FUNDAMENTA
MATHEMATICAE
219 (2012)

On hereditarily normal topological groups

by

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Abstract. We investigate hereditarily normal topological groups and their subspaces. We prove that every compact subspace of a hereditarily normal topological group is metrizable. To prove this statement we first show that a hereditarily normal topological group with a non-trivial convergent sequence has G_δ-diagonal. This implies, in particular, that every countably compact subspace of a hereditarily normal topological group with a non-trivial convergent sequence is metrizable. Another corollary is that under the Proper Forcing Axiom, every countably compact subspace of a hereditarily normal topological group is metrizable.

1. Introduction. It is a known fact that a hereditarily normal compact topological group is metrizable. This fact follows from the theorem of R. Engelking [5] that every compact topological group contains a subspace homeomorphic to $\{0,1\}^\tau$, where τ is the weight of the group. It is also a corollary to deep work of other mathematicians. For a proof of this theorem and historical development around it we refer to [1, Theorem 4.2.1] and [9].

In this paper we show that not only compact hereditarily normal topological groups are metrizable, but any compact subset of any hereditarily normal topological group is metrizable as well (Theorem 2.9). Thus, in the class of compact spaces, only metrizable ones can be embedded into hereditarily normal topological groups.

After it was established that every compact hereditarily normal topological group is metrizable it was natural to wonder if compactness could be relaxed to countable compactness. The example of Hajnal and Juhász in [6] closed this door in ZFC. More precisely, assuming the Continuum Hypothesis, Hajnal and Juhász constructed a hereditarily normal hereditarily separable countably compact topological group which is not compact, hence not metrizable. The example has many additional features and one of them is that it lacks non-trivial convergent sequences. In [3, Theorem 3],

2010 *Mathematics Subject Classification*: Primary 54H11, 22A05, 54D15.
Key words and phrases: topological group, hereditary normal space.

DOI: 10.4064/fm219-3-3 © Instytut Matematyczny PAN, 2012

T. Eisworth showed that this feature is not an accessory but a necessity. Eisworth's result is a corollary to one of our main theorems. Namely, we prove that every hereditarily normal topological group with a non-trivial convergent sequence has a G_δ-diagonal (Theorem 2.3). This result and the theorem of Chaber [2] imply that every countably compact subset of a hereditarily normal topological group with a non-trivial convergent sequence is metrizable. In [8], Nyikos, Soukup, and Veličković proved that under the Proper Forcing Axiom, every countably compact hereditarily normal space is sequentially compact. This result and our Theorem 2.3 imply that under the Proper Forcing Axiom, every countably compact subset of a hereditarily normal topological group is metrizable. This, in its turn, implies another earlier result of Eisworth [3, Corollary 10] that under PFA, every countably compact hereditarily normal topological group is metrizable. We would like to mention that this work was inspired by the well-known necessity condition of Katětov for hereditary normality of the product of two spaces [7, Theorem 1].

In notation and terminology we will follow [4]. We reserve the symbol $\star$ for the binary group operation of a group G, and the letter e for the neutral element of G. Following a group theory convention, we will omit the group binary operation symbol in standard situations. In particular, for elements $a, b \in G$ we will write ab instead of $a \star b$. However, in a few places in this paper, the use of the symbol $\star$ will be necessary for the sake of clarity and in some cases to stress the relation of Katětov's argument to our work.

A space X has a *G_δ-diagonal* if the diagonal $\{\langle x, x\rangle : x \in X\}$ is the intersection of a countable family of its open neighborhoods in $X \times X$. A *non-trivial convergent sequence* is a space homeomorphic to the subspace of the reals $\{0\} \cup \{1/n : n = 1, 2, \ldots\}$. Since we will often switch from a given space to its subspaces and vice versa we agree that when dealing with a space X, its subspace Y, and subsets $S \subset X$ and $P \subset Y$, we denote by $\bar{S}$ the closure of S in X and by $\mathrm{cl}_Y(P)$ the closure of P in Y. All spaces are assumed to be T_1.

2. Results. For our argument we will need the following extract from Katětov's proof of his theorem in [7]. For convenience, we also give a sketch of Katětov's proof with some notational changes that fit our goal.

THEOREM 2.1 (Katětov [7, extract from Theorem 1]). *Let T be a topological space and let $t \in T$ have uncountable pseudocharacter in T. Also let S be a separable topological space and let $s \in S$ be a limit point of S. Then $A = [S \times \{t\}] \setminus \{\langle s, t\rangle\}$ and $B = [\{s\} \times T] \setminus \{\langle s, t\rangle\}$ are closed and disjoint sets in $Z = [S \times T] \setminus \{\langle s, t\rangle\}$ that cannot be separated by open neighborhoods in Z.*

Proof. (Follows Katětov's argument.) Closedness and disjointness are clear. Let U be a neighborhood of A in Z. We need to show that $\mathrm{cl}_Z(U)$ meets B. Fix a countable set $D \subset S$ which is dense in S. For each $x \in D \setminus \{s\}$, fix a neighborhood U_x of t in T such that $\{x\} \times U_x \subset U$. Since t has uncountable pseudocharacter, we conclude that there exists $y \in \bigcap\{U_x : x \in D \setminus \{s\}\}$ distinct from t. This means that $[D \setminus \{s\}] \times \{y\}$ is in U. Since D is dense in S, we conclude that $\langle s, y\rangle \in \mathrm{cl}_Z(U) \cap B$. ∎

Lemma 2.2 (Folklore). *Folklore Let G be a topological group. If the diagonal $\{\langle g, g\rangle : g \in G\}$ has uncountable pseudocharacter in $G \times G$, then the neutral element e of G has uncountable pseudocharacter in G.*

Proof. The conclusion follows from the fact that $\star^{-1}(e) = \{\langle g, g^{-1}\rangle : g \in G\}$ and the set on the right has the same pseudocharacter in $G \times G^{-1}$ as the diagonal $\{\langle g, g\rangle : g \in G\}$ in $G \times G$. ∎

Theorem 2.3. *A hereditarily normal topological group with a non-trivial convergent sequence has G_δ-diagonal.*

Proof. Let G be a topological group with a non-trivial convergent sequence. Assume that the diagonal $\{\langle g, g\rangle : g \in G\}$ is not a G_δ-set in $G \times G$. We need to show that G is not hereditarily normal. By Lemma 2.2, the pseudocharacter of the neutral element e is uncountable. Let $\{e_n : n \in \omega\}$ be a sequence that converges to e such that $e_n \neq e$ for every $n \in \omega$. Such a sequence exists due to homogeneity of G and the theorem's hypothesis. For every $n \in \omega$, select an open neighborhood U_n of e whose closure does not meet $\{e_n, e_n^{-1}\}$. Put $T = \bigcap_{n\in\omega} \overline{U}_n$. Since the pseudocharacter of e is uncountable, we conclude that e is a limit point for the closed set T. Since T is a G_δ-set in G and $\{e\}$ is not, we conclude that e has uncountable pseudocharacter in T. Put $S = \{e\} \cup \{e_n : n \in \omega\}$. The following three sets are the key objects for the remainder of our argument:

$$Z = (S \times T) \setminus \{\langle e, e\rangle\}, \quad A = \{\langle e_n, e\rangle : n \in \omega\}, \quad B = \{\langle e, y\rangle : y \in T \setminus \{e\}\}.$$

Since e has uncountable pseudocharacter in T, by Katětov's theorem (Theorem 2.1), A and B are closed and disjoint subsets of Z that cannot be separated by open sets in Z. To finish the proof, it suffices to show that $\star(A) = \{e_n : n \in \omega\}$ and $\star(B) = T \setminus \{e\}$ are closed and disjoint subsets of $\star(Z) = ST \setminus \{e\}$.

Let us first prove that $\{e_n : n \in \omega\}$ is closed in $ST \setminus \{e\}$. For this observe that e is the only limit point of $\{e_n : n \in \omega\}$ that does not belong to $\{e_n : n \in \omega\}$. Since e does not belong to $ST \setminus \{e\}$ either, we conclude that $\{e_n : n \in \omega\}$ is closed in $ST \setminus \{e\}$. The proof that $T \setminus \{e\}$ is closed in $ST \setminus \{e\}$ is analogous.

Now let us show that $\{e_n : n \in \omega\}$ and $T \setminus \{e\}$ are disjoint. For this recall that $T = \bigcap_{n\in\omega} \overline{U}_n$, where $\overline{U}_n$ misses $\{e_n\}$ for each n. ∎

Chaber proved in [2] that a countably compact space with G_δ-diagonal is metrizable. Chaber's result and Theorem 2.3 imply the following statements.

Corollary 2.4. *Every countably compact subspace of a hereditarily normal topological group that contains a non-trivial convergent sequence is metrizable.*

Corollary 2.5 ([3, Corollary 10]). *Every countably compact hereditarily normal topological group that contains a non-trivial convergent sequence is metrizable.*

In [8], Nyikos, Soukup, and Veličković proved that under the Proper Forcing Axiom, every countably compact hereditarily normal space is sequentially compact. This theorem and Corollary 2.4 imply the following statement.

Corollary 2.6. *Assume the Proper Forcing Axiom. Then every countably compact subspace of a hereditarily normal topological group is metrizable.*

Next we shall prove that every compact subset of a hereditarily normal topological group is metrizable. We start with the following lemma.

Lemma 2.7. *Let G be a hereditarily normal topological group and let S and T be its compact subspaces. Suppose that S is separable, s is a limit point of S, and $t \in T$ has uncountable character in T. Then there exists a compactum $C \subset T$ such that t has uncountable character in C and $sC \subset St$.*

Proof. Let us show that $C = \{g \in T : sg \in St\}$ is as desired. Since S and T are compact, we conclude that C is compact. Clearly $t \in C$. It is left to show that t has uncountable character in C. We assume the contrary. Then there exists a countable family $\{U_n : n \in \omega\}$ of neighborhoods of t in T such that $F = \bigcap\{\overline{U}_n : n \in \omega\}$ misses $C \setminus \{t\}$. Since $\chi(t, T)$ is uncountable, $\chi(t, F)$ is uncountable as well. To reach a contradiction it suffices to find $g \in F \setminus \{t\}$ such that $sg \in St$.

For this put $A = [S \times \{t\}] \setminus \{\langle s, t\rangle\}$ and $B = [\{s\} \times F] \setminus \{\langle s, t\rangle\}$. By Katětov's theorem (Theorem 2.1), the sets A and B are closed and disjoint subsets of $Z = [S \times F] \setminus \{\langle s, t\rangle\}$ that cannot be separated by disjoint open neighborhoods in Z. Put $Z_1 = Z \setminus \star^{-1}(st)$. Clearly Z_1 is open in Z and contains both A and B. Therefore A and B cannot be separated by open neighborhoods in Z_1 either. Since G is hereditarily normal, the closures of $\star(A) = St \setminus \{st\}$ and $\star(B) = sF \setminus \{st\}$ in $\star(Z_1) \subset G \setminus \{st\}$ must meet. Since St is compact and $st \notin G \setminus \{st\}$, we conclude that $St \setminus \{st\}$ is closed in $G \setminus \{st\}$. The proof that $sF \setminus \{st\}$ is closed in $G \setminus \{st\}$ is analogous. Since $St \setminus \{st\}$ and $sF \setminus \{st\}$ are closed in $G \setminus \{st\}$, we conclude that they meet. Therefore, we can find $g \in F \setminus \{t\}$ such that $sg \in St$. ∎

We are now ready to prove our main result. For reference we will first formulate an often-used corollary from the following well-known result of Shapirovskiĭ [10, Theorem 2], in which a *π-base at $x \in X$* is a collection $\mathcal{U}$ of non-void open subsets of X such that every open neighborhood of x in X contains an element of $\mathcal{U}$.

THEOREM 2.8 (Shapirovskiĭ, [10, Corollary to Theorem 2]). *Any hereditarily normal compact space has a point with countable π-base.*

THEOREM 2.9. *Every compact subset of a hereditarily normal topological group is metrizable.*

Proof. Let G be a hereditarily normal topological group and let X be its compact subset. We may assume that X is infinite. By virtue of Theorem 2.3 it suffices to find a non-trivial convergent sequence in G. Assume that no such sequences exist in G.

CLAIM 1. *If Z is an infinite compact subspace of G then its derived set Z' has no isolated points.*

If Z' has an isolated point p then p is the limit of a convergent sequence from $Z \setminus Z'$. Since no such sequences exist, the claim is proved.

CLAIM 2. *There exist separable compact subsets A and B of G such that $A \cap B = \{e\}$ and e is a limit point for both A and B.*

Let Z be an infinite separable compact subset of X. By Claim 1, the derived set Z' is an infinite compactum without isolated points. By Shapirovskiĭ's theorem there exists an element in Z' that has a countable π-base in Z'. By homogeneity of G, we can find such Z with an additional requirement that the neutral element e is in Z' and has a countable π-base in Z'. Fix a collection $\{P_n : n \in \omega\}$ of subsets of Z' that form a π-base at e in Z'. Put $F_n = \overline{P}_n \overline{P}_n^{-1}$.

SUBCLAIM. *$\{F_n : n \in \omega\}$ is a network at e consisting of compact sets, and e is a limit point for every F_n.*

The set F_n is compact because $\overline{P}_n$ is a closed subset of Z', which is compact. Thus, F_n is the image of a compactum under the continuous map $\star$. To show that F_n's form a network, fix any neighborhood U at e. One can find an open neighborhood V of e such that $VV^{-1} \subset U$. Then there exists an element P_n of our π-base such that $\overline{P}_n \subset V$. We then have $e \in F_n = \overline{P}_n \overline{P}_n^{-1} \subset V_n V_n^{-1} \subset U$.

Finally, to show that e is a limit point for F_n, recall that P_n is a non-void open set of Z' and Z' has no isolated points. Pick any $p \in P_n$. Then $P_n p^{-1}$ contains e and is a subset of F_n. Clearly e is a limit point for $P_n p^{-1}$. The proof of the Subclaim is complete.

We are now ready to construct the desired sets A and B. For this we consider two cases.

Case 1: $F_0 \cap \cdots \cap F_n \neq \{e\}$ *for all* $n \in \omega$. Then for each $n \in \omega$, we can pick $x_n \in F_0 \cap \cdots \cap F_n$ distinct from e. Clearly, $x_n \to e$, which contradicts our assumption that G has no non-trivial convergent sequences.

Case 2: *Negation of Case 1.* Then there exists the smallest n such that e is not a limit point for $F_0 \cap \cdots \cap F_{n+1}$. Hence there exists a neighborhood U of e such that $\overline{U} \cap F_0 \cap \cdots \cap F_{n+1} = \{e\}$. Put $C = \overline{U} \cap F_0 \cap \cdots \cap F_n$ and $D = \overline{U} \cap F_{n+1}$. We have $C \cap D = \{e\}$. By the property of n and Subclaim, e is a limit point for both C and D. To finish the proof of existence of sets with the desired properties it suffices to place C and D in separable compact subsets of G whose only common element is e. For this recall that $F_0, \dots, F_{n+1} \subset ZZ^{-1}$ and Z is a separable compactum. Further, $(ZZ^{-1}) \setminus \{e\}$ is normal. Therefore, there exist open neighborhoods V and W of C and D, respectively, in ZZ^{-1} whose closures in ZZ^{-1} have only one point in common, namely, e. Put $A = \overline{V}$ and $B = \overline{W}$. Since Z is separable, A and B are separable. The rest of the desired properties are obvious. This completes Case 2 and proves the claim.

Let A and B be as in Claim 2. Recall that our assumption is that G does not have any non-trivial convergent sequence. Therefore, the fact that e is a limit point of the compact sets A and B implies that e has uncountable character both in A and in B.

We will finish our argument by two consecutive applications of Lemma 2.7 as follows: Put $S = A$, $T = A$, and $s = t = e$. The sets S and T together with the points s and t satisfy the hypothesis of Lemma 2.7. Therefore, there exists a compact set $A_1 \subset A$ such that:

(1) $t = e$ has uncountable character in A_1.
(2) $sA_1 \subset St$.

Note that (2) implies

(3) $A_1 \subset A$.

Next, for our second application of Lemma 2.7 we put $S = B$, $T = A_1$, and $s = t = e$. By (1) and the properties of B listed in Claim 2, these sets and points satisfy the hypothesis of Lemma 2.7. Therefore, there exists a compact set A_2 such that:

(4) $A_2 \subset A_1$.
(5) $t = e$ has uncountable character in A_2.
(6) $sA_2 \subset St$.

Note that (6) implies

(7) $A_2 \subset B$.

By (5), there exists $a \in A_2$ distinct from e. By (7), $a \in B$. By (3) and (4), $a \in A$. Therefore, $a \in A \cap B$ and is distinct from e, which contradicts the fact that $A \cap B = \{e\}$. This contradiction completes the proof. ■

Observe that the first paragraph of the proof of Theorem 2.9 suggests rephrasing the theorem in a way that is more descriptive of the internal structure (with regard to convergence) of hereditarily normal topological groups.

THEOREM 2.10. *Let G be a hereditarily normal topological group. Then either G has a non-trivial convergent sequence and a G_δ-diagonal, or G has no non-trivial convergent sequences and every compact subset of G is finite. In either case, every compact subset of G is metrizable.*

Acknowledgments. The author would like to thank the referee for many helpful remarks, corrections, and suggestions.

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Received 3 April 2012;
in revised form 10 September 2012