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# The growth rate and dimension theory of beta-expansions

by

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**Abstract.** In a recent paper of Feng and Sidorov they show that for $\beta \in (1,(1+\sqrt{5})/2)$ the set of $\beta$-expansions grows exponentially for every $x \in (0,1/(\beta-1))$. In this paper we study this growth rate further. We also consider the set of $\beta$-expansions from a dimension theory perspective.

**1. Introduction.** Let $1<\beta<2$ and $I_\beta=[0,1/(\beta-1)]$. Each $x \in I_\beta$ can be expressed as

$$x=\sum_{n=1}^{\infty}\frac{\epsilon_n}{\beta^n}$$

for some $(\epsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. We call such a sequence a $\beta$-*expansion* for $x$. We define

$$\Sigma_\beta(x)=\left\{(\epsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : \sum_{n=1}^{\infty}\frac{\epsilon_n}{\beta^n}=x\right\}.$$

In [2] it is shown that for $\beta \in (1,(1+\sqrt{5})/2)$ and $x \in (0,1/(\beta-1))$ the set $\Sigma_\beta(x)$ is uncountable. In [6, 7] Sidorov considers the case where $\beta \in [(1+\sqrt{5})/2,2)$. He shows that for Lebesgue almost every $x \in I_\beta$ the set $\Sigma_\beta(x)$ is uncountable.

To describe the growth rate of $\beta$-expansions we set

$$\mathcal{E}_k(x,\beta)=\left\{(\epsilon_1,\ldots,\epsilon_k) \in \{0,1\}^k : \exists(\epsilon_{k+1},\epsilon_{k+2},\ldots) \in \{0,1\}^{\mathbb{N}}, \sum_{n=1}^{\infty}\frac{\epsilon_n}{\beta^n}=x\right\}.$$

Each element of $\mathcal{E}_k(x,\beta)$ will be called a $k$-*prefix* for $x$. Moreover, we let

$$\mathcal{N}_k(x,\beta)=\#\mathcal{E}_k(x,\beta)$$

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and define the *growth rate* of $\mathcal{N}_k(x,\beta)$ to be

$$\lim_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k},$$

when this limit exists. When the limit does not exist we can consider the *lower* and *upper growth rates* of $\mathcal{N}_k(x,\beta)$, defined to be

$$\liminf_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} \quad \text{and} \quad \limsup_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k}$$

respectively.

In this paper we also consider $\Sigma_\beta(x)$ from a dimension theory perspective. We endow $\{0,1\}^{\mathbb{N}}$ with the metric $d(\cdot,\cdot)$ defined as follows:

$$d(x,y) = \begin{cases} 2^{-n(x,y)} & \text{if } x \neq y, \text{ where } n(x,y) = \inf\{i : x_i \neq y_i\}, \\ 0 & \text{if } x = y. \end{cases}$$

We consider the Hausdorff dimension of $\Sigma_\beta(x)$ with respect to this metric. It is a simple exercise to show that

$$\dim_{\mathrm{H}}(\Sigma_\beta(x)) \leq \liminf_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} \leq \limsup_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k}. \tag{1}$$

In [4] the following theorem was shown to hold.

Theorem 1.1. *Let $\beta \in (1,(1+\sqrt{5})/2)$. Then*

$$\liminf_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} \geq \kappa(\beta) > 0 \quad \textit{for all } x \in \left(0, \frac{1}{\beta-1}\right),$$

*where $\kappa(\beta)$ is given explicitly by the formula*

$$\kappa(\beta) = \begin{cases} \frac{1}{2}\left(\left[\log_\beta\left(\frac{\beta^2-1}{1+\beta-\beta^2}\right)\right]+1\right)^{-1} & \textit{if } \beta > \sqrt{2}, \\ \frac{1}{2}\left(\left[\log_\beta\left(\frac{1}{\beta-1}\right)\right]+1\right)^{-1} & \textit{if } \beta \leq \sqrt{2}. \end{cases}$$

The growth rate of $\mathcal{N}_k(x,\beta)$ is addressed from the measure-theoretic point of view in [5]. The following result is implicit there.

Theorem 1.2. *For almost every $\beta \in (1,2)$ and almost every $x \in I_\beta$,*

$$\limsup_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} = \log_2\left(\frac{2}{\beta}\right).$$

*Moreover, for almost every $\beta \in (1,\sqrt{2})$ and almost every $x \in I_\beta$,*

$$\lim_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} = \log_2\left(\frac{2}{\beta}\right).$$

We remark that the bounds given in Theorem 1.1 are somewhat weak. We observe that $\kappa(\beta) \to 0$ as $\beta \to 1$, contrary to what we would expect. As $\beta \to 1$ we would expect the number of $k$-prefixes for $x$ to grow and the growth rate of $\mathcal{N}_k(x,\beta)$ to increase. In this paper we show that the following theorem holds.

THEOREM 1.3. *There exists a strictly decreasing sequence* $(\omega_m)_{m=1}^{\infty}$ *converging to* 1 *such that if* $\beta\in(1,\omega_m]$ *then*

$$\dim_{\mathrm{H}}(\Sigma_\beta(x)) \geq \frac{2m}{2m+1} \quad \text{for all } x\in\left(0,\frac{1}{\beta-1}\right).$$

As an immediate corollary of Theorem 1.3 we deduce that

$$\inf_{x\in(0,\frac{1}{\beta-1})} \dim_{\mathrm{H}}(\Sigma_\beta(x)) \to 1 \quad \text{as } \beta\to 1.$$

By (1) similar statements hold for the lower and upper growth rate of $\mathcal{N}_k(x,\beta)$.

We also improve on the bounds given in Theorem 1.1. We show that the following theorem holds.

THEOREM 1.4. *There exists a strictly increasing sequence* $(\lambda_m)_{m=1}^{\infty}$ *converging to* $(1+\sqrt{5})/2$ *such that for* $\beta\in(1,\lambda_m]$,

$$\dim_{\mathrm{H}}(\Sigma_\beta(x)) \geq \frac{1}{m+2} \quad \text{for any } x\in\left(0,\frac{1}{\beta-1}\right).$$

In Section 2 we prove Theorem 1.3, and in Section 3 we prove Theorem 1.4 by a similar argument. In Section 4 we give some bounds for the upper growth rate of $\mathcal{N}_k(x,\beta)$, while in Section 5 we use our results to obtain bounds for the local dimension of Bernoulli convolutions.

**2. Proof of Theorem 1.3.** To prove Theorem 1.3 we devise an algorithm for generating $\beta$-expansions. We then show that the Hausdorff dimension of the set of expansions generated by this algorithm is greater than or equal to $2m/(2m+1)$ for $\beta\in(1,\omega_m]$. Before giving the details of this algorithm we provide a useful reinterpretation of $\mathcal{N}_k(x,\beta)$ and define the sequence $(\omega_m)_{m=1}^{\infty}$.

**2.1.** $B_k(x,\beta)$ **and the sequence** $(\omega_m)_{m=1}^{\infty}$

**2.1.1.** *Reinterpretation of* $\mathcal{N}_k(x,\beta)$. Fix $T_{0,\beta}(x)=\beta x$ and $T_{1,\beta}(x)=\beta x-1$. We let

$$\Omega_k=\{a=(a_n)_{n=1}^{k}\in\{T_{0,\beta},T_{1,\beta}\}^k\}.$$

For $x\in I_\beta$ and $a\in\Omega_k$, we denote $a_k\circ a_{k-1}\circ\cdots\circ a_1(x)$ by $a(x)$. We let

$$|a|_0=\#\{1\leq n\leq k: a_n=T_{0,\beta}\}, \qquad |a|_1=\#\{1\leq n\leq k: a_n=T_{1,\beta}\}.$$

Finally we define

$$T_k(x,\beta)=\{a\in\Omega_k: a(x)\in I_\beta\}, \qquad B_k(x,\beta)=\#T_k(x,\beta).$$

PROPOSITION 2.1. $\mathcal{N}_k(x,\beta)=B_k(x,\beta)$.

*Proof.* Following [4] we observe that

$$\begin{aligned}\mathcal{E}_k(x,\beta) &= \Big\{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k : x-\frac{1}{\beta^k(\beta-1)}\le\sum_{n=1}^{k}\frac{\epsilon_n}{\beta^n}\le x\Big\}\\ &= \Big\{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k : 0\le x-\sum_{n=1}^{k}\frac{\epsilon_n}{\beta^n}\le\frac{1}{\beta^k(\beta-1)}\Big\}\\ &= \Big\{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k : 0\le \beta^k x-\sum_{n=1}^{k}\epsilon_n\beta^{k-n}\le\frac{1}{\beta-1}\Big\}\\ &= \Big\{(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k : 0\le (T_{\epsilon_1,\beta},\dots,T_{\epsilon_k,\beta})(x)\le\frac{1}{\beta-1}\Big\}.\end{aligned}$$

Our result follows immediately. ∎

By Proposition 2.1 we can identify elements of $T_k(x,\beta)$ with elements of $\mathcal{E}_k(x,\beta)$, so that we can also call elements of $T_k(x,\beta)$ *$k$-prefixes* of $x$. To help with our later calculations we include the following technical lemmas.

LEMMA 2.2. *For all $k,n\in\mathbb{N}$,*

$$T^k_{1,\beta}\bigg(\frac{\beta^n}{\beta^2-1}\bigg)=\frac{\beta^{n+k}-\beta^{k+1}-\beta^k+\beta+1}{\beta^2-1}.$$

The proof of this lemma is trivial and hence omitted.

LEMMA 2.3. *Assume $a\in\Omega_{2k+1}$ and $|a|_0\ge k+1$. Then for all $x\in\mathbb{R}$,*

$$a(x)\ge(\overbrace{T_{1,\beta},\dots,T_{1,\beta}}^{k},\overbrace{T_{0,\beta},\dots,T_{0,\beta}}^{k+1})(x).$$

*Similarly, if $|a|_1\ge k+1$ then*

$$a(x)\le(\overbrace{T_{0,\beta},\dots,T_{0,\beta}}^{k},\overbrace{T_{1,\beta},\dots,T_{1,\beta}}^{k+1})(x).$$

*Proof.* Suppose $a\in\Omega_{2k+1}$ and $|a|_0\ge k+1$. By a simple calculation we have

$$a(x)=\beta^{2k+1}x-\sum_{n=1}^{2k+1}\chi_n(a)\beta^{2k+1-n},$$

where $\chi_n(a)=1$ if $a_n=T_{1,\beta}$ and 0 otherwise. Since $|a|_0\ge k+1$ we have $\chi_n(a)=1$ for at most $k$ different values of $n$. It follows that

$$a(x)\ge\beta^{2k+1}x-\beta^{2k}-\dots-\beta^{k+1}.$$

Since

$$\beta^{2k+1}x-\beta^{2k}-\dots-\beta^{k+1}=(\overbrace{T_{1,\beta},\dots,T_{1,\beta}}^{k},\overbrace{T_{0,\beta},\dots,T_{0,\beta}}^{k+1})(x)$$

the desired result follows. The second inequality is proved similarly. ∎

**2.1.2.** *The sequence* $(\omega_m)_{m=1}^{\infty}$. We now define our sequence $(\omega_m)_{m=1}^{\infty}$. For each $m \in \mathbb{N}$ we consider the polynomials

$$\begin{aligned}
P_m^1(x) &= x^{4m+3} - x^{2m+2} - x^{m+2} - x^{m+1} + x + 1,\\
P_m^2(x) &= x^{2m+3} - x^{2m+2} - x^2 + 1,\\
P_m^3(x) &= x^{2m+3} - x - 1.
\end{aligned}$$

We define $\omega_m^{(i)}$ to be the smallest real root of $P_m^i(x)$ greater than 1. Clearly $P_m^3(x)$ has such a root; so do $P_m^1(x)$ and $P_m^2(x)$, as can be seen by observing that $P_m^i(1) = 0$ and $(P_m^i)'(1) < 0$ for $i = 1, 2$. We define $\omega_m = \min\{\omega_m^{(1)}, \omega_m^{(2)}, \omega_m^{(3)}\}$. We now state some properties of $(\omega_m)_{m=1}^{\infty}$ that will be useful in our later analysis.

Property 2.4. *For* $\beta \in (1, \omega_m]$,

$$\beta^{4m+3} - \beta^{2m+2} - \beta^{m+2} - \beta^{m+1} + \beta + 1 \le 0.$$

*Proof.* This follows since $\omega_m \le \omega_m^{(1)}$, $P_m^1(1) = 0$ and $(P_m^1)'(1) < 0$. ∎

Property 2.5. *For* $\beta \in (1, \omega_m]$,

$$T_{0,\beta}^{2m+1}\left(\frac{\beta}{\beta^2 - 1}\right) = \frac{\beta^{2m+2}}{\beta^2 - 1} \in I_\beta.$$

*Proof.* It suffices to show that

$$\frac{\beta^{2m+2}}{\beta^2 - 1} \le \frac{1}{\beta - 1}.$$

This is equivalent to

$$\beta^{2m+3} - \beta^{2m+2} - \beta^2 + 1 \le 0,$$

which is true for $\beta \in (1, \omega_m]$ since $\omega_m \le \omega_m^{(2)}$, $P_m^2(1) = 0$ and $(P_m^2)'(1) < 0$. ∎

Property 2.6. *For* $\beta \in (1, \omega_m]$,

$$T_{1,\beta}^{2m+1}\left(\frac{\beta}{\beta^2 - 1}\right) = \frac{-\beta^{2m+1} + \beta + 1}{\beta^2 - 1} \in I_\beta.$$

*Proof.* The equality follows by Lemma 2.2. It remains to show that

$$\frac{-\beta^{2m+1} + \beta + 1}{\beta^2 - 1} \ge 0.$$

This is equivalent to

$$\beta^{2m+1} - \beta - 1 \le 0,$$

which is true for $\beta \in (1, \omega_m]$ since $\omega_m \le \omega_m^{(3)}$ and $P_m^3(1) < 0$. ∎

Property 2.7. *The sequence* $(\omega_m)_{m=1}^{\infty}$ *is strictly decreasing and converging to* 1.

This follows from the fact that $(\omega_m^{(i)})_{m=1}^{\infty}$ is strictly decreasing and converging to 1 for $i = 1, 2, 3$.

In Section 6 we include a table of values for $(\omega_m)_{m=1}^{\infty}$.

**2.2. Algorithm for generating $\beta$-expansions.** We now give details of our algorithm for generating $\beta$-expansions that we mentioned at the start of this section. In what follows we assume $\beta \in (1, \omega_m]$ and $x \in (0, 1/(\beta-1))$.

We define

$$\begin{aligned}\mathcal{I}_m &= \left[T_{1,\beta}^{2m+1}\left(\frac{1}{\beta^2-1}\right), T_{0,\beta}^{2m+1}\left(\frac{\beta}{\beta^2-1}\right)\right]\\ &= \left[\frac{-\beta^{2m+2}+\beta+1}{\beta^2-1}, \frac{\beta^{2m+2}}{\beta^2-1}\right].\end{aligned}$$

Then $\mathcal{I}_m \subset I_\beta$ by Properties 2.5 and 2.6.

Remark 2.8. The significance of the points $1/(\beta^2-1)$ and $\beta/(\beta^2-1)$ is that

$$T_{0,\beta}\left(\frac{1}{\beta^2-1}\right) = \frac{\beta}{\beta^2-1} \quad \text{and} \quad T_{1,\beta}\left(\frac{\beta}{\beta^2-1}\right) = \frac{1}{\beta^2-1}.$$

Therefore it is not possible for a point to pass over the interval $\left[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}\right]$ without landing in it.

Step 1. There exists a minimal number $j(x)$ of transformations that map the point $x$ into $\mathcal{I}_m$. This follows from the fact that $\left[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}\right] \subset \mathcal{I}_m$ and Remark 2.8. We choose a sequence $a \in \Omega_{j(x)}$ of transformations such that $a(x) \in \mathcal{I}_m$. We fix the first $j(x)$ entries in our $\beta$-expansion of $x$ to be those uniquely determined by $a$.

Step 2. If

$$a(x) \in \left[\frac{1}{\beta^2-1}, \frac{\beta^{2m+2}}{\beta^2-1}\right]$$

then we can extend the $j(x)$-prefix $a$ to a $(j(x)+2m+1)$-prefix by concatenating $a$ with any element $a^{(1)} \in \Omega_{2m+1}$ such that $|a^{(1)}|_1 \geq m+1$. To see why $aa^{(1)}$ is a $(j(x)+2m+1)$-prefix for $x$ we observe that

$$\begin{aligned}\frac{-\beta^{2m+2}+\beta+1}{\beta^2-1} &= T_{1,\beta}^{2m+1}\left(\frac{1}{\beta^2-1}\right) \leq aa^{(1)}(x) \leq aa^{(1)}\left(\frac{\beta^{2m+2}}{\beta^2-1}\right)\\ &\leq (\overbrace{T_{0,\beta}, \ldots, T_{0,\beta}}^{m}, \overbrace{T_{1,\beta}, \ldots, T_{1,\beta}}^{m+1})\left(\frac{\beta^{2m+2}}{\beta^2-1}\right)\\ &= T_{1,\beta}^{m+1}\left(\frac{\beta^{3m+2}}{\beta^2-1}\right) = \frac{\beta^{4m+3}-\beta^{m+2}-\beta^{m+1}+\beta+1}{\beta^2-1},\end{aligned}$$

by Lemmas 2.2 and 2.3. The inequality

$$\frac{\beta^{4m+3}-\beta^{m+2}-\beta^{m+1}+\beta+1}{\beta^2-1}\le\frac{\beta^{2m+2}}{\beta^2-1}$$

is equivalent to $\beta^{4m+3}-\beta^{2m+2}-\beta^{m+2}-\beta^{m+1}+\beta+1\le 0$, which is true for $\beta\in(1,\omega_m]$ by Property 2.4. Therefore $aa^{(1)}(x)\in\mathcal{I}_m$ for all $a^{(1)}\in\Omega_{2m+1}$ such that $|a^{(1)}|_1\ge m+1$, which by the remarks following Proposition 2.1 implies that $aa^{(1)}$ is a $(j(x)+2m+1)$-prefix for $x$.

If

$$a(x)\in\left[\frac{-\beta^{2m+2}+\beta+1}{\beta^2-1},\frac{1}{\beta^2-1}\right]$$

then we consider elements $a^{(1)}\in\Omega_{2m+1}$ such that $|a^{(1)}|_0\ge m+1$. By a similar argument it can be shown that $aa^{(1)}(x)\in\mathcal{I}_m$ for all $a^{(1)}\in\Omega_{2m+1}$ such that $|a^{(1)}|_0\ge m+1$. As

$$\#\{a\in\Omega_{2m+1}:|a^{(1)}|_1\ge m+1\}=\#\{a\in\Omega_{2m+1}:|a^{(1)}|_0\ge m+1\}=2^{2m},$$

our algorithm generates $2^{2m}$ $(j(x)+2m+1)$-prefixes for $x$.

Step 3. We proceed inductively: if $aa^{(1)}(x)\in\left[\frac{1}{\beta^2-1},\frac{\beta^{2m+2}}{\beta^2-1}\right]$ then we extend the $(j(x)+2m+1)$-prefix $aa^{(1)}$ to a $(j(x)+4m+2)$-prefix for $x$ by concatenating $aa^{(1)}$ with any element $a^{(2)}\in\Omega_{2m+1}$ such that $|a^{(2)}|_1\ge m+1$. Similarly, if $aa^{(1)}(x)\in\left[\frac{-\beta^{2m+2}+\beta+1}{\beta^2-1},\frac{1}{\beta^2-1}\right]$ then we consider elements $a^{(2)}\in\Omega_{2m+1}$ such that $|a^{(2)}|_0\ge m+1$. We repeat this process indefinitely.

Remark 2.9. It is clear that by proceeding inductively our algorithm generates $2^{2km}$ $(j(x)+k(2m+1))$-prefixes for $x$, for each $k\in\mathbb{N}$.

Remark 2.10. The construction of our interval $\mathcal{I}_m$ is somewhat arbitrary. We could have begun by choosing any interval of the form $[z,\beta z]$ for some $z\in(0,1/(\beta-1))$. We then construct the interval $[T^{2m+1}_{1,\beta}(z),T^{2m+1}_{0,\beta}(\beta z)]$ to perform the role of $\mathcal{I}_m$. By defining a similar set of polynomials to $P^1_m(x)$, $P^2_m(x)$ and $P^3_m(x)$ and assuming our $\beta$ satisfies certain restrictions imposed by these polynomials we can ensure that $[T^{2m+1}_{1,\beta}(z),T^{2m+1}_{0,\beta}(\beta z)]\subset I_\beta$ and if $x\in[T^{2m+1}_{1,\beta}(z),T^{2m+1}_{0,\beta}(\beta z)]$ then $a(x)\in[T^{2m+1}_{1,\beta}(z),T^{2m+1}_{0,\beta}(\beta z)]$ for $2^{2m}$ elements of $\Omega_{2m+1}$. It would be interesting to know whether $\mathcal{I}_m$ is the most efficient choice of interval for this method.

We denote by $\Sigma_\beta(x,m)$ the set of $\beta$-expansions generated by this algorithm, and by $\Sigma_\beta(x,m,k)$ the set of $k$-prefixes for $x$ generated by the algorithm.

Property 2.11. *The cardinality of $\Sigma_\beta(x,m,k)$ is increasing with $k$.*

PROPERTY 2.12. *Let $s, s' \in \mathbb{N}$ and $s > s'$. If $b \in \Sigma_\beta(x, m, j(x) + s'(2m+1))$ then*

$$\#\{a \in \Sigma_\beta(x, m, j(x) + s(2m+1)) : a_n = b_n \text{ for } 1 \le n \le j(x) + s'(2m+1)\} = 2^{2m(s-s')}.$$

We now prove two technical lemmas.

LEMMA 2.13. *Let $\beta \in (1, \omega_m]$ and $x \in (0, 1/(\beta-1))$. If $k \ge j(x)$ then*

$$\#\Sigma_\beta(x, m, k) \ge 2^{2m(\frac{k-j(x)}{2m+1} - 1)}.$$

*Proof.* We have

$$\#\Sigma_\beta(x, m, k) \ge \#\Sigma_\beta\left(x, m, \left((2m+1)\left[\frac{k-j(x)}{2m+1}\right] + j(x)\right)\right) \ge 2^{2m[\frac{k-j(x)}{2m+1}]} \ge 2^{2m(\frac{k-j(x)}{2m+1} - 1)}. \blacksquare$$

LEMMA 2.14. *Let $l \ge j(x)$ and $b \in \Sigma_\beta(x, m, l)$. Then for $k \ge l$,*

$$\#\{a = (a_n)_{n=1}^k \in \Sigma_\beta(x, m, k) : a_n = b_n \text{ for } 1 \le n \le l\} \le 2^{2m(\frac{k-l}{2m+1} + 2)}.$$

*Proof.* We remark that $j(x) + (2m+1)\left[\frac{l-j(x)}{2m+1}\right]$ is an integer of the form $j(x) + s(2m+1)$, not exceeding $l$, while $j(x) + (2m+1)\left(\left[\frac{k-j(x)}{2m+1}\right] + 1\right)$ is an integer of the form $j(x) + s(2m+1)$ greater than or equal to $k$. It follows immediately that

$$\begin{aligned}
&\#\{a = (a_n)_{n=1}^k \in \Sigma_\beta(x, m, k) : a_n = b_n \text{ for } 1 \le n \le l\} \\
&\le \#\Bigg\{a = (a_n)_{n=1}^{j(x)+(2m+1)([\frac{k-j(x)}{2m+1}]+1)} \\
&\qquad \in \Sigma_\beta\left(x, m, j(x) + (2m+1)\left(\left[\frac{k-j(x)}{2m+1}\right] + 1\right)\right) : \\
&\qquad\qquad a_n = b_n \text{ for } 1 \le n \le j(x) + (2m+1)\left[\frac{l-j(x)}{2m+1}\right]\Bigg\} \\
&\le 2^{2m([\frac{k-j(x)}{2m+1}] + 1 - [\frac{l-j(x)}{2m+1}])} \le 2^{2m(\frac{k-j(x)}{2m+1} + 1 - \frac{l-j(x)}{2m+1} + 1)} = 2^{2m(\frac{k-l}{2m+1} + 2)},
\end{aligned}$$

by Properties 2.11 and 2.12. ■

**2.2.1.** *Proof of Theorem 1.3.* The proof is based upon the argument given in Example 2.7 of [3].

By the monotonicity of Hausdorff dimension with respect to inclusion it suffices to show that $\dim_{\mathrm{H}}(\Sigma_\beta(x, m)) \ge 2m/(2m+1)$. Further, it suffices to show that for any sufficiently small cover $\{U_i\}_{i=1}^\infty$ of $\Sigma_\beta(x, m)$ we can bound $\sum_{i=1}^\infty \mathrm{Diam}(U_i)^{2m/(2m+1)}$ below by a positive constant independent of our choice of cover.

It is a simple exercise to show that $\Sigma_\beta(x,m)$ is a compact set; hence we may restrict to finite covers. Let $\{U_i\}_{i=1}^N$ be a finite cover of $\Sigma_\beta(x,m)$. We may assume that $\mathrm{Diam}(U_i) < 2^{-j(x)}$ for all $i$. For each $U_i$ there exists $l(i)$ such that

$$2^{-(l(i)+1)} \le \mathrm{Diam}(U_i) < 2^{-l(i)}.$$

It follows that there exists $z^{(i)} \in \{0,1\}^{l(i)}$ such that $y_n = z_n^{(i)}$ for all $y \in U_i$, for $1 \le n \le l(i)$. We may assume that $z^{(i)} \in \Sigma_\beta(x,m,l(i))$, since otherwise $\Sigma_\beta(x,m) \cap U_i = \emptyset$ and we can remove $U_i$ from our cover. We set

$$C_i = \{(\epsilon_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}} : \epsilon_n = z_n^{(i)} \text{ for } 1 \le n \le l(i)\}.$$

Clearly $U_i \subset C_i$ and therefore $\{C_i\}_{i=1}^N$ is a cover of $\Sigma_\beta(x,m)$.

Since there are only finitely many elements in our cover, there exists $J$ such that $2^{-J} \le \mathrm{Diam}(U_i)$ for all $i$. We now consider the set $\Sigma_\beta(x,m,J)$. Since $\{C_i\}_{i=1}^N$ is a cover of $\Sigma_\beta(x,m)$, each $a \in \Sigma_\beta(x,m,J)$ satisfies $a_n = z_n^{(i)}$ for $1 \le n \le l(i)$, for some $i$. Therefore

$$\#\Sigma_\beta(x,m,J) \le \sum_{i=1}^N \#\{a \in \Sigma_\beta(x,m,J) : a_n = z_n^{(i)} \text{ for } 1 \le n \le l(i)\}.$$

By counting the elements of $\Sigma_\beta(x,m,J)$ and by Lemmas 2.13 and 2.14 we obtain

$$\begin{aligned}
2^{2m(\frac{J-j(x)}{2m+1}-1)} &\le \#\Sigma_\beta(x,m,J) \\
&\le \sum_{i=1}^N \#\{a \in \Sigma_\beta(x,m,J) : a_n = z_n^{(i)} \text{ for } 1 \le n \le l(i)\} \\
&\le \sum_{i=1}^N 2^{2m(\frac{J-l(i)}{2m+1}+2)} = 2^{\frac{2mJ+1}{2m+1}+4m} \sum_{i=1}^N 2^{\frac{-2m(l(i)+1)}{2m+1}} \\
&\le 2^{\frac{2mJ+1}{2m+1}+4m} \sum_{i=1}^N \mathrm{Diam}(U_i)^{\frac{2m}{2m+1}}.
\end{aligned}$$

Dividing through by $2^{\frac{2mJ+1}{2m+1}+4m}$ yields

$$\sum_{i=1}^N \mathrm{Diam}(U_i)^{\frac{2m}{2m+1}} \ge 2^{\frac{-12m^2-(6+2j(x))m-1}{2m+1}},$$

as desired.

**3. Proof of Theorem 1.4.** The proof is analogous to that of Theorem 1.3, so we only give the details where appropriate. As before, we devise an algorithm for generating $\beta$-expansions; the Hausdorff dimension of the set of expansions generated by this algorithm will be greater than $1/(m+2)$ for $\beta \in (1, \lambda_m]$.

**3.1. The sequence $(\lambda_m)_{m=1}^{\infty}$.** Let $\lambda_m$ be the smallest real root of the equation

$$x^{m+3} - x^{m+2} - x^{m+1} + 1 = 0$$

greater than 1. The sequence $(\lambda_m)_{m=1}^{\infty}$ is well known (see [1] for the details).

REMARK 3.1. To see that $P_m(x) = x^{m+3} - x^{m+2} - x^{m+1} + 1$ has a real root greater than 1, it suffices to observe that $P_m(1) = 0$ and $P_m'(1) < 0$.

PROPERTY 3.2. *For* $\beta \in (1, \lambda_m]$,

$$\beta^{m+3} - \beta^{m+2} - \beta^{m+1} + 1 \leq 0.$$

REMARK 3.3. Each $\lambda_m$ is a Pisot number, i.e. a real algebraic integer greater than 1 whose Galois conjugates are of modulus strictly less than 1. Moreover, $\lambda_1$ is the greatest real root of $x^3 - x - 1 = 0$, the first Pisot number.

PROPERTY 3.4. *The sequence* $(\lambda_m)_{m=1}^{\infty}$ *is strictly increasing and converges to* $(1+\sqrt{5})/2$ *as* $m \to \infty$.

In Section 6 we include a table of values for $(\lambda_m)_{m=1}^{\infty}$.

**3.2. Algorithm for generating $\beta$-expansions.** We define

$$\mathcal{I} = \left[T_{1,\beta}\left(\frac{1}{\beta^2-1}\right), T_{0,\beta}\left(\frac{\beta}{\beta^2-1}\right)\right] = \left[\frac{1+\beta-\beta^2}{\beta^2-1}, \frac{\beta^2}{\beta^2-1}\right].$$

For $1 < \beta < (1+\sqrt{5})/2$ the interval $\mathcal{I}$ is contained in $I_\beta$. This interval will play a similar role to $\mathcal{I}_m$. Before giving the details of our algorithm we require the following technical lemma.

LEMMA 3.5. *For* $\beta \in (1, \lambda_m]$,

$$T_{0,\beta}^{m+1}\left(\frac{1+\beta-\beta^2}{\beta^2-1}\right) \geq \frac{1}{\beta^2-1} \quad \text{and} \quad T_{1,\beta}^{m+1}\left(\frac{\beta^2}{\beta^2-1}\right) \leq \frac{\beta}{\beta^2-1}.$$

*Proof.* It is a simple exercise to show that

$$T_{0,\beta}^{m+1}\left(\frac{1+\beta-\beta^2}{\beta^2-1}\right) = \frac{\beta^{m+1}+\beta^{m+2}-\beta^{m+3}}{\beta^2-1}.$$

This is greater than or equal to $\frac{1}{\beta^2-1}$ precisely when $\beta^{m+3} - \beta^{m+2} - \beta^{m+1} + 1 \leq 0$, which is true by Property 3.2. The second inequality is proved similarly. ∎

We now formalise our algorithm for generating expansions.

STEP 1. Let $x \in (0, 1/(\beta-1))$. By Remark 2.8 there exists a minimal number $g(x)$ of transformations that map $x$ into the interval $\mathcal{I}$. We choose a sequence $a \in \Omega_{g(x)}$ of transformations such that $a(x) \in \mathcal{I}$. We fix the first $g(x)$ entries in our $\beta$-expansion to be those uniquely determined by $a$.

STEP 2. If $a(x) \in \left[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}\right]$ then we can extend $a(x)$ to a $(g(x)+1)$-prefix by either $T_{0,\beta}$ or $T_{1,\beta}$; we then choose $a^{(i)}(x) \in \Omega_{m+1}$ such that $a^{(i)} \circ T_{i,\beta} \circ a(x) \in \mathcal{I}$, for $i = 0, 1$. This defines two prefixes of length $g(x)+m+2$ for $x$.

If $a(x) \in \left[\frac{1+\beta-\beta^2}{\beta^2-1}, \frac{1}{\beta^2-1}\right]$ we iterate $T_{0,\beta}$ until $T_{0,\beta}^k \circ a(x) \in \left[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}\right]$. By Lemma 3.5 and the monotonicity of $T_{0,\beta}$ we have $k \le m+1$. The transformation $T_{0,\beta}^k \circ a(x)$ defines a $(g(x)+k)$-prefix. We can extend it to a $(g(x)+k+1)$-prefix by either $T_{0,\beta}$ or $T_{1,\beta}$; we then choose $a^{(i)} \in \Omega_{m+1-k}$ such that $a^{(i)} \circ T_{i,\beta} \circ T_{0,\beta}^k \circ a(x) \in \mathcal{I}$. This defines two prefixes of length $g(x)+m+2$ for $x$.

If $a(x) \in \left[\frac{\beta}{\beta^2-1}, \frac{\beta^2}{\beta^2-1}\right]$ then by a similar argument to the previous case we can formalise a method for choosing $a^{(0)}, a^{(1)} \in \Omega_{m+2}$ such that $a^{(i)} \circ a(x) \in \mathcal{I}$, for $i = 0, 1$.

At this stage our algorithm has generated two prefixes of length $g(x) + m + 2$ for $x$.

STEP 3. The two prefixes defined in Step 2 map $x$ into $\mathcal{I}$, so we can apply Step 2 to the image of $x$ under the transformations corresponding to those prefixes. This defines four prefixes of length $g(x)+2(m+2)$. We repeat this process indefinitely.

REMARK 3.6. Proceeding inductively our algorithm generates $2^k$ prefixes of length $g(x) + k(m+2)$ for $x$, for all $k \in \mathbb{N}$.

Repeating the arguments given in Section 2 we can show that analogues of Property 2.12, Lemma 2.13 and Lemma 2.14 all hold. Theorem 1.4 then follows by an analogous argument to the one given in the proof of Theorem 1.3.

REMARK 3.7. As in the proof of Theorem 1.3, our choice of interval $\mathcal{I}$ is somewhat arbitrary. It would be interesting to know whether $\mathcal{I}$ is the most efficient choice of interval for this method.

**4. Upper bounds for the upper growth rate of $\mathcal{N}_k(x,\beta)$.** Our main result in this section is the following theorem.

THEOREM 4.1. *The supremum of the upper growth rates converges to* 0 *as* $\beta \to 2$, *i.e.*,

$$\sup_{x \in (0, \frac{1}{\beta-1})} \left\{ \limsup_{k\to\infty} \frac{\log_2 \mathcal{N}_k(x,\beta)}{k} \right\} \to 0 \qquad \textit{as } \beta \to 2.$$

By (1) similar statements hold for $\dim_{\mathrm{H}}(\Sigma_\beta(x))$ and the lower growth rate of $\mathcal{N}_k(x,\beta)$. Theorem 4.1 can be interpreted as an analogue of Theorem 1.3 in the case where $\beta$ is close to 2.

*Proof of Theorem 4.1.* Fix $m \in \mathbb{N}$. Recall that

$$(2)\quad \mathcal{N}_m(x,\beta) = \#\Big\{(\epsilon_1,\dots,\epsilon_m)\in\{0,1\}^m : x-\frac{1}{\beta^m(\beta-1)} \le \sum_{n=1}^m \frac{\epsilon_n}{\beta^n} \le x\Big\}.$$

Let

$$L(m,\beta) = \Big\{\sum_{n=1}^m \frac{\epsilon_n}{\beta^n} : (\epsilon_1,\dots,\epsilon_m)\in\{0,1\}^m\Big\}.$$

Since $L(m,2)$ is the set of dyadic rationals of degree $m$, we have $|x-y| \ge 2^{-m}$ for all $x,y\in L(m,2)$ such that $x\ne y$. By continuity, for each $m\in\mathbb{N}$ there exists $\delta(m)>0$ such that, for all $\beta\in(2-\delta(m),2)$,

$$|x-y| > \frac{1}{2\beta^m(\beta-1)} \quad \text{for all } x,y\in L(m,\beta) \text{ with } x\ne y.$$

It follows that any interval of length $1/(\beta^m(\beta-1))$ contains at most two elements of $L(m,\beta)$. By (2) for $\beta\in(2-\delta(m),2)$ any $x\in(0,1/(\beta-1))$ has at most two prefixes of length $m$. Proceeding inductively we can deduce that for $\beta\in(2-\delta(m),2)$ and $x\in(0,1/(\beta-1))$,

$$\mathcal{N}_{km}(x,\beta)\le 2^k \quad \text{for all } k\in\mathbb{N}.$$

By a simple argument it follows that

$$\limsup_{k\to\infty}\frac{\log_2\mathcal{N}_k(x,\beta)}{k}\le\frac{1}{m}.$$

As $m$ was arbitrary, we can deduce our result. ∎

The following result gives an upper bound for the upper growth rate of $\mathcal{N}_k(x,\beta)$ for $\beta$ close to 1.

THEOREM 4.2. *Let $m\in\mathbb{N}$ and $m\ge 2$. For $\beta\in(2^1/m,2)$,*

$$\limsup_{k\to\infty}\frac{\log_2\mathcal{N}_k(x,\beta)}{k}\le\frac{\log_2(2^m-1)}{m} \quad \textit{for all } x\in\Big(0,\frac{1}{\beta-1}\Big).$$

*Proof.* It is a simple exercise to show that

$$T_{1,\beta}^{-m}(0)=\frac{\beta^m-1}{\beta^m(\beta-1)} \quad\text{and}\quad T_{0,\beta}^{-m}\Big(\frac{1}{\beta-1}\Big)=\frac{1}{\beta^m(\beta-1)}.$$

By a simple manipulation $T_{1,\beta}^{-m}(0)>T_{0,\beta}^{-m}(1/(\beta-1))$ is equivalent to $\beta^m>2$. Let $\beta\in(2^{1/m},2)$ and $x\in(0,1/(\beta-1))$. Then by the above and the monotonicity of the maps $T_{0,\beta}$ and $T_{1,\beta}$, either $T_{0,\beta}^m(x)$ or $T_{1,\beta}^m(x)$ will lie outside the interval $I_\beta$. It follows that any $x\in(0,1/(\beta-1))$ can have at most $2^m-1$

$m$-prefixes. By an inductive argument it follows that

$$\mathcal{N}_{km}(x,\beta) \le (2^m-1)^k \quad \text{for all } k \in \mathbb{N}.$$

Our result follows immediately. ∎

**5. Application to Bernoulli convolutions.** Given $1 < \beta < 2$ we define the *Bernoulli convolution* $\mu_\beta$ as follows. Let $E \subset \mathbb{R}$ be a Borel set. Then

$$\mu_\beta(E) = \mathbb{P}\biggl(\biggl\{(a_1,a_2,\ldots) \in \{0,1\}^{\mathbb{N}} : \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} \in E\biggr\}\biggr),$$

where $\mathbb{P}$ is the $(1/2,1/2)$ Bernoulli measure. For $x \in I_\beta$ we define the *local dimension* of $\mu_\beta$ at $x$ by

$$d(\mu_\beta,x) = \lim_{r\to 0} \frac{\log \mu_\beta([x-r,x+r])}{\log r},$$

when this limit exists. When the limit does not exist we can consider the *lower* and *upper local dimensions* of $\mu_\beta$ at $x$, defined as

$$\underline{d}(\mu_\beta,x) = \liminf_{r\to 0} \frac{\log \mu_\beta([x-r,x+r])}{\log r},$$

and

$$\overline{d}(\mu_\beta,x) = \limsup_{r\to 0} \frac{\log \mu_\beta([x-r,x+r])}{\log r}$$

respectively. In [4] the authors show that the following result holds.

THEOREM 5.1. *For any $\beta \in (1,(1+\sqrt{5})/2)$ we have*

$$\overline{d}(\mu_\beta,x) \le (1-\kappa(\beta)) \log_\beta 2 \quad \textit{for all } x \in \left(0, \frac{1}{\beta-1}\right),$$

*where $\kappa(\beta)$ is as in Theorem* 1.1.

Replicating the arguments given in [4] and using the improved bounds given by Theorems 1.3 and 1.4 we obtain

THEOREM 5.2. *If $\beta \in (1,\omega_m]$, then*

$$\overline{d}(\mu_\beta,x) \le \frac{1}{2m+1} \log_\beta 2 \quad \textit{for every } x \in \left(0, \frac{1}{\beta-1}\right).$$

*Similarly, if $\beta \in (1,\lambda_m]$, then*

$$\overline{d}(\mu_\beta,x) \le \frac{m+1}{m+2} \log_\beta 2 \quad \textit{for every } x \in \left(0, \frac{1}{\beta-1}\right).$$

**6. Open questions and tables for $(\omega_m)_{m=1}^{\infty}$ and $(\lambda_m)_{m=1}^{\infty}$.** Here are a few open questions:

- Does the positivity of the lower growth rate of $\mathcal{N}_k(x,\beta)$ imply the Hausdorff dimension of $\Sigma_\beta(x)$ is positive?
- Do we have equality in (1)?
- Under what conditions do we have equality in (1)?
- Is our choice of interval $\mathcal{I}_m$ in the proof of Theorem 1.3 the most efficient?
- Is our choice of interval $\mathcal{I}$ in the proof of Theorem 1.4 the most efficient?

The following tables list certain values of $\omega_m$ and $\lambda_m$ and their associated polynomials.

**Table 1.** Table of values for the sequence $(\omega_m)_{m=1}^{\infty}$

| $m$ | $\omega_m$ (to 5 DP) | Associated polynomials |
|---|---|---|
| 1 | 1.07445 | $P_1^1(x) = x^7 - x^4 - x^3 - x^2 + x + 1$ |
| | | $P_1^2(x) = x^5 - x^4 - x^2 + 1$ |
| | | $P_1^3(x) = x^5 - x - 1$ |
| 2 | 1.02838 | $P_2^1(x) = x^{11} - x^6 - x^4 - x^3 + x + 1$ |
| | | $P_2^2(x) = x^7 - x^6 - x^2 + 1$ |
| | | $P_3^2(x) = x^7 - x - 1$ |
| 3 | 1.01492 | $P_3^1(x) = x^{15} - x^8 - x^5 - x^4 + x + 1$ |
| | | $P_3^2(x) = x^9 - x^8 - x^2 + 1$ |
| | | $P_3^3(x) = x^9 - x - 1$ |
| 10 | 1.00172 | $P_{10}^1(x) = x^{43} - x^{22} - x^{12} - x^{11} + x + 1$ |
| | | $P_{10}^2(x) = x^{23} - x^{22} - x^2 + 1$ |
| | | $P_{10}^3(x) = x^{23} - x - 1$ |
| 100 | 1.00003 | $P_{100}^1(x) = x^{403} - x^{202} - x^{102} - x^{101} + x + 1$ |
| | | $P_{100}^2(x) = x^{203} - x^{202} - x^2 + 1$ |
| | | $P_{100}^3(x) = x^{203} - x - 1$ |

**Table 2.** Table of values for the sequence $(\lambda_m)_{m=1}^{\infty}$

| $m$ | $\lambda_m$ (to 5 DP) | Associated polynomial |
|---|---|---|
| 1 | 1.32472 (first Pisot number) | $x^4 - x^3 - x^2 + 1$ |
| 2 | 1.46557 | $x^5 - x^4 - x^3 + 1$ |
| 3 | 1.53416 | $x^6 - x^5 - x^4 + 1$ |
| 10 | 1.61575 | $x^{13} - x^{12} - x^{11} + 1$ |
| 100 | 1.61804 | $x^{103} - x^{102} - x^{101} + 1$ |

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