varprojlim \boldsymbol{X}$ and $f = \varprojlim \boldsymbol{f}$, which proves (ii).

(ii)⇒(iii). Since Y is a compact connected group, (Y, e) can be presented as the inverse limit of a pointed inverse system $\boldsymbol{Y}_* = ((Y_\lambda, e_\lambda), q_{\lambda\lambda'}, \Lambda)$, where each Y_λ is a compact connected Lie group with the identity e_λ, each bonding map $q_{\lambda\lambda'} : Y_{\lambda'} \to Y_\lambda$ and each projection $q_\lambda : Y \to Y_\lambda$ are epimorphisms of topological groups (see [1, Lemma 2.12]). Note that each Y_λ is an ANR and by (ii), f is the inverse limit of an ANR-pull-back expansion consisting of pointed s-sheeted covering maps $f_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, e_\lambda)$, $\lambda \geq \lambda_0$, with connected total space. Moreover, each X_λ is an ANR ([10, Remark 7]).

We claim that the projections $p_\lambda : X \to X_\lambda$ are surjections. Take an arbitrary $x'_\lambda \in X_\lambda$. Since the projection q_λ is surjective, there is $y \in Y$ such that $q_\lambda(y) = f_\lambda(x'_\lambda)$. D_λ is a pull-back diagram and, according to [10, Lemma 6], $p_\lambda|f^{-1}(y) : f^{-1}(y) \to f_\lambda^{-1}(q_\lambda(y))$ is a bijection. Hence, there is $x \in f^{-1}(y) \subseteq X$ such that $p_\lambda(x) = x'_\lambda$, which proves p_λ is surjective.

By [13, Theorem 79], each (X_λ, x_λ), $\lambda \geq \lambda_0$, admits a (unique) topological group structure with the identity x_λ and each $f_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, e_\lambda)$, $\lambda \geq \lambda_0$, becomes a covering homomorphism. Furthermore, each bonding map $p_{\lambda\lambda'} : (X_{\lambda'}, x_{\lambda'}) \to (X_\lambda, x_\lambda)$, $\lambda' \geq \lambda \geq \lambda_0$, becomes a homomorphism of topological groups ([1, Lemma 2.3]), which induces a topological group structure with the identity x_0 on the inverse limit space $(X, x_0) = \varprojlim((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \lambda \geq \lambda_0)$ and $f = \varprojlim \boldsymbol{f}$ becomes a homomorphism of topological groups. The groups X_λ, $\lambda \geq \lambda_0$, and the group X are locally compact.

It remains to prove that X is connected. According to [7, Corollary 7.9], it is sufficient to prove that $X = \bigcup_{n=1}^{\infty} U^n$ for each open neighborhood U of the identity x_0 of X. Fix such a U. As X is the inverse limit of $(X_\lambda, p_{\lambda\lambda'}, \lambda \geq \lambda_0)$ there is an index $\lambda \geq \lambda_0$ and an open set $U_\lambda \subseteq X_\lambda$ such that $x_0 \in p_\lambda^{-1}(U_\lambda) \subseteq U$. Note that U_λ is an open neighborhood of the identity $x_\lambda \in X_\lambda$ of the connected locally compact group X_λ, which implies $X_\lambda = \bigcup_{n=1}^{\infty} (U_\lambda)^n$. Let $x \in X$ be arbitrary. Then $p_\lambda(x) = u_1 \cdots u_n$ for some $u_1, \ldots, u_n \in U_\lambda$. The projection p_λ is surjective, which implies $U_\lambda = p_\lambda(p_\lambda^{-1}(U_\lambda))$. Then there are $v_1, \ldots, v_n \in p_\lambda^{-1}(U_\lambda)$ such that $p_\lambda(v_i) = u_i$ for each $i = 1, \ldots, n$. Hence $p_\lambda(x) = u_1 \cdots u_n = p_\lambda(v_1 \cdots v_n)$, $x(v_1 \cdots v_n)^{-1} \in \ker p_\lambda \subseteq p_\lambda^{-1}(U_\lambda)$ and we conclude $x \in p_\lambda^{-1}(U_\lambda)v_1 \cdots v_n \subseteq p_\lambda^{-1}(U_\lambda)(p_\lambda^{-1}(U_\lambda))^n \subseteq U^{n+1}$.

(iii)⇒(i). By Corollary 2.3. ∎

Note that every compact connected group can be represented as the inverse limit of a pointed inverse system of compact connected ANR's in different ways, but the statement (ii) does not depend on the choice of such a system.

The previous theorem can be rephrased as follows.

Corollary 2.5. *Let $f : X \to Y$ be a covering map over a compact connected group Y. Then X is a connected space and admits a topological group structure such that f is a covering homomorphism if and only if f is an indecomposable overlay map.*

We remark that compact connected groups Y are another class of spaces for which the indecomposability of overlay maps $f : X \to Y$ implies the connectedness of the total space since in Theorem 2.4 there are no requirements on local properties of Y or on the number s of sheets of f. Since we are interested only in covering maps from connected spaces, we are now able to omit the assumption on indecomposability of f in Theorem 2.4 without losing generality. So, taking a covering map $f : X \to Y$ from a connected space X, we get the following versions of Theorem 2.4 and Corollary 2.5.

THEOREM 2.6. *Let Y be a compact connected group with the identity e, X a connected space, and $f : (X, x_0) \to (Y, e)$ a pointed covering map. Then the following statements are equivalent:*

(i) *f is a pointed s-sheeted overlay map.*
(ii) *If (Y, e) is the inverse limit of a pointed inverse system $((Y_\lambda, e_\lambda), q_{\lambda\lambda'}, \Lambda)$, where each Y_λ is a compact connected ANR, then f is the inverse limit of a pointed ANR-pull-back expansion $\boldsymbol{E}_*$ consisting of pointed s-sheeted covering maps $f_\lambda : (X_\lambda, x_\lambda) \to (Y_\lambda, e_\lambda), \lambda \geq \lambda_0$, with the connected total space.*
(iii) *There exists a multiplication $\cdot$ on X such that $(X, \cdot)$ is a topological group with the identity x_0 and f is an s-sheeted covering homomorphism.*

COROLLARY 2.7. *Let Y be a compact connected group, X a connected space, and $f : X \to Y$ a covering map. Then X admits a topological group structure such that f is a covering homomorphism if and only if f is an overlay map.*

By a *solenoid* we mean a compact connected 1-dimensional abelian group. It is known that any solenoid is the inverse limit of finite coverings of circles whose covering numbers are primes. Precisely, for a solenoid Σ there exists a sequence $\boldsymbol{P} = \langle p_0, p_1, \ldots \rangle$ of primes such that $\Sigma = \varprojlim(S_n, g_n, n < \omega)$, where $S_n = S^1$ is the unit circle and $g_n : S^1 \to S^1$, $g_n(z) = z^{p_n}$ for each $n \geq 0$. We say that the sequence $\boldsymbol{P}$ is *related to* Σ (or that Σ is *generated by the sequence* $\boldsymbol{P}$) and denote Σ by $\Sigma_{\boldsymbol{P}}$,

$$S^1 \xleftarrow{p_0} S^1 \xleftarrow{p_1} S^1 \leftarrow \cdots \leftarrow \Sigma_{\boldsymbol{P}}$$

Two sequences $\boldsymbol{P} = \langle p_0, p_1, \ldots \rangle$ and $\boldsymbol{Q} = \langle q_0, q_1, \ldots \rangle$ of primes are said to be equivalent, written $\boldsymbol{P} \sim \boldsymbol{Q}$, provided it is possible to delate a finite number of terms from each so that every prime occurs the same number of times in each of the resulting sequences. It is a well-known result that solenoids $\Sigma_{\boldsymbol{P}}$ and $\Sigma_{\boldsymbol{Q}}$ are homeomorphic if and only if $\boldsymbol{P} \sim \boldsymbol{Q}$ (see [12, §2] or [8, Theorem 17]). We see that, for any solenoid Σ, the related sequence $\boldsymbol{P}$ is unique up to the equivalence $\sim$ of sequences of primes.

Assume that X is a connected space and $f : X \to \Sigma$ is an infinite-sheeted covering map over a solenoid Σ. Applying Theorem 2.4 we get the following corollary.

COROLLARY 2.8. *Let X be a connected space and let $f : X \to \Sigma$ be an infinite-sheeted covering map over a solenoid Σ. Then X does not admit a topological group structure such that f is a covering homomorphism.*

Proof. Assume the contrary. Let $\boldsymbol{P} = \langle p_0, p_1, \ldots \rangle$ be the sequence of primes which is related to Σ and let $\cdot$ be a multiplication on X such that

X is a topological group with the identity x_0 and $f : (X, x_0) \to (\Sigma_{\boldsymbol{P}}, e)$ is an infinite-sheeted covering homomorphism. By Theorem 2.6(ii), f is the inverse limit of a pointed ANR-pull-back expansion $\boldsymbol{E}_*$ consisting of pointed infinite-sheeted covering maps $f_n : (X_n, x_n) \to (S^1, 1)$, $n \geq n_0$, with connected total space. Then $H_n = (f_n)_\#(\pi_1(X_n, x_n)) = \{0\}$, $n \geq n_0$, the quotient set $\pi_1(S^1, 1)/H_n$ equals $\mathbb{Z}$ and a function $r_{n,n+1} : \pi_1(S^1, 1)/H_{n+1} \to \pi_1(S^1, 1)/H_n$ induced by $(q_{n,n+1})_\#$ is given by $r_{n,n+1}(z) = p_n \cdot z$. Since all $D_{n,n+1}$, $n \geq n_0$, are pull-back diagrams, Lemma 10 of [10] implies that each $r_{n,n+1}$ is a bijection. Hence, p_n has to be 1 for each $n \geq n_0$ and we get a contradiction. ∎

Note that Corollary 2.8 implies that the solenoid Σ does not admit an infinite-sheeted overlay map with connected total space.

The next lemma asserts that we may confine the sequences of primes related to the construction of solenoids to those with some additional divisibility property. This lemma will be used in the proof of Theorem 3.1 in the next section.

Lemma 2.9. *For any solenoid Σ there exists a sequence $\boldsymbol{P} = \langle p_0, p_1, \ldots \rangle$ of primes such that $\Sigma_{\boldsymbol{P}}$ is homeomorphic to Σ and p_n is prime to $\sum_{i=0}^{n-1} p_i + 1$ for every $n \geq 1$.*

Proof. Let $\boldsymbol{Q} = \langle q_0, q_1, \ldots \rangle$ be the sequence of primes which is related to Σ. We divide our proof into two cases.

Case 1: *Infinitely many different primes appear in $\boldsymbol{Q}$.* By induction on $n \geq 0$, we will define a permutation $\tau : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that each $q_{\tau(n)}$, $n \geq 1$, is prime to $\sum_{k=0}^{n-1} q_{\tau(k)} + 1$. Then the sequence $\boldsymbol{P} = \langle p_0, p_1, \ldots \rangle$, where each $p_n = q_{\tau(n)}$, has both the required properties.

Let $\tau(0) = 0$. Suppose that we have defined $\tau(n)$ so that the condition is satisfied. Let i be the least positive integer such that i does not belong to the image of τ. If q_i does not divide $\sum_{i=0}^{n} q_{\tau(i)} + 1$, then let $\tau(n+1) = i$. Otherwise, we have q_j such that j does not belong to the image of τ and $q_j > \sum_{k=0}^{n} q_{\tau(k)} + 1$. We let $\tau(n+1) = j$. Note that in both cases $q_{\tau(n+1)}$ is prime to $\sigma_{k=0}^{n} q_{\tau(k)} + 1$ and the inductive step is done.

We claim that τ is a bijection. By construction τ is injective. Assume that τ is not surjective. Then we have the least positive integer m which does not belong to the image of τ. Choose n so that $\{0, \ldots, m-1\} \subseteq \{\tau(i) : i < n\}$. Since $m \neq \tau(n)$, q_m divides $\sum_{i=0}^{n-1} q_{\tau(i)} + 1$, but $q_{\tau(n)}$ does not. Hence, q_m does not divide $\sum_{i=0}^{n} q_{\tau(i)} + 1$ and $\tau(n+1)$ should be m, which is a contradiction.

Case 2: *Only finitely many different primes appear in $\boldsymbol{Q}$.* First we delete from $\boldsymbol{Q}$ all those primes which appear in the sequence only finitely many times. We get a sequence $\boldsymbol{Q}'$ and the solenoid $\Sigma_{\boldsymbol{Q}'}$ is homeomorphic to Σ. Note that each prime in $\boldsymbol{Q}'$ appears infinitely many times. Then we define a permutation τ related to $\boldsymbol{Q}'$ in a similar way as above. Let $\tau(0) = 0$. Suppose that we have defined $\tau(n)$ so that the condition is satisfied. Let i be the least positive integer such that i does not belong to the image of τ. If q_i does not divide $\sum_{k=0}^{n} q_{\tau(k)} + 1$, then let $\tau(n+1) = i$. Otherwise, since $q_{\tau(n)}$ does not divide $\sum_{k=0}^{n} q_{\tau(k)} + 1$ and $q_{\tau(n)}$ appears infinitely many times in $\boldsymbol{Q}'$, we can choose $\tau(n+1)$ distinct from $\tau(0), \ldots, \tau(n)$ such that $q_{\tau(n+1)} = q_{\tau(n)}$. Then we can show that $p_n = q_{\tau(n)}$ are as desired, similarly to the first case. ■

REMARK 2.10. Since solenoids are compact topological abelian groups, discrete abelian groups correspond to them by the Pontryagin duality. Solenoids are one-dimensional and connected, hence their Pontryagin duals are isomorphic to subgroups of the rational group $\mathbb{Q}$. The subgroups of $\mathbb{Q}$ are classified into types [4, p. 110] and the classification is essentially the same as that given prior to Lemma 2.9.

3. Infinite-sheeted covering maps over the solenoids $\Sigma_{\boldsymbol{P}}$. In this section we prove the next main theorem.

THEOREM 3.1. *For each solenoid Σ there exists an infinite-sheeted covering map over Σ with connected total space.*

Together with Corollary 2.8 we get the following negative answer to the question stated in the introduction.

COROLLARY 3.2. *For any solenoid there exists a connected covering space which does not admit a topological group structure so that the covering map becomes a homomorphism between topological groups.*

Before proving Theorem 3.1, we give another description of Σ, more suitable for our purpose. Let $\boldsymbol{P}$ be a sequence of primes which is related to Σ. We define a $\boldsymbol{P}$-adic group $\mathbb{J}_{\boldsymbol{P}}$ and a quotient space $\mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]/\sim$ which is homeomorphic to $\Sigma_{\boldsymbol{P}}$. For a nonnegative integer n, i.e. $n < \omega$, let $C_n = \mathbb{Z}/(\prod_{i=0}^{n} p_i)\mathbb{Z}$ and define $h_n : C_{n+1} \to C_n$ by $h_n([u]_{\prod_{i=0}^{n+1} p_i}) = [u]_{\prod_{i=0}^{n} p_i}$ for $u \in \mathbb{Z}$. Then we have an inverse sequence $(C_n, h_n : n < \omega)$ of discrete compact abelian groups. Let $\mathbb{J}_{\boldsymbol{P}}$ be the inverse limit $\varprojlim(C_n, h_n : n < \omega)$. Recall that elements of $\mathbb{J}_{\boldsymbol{P}}$ are all sequences $(u_n : n < \omega) \in \prod_{n<\omega} C_n$ such that $h_n(u_{n+1}) = u_n$ for $n < \omega$. Then $\mathbb{J}_{\boldsymbol{P}}$ is a compact, totally disconnected topological abelian group, where the group operation is coordinatewise addition and the topology is induced from the product topology. The canonical projection from $\mathbb{J}_{\boldsymbol{P}}$ to C_n is denoted by ρ_n, i.e. $\rho_n((u_n : n < \omega)) = u_n$. The notation $\mathbb{J}_{\boldsymbol{P}}$ comes from the p-adic integer group $\mathbb{J}_p$ for a prime p. When our notation starts to be rather complicated, we recommend the reader to replace p_n by the constant prime p and the situation will be clearer.

Let $\mathrm{Seq}(\boldsymbol{P})$ be the set of finite sequence $\boldsymbol{s} = \langle s_0, \ldots, s_{n-1} \rangle$ such that $0 \le s_i < p_i$ for $0 \le i \le n-1$ and let $\mathrm{lh}(\boldsymbol{s})$ be the length of $\boldsymbol{s}$, i.e. n. We use $*$ for concatenation of finite sequences. In particular, we define $\mathbf{0}_n, \boldsymbol{l}_n \in \mathrm{Seq}(\boldsymbol{P})$ as follows: $\mathrm{lh}(\mathbf{0}_n) = \mathrm{lh}(\boldsymbol{l}_n) = n$ and $\mathbf{0}_{n,i} = 0$ and $\boldsymbol{l}_{n,i} = p_i - 1$ for $0 \le i < n$. Since each element of C_n corresponds to a finite sequence $\boldsymbol{s} \in \mathrm{Seq}(\boldsymbol{P})$ of length $n+1$, we identify them. For instance $\boldsymbol{s}+1$ and $\boldsymbol{s}-1$ are elements of C_n and also the corresponding finite sequences. Since $h_n([1]_{\prod_{i=0}^{n+1} p_i}) = [1]_{\prod_{i=0}^{n} p_i}$ for every n, we use the symbol 1 for the elements in $\mathbb{J}_{\boldsymbol{P}}$ and C_n. For $\boldsymbol{s} \in \mathrm{Seq}(\boldsymbol{P})$ with $\mathrm{lh}(\boldsymbol{s}) = n+1$, we define $U_{\boldsymbol{s}} = \{u \in \mathbb{J}_{\boldsymbol{P}} : \rho_n(u) = \boldsymbol{s}\}$. Note that $U_{\boldsymbol{s}}$ are basic open sets of $\mathbb{J}_{\boldsymbol{P}}$.

Denote by $X_n = C_n \times [0, 2\pi]/\sim_n$ the quotient space obtained by the identifications $(u, 0) \sim_n (u-1, 2\pi)$ for $u \in C_n$. Define $\overline{h_n} : X_{n+1} \to X_n$ by $\overline{h_n}((u,\theta)) = (h_n(u), \theta)$ for $(u,\theta) \in X_{n+1}$. Since $(h_n(u), 0) \sim_n (h_n(u) - 1, 2\pi)$ and $h_n(u) + 1 = h_n(u) + h_n(1) = h_n(u+1)$, it follows that $(h_n(u), 0) \sim_n (h_n(u-1), 2\pi)$ and consequently $\overline{h_n}$ is well-defined. Since each X_n is homeomorphic to the unit circle S^1 and $\overline{h_n}$ is a p_{n+1}-sheeted covering map, $\varprojlim(X_n, \overline{h_n} : n < \omega)$ is homeomorphic to $\Sigma_{\boldsymbol{P}}$.

Define $(u, 0) \sim (u-1, 2\pi)$ for $u \in \mathbb{J}_{\boldsymbol{P}}$. Then $(u,0) \sim (u-1, 2\pi)$ if and only if $(\rho_n(u), 0) \sim_n (\rho_n(u) - 1, 2\pi)$ for every n. Hence the quotient space $\mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]/\sim$ is homeomorphic to $\varprojlim(X_n, \overline{h_n} : n < \omega)$ and also to $\Sigma_{\boldsymbol{P}}$.

Proof of Theorem 3.1. We shall define a total space $X_{\boldsymbol{P}}$ to be a quotient space obtained by certain identifications in the countable disjoint union $\bigsqcup_{i=1}^{\infty} Z^i$, where each Z^i is a copy of $\mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]$. Then we have a map $\overline{\sigma} : X_{\boldsymbol{P}} \to \Sigma_{\boldsymbol{P}}$ induced from a natural map $\sigma : \bigsqcup_{i=1}^{\infty} Z^i \to \mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]$, which is an infinite-sheeted covering map. Next, we prove the connectedness of $X_{\boldsymbol{P}}$, where the property of a sequence of primes in Lemma 2.9 actually intervenes.

An element of Z^i which corresponds to $(u, \theta) \in \mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]$ is denoted by $(u,\theta)^i$. We define an identification $\approx$ on $\bigsqcup_{i=1}^{\infty} Z^i$ as follows.

To simplify index sets, let $I_0 = 0$ and $I_n = \sum_{i=0}^{n-1} p_i$ for $n \ge 1$. If k is a positive integer such that $I_n + 1 \le k \le I_{n+1}$, then $0 \le I_{n+1} - k \le p_n - 1$ and $\boldsymbol{l}_n * \langle I_{n+1} - k\rangle \in \mathrm{Seq}(\boldsymbol{P})$. In particular, $\boldsymbol{l}_0 * \langle I_1 - k\rangle = \langle p_0 - k\rangle$.

First, for each $k \ge 1$ and $n \ge 0$ such that $I_n + 1 \le k \le I_{n+1}$, we put $(u, 2\pi)^k \approx (u+1, 0)^{k+1}$ and $(u, 2\pi)^{k+1} \approx (u+1, 0)^k$ for $u \in U_{\boldsymbol{l}_n * \langle I_{n+1} - k\rangle}$. Next, in case neither $(u, 2\pi)^j \approx (u+1, 0)^{j+1}$ nor $(u, 2\pi)^j \approx (u+1, 0)^{j-1}$, we put $(u, 2\pi)^j \approx (u+1, 0)^j$.

We remark the following. The identification rule on the first p_0 copies $Z^1, \ldots, Z^{p_0}$ depends on sequences of length 1, on the next p_1 copies $Z^{p_0+1}, \ldots, Z^{p_0+p_1}$ on sequences of length 2 and so on. Further, for each $(u, 0)^j$ there exists a unique k such that $(u, 0)^j \approx (u-1, 2\pi)^k$ and $k = j-1, j$ or $j+1$.

Before proceeding, let us explain our construction geometrically. We take infinitely many copies of $\Sigma_{\boldsymbol{P}}$, say $\Sigma_{\boldsymbol{P}}^i$. We slit a part of the first copy $\Sigma_{\boldsymbol{P}}^1$ and the corresponding part of the second copy $\Sigma_{\boldsymbol{P}}^2$ and switch the connections.

Except the first copy $\Sigma^1_{\boldsymbol{P}}$ we slit two parts of each $\Sigma^i_{\boldsymbol{P}}$, and by switchings one is connected to $\Sigma^{i-1}_{\boldsymbol{P}}$ and the other is connected to $\Sigma^{i+1}_{\boldsymbol{P}}$. The rule of these slittings is given in the definition of $\approx$. If $I_n+2 \le i \le I_{n+1}$, the sizes of the two parts are the same, but otherwise, i.e. if $i = I_n+1$, the sizes of the slit parts are different. Since each Z^i is a copy of $\mathbb{J}_{\boldsymbol{P}} \times [0,2\pi]$, we have a natural map $\sigma : \bigsqcup_{i=1}^{\infty} Z^i \to \mathbb{J}_{\boldsymbol{P}} \times [0,2\pi]$, which is obviously an infinite sheeted cover over $\mathbb{J}_{\boldsymbol{P}} \times [0,2\pi]$. Let $X_{\boldsymbol{P}} = \bigsqcup_{i=1}^{\infty} Z^i/{\approx}$. Then, via the quotients by $\approx$ and $\sim$ we get the induced map $\overline{\sigma} : X_{\boldsymbol{P}} \to \Sigma_{\boldsymbol{P}}$.

We claim that $\overline{\sigma}$ evenly covers $\Sigma_{\boldsymbol{P}}$. Since $\mathbb{J}_{\boldsymbol{P}} \times (0,2\pi) = \mathbb{J}_{\boldsymbol{P}} \times (0,2\pi)/{\sim}$, there is no difficulty for this case. We need to examine $\mathbb{J}_{\boldsymbol{P}} \times ([0,\pi) \cup (\pi,2\pi])/{\sim}$.

To analyze $\overline{\sigma}^{-1}(\mathbb{J}_{\boldsymbol{P}} \times ([0,\pi) \cup (\pi,2\pi])/{\sim})$, we consider $\sigma^{-1}(\mathbb{J}_{\boldsymbol{P}} \times ([0,\pi) \cup (\pi,2\pi]))$.

First let A_1 be the set

$$(U_{\langle p_0-1\rangle} \times (\pi,2\pi])^2 \cup ((\mathbb{J}_{\boldsymbol{P}} \setminus U_{\langle p_0-1\rangle}) \times (\pi,2\pi])^1 \cup (\mathbb{J}_{\boldsymbol{P}} \times [0,\pi))^1.$$

For $I_n+2 \le k \le I_{n+1}$, let A_k be the set

$$\begin{aligned}(U_{\boldsymbol{l}_n * \langle I_{n+1}-k\rangle} \times (\pi,2\pi])^{k+1} \cup (U_{\boldsymbol{l}_n * \langle I_{n+1}-k+1\rangle} \times (\pi,2\pi])^{k-1} \\ \cup ((\mathbb{J}_{\boldsymbol{P}} \setminus (U_{\boldsymbol{l}_n * \langle I_{n+1}-k\rangle} \cup U_{\boldsymbol{l}_n * \langle I_{n+1}-k+1\rangle})) \times (\pi,2\pi])^k \cup (\mathbb{J}_{\boldsymbol{P}} \times [0,\pi))^k.\end{aligned}$$

For $k = I_n+1$ $(n \ge 1)$, let A_k be

$$\begin{aligned}(U_{\boldsymbol{l}_n * \langle p_n-1\rangle} \times (\pi,2\pi])^{k+1} \cup (U_{\boldsymbol{l}_{n-1} * \langle 0\rangle} \times (\pi,2\pi])^{k-1} \\ \cup ((\mathbb{J}_{\boldsymbol{P}} \setminus (U_{\boldsymbol{l}_n * \langle p_n-1\rangle} \cup U_{\boldsymbol{l}_{n-1} * \langle 0\rangle})) \times (\pi,2\pi])^k \cup (\mathbb{J}_{\boldsymbol{P}} \times [0,\pi))^k.\end{aligned}$$

We remark that the restriction of $\overline{\sigma}$ to $A_k/{\approx}$ is a homeomorphism onto $\mathbb{J}_{\boldsymbol{P}} \times ([0,\pi) \cup (\pi,2\pi])/{\sim}$. Since $\sigma^{-1}(\mathbb{J}_{\boldsymbol{P}} \times [0,\pi) \cup (\pi,2\pi])$ is the disjoint union of A_k's, $\overline{\sigma}$ evenly covers $(\mathbb{J}_{\boldsymbol{P}} \times [0,\pi) \cup (\pi,2\pi])/{\sim}$ and we conclude that $\overline{\sigma}$ is an infinite sheeted covering map.

Showing the connectedness of $X_{\boldsymbol{P}}$ is a delicate and long part of this proof. First we define some connection between subsets $(U_{\boldsymbol{s}} \times \{0\})^m$ of $\bigsqcup_{i=1}^{\infty} Z^i$.

Let $m : \boldsymbol{s}$ denote the subset $(U_{\boldsymbol{s}} \times \{0\})^m$. We call $n : \boldsymbol{t}$ a *successor* of $m : \boldsymbol{s}$ if $\boldsymbol{t} = \boldsymbol{s}+1$ and, for each $u \in U_{\boldsymbol{s}}$, $(u,2\pi)^m \approx (u+1,0)^n$. Note that $m : \boldsymbol{s}$ may not have a successor, but there is at most one successor. However, if the length of $\boldsymbol{s}$ is larger than m, then $m : \boldsymbol{s}$ has its successor. Here we give some examples. The successor of $1 : \langle p_0-1,0\rangle$ is $2 : \langle 0,1\rangle$, the successor of $2 : \langle p_0-1,0\rangle$ is $1 : \langle 0,1\rangle$, the successor of $p_0+1 : \langle p_0-1,p_1-1,0\rangle$ is $p_0+2 : \langle 0,0,1\rangle$, the successor of $p_0 : \langle p_0-1,0,1\rangle$ is $p_0 : \langle 0,1,1\rangle$. But $p_0+1 : \langle p_0-1\rangle$ has no successor.

If there exist $n_i : \boldsymbol{t}_i$ $(0 \le i \le k)$ such that $m = n_0$ and $\boldsymbol{s} = \boldsymbol{t}_0$, $n = n_k$ and $\boldsymbol{t} = \boldsymbol{t}_k$, and each $n_{i+1} : \boldsymbol{t}_{i+1}$ is a successor of $n_i : \boldsymbol{t}_i$, we call $n : \boldsymbol{t}$ the *kth successor* of $m : \boldsymbol{s}$ and the related chain $(n_i : \boldsymbol{t}_i \mid 0 \le i \le k)$ a *path.*

In this terminology a successor of $m : \boldsymbol{s}$ is the first successor of $m : \boldsymbol{s}$. The subset $m : \boldsymbol{s}$ is a starting 0-position and $n : \boldsymbol{t}$ is a final k-position of the path $(n_i : \boldsymbol{t}_i \mid 0 \leq i \leq k)$. Since points in $n_i : \boldsymbol{t}_i$ are connected by paths in $X_{\boldsymbol{P}}$ to points in $n_{i+1} : \boldsymbol{t}_{i+1}$, points in $m : \boldsymbol{s}$ are connected by paths in $X_{\boldsymbol{P}}$ to points in $n : \boldsymbol{t}$. Taking the successor of a position $m : \boldsymbol{s}$ is called a *step*. Hence, starting from $m : \boldsymbol{s}$, after k steps we reach the kth successor of $m : \boldsymbol{s}$. An example of a path with the starting position $p_0 + 1 : \langle 0, 0\rangle$ and the final position $1 : \langle 1, 1\rangle$ is the following:

$$p_0 + 1 : \langle 0, 0\rangle, p_0 : \langle 1, 0\rangle, p_0 - 1 : \langle 2, 0\rangle, \ldots, 2 : \langle p_0 - 1, 0\rangle, 1 : \langle 0, 1\rangle, 1 : \langle 1, 1\rangle.$$

It is possible to give a certain geometrical meaning to a successor and a step. For this purpose we use points in $\mathbb{J}_{\boldsymbol{P}} \times [0, 2\pi]$ and $\bigsqcup_{i=1}^{\infty} Z^i$ to express points in $\Sigma_{\boldsymbol{P}}$ and $X_{\boldsymbol{P}}$ respectively. One round in $\Sigma_{\boldsymbol{P}}$ corresponds to $+1$ or -1 in $\mathbb{J}_{\boldsymbol{P}}$. We fix a direction such that a clockwise round corresponds to $+1$. Now we consider points $(u, 0) \in \Sigma_{\boldsymbol{P}}$ and $(u, 0)^m \in X_{\boldsymbol{P}}$. Since $\overline{\sigma}$ is a covering map, a clockwise round from $(u, 0)$ to $(u + 1, 0)$ by a path is lifted to a path from $(u, 0)^m$ to $(u + 1, 0)^k$ for some k. This k may be $m - 1$, m or $m + 1$. If $v \in U_{\boldsymbol{s}}$ and a clockwise round from $(v, 0)$ to $(v+1, 0)$ is lifted to the path from $(v, 0)^m$ to $(v+1, 0)^k$ for every $v \in U_{\boldsymbol{s}}$, $k : \boldsymbol{s}+1$ is the successor of $m : \boldsymbol{s}$. This is a step from $m : \boldsymbol{s}$ to $k : \boldsymbol{s} + 1$ and clockwise rounds correspond to steps.

Claim. *Let $\boldsymbol{P}$ be a sequence of primes which satisfies the property in Lemma 2.9. Then for each $n \geq 0$ the following holds:*

($*_n$) $\sum_{i=0}^{n-1} p_i + 2 : \mathbf{0}_{n+1}$ *is the* $((\prod_{i=0}^{n} p_i)(\sum_{i=0}^{n-1} p_i + 1))$*th successor of* $\sum_{i=0}^{n-1} p_i + 1 : \mathbf{0}_{n+1}$ *and* $k : \boldsymbol{x}$ *appears on that path for any* $k \leq \sum_{i=0}^{n-1} p_i + 1$ *and any* $\boldsymbol{x} \in \mathrm{Seq}(\boldsymbol{P})$ *having* $\mathrm{lh}(\boldsymbol{x}) = n + 1$.

We prove the claim by induction on $n \geq 0$. First we show $(*_0)$. We have a path $1 : \langle 0\rangle$, $1 : \langle 1\rangle$, $1 : \langle 2\rangle, \ldots, 1 : \langle p_0 - 1\rangle$, $2 : \langle 0\rangle$. Hence, $2 : \langle 0\rangle$ is the p_0th successor of $1 : \langle 0\rangle$ and for each x, $0 \leq x \leq p_0 - 1$, $1 : \langle x\rangle$ appears on that path. Thus $(*_0)$ is proven.

Now suppose that $(*_n)$ holds. The 0th position is $\sum_{i=0}^{n} p_i + 1 : \mathbf{0}_{n+2}$. The $(\prod_{i=0}^{n-1} p_i - 1)$th successor is $\sum_{i=0}^{n} p_i + 1 : (\boldsymbol{l}_n * \langle 0, 0\rangle)$ and its successor is $\sum_{i=0}^{n} p_i : (\mathbf{0}_n * \langle 1, 0\rangle)$. For each $\sum_{i=0}^{n-1} p_i + 2 \leq k \leq \sum_{i=0}^{n} p_i + 1$ we count $\prod_{i=0}^{n-1} p_i$ steps and reach $\sum_{i=0}^{n-1} p_i + 1 : (\mathbf{0}_n * \langle 0, 1\rangle)$ as the $(p_n \cdot \prod_{i=0}^{n-1} p_i)$th successor. Then, by induction hypothesis, we have $\sum_{i=0}^{n-1} p_i + 2 : \mathbf{0}_{n+1} * \langle [2]_{p_{n+1}}\rangle$ as the $(p_n \prod_{i=0}^{n-1} p_i + \prod_{i=0}^{n} p_i(\sum_{i=0}^{n-1} p_i + 1))$th successor.

Then, we count $(p_n - 1) \prod_{i=0}^{n-1} p_i$ steps for each $\sum_{i=0}^{n-1} p_i + 2 \leq k \leq \sum_{i=0}^{n} p_i$ and we have $\sum_{i=0}^{n} p_i + 1 : (\mathbf{0}_n * \langle 1, a - 1\rangle)$ as the Sth successor, where

$$S = (p_n-1)(p_n-1)\prod_{i=0}^{n-1} p_i + p_n \prod_{i=0}^{n-1} p_i + \Big(\prod_{i=0}^{n} p_i\Big)\Big(\sum_{i=0}^{n-1} p_i + 1\Big)$$
$$= -(p_n-1)\prod_{i=0}^{n-1} p_i + \Big(\prod_{i=0}^{n} p_i\Big)\Big(\sum_{i=0}^{n} p_i + 1\Big)$$

and $a = [\sum_{i=0}^n p_i + 1]_{p_{n+1}}$.

Hence $\sum_{i=0}^n p_i + 1 : \mathbf{0}_n * \langle 0, a\rangle$ is the $((p_n-1)\prod_{i=0}^{n-1} p_i)$th successor of $\sum_{i=0}^n p_i + 1 : \mathbf{0}_n * \langle 1, 0\rangle$, so that $\sum_{i=0}^n p_i + 1 : \mathbf{0}_n * \langle 0, a\rangle$ is the $(\prod_{i=0}^n p_i(\sum_{i=0}^n p_i + 1))$th successor of $\sum_{i=0}^n p_i + 1 : \mathbf{0}_{n+2}$.

By our assumption on $\boldsymbol{P}$ we have $0 < a < p_{n+1}$. We remark that the $n+1$-digit varies, where the *i-digit* of $\langle s_0, \dots, s_{n+1}\rangle$ is s_i, before we reach $\sum_{i=0}^n p_i + 1 : \mathbf{0}_n * \langle 1, a\rangle$, but the $n+1$-digit possibly affects successors on the path only when we are in $\sum_{i=0}^n p_i + 1 : \boldsymbol{s}$ for some $\boldsymbol{s}$.

Then we continue similarly and as the $(2\prod_{i=0}^n p_i(\sum_{i=0}^n p_i + 1))$th successor we have $\sum_{i=0}^n p_i + 1 : \mathbf{0}_n * \langle 0, [2a]_{p_{n+1}}\rangle$. Since $0 < a < p_{n+1}$, for $0 < k < p_{n+1}$ we have $[ka]_{p_{n+1}} \neq 0$ and certainly have $[p_{n+1}a]_{p_{n+1}} = 0$. This means that as the $(p_{n+1}\prod_{i=0}^n p_i(\sum_{i=0}^n p_i + 1) - 1)$th successor we have $\sum_{i=0}^n p_i + 1 : \boldsymbol{l}_{n+2}$ and as the $(p_{n+1}\prod_{i=0}^n p_i(\sum_{i=0}^n p_i + 1))$th successor we have $\sum_{i=0}^n p_i + 2 : \mathbf{0}_{n+2}$.

Since the successor is determined by a position and the operation of taking the successor is invertible and, in addition, we have counted $\prod_{i=0}^{n+1} p_i \cdot (\sum_{i=0}^n p_i + 1)$ steps and have the new position $\sum_{i=0}^n p_i + 2 : \mathbf{0}_{n+2}$, every $k : \boldsymbol{x}$ appears on the way for $k \le \sum_{i=0}^n p_i + 1$ and $\boldsymbol{x} \in \mathrm{Seq}(\boldsymbol{P})$ with $\mathrm{lh}(\boldsymbol{x}) = n+2$. We have shown $(*_n)$ and have proved the claim.

Finally we show the connectedness of $X_{\boldsymbol{P}}$. Without loss of generality, we may assume that $\boldsymbol{P}$ has the property in Lemma 2.9. Assume that there is a non-trivial clopen set W in $X_{\boldsymbol{P}}$. Then we have a basic set $(U_{s_0} \times \{0\})^{n_0}$ in W and another basic set $(U_{s_1} \times \{0\})^{n_1}$ in its complement $X_{\boldsymbol{P}} \setminus W$. Take a sufficiently large n such that $\mathrm{lh}(s_0), \mathrm{lh}(s_1) \le n+1$ and $n_0, n_1 \le \sum_{i=0}^{n-1} p_i + 1$ and extend s_0 and s_1 to sequences s_0^* and s_1^* of length $\mathrm{lh}(s_0^*) = \mathrm{lh}(s_1^*) = n+1$. Then $(*_n)$ implies the existence of a path between $n_0 : s_0^*$ and $n_1 : s_1^*$, which means that there is an arc in $X_{\boldsymbol{P}}$ connecting a point in $(U_{s_0} \times \{0\})^{n_0}$ to a point in $(U_{s_1} \times \{0\})^{n_1}$. This contradiction proves Theorem 3.1.

Remark 3.3. Finally, let us remark that every half-line contained in a solenoid $\Sigma_{\boldsymbol{P}}$ is dense in $\Sigma_{\boldsymbol{P}}$, but this does not hold for $X_{\boldsymbol{P}}$. We show this using the proof of $(*_n)$. Let $u \in \mathbb{J}_{\boldsymbol{P}}$ be the element defined by $\rho_n(u) = \sum_{i=1}^n \prod_{j=0}^{i-1} p_j$. Define $\boldsymbol{s}_{n+1}$ to be the first position of the form $1 : \boldsymbol{s}_{n+1}$ starting from $\sum_{i=0}^{n-1} p_i + 1 : \mathbf{0}_{n+1}$. Then, by $(*_i)$ for $0 \le i \le n$ we have $\boldsymbol{s}_{n+1,i} = 1$ for $i \ge 1$ and $\boldsymbol{s}_{n+1,0} = 0$ and consequently $u \in \bigcap_{n=0}^\infty U_{\boldsymbol{s}_{n+1}}$. Considering the half-line from $(u, 0)$ tracing back steps, we see that this half-line intersects $(U_{\mathbf{0}_{n+1}})^{\sum_{i=0}^{n} p_i}$, but does not intersect $(U_{\langle 1\rangle})^1$. On the other hand every line in $X_{\boldsymbol{P}}$ is dense in $X_{\boldsymbol{P}}$ as in the case of a solenoid $\Sigma_{\boldsymbol{P}}$. To see this, since every line in $X_{\boldsymbol{P}}$ contains a point $(u,0)^{i_0}$ we fix such a point. Every open set in $X_{\boldsymbol{P}}$ contains a subset of the form $(U_{\boldsymbol{s}} \times (\alpha,\beta))^{j_0}$. By extending $\boldsymbol{s}$ we may assume $i_0, j_0 \le \sum_{i=0}^{n-1} p_i + 1$ for $n+1 = \mathrm{lh}(\boldsymbol{s})$. Applying $(*_n)$, we can see that the line containing $(u,0)^{i_0}$ intersects $(U_{\mathbf{0}_{n+1}} \times \{0\})^{\sum_{i=0}^{n-1} p_i+1}$ and consequently intersects $(U_{\boldsymbol{s}} \times \{0\})^{j_0}$ and $(U_{\boldsymbol{s}} \times (\alpha,\beta))^{j_0}$.

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