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Orbit spaces, Quillen's Theorem A and Minami's formula for compact Lie groups

by

Assaf Libman (Aberdeen)

Abstract. Let G be a compact Lie group. We present a criterion for the orbit spaces of two G-spaces to be homotopy equivalent and use it to obtain a quick proof of Webb's conjecture for compact Lie groups. We establish two Minami type formulae which present the p-localised spectrum $\Sigma^\infty BG_+$ as an alternating sum of p-localised spectra $\Sigma^\infty BH_+$ for subgroups H of G. The subgroups H are calculated from the collections of the non-trivial elementary abelian p-subgroups of G and the non-trivial p-radical subgroups of G. We also show that the Bousfield–Kan spectral sequences of the normaliser decompositions associated to these collections and to any p-local cohomology theory h^* collapse at their E_2-pages to their vertical axes, and converge to $h^*(BG)$. An important tool is a topological version of Quillen's Theorem A which we prove.

1. The main results. Let G be a compact Lie group. A *collection* $\mathcal{H}$ in G is a union of conjugacy classes (H) of subgroups H of G. We topologise $\mathcal{H}$ as the disjoint union of its conjugacy classes and let G act on $\mathcal{H}$ in this way. More generally, $(H_0 \leq \cdots \leq H_n)$ denotes the conjugacy class of a chain of inclusions of subgroups $H_i \in \mathcal{H}$. As a G-orbit it is homeomorphic to $G/\bigcap_i N_G(H_i)$.

Inclusion of subgroups endows $\mathcal{H}$ with a G-invariant partial order. This yields a G-simplicial complex $|\mathcal{H}|$ whose n-simplices are $\Delta^n \times (H_0 < \cdots < H_n)$. Faces are formed by removing elements from the chain $H_0 < \cdots < H_n$. A more elaborate discussion is deferred to §5.

Throughout we shall adopt the convention that $\mathcal{H}^0$ denotes the subcollection of $\mathcal{H}$ from which the trivial subgroup is removed. Recall that a *p-toral group* is an extension of a torus by a finite p-group. Every compact Lie group G contains a maximal normal p-toral subgroup $O_p(G)$. See Appendix A.1.

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Definition 1.1. Let $\mathcal{S}_p(G)$ denote the collection of all the p-toral subgroups of G. The collection of all the elementary abelian p-subgroups of G is denoted $\mathcal{E}_p(G)$.

A p-toral subgroup P of G is called *p-radical* if $O_p(N_G P) = P$. The collection of all the p-radical subgroups is denoted $\mathcal{B}_p(G)$.

The collections $\mathcal{S}_p^0(G), \mathcal{E}_p^0(G)$ and $\mathcal{B}_p^0(G)$ are obtained by removing the trivial subgroup.

The collection $\mathcal{B}_p(G)$ should be compared with the smaller collection $\mathcal{R}_p(G)$ of the p-stubborn subgroups which was studied by Jackowski, McClure and Oliver in [23]. When G is finite, $\mathcal{B}_p(G)$ is Bouc's collection (e.g. [4, pp. 222]).

The starting point of this paper is Theorem A below. For finite groups it was obtained by Minami in [29, Theorem 6.5]. In the present form it was announced by Martino and Priddy in [27], with the caveat that they use the collection $\mathcal{R}_p^0(G)$ instead of $\mathcal{B}_p^0(G)$. Martino and Priddy's argument hinges on [27, Theorem 6.1] whose statement and proof was criticised by the reviewer of their article for being "not explicit". Specifically, they apply Webb's results from [40] to "Mackey functors" which take values in the homotopy category of spectra which is only an additive category, rather than an abelian one, as Webb requires. Another flaw in Martino and Priddy's argument in [27, §4] is the application of Quillen's Theorem A [32] to posets which carry non-trivial topologies. As we show in Example 5.4, Quillen's theorem fails in these cases.

Let $\Sigma^\infty_+ X$ denote the suspension spectrum of a space X to which a disjoint basepoint is added.

Theorem A. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup. Let $\mathcal{C}$ denote either the collection $\mathcal{E}_p^0(G)$ or $\mathcal{B}_p^0(G)$. Then $\mathcal{C}$ contains finitely many conjugacy classes of chains of the form $P_0 < \cdots < P_k$ and after localisation at the Moore spectrum $S\mathbb{Z}_{(p)}$ of type $(\mathbb{Z}_{(p)}, 0)$ one has an equivalence of spectra*

$$(\Sigma^\infty_+ BG)_{S\mathbb{Z}_{(p)}} \simeq \sum_{(P_0<\cdots<P_k)} (-1)^k (\Sigma^\infty_+ BG_{P_0<\cdots<P_k})_{S\mathbb{Z}_{(p)}}$$

where $G_{P_0<\cdots<P_k} = \bigcap_{i=0}^k N_G P_i$ and the sum runs through all the conjugacy classes of $P_0 < \cdots < P_k$ in $\mathcal{C}$.

Bousfield's localisation of spectra [8] is recalled in §A.3. Proposition A.3.2 shows that the localisation $(-)_{S\mathbb{Z}_{(p)}}$ used in the theorem is equivalent to $H_*(-;\mathbb{Z}_{(p)})$-localisation. By transferring the negative terms in the sum to the left hand side one obtains a genuine equivalence of spectra, and the statement of the theorem should be understood in this way.

Proof. There are finitely many conjugacy classes $(P_0 < \cdots < P_n)$ by Propositions 7.4(i), 8.2, 9.2 and Definition 5.11. The rest is the content of Corollaries 8.4(c) and 9.6(c). ■

Webb showed that if G is a finite group then $|\mathcal{S}_p^0(G)|/G$ is a $\mathbb{Z}_{(p)}$-acyclic space. He conjectured that this space is in fact contractible. Webb's conjecture was resolved by Symonds in [36]. Another proof was given by Bux [11]. For compact Lie groups Słomińska shows in [34] that $|\mathcal{E}_p^0(G)|/G$ is contractible. In this paper we offer a generalisation of Symonds' theorem.

Following Słomińska's terminology, we say that $\mathcal{C}$ is a *concave collection of p-toral subgroups* of G if whenever Q is a p-toral subgroup of G and Q contains an element from $\mathcal{C}$ then $Q \in \mathcal{C}$.

THEOREM B. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup. If $\mathcal{C}$ is either a non-empty concave collection of p-toral subgroups of G or $\mathcal{C} = \mathcal{E}_p^0(G)$, then $|\mathcal{C}|/G$ is contractible.*

Proof. Corollaries 8.4(a) and 9.7. ■

The proof of Theorem B as well as the rest of the results depend on Lemma 3.2. It gives a criterion for the orbit spaces of two G-CW complexes to be homotopy equivalent by checking the subspaces fixed by p-toral subgroups of G only (!). Even though we do not state the lemma in this introduction, it is by all means the **key observation** of this paper.

The second goal of this paper is to generalise Dwyer's work on homology decompositions in [13] to compact Lie groups. The main tool we need is a topological version of Quillen's Theorem A, which we prove as Theorem 5.8. More importantly, we generalise Dwyer's results in [14] on sharp homology decompositions of finite groups to compact Lie groups. Dwyer's chain level arguments do not carry over to the compact Lie group case and we develop a new approach.

Notation. For an element $g \in G$ we let c_g denote the inner automorphism $x \mapsto gxg^{-1}$. If U is a subset of G, then ${}^gU$ denotes $c_g(U)$ and U^g denotes $c_{g^{-1}}(U)$. Given two subgroups H, K we let

$$N_G(H,K) = \{g \in G : {}^gH \le K\}.$$

For a set $\mathbf{H} = \{H_0, \ldots, H_k\}$ of subgroups of G denote $N_G(\mathbf{H}) = \bigcap_{i=0}^k N_G(H_i)$.

Notation. We fix once and for all a free G-CW complex EG on which G acts freely. The Borel construction X_{hG} on a G-space X is the orbit space $EG \times_G X$ of the diagonal action of G on $EG \times X$. The orbit space EG/H, where $H \le G$, is the classifying space BH of H.

We now fix a collection $\mathcal{H}$ in a compact Lie group G. Throughout, $G\mathcal{T}$ denotes the category of compactly generated G-spaces.

1.2. *The subgroup decomposition.* Let $\mathcal{O}_G(\mathcal{H})$ denote the full subcategory of the category $G\mathcal{T}$ of G-spaces whose objects are the orbits G/H for all $H \in \mathcal{H}$ and whose morphism spaces are $\mathrm{map}_G(G/H, G/H')$. By abuse of notation we identify the objects G/H with the subgroups $H \in \mathcal{H}$. Clearly $\mathcal{O}_G(\mathcal{H})$ is a topological category in the sense that morphism sets carry non-trivial topologies (see 5.1). Let $\tilde{\beta}_{\mathcal{H}}$ denote the inclusion of $\mathcal{O}_G(\mathcal{H})$ into $G\mathcal{T}$. The *subgroup decomposition* functor $\beta_{\mathcal{H}}$ is defined using the Borel construction by

$$\beta_{\mathcal{H}} := (\tilde{\beta}_{\mathcal{H}}) \times_G EG \equiv (\tilde{\beta}_{\mathcal{H}})_{hG}.$$

Note that $\beta_{\mathcal{H}}(H) \simeq BH$ and the natural cone $\tilde{\beta}_{\mathcal{H}} \to *$ [26, §III.3] induces a natural cone $\beta_{\mathcal{H}} \to BG$, whence a natural map

$$\operatorname*{hocolim}_{\mathcal{O}_G(\mathcal{H})} \beta_{\mathcal{H}} \to BG.$$

1.3. *The centraliser decomposition.* Let $\mathcal{A}_G(\mathcal{H})$ denote the topological category whose objects are the elements H of $\mathcal{H}$ and whose morphism spaces are

$$\mathrm{Hom}_G(H, H') := \{c_g : H \to H' \ : \ g \in G\}.$$

Define a functor $\tilde{\alpha}_{\mathcal{H}} : \mathcal{A}_G(\mathcal{H})^{\mathrm{op}} \to G\mathcal{T}$ by

$$\tilde{\alpha}_{\mathcal{H}}(H) = \mathrm{Hom}_G(H, G).$$

Note that $\mathrm{Hom}_G(H, G) \approx G/C_G H$. Define the centraliser decomposition $\alpha_{\mathcal{H}}$ by $\alpha_{\mathcal{H}} := (\tilde{\alpha}_{\mathcal{H}})_{hG}$. Clearly $\alpha_{\mathcal{H}}(H) \simeq BC_G(H)$ and the cone $\tilde{\alpha}_{\mathcal{H}} \to *$ gives rise to a cone $\alpha_{\mathcal{H}} \to BG$ and a natural map

$$\operatorname*{hocolim}_{\mathcal{A}_G(\mathcal{H})^{\mathrm{op}}} \alpha_{\mathcal{H}} \to BG.$$

1.4. *The normaliser decomposition.* Let $\bar{s}(\mathcal{H})$ denote the poset whose underlying set is the set of conjugacy classes $(H_0 < \cdots < H_n)$ of chains of *proper* inclusions in $\mathcal{H}$. Denote such a chain by $\mathbf{H}$ and its conjugacy class by $(\mathbf{H})$. Define a unique morphism $(\mathbf{H}) \to (\mathbf{H}')$ in $\bar{s}(\mathcal{H})$ if $\mathbf{H}'$ is conjugate to a subchain of $\mathbf{H}$. Note that the subchain of $\mathbf{H}$, if it exists, is unique and is determined by the dimensions and the number of components of the groups in the chains $\mathbf{H}$ and $\mathbf{H}'$. There is a tautological functor

$$\tilde{\delta}_{\mathcal{H}} : \bar{s}(\mathcal{H}) \to G\mathcal{T}$$

which sends the object $(\mathbf{H})$ to the G-space $(\mathbf{H})$. A morphism $(\mathbf{H}) \to (\mathbf{H}')$ in $\bar{s}(\mathcal{H})$ is carried to the obvious G-map $(\mathbf{H}) \to (\mathbf{H}')$ which sends $\mathbf{H}$ to its unique subchain that belongs to $(\mathbf{H}')$. We may assume that $\mathbf{H}' \subset \mathbf{H}$, and $\tilde{\delta}_{\mathcal{H}}((\mathbf{H})) \to \tilde{\delta}_{\mathcal{H}}((\mathbf{H}'))$ correspond to the quotient map $G/N_G(\mathbf{H}) \to G/N_G(\mathbf{H}')$.

The *normaliser decomposition* $\delta_{\mathcal{H}} : \bar{s}(\mathcal{H}) \to \mathcal{T}$ is defined by $\delta_{\mathcal{H}} = (\tilde{\delta}_{\mathcal{H}})_{hG}$. Clearly $\delta_{\mathcal{H}}((\mathbf{H})) \simeq BN_G(\mathbf{H})$ and the natural cone $\tilde{\delta}_{\mathcal{H}} \to *$ gives rise to a natural map

$$\operatorname*{hocolim}_{\bar{s}(\mathcal{H})} \delta_{\mathcal{H}} \to BG.$$

When G is finite the following result is due to Dwyer [13]. We generalise it to compact Lie groups.

THEOREM C. *Fix a collection $\mathcal{H}$ in a compact Lie group G. Then there are natural weak homotopy equivalences*

$$\operatorname*{hocolim}_{\mathcal{O}_G(\mathcal{H})} \beta_{\mathcal{H}},\ \operatorname*{hocolim}_{\mathcal{A}_G(\mathcal{H})^{\mathrm{op}}} \alpha_{\mathcal{H}},\ \operatorname*{hocolim}_{\bar{s}(\mathcal{H})} \delta_{\mathcal{H}} \to |\mathcal{H}|_{hG}.$$

Proof. See §10. ∎

DEFINITION 1.5 (Dwyer, [13, 1.3]). Fix a generalised cohomology theory h^*. A collection $\mathcal{H}$ is *h^*-ample* if the natural map $|\mathcal{H}|_{hG} \to BG$ is an h^*-isomorphism.

In light of Theorem C, either the homotopy colimits of all three decompositions 1.2, 1.3 and 1.4 are h^*-equivalent to BG or none of them is.

The topology on the morphism spaces of $\mathcal{O}_G(\mathcal{H})$ and $\mathcal{A}_G(\mathcal{H})$ is in general non-trivial. When $\mathcal{O}_G(\mathcal{H})$ has discrete morphism spaces we obtain a Bousfield Kan spectral sequence [9, Ch. XII.5.8]

$$E_2^{i,j} = \varprojlim_{\mathcal{O}_G(\mathcal{H})^{\mathrm{op}}}{}^{i}\ h^j(\beta_{\mathcal{H}}) \ \Rightarrow\ h^{i+j}(\operatorname{hocolim} \beta_{\mathcal{H}}) \approx h^{i+j}(|\mathcal{H}|_{hG}).$$

The isomorphism of the abutment modules follows from Theorem C. Similarly, when $\mathcal{A}_G(\mathcal{H})$ has discrete morphism spaces we get a spectral sequence

$$E_2^{i,j} = \varprojlim_{\mathcal{A}_G(\mathcal{H})}{}^{i}\ h^j(\alpha_{\mathcal{H}}) \ \Rightarrow\ h^{i+j}(\operatorname{hocolim} \alpha_{\mathcal{H}}) \approx h^{i+j}(|\mathcal{H}|_{hG}).$$

This happens, for example, when $\mathcal{H} = \mathcal{E}_p^0(G)$, and more generally when the elements of $\mathcal{H}$ are finite groups. By definition $\bar{s}\mathcal{H}$ is a poset so we always have a spectral sequence for the normaliser decomposition

$$E_2^{i,j} = \varprojlim_{\bar{s}(\mathcal{H})^{\mathrm{op}}}{}^{i}\ h^j(\delta_{\mathcal{H}}) \ \Rightarrow\ h^{i+j}(\operatorname{hocolim} \delta_{\mathcal{H}}) \approx h^{i+j}(|\mathcal{H}|_{hG}).$$

DEFINITION 1.6. A collection $\mathcal{H}$ is *subgroup sharp* (resp. *centraliser sharp*, *normaliser sharp*) for h^* if it is h^*-ample and if the Bousfield–Kan spectral sequence of $\beta_{\mathcal{H}}$ (resp. $\alpha_{\mathcal{H}}$, $\delta_{\mathcal{H}}$) collapses at its E_2-page to the vertical axis. That is, all the higher derived functors $\varprojlim^i$ vanish for $i > 0$.

DEFINITION 1.7. A cohomology theory h^* is *p-local* if $\pi_* h$ are $\mathbb{Z}_{(p)}$-modules where h is the spectrum which represents $\tilde{h}^*$.

Equivalently, by Proposition A.3.1, h is $S\mathbb{Z}_{(p)}$-local.

THEOREM D. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup. Then the collections $\mathcal{E}_p^0(G)$ and $\mathcal{B}_p^0(G)$ are normaliser sharp* *for any p-local cohomology theory h^*. The collection $\mathcal{E}_p^0(G)$ is centraliser sharp for any p-local cohomology theory h^*.*

Proof. This is immediate from Corollaries 7.11, 8.4(b) and 9.6(b). ∎

A special case of the first assertion of Theorem D was obtained by Dwyer in [14] for finite groups and the cohomology theory $H^*(-,\mathbb{Z}/p)$. Our extension of his result requires totally different methods. The second assertion was obtained by Jackowski and McClure in [22] but we use different methods.

2. Mackey functors and Bredon cohomology. Fix a compact Lie group G. Recall that a G-CW *complex* is a G-space X together with a filtration $X_0 \subseteq X_1 \subseteq \cdots$ such that $X = \bigcup_n X_n$ and the following holds. There exists a sequence of G-spaces $\mathfrak{X}_0, \mathfrak{X}_1, \ldots$, each of which is a coproduct of orbits, that is, $\mathfrak{X}_n/G$ is a discrete space, and for every $n \geq 0$ there is a pushout square

$$\begin{tikzcd} \partial\Delta^n \times \mathfrak{X}_n \arrow[r, "\varphi_n"] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ \Delta^n \times \mathfrak{X}_n \arrow[r] & X_n \end{tikzcd} \tag{2.1}$$

The φ_n's are called the *attaching maps* and they are part of the structure of X as a G-CW complex. The spaces $\mathfrak{X}_n$ are called the *spaces of n-cells* and X_{-1} is by convention the empty space.

We shall denote the category of G-spectra by $G\mathcal{S}$ as constructed by Lewis May and Steinberger in the encyclopedic account [25]. A more readable exposition can be found in [28] and also in [17]. Throughout we shall assume familiarity with the terminology of these sources. To every G-space X there is an associated G-spectrum X_+. The homotopy category $\bar{h}G\mathcal{S}$ of $G\mathcal{S}$ is a triangulated category, and in particular an additive one.

The category $G\mathcal{O}$ of stable orbits is the full subcategory of $G\mathcal{S}$ whose objects are the suspension G-spectra G/H_+ for all $H \leq G$. Its homotopy category is the full subcategory $hG\mathcal{O}$ of $\bar{h}G\mathcal{S}$.

Fix a commutative ring k. Recall that a *Mackey functor* is a contravariant additive functor $M : hG\mathcal{O}^{\mathrm{op}} \to k\text{-}\mathfrak{mod}$. Mackey functors form an abelian category and they give rise to Eilenberg–Mac Lane G-spectra HM of type $(M,0)$ as explained in [28, Ch. XIII, Theorem 4.1]. Explicitly, $\pi_0^K(HM) = M(G/K_+)$ and $\pi_{i\neq 0}^K(HM) = 0$.

Definition 2.2. The G-equivariant cohomology theory represented by HM is called the *reduced Bredon cohomology* with coefficients M and is denoted $\tilde{H}_G^*(-;M)$. Thus, $\tilde{H}_G^V(-;M) = [\Sigma^V -, HM]_G$ for every G-representation V. There is an unreduced theory $H^*_G(-;M) := \tilde{H}^*_G(-_+;M)$ defined on G-spaces.

Note that $*$ can be any finite-dimensional representation of G. We shall however restrict attention to trivial representations, i.e. $*$ is an integer. Also observe that by definition $H^0_G(G/H;M) = M(G/H_+)$ and $H^{*\neq 0}_G(G/H;M) = 0$. Fix a G-CW complex X and consider the chain complex of Mackey functors $\underline{C}_*(X)$ defined by

$$\underline{C}_n(X)(G/H_+) = \bar{h}G\mathcal{S}(S^n \wedge G/H_+, X_n/X_{n-1}) = \pi^H_n(X_n/X_{n-1}).$$

The differentials are induced by the connecting maps $X_n/X_{n-1} \to \Sigma X_{n-1}/X_{n-2}$ associated to the triple (X_n, X_{n-1}, X_{n-2}). According to [28, Ch. X, §4], the cohomology groups $H^*_G(X;M)$ are isomorphic to the cohomology groups of the cochain complex $\mathrm{Hom}_{hG\mathcal{O}}(\underline{C}_*(X), M)$. It follows immediately that the skeletal filtration of X gives an isomorphism

$$H^*_G(X;M) = \varprojlim_k H^*_G(X_k;M). \tag{2.3}$$

In fact for each $*$ the tower stabilises. More generally, X_k can be replaced with any subcomplex which contains the k-skeleton of X. It also follows that the cohomology groups vanish for $* < 0$.

Proposition 2.4. *If Y is a G-subcomplex of X and $M(G/H_+) = 0$ for every orbit G/H in $X \setminus Y$ then $H^*_G(Y;M) \approx H^*_G(X;M)$.*

Proof. By (2.3) it suffices to prove the isomorphism for the inclusion of the skeleta $Y_n \subseteq X_n$. Equivalently, we have to show that $\tilde{H}^*_G(X_n/Y_n;M) = 0$. This follows easily by induction using the cofibre sequences $X_{n-1}/Y_{n-1} \to X_n/Y_n \to X_n/Y_n \cup X_{n-1} \simeq \bigvee_\alpha (S^n \wedge G/H_{\alpha+})$ where $G/H_\alpha \subseteq X \setminus Y$. ∎

We shall now define a class of Mackey functors which is central to this paper. Fix a G-free G-CW complex EG. Define functors $\underline{B}, \underline{S} : hG\mathcal{O} \to \bar{h}\mathcal{S}$, where $\bar{h}\mathcal{S}$ is the homotopy category of spectra, as follows:

$$\underline{B}(-) = EG_+ \wedge_G -, \qquad \underline{S}(-) = S.$$

Here S denotes the sphere spectrum.

Definition 2.5. Fix a ring k and a spectrum h such that $\pi_0 h$ is a k-module. Define functors $\mathcal{M}_h, \mathrm{Const}_h : hG\mathcal{O}^{\mathrm{op}} \to k\text{-}\mathfrak{mod}$ by

$$\mathcal{M}_h(-) = \tilde{h}^0 \underline{B}(-), \qquad \mathrm{Const}_h(-) = \tilde{h}^0 \underline{S}(-).$$

Note that $\mathcal{M}_h(G/H_+) \approx h^0(BH)$ and $\mathrm{Const}_h(G/H) = \pi_0 h$.

Proposition 2.6. *The functors $\mathcal{M}_h$ and Const_h are Mackey functors. Moreover, there is a morphism of Mackey functors $\mathcal{M}_h \to \mathrm{Const}_h$ which is a split surjection at every object of $hG\mathcal{O}$.*

REMARK. The basic idea of the proof, due to John Greenlees, is to show that $\mathcal{M}_h(-) = [-, F(EG_+, h)]_G$, that is, $\mathcal{M}_h$ is a restriction of a representable functor on $\bar{h}G\mathcal{S}$. Then we show that $\mathrm{Const}_h = \mathcal{M}_k$ where $\tilde{k}^* = \tilde{H}^*(-;\pi_0 h)$.

Proof. Step 1. Fix a complete G-universe U and let $i : U^G \to U$ denote the inclusion [28, §XII]. We view h as a naive G-spectrum, i.e. an object of $G\mathcal{S}U^G$ with a trivial G-action. Let i_*h denote the G-spectrum in $G\mathcal{S}U$ which is built out of the naive G-spectrum h by building non-trivial representations (see [28, p. 163]). Recall from [28, bottom of p. 165] that EG_+ is a G-free G-spectrum and consider the G-spectrum $F(EG_+, i_*h)$. Observe that for every G/K_+ in $hG\mathcal{O}$ we have

$$\begin{aligned}\bar{h}G\mathcal{S}(G/K_+, F(EG_+, i_*h)) &= \pi_0^G\, F(G/K_+, F(EG_+, i_*h))\\ &= \pi_0^G\, F(G/K_+ \wedge EG_+, i_*h)\\ &= \bar{h}G\mathcal{S}(i_*(G/K_+ \wedge EG_+), i_*h)\\ &= \bar{h}\mathcal{S}(EG/K_+, h) = \tilde{h}^0\underline{B}(G/K_+) = \mathcal{M}_h(G/K_+).\end{aligned}$$

The second equality follows from [28, Ch. XI, §4] and the third because the functor $\Sigma^\infty : G\mathcal{T} \to G\mathcal{S}U$ factors as $G\mathcal{T} \xrightarrow{-_+} G\mathcal{S}U^G \xrightarrow{i_*} G\mathcal{S}U$. The fourth equality follows from [17, Theorem 4.14]. We see that

$$\mathcal{M}_h(-) = [-, F(EG_+, i_*h)]_G. \tag{1}$$

In particular it is a Mackey functor.

Step 2. Let HA denote the Eilenberg–Mac Lane spectrum of type $(0, A)$ where $A = \pi_0 h$. Since BH is a connected space for any $H \le G$ there are isomorphisms

$$\mathcal{M}_k(G/H_+) = \tilde{k}^0(BH_+) = \tilde{H}^0(BH_+; A) \approx \tilde{H}^0(S^0; A) = \mathrm{Const}_h(G/H_+).$$

This shows that $\mathrm{Const}_h = \mathcal{M}_k$ and it is therefore also a Mackey functor.

Step 3. We now construct $\mathcal{M}_h \to \mathrm{Const}_h$. Let ℓ be the connected cover of h. That is, $\pi_*\ell = 0$ for all $* < 0$ and there is a map $\ell \to h$ which induces an isomorphism in all non-negative homotopy groups. By induction, it is easy to show that $\ell^*(X) \to h^*(X)$ is an isomorphism for all $* \le 0$ and for all finite-dimensional CW-complexes X. It follows that $\ell^*(X) \approx h^*(X)$ for all $* \le 0$ and all CW-complexes X by filtering X by its skeleta and using Milnor's $\varprojlim^1$ short exact sequence [1, Ch. III, Proposition 8.1]. In particular we obtain an isomorphism

$$\mathcal{M}_\ell \approx \mathcal{M}_h. \tag{2}$$

Let k denote the 0th Postnikov piece of ℓ and note that it is an Eilenberg–Mac Lane spectrum of type $(\pi_0 h, 0)$. We have seen that $\mathcal{M}_k = \mathrm{Const}_\ell = \mathrm{Const}_h$.

The maps of spectra $h \leftarrow \ell \rightarrow k$ induce morphisms of G-spectra

$$F(EG_+, i_*h) \leftarrow F(EG_+, i_*\ell) \rightarrow F(EG_+, i_*k),$$

hence, by (1), natural morphisms of Mackey functors

$$\mathcal{M}_h \xleftarrow{\approx} \mathcal{M}_\ell \rightarrow \mathcal{M}_k = \mathrm{Const}_h.$$

We have thus constructed a natural transformation $\mathcal{M}_h \rightarrow \mathrm{Const}_h$.

Let e denote the trivial subgroup of G. For every object G/K_+ of $hG\mathcal{O}$ we can choose a map $G/e_+ \rightarrow G/K_+$ and obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_\ell(G/K_+) \arrow[r] \arrow[d] & \mathcal{M}_k(G/K_+) \arrow[r, equal] \arrow[d, "\approx"] & \mathrm{Const}_h(G/K_+) \\
\mathcal{M}_\ell(G/e) \arrow[r, "\approx"] & \mathcal{M}_k(G/e) &
\end{tikzcd}$$

where the arrow at the bottom is an isomorphism because $\pi_0\ell = \pi_0 k$ and the vertical arrow on the right is an isomorphism because BK is a connected space and $\tilde{k}^0(-) = \tilde{H}^0(-;\pi_0 h)$. We see from (2) that $\mathcal{M}_h(G/K_+) \rightarrow \mathrm{Const}_h(G/K_+)$ can be identified with $\mathcal{M}_\ell(G/K_+) \rightarrow \mathcal{M}_\ell(G/e)$, that is, with $\ell^0(BK) \rightarrow \ell^0(Be)$. This map clearly has a left inverse $\ell^0(BK \rightarrow *)$. ∎

Proposition 2.6 justifies the definition of a new Mackey functor $\widetilde{\mathcal{M}}_h$ and the proposition below, where

$$\widetilde{\mathcal{M}}_h = \mathrm{Ker}(\mathcal{M}_h \rightarrow \mathrm{Const}_h). \tag{2.7}$$

Proposition 2.8. *There is a short exact sequence of Mackey functors*

$$0 \rightarrow \widetilde{\mathcal{M}}_h \rightarrow \mathcal{M}_h \rightarrow \mathrm{Const}_h \rightarrow 0.$$

Definition 2.9. A Mackey functor M on G is *p-constrained* if an inclusion $H \le K$ of subgroups of G induces an isomorphism $M(G/K_+) \xrightarrow{\approx} M(G/H_+)$ whenever H and K contain a p-toral subgroup which is maximal in both.

Lemma 2.10. *Let G be a p-toral group* (A.1) *and fix a p-local cohomology theory h^** (1.7). *Then $\mathcal{M}_h$,* Const_h *and $\widetilde{\mathcal{M}}_h$ are p-constrained Mackey functors.*

Proof. The Mackey functor Const_h is trivially p-constrained. The short exact sequence of Mackey functors in 2.8 implies that $\widetilde{\mathcal{M}}_h$ is p-constrained if M_h is p-constrained, which we now prove.

Let P be a p-toral group which is maximal in both H and K. Proposition A.1.3 shows that $P_0 = H_0 = K_0$ and that there are isomorphisms

$$H/O_{p'}(H) \cong K/O_{p'}(K) \cong P/P_0,$$

the first of which is induced by the inclusion of $O_{p'}(H)$ in $O_{p'}(K)$. As a result we obtain a morphism of fibre sequences

$$
\begin{tikzcd}
BO_{p'}(H) \arrow[r] \arrow[d] & BH \arrow[r] \arrow[d] & B\frac{P}{P_0} \arrow[d, "="] \\
BO_{p'}(K) \arrow[r] & BK \arrow[r] & B\frac{P}{P_0}
\end{tikzcd}
\tag{1}
$$

To prove that $h^*BH \approx h^*BK$ it suffices, by Proposition A.3.3, to prove that $BH \to BK$ is an $H_*(-;\mathbb{Z}_{(p)})$-equivalence. By comparing the E^2-pages of the associated Serre spectral sequences it suffices to show that $BO_{p'}(H) \to BO_{p'}(K)$ is an $H_*(-;\mathbb{Z}_{(p)})$-isomorphism. This is indeed the case because A.1.3 implies that $O_{p'}(H) \leq O_{p'}(K)$ are abelian compact Lie groups of the form $P_0 \times \Gamma'$ where Γ' is a finite abelian group of order prime to p. ∎

3. The key lemma. Fix a collection $\mathcal{F}$ of subgroups of a compact Lie group G. A G-space has *orbit type* $\mathcal{F}$ if the collection $\mathrm{Iso}_G(X)$ of the isotropy groups of the points of X is contained in $\mathcal{F}$. The collection of the maximal p-toral subgroups of the elements of $\mathcal{F}$ is denoted $\mathrm{Syl}_p(\mathcal{F})$.

Definition 3.1. Fix a collection $\mathcal{P}$ of p-toral subgroups of G. The *p-type* of a collection $\mathcal{F}$ is $\mathcal{P}$ if $\mathrm{Syl}_p(\mathcal{F}) \subseteq \mathcal{P}$. It has a *finite* p-type if $\mathrm{Syl}_p(\mathcal{F})/G < \infty$. A G-space has p-type $\mathcal{P}$ (resp. finite p-type) if $\mathrm{Iso}_G(X)$ has the corresponding properties.

Here is our key lemma.

Lemma 3.2. *Let $f : X \to Y$ be a G-map of G-CW complexes of finite p-type $\mathcal{P}$. Assume that f induces an NP-homotopy equivalence $X^P \to Y^P$ for every $P \in \mathcal{P}$. Then*

(i) *f induces a homotopy equivalence $X/G \to Y/G$.*
(ii) *f induces an isomorphism $H^*_G(Y;M) \to H^*_G(X;M)$ for every p-constrained Mackey functor M (2.9).*

Recall that X^H is an NH-space for every G-space X and $H \leq G$. Define

$$Z_H(X) = G \times_{NH} X^H. \tag{3.3}$$

The assignment $X \mapsto Z_H(X)$ is clearly functorial in X and the assignment $(g \times_{NH} x) \mapsto gx$ yields a natural map $\epsilon : Z_H(X) \to X$.

Note that when $i : A \hookrightarrow X$ is a closed inclusion of G-spaces and $f : A \to Y$ is a G-map, then the pushout $X \sqcup_A Y$ is setwise the disjoint union of Y and $X \setminus A$. It follows that the natural map $X^H \sqcup_{A^H} Y^H \to (X \sqcup_A Y)^H$ is a bijection. One also easily checks that this map is closed because X^H and Y^H are closed subspaces of X and Y, whence it is a homeomorphism. See e.g. [38, pp. 95–96].

PROPOSITION 3.4. *Let H be a subgroup of G and set $Z(-) = Z_H(-)$. Then $\epsilon : Z(X) \to X$ is a closed map. If X' is a closed subspace of X then $Z(X')$ is a closed subspace of $Z(X)$.*

(a) *Consider a G-map $f : A \to Y$ and an inclusion $i : A \hookrightarrow X$ of G-subspaces. Then $Z(X) \sqcup_{Z(A)} Z(Y)$ is homeomorphic to $Z(X \sqcup_A Y)$ via the natural map.*
(b) *If $X_0 \subseteq X_1 \subseteq \cdots$ is a telescope of closed inclusions then $Z(\operatorname{colim}_n X_n) \cong \operatorname{colim}_n Z(X_n)$.*

Proof. A closed subset F of $Z(X)$ is the NH-orbit of a closed subspace A of $G \times X^H$ which is closed in $G \times X$. Now, $\epsilon(F)$ is closed in X as the image of A under the action map $G \times X \to X$ which is a closed map by [10, Theorem I.1.2].

If X' is closed in X then $G \times X'^H$ is a closed subspace of $G \times X^H$ and the NH-orbit space $Z(X')$ is a closed subspace of $Z(X)$ by [10, Theorem I.3.1].

Note that $G \times_{NH} -$ is a left adjoint functor. Therefore it commutes with direct limits. Given $i : A \hookrightarrow X$ and $f : A \to Y$ we know that $X^H \sqcup_{A^H} Y^H \cong (X \sqcup_A Y)^H$ so applying $G \times_{NH} -$ yields $Z(X) \sqcup_{Z(A)} Z(Y) \cong Z(X \sqcup_A Y)$.

Consider a telescope $X_0 \subseteq X_1 \subseteq \cdots$. Since by definition a subset A of $X = \operatorname{colim}_n X_n$ is closed if and only if $X_n \cap A$ is closed in X_n and since X_n^H is closed in X_n, it easily follows that the natural map $\operatorname{colim}_n X_n^H \to (\operatorname{colim}_n X_n)^H$ is closed. It is also bijective, hence a homeomorphism. ∎

The proposition implies that if A and B are closed G-subspaces of X then $Z_H(A \cup B) = Z_H(A) \cup Z_H(B)$. Here are two simple but important observations.

PROPOSITION 3.5. *Let P be a p-toral subgroup of $H \le G$. Then P is maximal p-toral in H if and only if it is maximal p-toral in $N_H P$.*

Proof. Extend P to a maximal p-toral subgroup of H and apply Lemma A.1.1. ∎

LEMMA 3.6. *Fix a compact Lie group G. Fix $H \le G$ and let P be a maximal p-toral subgroup of H. Then $(G/H)^P$ is homeomorphic to the NP-orbit $NP/N_H(P)$.*

Proof. If $gH \in (G/H)^P$ then $P^g \le H$. Since P is maximal p-toral in H, $P^g = P^h$ for some $h \in H$, hence $g \in NP \cdot H$. This shows that $G/H^P \subseteq NP \cdot H/H$, and the opposite inclusion is obvious. It follows that $G/H^P \approx NP/H \cap NP$. ∎

PROPOSITION 3.7. *Fix a p-toral subgroup P of G. Let X' be a subcomplex of a G-CW complex X and assume that $X \setminus X'$ has p-type $\{(P)\}$, that is,*

P is conjugate to a maximal p-toral subgroup of the isotropy group G_x of any $x \in X \setminus X'$. Let Z denote $Z_P(X)$ and let Z' denote $Z_P(X')$. Then

(a) *(Z, Z') is a relative G-CW complex.*
(b) *The map $\epsilon : (Z/G, Z'/G) \to (X/G, X'/G)$ is a relative homeomorphism of relative G-CW complexes.*
(c) *The map $\epsilon : (Z, Z') \to (X, X')$ induces an isomorphism $H^*_G(X, X'; M) \approx H^*_G(Z, Z'; M)$ for every p-constrained Mackey functor M (2.9).*

Proof. Following the notation of (2.1) let $\mathfrak{X}'_n \subset \mathfrak{X}_n$ denote the spaces of n-cells of $X' \subseteq X$. Let $\bar{\mathfrak{X}}_n$ denote $\mathfrak{X}_n \setminus \mathfrak{X}'_n$ and observe that there are pushout squares

$$\begin{tikzcd} S^{n-1} \times \bar{\mathfrak{X}}_n \arrow[r] \arrow[d, hook] & X_{n-1} \cup X' \arrow[d, hook] \\ D^n \times \bar{\mathfrak{X}}_n \arrow[r] & X_n \cup X' \end{tikzcd} \tag{1}$$

Proposition 3.4(a) shows that there are pushout squares

$$\begin{tikzcd} S^{n-1} \times Z_P(\bar{\mathfrak{X}}_n) \arrow[r] \arrow[d, hook] & Z_P(X_{n-1}) \cup Z_P(X') \arrow[d, hook] \\ D^n \times Z_P(\bar{\mathfrak{X}}_n) \arrow[r] & Z_P(X_n) \cup Z_P(X') \end{tikzcd} \tag{2}$$

Now, $Z_P(\bar{\mathfrak{X}}_n)$ is a disjoint union of orbits by Lemma 3.6 so the pushout squares (2) and Proposition 3.4(b) imply that (Z, Z') is a relative G-CW complex with skeletal filtration $Z_P(X_n) \cup Z'$ whose n-cells are $D^n \times Z_P(\bar{\mathfrak{X}}_n)$. This proves point (a).

For convenience, let X_G denote the orbit space of a G-space X. Clearly ϵ induces morphisms of pushout squares $\bar{\epsilon} : (2) \to (1)$. Since the p-type of the $\bar{\mathfrak{X}}_n$'s is (P), Proposition 3.6 shows that $Z_P(\bar{\mathfrak{X}}_n)/G \to \bar{\mathfrak{X}}_n/G$ is a bijection of sets.

Note that $\epsilon : Z \to X$ carries $Z \setminus Z'$ into $X \setminus X'$ and Z' into X'. Since taking orbit spaces commutes with pushouts, induction on n easily shows using the morphisms of pushout diagrams $\bar{\epsilon}_G : (2)/G \to (1)/G$ that

$$(Z_P(X_n) \setminus Z_P(X'))_G \to (X_n \setminus X')_G$$

is a bijection. Orbit spaces also commute with telescopes so we conclude that $\bar{\epsilon} : (Z \setminus Z')_G \to (X \setminus X')_G$ is a bijection. It is a homeomorphism because it is a closed map by [10, Theorem I.3.1], Proposition 3.4 and the fact that $\epsilon(Z') \subseteq X'$. This proves point (b).

We use induction on n to prove that $\epsilon : Z_P(X_n) \to X_n$ induces isomorphisms

$$H^*_G(X_n \cup X', X'; M) \approx H^*_G(Z_P(X_n) \cup Z', Z'; M).$$

For $n = 0$ this is a triviality. Assume the result for $n-1 \geq 0$. Propositions 3.5 and 3.6, the pushout squares (1) and (2) and the fact that M is p-constrained imply that ϵ induces an isomorphism

$$\begin{aligned}\tilde{H}^*_G(X_n \cup X'/X_{n-1} \cup X'; M) &\approx \tilde{H}^*_G(S^n \wedge \bar{\mathfrak{X}}_{n+}; M)\\ &\approx \tilde{H}^*_G(S^n \wedge Z_P(\bar{\mathfrak{X}}_n)_+; M) \approx \tilde{H}^*_G(Z_P(X_n) \cup Z'/Z_P(X_{n-1}) \cup Z'; M).\end{aligned}$$

The second and third isomorphism follow from [38, II.1.1]. The induction step follows by comparing the long exact sequences in cohomology of

$$(Z_P(X_n) \cup Z',\ Z_P(X_{n-1}) \cup Z') \xrightarrow{\epsilon} (X_n \cup X',\ X_{n-1} \cup X').$$

Finally,

$$\begin{aligned}\tilde{H}^*_G(X/X'; M) &\approx \varprojlim_n \tilde{H}^*_G(X_n \cup X'/X'; M)\\ &\approx \varprojlim_n \tilde{H}^*_G(Z_P(X_n) \cup Z'/Z'; M) \approx \tilde{H}^*_G(Z/Z'; M)\end{aligned}$$

by using (2.3). ∎

Definition 3.8. A subcollection $\mathcal{F}'$ of $\mathcal{F}$ is called *concave* if whenever $H' \leq H$ are subgroups such that $H' \in \mathcal{F}'$ and $H \in \mathcal{F}$, then $H \in \mathcal{F}'$.

In [38] tom Dieck calls $\mathcal{F}'$ closed in $\mathcal{F}$. The term "concave" which is used by Słomińska in [34] seems more appropriate because later on the collection $\mathcal{F}$ will be considered as a topological space and by using the word "closed" we run into a possible risk of confusion.

Definition 3.9. Given a G-space X and a collection $\mathcal{F}$ let $X_{\mathcal{F}}$ denote the subspace of X consisting of the points x such that $G_x \in \mathcal{F}$.

Definition 3.10. Fix a collection $\mathcal{P}$ of p-toral subgroups of G. Let type-$\mathcal{P}$ denote the collection of those subgroups of G whose maximal p-toral subgroups belong to $\mathcal{P}$.

Proposition 3.11. *Let $\mathcal{P}$ be a collection of p-toral subgroups of G and let $\mathcal{P}'$ be a concave subcollection. Then $X \mapsto X_{\text{type-}\mathcal{P}'}$ defines a functor*

$$\{G\text{-spaces of } p\text{-type } \mathcal{P}\} \xrightarrow{X \mapsto X'} \{G\text{-spaces of } p\text{-type } \mathcal{P}'\}.$$

which preserves G-homotopic maps and G-homotopy equivalences. It carries G-CW complexes to G-CW complexes.

Proof. Lemma A.1.1 clearly implies that $\mathcal{F}' := \text{type-}\mathcal{P}'$ is a concave subcollection of $\mathcal{F} := \text{type-}\mathcal{P}$. T. tom Dieck shows in [38, I.§6] that $X' := X_{\mathcal{F}'}$ is a closed subspace of X. Since G-maps must increase orbit type we also see that any G-map $f : X \to Y$ restricts to a G-map $f' : X' \to Y'$. Similarly a homotopy $h : X \times I \to Y$ restricts to $h' : X' \times I \to Y'$. If X is a G-CW complex then in the notation of (2.1) one easily checks that X' is a G-CW complex whose spaces of n-cells are $\mathfrak{X}'_n$. See e.g. [38, II.1.12]. ∎

Remark 3.12. Well order the set $\mathbb{N}\times\mathbb{N}$ lexicographically, that is, $(n,m)\prec(n',m')$ if and only if $n<n'$ or $n=n'$ and $m<m'$. Define the *order* of a compact Lie group $o(G)$ as the pair $(\dim G, \#\pi_0 G)$. It is not hard to see that if $G\le G'$ then $o(G)\preceq o(G')$ and equality holds if and only if $G=G'$.

As a consequence every compact Lie group satisfies the descending chain condition for subgroups. In particular, every collection of subgroups of G has a minimal element.

For $H\le G$ let WH denote NH/H and recall X^H is a WH-space. Also recall that $\mathcal{S}_p(G)$ denotes the collection of all the p-toral subgroups of G.

Proposition 3.13. *Consider a non-empty collection $\mathcal{P}$ of p-toral subgroups of G and let $\mathcal{P}'$ denote $\mathcal{P}\setminus\{(P)\}$ where P is a minimal element in $\mathcal{P}$. Let X be a G-space of p-type $\mathcal{P}$ and consider $X':=X_{\text{type-}\mathcal{P}'}$ as in Proposition 3.11. Then ${X'}^P=(X^P)_{\text{type-}\mathcal{S}_p^0(WP)}$.*

Proof. Clearly $\mathcal{S}_p^0(WP)$ is concave in $\mathcal{S}_p(WP)$, whence Proposition 3.11 applies to X^P. Note that the maximal p-toral subgroup of a compact Lie group K is trivial if and only if K is finite of order prime to p. In light of Proposition 3.5 and the fact that the class of p-toral groups is closed under extensions, the following statements are equivalent:

(a) $x\in X^P\setminus {X'}^P$.
(b) $x\in X^P$ and G_x contains P as a maximal p-toral subgroup.
(c) $x\in X^P$ and $N_{G_x}(P)$ contains P as a maximal p-toral subgroup.
(d) $x\in X^P$ and the maximal p-toral subgroup of $N_{G_x}P/P$ is trivial.
(e) $x\in X^P$ and $WP_x\notin \text{type-}\mathcal{S}_p^0(WP)$, i.e. $x\notin (X^P)_{\text{type-}\mathcal{S}_p^0(WP)}$. ∎

Proof of Lemma 3.2. We prove the result by inducting on the size of $\mathcal{P}/G$. If $\mathcal{P}$ is empty then so are X and Y and the result is trivial.

Assume that the result holds whenever $|\mathcal{P}/G|=n-1\ge 0$ and assume that $|\mathcal{P}/G|=n$. Let P be a minimal element in $\mathcal{P}$ and set $\mathcal{P}':=\mathcal{P}\setminus\{(P)\}$. Clearly $\mathcal{P}'$ is concave in $\mathcal{P}$ and we denote $X':=X_{\text{type-}\mathcal{P}'}$ and $Y':=Y_{\text{type-}\mathcal{P}'}$. Note that f induces a morphism of commutative squares

$$(1)\qquad \begin{array}{ccc} G\times_{NP}{X'}^P & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \\ G\times_{NP}X^P & \longrightarrow & X \end{array} \quad\xrightarrow{f}\quad \begin{array}{ccc} G\times_{NP}{Y'}^P & \longrightarrow & Y' \\ \big\downarrow & & \big\downarrow \\ G\times_{NP}Y^P & \longrightarrow & Y \end{array}$$

Clearly, $\mathrm{Syl}_p(\mathrm{Iso}_G(X\setminus X'))\subseteq\{(P)\}$. Since P is minimal in $\mathcal{P}$, if $Q\in\mathcal{P}'$ then $(X\setminus X')^Q$ is empty because $(Q)\not\le(P)$. We see that ${X'}^Q=X^Q$ and similarly ${Y'}^Q=Y^Q$ for every $Q\in\mathcal{P}'$ and therefore by hypothesis, ${X'}^Q\to {Y'}^Q$ is an NQ-homotopy equivalence. Note that the p-type of X' and Y' is contained in $\mathcal{P}'$. The induction hypothesis applies to $f':X'\to Y'$ and $\mathcal{P}'$ and therefore there is a homotopy equivalence and an isomorphism

$$X'/G \xrightarrow{\simeq} Y'/G, \tag{2}$$

$$H_G^*(Y';M) \approx H_G^*(X';M). \tag{3}$$

By hypothesis $X^P \to Y^P$ is an NP/P-homotopy equivalence, therefore there is a homotopy equivalence and an isomorphism

$$X^P/NP \xrightarrow{\simeq} Y^P/NP, \tag{4}$$

$$H_G^*(G \times_{NP} Y^P;M) \approx H_G^*(G \times_{NP} X^P;M). \tag{5}$$

Propositions 3.13 and 3.11 now show that ${X'}^P \to {Y'}^P$ is an NP-homotopy equivalence and therefore there is a homotopy equivalence and an isomorphism

$${X'}^P/NP \xrightarrow{\simeq} {Y'}^P/NP, \tag{6}$$

$$H_G^*(G \times_{NP} {Y'}^P;M) \approx H_G^*(G \times_{NP} {X'}^P;M). \tag{7}$$

Consider the left hand square of (1) and note that the spaces in the left hand column are $Z_P(X') \subseteq Z_P(X)$. Note that $Z_P(X) \to X$ and $X' \hookrightarrow X$ are closed maps. It immediately follows from [10, Theorem I.3.1] that

$$(Z_P(X) \sqcup_{Z_P(X')} X')/G \to X/G$$

is a closed map. It is also a bijection by Proposition 3.7(b), hence a homeomorphism. This shows that by taking orbit spaces in (1), f/G induces a morphism of pushout squares. Since the vertical arrows in (1)$/G$ are cofibrations by Proposition 3.7(a), this is in fact a morphism of homotopy pushout squares. Now (2), (4) and (6) show that $X/G \to Y/G$ is a homotopy equivalence.

The isomorphisms (5) and (7) and Proposition 3.7(a) give an isomorphism

$$\tilde{H}_G^*(Z_P(Y)/Z_P(Y');M) \approx \tilde{H}_G^*(Z_P(X)/Z_P(X');M).$$

Proposition 3.7(c) now implies that

$$\tilde{H}_G^*(Y/Y';M) \approx \tilde{H}_G^*(X/X';M)$$

and together with (3) and the long exact sequences in cohomology we obtain the desired isomorphism $H_G^*(Y;M) \approx H_G^*(X;M)$. ∎

4. The transfer and G-acyclicity

Definition 4.1. A G-space X is called *G-acyclic* for the Mackey functor M if the map $X \to *$ induces an $H_G^*(-;M)$-isomorphism.

The main results of this section are Proposition 4.2, Theorem 4.3 and Lemma 4.7 below.

Proposition 4.2. *Fix a p-local cohomology theory* h^* (*see* 1.7) *and a compact Lie group* G. *Let* X *be a* G-*CW complex and assume that*

(i) $\tilde{H}^*(X/G;\mathbb{Z}_{(p)})=0$,
(ii) X *is* P-*acyclic for* $\widetilde{\mathcal{M}}_h$ (2.7) *where* P *is a maximal* p-*toral subgroup of* G.

Then X *is* G-*acyclic for* $\mathcal{M}_h$ (2.5).

Theorem 4.3. *Let* G *be a compact Lie group which contains a non-trivial* p-*toral subgroup and let* X *be a* G-*CW complex of finite orbit type. Assume that*

(a) *each isotropy group of a point of* X *contains a non-trivial* p-*toral subgroup,*
(b) X^K *is contractible for every* $K\le G$ *such that* $O_p(K)\neq 1$ (A.1.2).

Then

(i) X/G *is contractible.*
(ii) X *is* G-*acyclic for every* p-*local coefficient functor* $\mathcal{M}_h$ (2.5).

This section is a convenient place to prove Lemma 4.7. The idea goes back to Dwyer [14] as the "method of discarded orbits".

Fix a closed subgroup K of a compact Lie group G. The inclusion $\alpha: K\to G$ induces a functor of equivariant stable categories $\alpha_*: K\mathcal{S}\to G\mathcal{S}$ by $G_+\wedge_K -$, see [25, p. 75]. It clearly respects the relation of homotopy and therefore induces a functor $\alpha_*: hK\mathcal{O}\to hG\mathcal{O}$. By definition, the *restriction* of a Mackey functor M on G to K is the functor $M{\downarrow}_K^G = M\circ\alpha_*$. That is, $M{\downarrow}_K^G(K/L_+)=M(G/L_+)$.

Example 4.4. Denote the Mackey functor $\mathcal{M}_h$ (resp. $\widetilde{\mathcal{M}}_h$) on G (2.5, 2.7) by $\mathcal{M}_h^G$ (resp. $\widetilde{\mathcal{M}}_h^G$). By inspection

$$(\mathcal{M}_h^G){\downarrow}_K^G=\mathcal{M}_h^K \quad\text{and}\quad (\widetilde{\mathcal{M}}_h^G){\downarrow}_K^G=\widetilde{\mathcal{M}}_h^K$$

One easily checks using induction on the skeleta that for a K-CW complex X there is a natural isomorphism

$$H_G^*(G\times_K X;M)\approx H_K^*(X;M{\downarrow}_K^G).$$

We denote $G\times_K X$ by $X{\uparrow}_K^G$ and call it the *induction* from K to G. Clearly if X is a K-CW complex then $X{\uparrow}_K^G$ is a G-CW complex.

When X is a G-CW complex then by restriction it is also a K-CW complex by Illman [21] and $X{\uparrow}_K^G$ is canonically homeomorphic to $X\times G/K$. The projection $X\times G/K\to X$ gives rise to a natural map $H_G^*(X;M)\xrightarrow{\mathrm{res}_K^G} H_K^*(X;M)$. In this section we shall recall and exploit the transfer map

$$H_K^*(X;M)\xrightarrow{\mathrm{tr}_K^G} H_G^*(X;M).$$

Definition 4.5. A collection $\mathcal{C}$ of subgroups of G is *convex* if it is closed under formation of subgroups. The *complement* of $\mathcal{C}$ is the collection of all subgroups of G not in $\mathcal{C}$ and is denoted by $\overline{\mathcal{C}}$. For a subgroup $K \leq G$ we let $\mathcal{C} \cap K$ denote the collection of all the subgroups of K which belong to $\mathcal{C}$.

We remark that $\mathcal{C}$ is convex if and only if $\overline{\mathcal{C}}$ is concave (3.8). Furthermore $\mathcal{C} \cap K$ is convex in K if $\mathcal{C}$ is convex in G.

Definition 4.6. Fix collections $\mathcal{F}$ and $\mathcal{D}$ in G where $\mathcal{D}$ is convex. A Mackey functor M is $(\mathcal{F}, \mathcal{D})$-*discardable* if for every stable G-map $\xi : G/H_+ \to G/D_+$ where $H \in \mathcal{F}$ and $D \in \mathcal{D}$, the induced homomorphism $M(\xi)$ is the trivial homomorphism $M(G/D_+) \xrightarrow{0} M(G/H_+)$.

Lemma 4.7. *Let M be an $(\mathcal{F}, \mathcal{D})$-discardable Mackey functor on G and let X be a G-CW complex of orbit type $\mathcal{F}$. Fix a subgroup $P \leq G$ and let $\mathcal{C}$ denote the P-collection $\overline{P \cap \mathcal{D}}$. Then $\mathcal{C}$ is concave in P and the transfer map $\mathrm{tr}_P^G : H_P^*(X; M) \to H_G^*(X; M)$ factors as follows:*

$$H_P^*(X; M) \xrightarrow{i^*} H_P^*(X_{\mathcal{C}}; M) \to H_G^*(X; M)$$

where i is the inclusion of the P-subcomplex $X_{\mathcal{C}}$ of X (3.9).

We recall that for every finite G-CW complex F there is an associated stable transfer map $\tau(F) : S \to F_+$ in $G\mathcal{S}$ where S is the sphere spectrum. It is the composition [25, Ch. XVII (1.2)]

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{1 \wedge \Delta} DX \wedge X \wedge X \xrightarrow{\epsilon} S \wedge X \simeq X.$$

where X is the suspension spectrum F_+ and DX is its Spanier–Whitehead dual. By definition $DX = F(X, S)$ where G acts by "conjugation". Therefore the map $\tau(F)\downarrow_K^G$ which is obtained by restriction of the action to a subgroup K, is the transfer map of F considered as a finite K-CW complex.

Consider the projection $\xi : X \times F \to X$ where X is another G-CW complex. There is an associated transfer map $\tau(\xi)$ in $\bar{h}G\mathcal{S}$ defined by

$$X_+ \xrightarrow{1 \wedge \tau(F)} X_+ \wedge F_+.$$

See [28, pp. 191] for details; in the notation there we use $\Pi = 1$.

By applying the Borel construction we may consider the fibre bundle of spaces $\xi_{hG} : (X \times F)_{hG} \to X_{hG}$ with fibre F. It has an associated stable (non-equivariant) transfer map $\tau(\xi_{hG})$.

Proposition 4.8. *Fix a finite G-CW complex F and a projection $\xi : X \times F \to X$ where X is any G-CW complex. Then $\tau(\xi_{hG}) = \tau(\xi) \wedge_G EG_+$ in $\bar{h}G\mathcal{S}$.*

Proof. The fibre bundle $\xi_{hG} : (X \times F)_{hG} \to X_{hG}$ has structure group G. According to [28] the stable transfer $\tau(\xi_{hG})$ in $\bar{h}\mathcal{S}$ is constructed as follows. Fix a complete G universe U. First one looks at the bundle of free G-spaces

$X \times F \times EG \to X \times EG$. It has an associated transfer in $\bar{h}G\mathcal{S}$, which up to an equivalence is

$$X_+ \wedge EG_+ \xrightarrow{X_+ \wedge \tau(F) \wedge EG_+} X_+ \wedge F_+ \wedge EG_+.$$

Since these are free G-spectra, there is a map, by [28, Theorem XVI.5.3], unique up to homotopy, of naive G-spectra $\tilde{\tau} : X_+ \wedge EG_+ \to X_+ \wedge F_+ \wedge EG_+$, namely a map in $G\mathcal{S}U^G$, such that $i_*(\tilde{\tau}) = X_+ \wedge \tau(F) \wedge EG_+$, where $i : U^G \to U$ is the inclusion. By definition $\tau(\xi_{hG}) := \tilde{\tau}/G$. But this is exactly how $\tau(\xi)_{hG} = (1_{X_+} \wedge \tau(F)) \wedge_G EG_+$ is defined in [28, p. 165, last paragraph]. ∎

Note that $X \times G/K = X\uparrow_K^G$ for any G-space X. The projection $\xi : X \times G/K \to X$ and $\tau(\xi)$ give rise to maps

$$H^*_G(X;M) \xrightarrow{\xi^*} H^*_K(X;M) \xrightarrow{\tau(\xi)^*} H^*_G(X;M)$$

which are natural in X. They are denoted $\mathrm{res}_K^G := \xi^*$ and $\mathrm{tr}_K^G := \tau(\xi)^*$.

Proposition 4.9. *Let h^* be a p-local cohomology theory and let M denote one of the Mackey functors $\mathcal{M}_h$*, Const_h *or $\widetilde{\mathcal{M}}_h$* (2.5, 2.7). *Let P be a maximal p-toral subgroup of G. Then for every G-CW complex X the composition*

$$H^*_G(X;M) \xrightarrow{\mathrm{res}_P^G} H^*_P(X;M\downarrow_P^G) \xrightarrow{\mathrm{tr}_P^G} H^*_G(X;M)$$

is an isomorphism.

Proof. Consider first the case $M = \mathcal{M}_h$. The composition $X_+ \to X_+ \wedge G/P_+ \to X_+$ is natural with respect to X so by (2.3) it suffices to prove the isomorphism for the skeleta of X. Using induction and the cofibre sequence of G-CW complexes $X_{n-1+} \to X_{n+} \to X_n/X_{n-1} \simeq \bigvee_i (S^n \wedge G/H_{i+})$ we see that it suffices to prove the isomorphism for orbits, namely, that

$$(1) \qquad H^0_G(G/K;\mathcal{M}_h) \xrightarrow{\mathrm{tr}_P^G} H^0_P(G/K \times G/P;\mathcal{M}_h) \xrightarrow{\mathrm{res}_P^G} H^0_G(G/K;\mathcal{M}_h)$$

is an isomorphism. Set $\tau = \tau(G/P) : S \to G/P_+$ and let $\xi : G/K \times G/P \to G/K$ denote the projection. By definition, (1) is obtained by applying $\tilde{H}^0_G(-;\mathcal{M}_h)$ to the maps of G-spectra

$$G/K_+ \xrightarrow{1\wedge\tau} G/K_+ \wedge G/P_+ \xrightarrow{\xi_+} G/K_+.$$

Since this is a stable map of G-orbits, by the definition of $\mathcal{M}_h$ we obtain

$$\begin{aligned}\tilde{H}^0_G(\xi_+ \circ (1\wedge\tau);\mathcal{M}_h) &= \mathcal{M}_h(\xi_+ \circ (1\wedge\tau)) \\ &= \tilde{h}^0\big((\xi_+ \circ (1\wedge\tau)) \wedge_G EG_+\big) = \tilde{h}^0\big(\xi_{hG+} \circ ((1\wedge\tau)\wedge_G EG_+)\big).\end{aligned}$$

Proposition 4.8 now implies that

$$\tilde{H}^0_G(\xi_+ \circ (1\wedge\tau);\mathcal{M}_h) = \tilde{h}^0(\xi_{hG+} \circ \tau(\xi_{hG})).$$

Proposition A.3.3 shows that this map is an isomorphism if the composite map of spectra

$$BK_+ \xrightarrow{\tau(\xi_{hG})} (G/K \times G/P)_{hG+} \xrightarrow{\xi_{hG+}} BK_+$$

induces an $\tilde{H}_*(-;\mathbb{Z}_{(p)})$-isomorphism. This is indeed the case because in ordinary cohomology this map induces multiplication by $\chi(G/P)$ which is a unit in $\mathbb{Z}_{(p)}$. See, e.g., [2]. This completes the proof when $M = \mathcal{M}_h$.

If $M = \mathrm{Const}_h$ then we have seen in Proposition 2.6 that $M = \mathcal{M}_k$ where $\tilde{k}^* = \tilde{H}^*(-;\pi_0 h)$ and the result follows from the previous case. The case $M = \widetilde{\mathcal{M}}_h$ follows from the previous two cases by examination of the self-map $\mathrm{tr}_P^G \circ \mathrm{res}_P^G$ of the long exact sequence in Bredon cohomology which is associated to the short exact sequence of Mackey functors in Proposition 2.8. ∎

Proof of Proposition 4.2. First, X is G-acyclic for Const_h by hypothesis (i) because $H^*_G(X;\mathrm{Const}_h) = H^*(X/G;\pi_0 h)$. The map $X \to *$ induces a morphism of the long exact sequences in cohomology associated with the short exact sequence of Mackey functors (2.8) which shows that X is G-acyclic for $\mathcal{M}_h$ if and only if it is G-acyclic for $\widetilde{\mathcal{M}}_h$. The naturality of the transfer with respect to $X \to *$ gives a commutative diagram

$$(1)\qquad \begin{array}{ccccc} H^*_G(*;\widetilde{\mathcal{M}}_h) & \xrightarrow{\mathrm{res}_P^G} & H^*_P(*;\widetilde{\mathcal{M}}_h) & \xrightarrow{\mathrm{tr}_P^G} & H^*_G(*;\widetilde{\mathcal{M}}_h) \\ \downarrow & & \downarrow & & \downarrow \\ H^*_G(X;\widetilde{\mathcal{M}}_h) & \xrightarrow{\mathrm{res}_P^G} & H^*_P(X;\widetilde{\mathcal{M}}_h) & \xrightarrow{\mathrm{tr}_P^G} & H^*_G(X;\widetilde{\mathcal{M}}_h) \end{array}$$

where the composition along the rows are isomorphisms by Proposition 4.9. We see that the homomorphism $H^*_G(*;\widetilde{\mathcal{M}}_h) \to H^*_G(X;\widetilde{\mathcal{M}}_h)$ is a retract of $H^*_P(*;\widetilde{\mathcal{M}}_h) \to H^*_P(X;\widetilde{\mathcal{M}}_h)$ which is an isomorphism by hypothesis (ii). The result now follows because a retract of an isomorphism is an isomorphism. ∎

We quote from Palais [31, Corollary 1.7.29]

PROPOSITION 4.10. *Let H, K be subgroups of a compact Lie group G. Then the number of H-conjugacy classes of subgroups of H of the form $H \cap K^g$ is finite.*

Proof of Theorem 4.3. First we observe that if Q is a non-trivial p-toral subgroup of G then X^Q is NQ-contractible. To see this note that if K/Q is a subgroup of NQ/Q then $O_p(K) \neq 1$, hence, by hypothesis, $(X^Q)^{K/Q} = X^K$ is contractible. Now [38, Proposition II.2.7] implies that $X^Q \to *$ is an NQ/Q-homotopy equivalence, whence an NQ-homotopy equivalence.

(i) Consider the G-map $X \to *$. If Q is a maximal p-toral subgroup of $H \in \mathrm{Iso}_G(X \cup *)$ then $Q \neq 1$ by hypothesis (a) and the hypothesis on G.

We have seen that X^Q is NQ-contractible and we can therefore apply part (i) of the key Lemma 3.2 to $X \to *$ and conclude that X/G is contractible.

(ii) Fix a maximal p-toral subgroup P of G. Note that $X{\downarrow}_P^G$ has finite orbit type by 4.10 and it is a P-CW complex by [21]. With the notation of Example 4.4, Lemma 2.10 shows that $\widetilde{\mathcal{M}}_h^G{\downarrow}_P^G = \widetilde{\mathcal{M}}_h^P$ is p-constrained. In particular $\widetilde{\mathcal{M}}_h^P$ vanishes on orbits P/Q where Q is finite of order prime to p because $\widetilde{\mathcal{M}}_h$ vanishes on free orbits. Let Z denote the P-subcomplex of X consisting of all the orbits whose type is not a finite group of order prime to p, that is, $Z = X_{\text{type-}\mathcal{S}_p^0(P)}$. Proposition 2.4 implies that

$$(1) \qquad H_P^*(Z; \widetilde{\mathcal{M}}_h) \approx H_P^*(X; \widetilde{\mathcal{M}}_h).$$

Furthermore Z has finite orbit type and we now consider the map $Z \to *$. If Q is a maximal p-toral subgroup of some $H \in \mathrm{Iso}_P(Z \cup *)$ then $Q \neq 1$ because $P \neq 1$ and because a compact Lie group which is not finite of order prime to p contains a non-trivial p-toral subgroup. It immediately follows that $Z^Q = X^Q$ and the latter was shown to be NQ-contractible, hence N_PQ contractible. Now part (ii) of Lemma 3.2 applies to $Z \to *$, that is, Z and thanks to (1) also X, are P-acyclic for $\widetilde{\mathcal{M}}_h$. Part (i) shows that X/G is contractible, therefore Proposition 4.2 applies, namely X is G-acyclic for $\mathcal{M}_h$. ∎

In the remainder of this section we will prove Lemma 4.7.

4.11. Fix a G-representation V and recall that S^V denotes the one-point compactification of V. For a pointed G-CW complex (resp. a G-spectrum) X the smash product $X \wedge S^V$ is denoted $\Sigma^V X$. For a pointed G-space X there is an equivalence of G-spectra $\Sigma^\infty \Sigma^V X \simeq \Sigma^V \Sigma^\infty X$.

We now fix a pointed G-CW complex X and a Mackey functor M. The filtration $\Sigma^V X_n$ of $\Sigma^V X$ is exhaustive and gives rise to a convergent spectral sequence

$$(1) \qquad E_1^{i,j}(X) = \tilde{H}_G^{V+i+j}(\Sigma^V X_i/X_{i-1}; M) \;\Rightarrow\; \tilde{H}_G^{V+i+j}(\Sigma^V X; M) \\ \approx \tilde{H}_G^{i+j}(X; M).$$

Clearly, $\Sigma^V X_i/\Sigma^V X_{i-1} \simeq \bigvee_i (S^{V+n} \wedge G/H_{i+})$ and therefore the spectral sequence collapses to its horizontal axis giving rise to a cochain complex $C_G^{V,*}(X; M)$ where in the notation of (2.1),

$$(2) \qquad C_G^{V,n}(X; M) \approx \tilde{H}_G^{n+V}(\Sigma^V X_n/X_{n-1}; M) \approx \tilde{H}_G^0(\mathfrak{X}_{n+}; M).$$

Thus, $H_G^*(X; M)$ is isomorphic to the cohomology groups of $C_G^{V,*}(X; M)$. In fact, it is not difficult to check that the cochain complex $C_G^{V,*}(X; M)$ is isomorphic to the cochain complex $\mathrm{Hom}_{hG\mathcal{O}}(\underline{C}_*(X), M)$ which was described after Definition 2.2. The differentials are obtained by applying $\tilde{H}_G^{V+n}(-; M)$

to the map $\Sigma^V X_n/X_{n-1} \to \Sigma^{V+1} X_{n-1}/X_{n-2}$ in the cofibre sequence of the subspaces $X_n \supseteq X_{n-1} \supseteq X_{n-2}$. When $V = 0$ we simply write $C_G^*(X;M)$.

Let Y be another pointed G-CW complex and let $f : \Sigma^V X \to \Sigma^V Y$ be a G-map such that $f(\Sigma^V X_n) \subseteq \Sigma^V Y_n$. Oliver shows in [30, p. 546] that every G-map $f' : \Sigma^V X \to \Sigma^V Y$ is G-homotopic to a G-map f with the property above. Since f preserves the filtrations of $\Sigma^V X$ and $\Sigma^V Y$, it induces a morphism of spectral sequences (1) and consequently a morphism of cochain complexes

$$C_G^{V,*}(f;M) : C_G^{V,*}(Y) \to C_G^{V,*}(X).$$

In every degree i it is induced by the maps $f_i : \Sigma^V X_i/X_{i-1} \to \Sigma^V Y_i/Y_{i-1}$. By taking cohomology groups we obtain

$$HC_G^{V,*}(f;M) = \tilde{H}_G^{*+V}(f;M) = \tilde{H}_G^*(\Sigma^{-V} f;M) : \tilde{H}_G^*(Y;M) \to \tilde{H}_G^*(X;M).$$

Proof of Lemma 4.7. We leave it to the reader to verify that $\mathcal{C}$ is a concave collection of subgroups in P.

Claim. *Fix a G-representation V and consider $f : \Sigma^V X_+ \to \Sigma^V Y_+$ where Y is another G-CW complex. Then $\tilde{H}_G^*(\Sigma^{-V} f;M)$ factors through* (4.5, 3.9)

$$i^* : H_G^*(Y;M) \to H_G^*(Y_{\overline{\mathcal{D}}};M).$$

Proof. We may assume by [30, p. 546] that $f(\Sigma^V X_n) \subseteq \Sigma^V Y_n$ and consider $C_G^{V,*}(f,M)$ as in 4.11. In degree n it is obtained by applying $\tilde{H}_G^{n+V}(-;M)$ to the map $f_n : \Sigma^V X_n/X_{n-1} \to \Sigma^V Y_n/Y_{n-1}$. Thus, in the notation of 4.11(2),

$$C_G^{V,n}(f;M) = \tilde{H}_G^0(\Sigma^{-V-n} f_n;M) : \tilde{H}_G^0(\mathfrak{Y}_{n+};M) \to \tilde{H}_G^0(\mathfrak{X}_{n+};M).$$

Now consider the stable map $\varphi_n := \Sigma^{-V-n} f_n : \mathfrak{X}_{n+} \to \mathfrak{Y}_{n+}$. It follows from [28, Corollary XIX.3.2] that every stable map $G/H_+ \to \mathfrak{Y}_n$ is a $\mathbb{Z}$-linear combination of stable maps $G/H_+ \to G/K_+$ where $G/K \subseteq \mathfrak{Y}_n$. Since M is $(\mathcal{F},\mathcal{D})$-discardable, if $H \in \mathcal{F}$ then $H_G^0(\mathfrak{Y}_{n\mathcal{D}};M)$ is in the kernel of

$$\varphi_n^* : \tilde{H}_G^0(\mathfrak{Y}_{n+};M) \to \tilde{H}_G^0(G/H_+;M)$$

for any stable G-map $G/H_+ \to \mathfrak{Y}_{n+}$. Therefore there is a unique diagonal arrow which renders the following triangle commutative:

$$\text{(1)}\qquad \begin{array}{ccc} \tilde{H}_G^0(\mathfrak{Y}_{n+};M) & \xrightarrow{\tilde{H}_G^0(\varphi_n;M)} & \tilde{H}_G^0(\mathfrak{X}_{n+};M) \\ \Big\downarrow{\scriptstyle (i_+)^*} & \nearrow & \\ \tilde{H}_G^0((\mathfrak{Y}_{n\overline{\mathcal{D}}})_+;M) & & \end{array}$$

where i is the inclusion $(\mathfrak{Y}_n)_{\overline{\mathcal{D}}} \subseteq \mathfrak{Y}_n$. Since the maps $(i_+)^*$ are surjective, we obtain a factorisation of $C_G^{V,*}(f;M)$ through $i^* : C_G^{V,*}(Y;M) \to C_G^{V,*}(Y_{\overline{\mathcal{D}}};M)$. The result follows by taking the cohomology groups of these cochain complexes. $\blacksquare_{\text{Claim}}$

We recall that Becker and Gottlieb constructed the transfer map in a geometric way as follows (see [2] and also [28, p. 180]). Choose an embedding of G/P in a G-representation V and let $\nu_{G/P}$ be a tubular neighbourhood homeomorphic to the normal bundle. Then the Thom–Pontryagin construction gives a map $S^V \to G/P_+ \wedge S^V$. For a G-space X we obtain $f : \Sigma^V X_+ \to \Sigma^V X_+ \wedge G/P_+$. On the spectrum level $\Sigma^\infty f = \Sigma^V \tau(\xi)$ where $\xi : X \times G/P \to X$ is the projection.

The result now follows from the claim by observing that $(X \times G/P)_{\overline{\mathcal{D}}} = G \times_P X_{\mathcal{C}}$, hence $H^*_P(X_{\mathcal{C}}; M) \approx H^*_G((X \times G/P)_{\overline{\mathcal{D}}}; M)$. ∎

PROPOSITION 4.12. *Fix a compact Lie group G and let $\mathcal{D}$ denote the collection of all its finite subgroups of order prime to p. Then the constant Mackey functor* $\mathrm{Const}_{\mathbb{Z}/p}$ *is $(\mathcal{F}, \mathcal{D})$-discardable for any collection $\mathcal{F}$ contained in $\overline{\mathcal{D}}$.*

Proof. Recall from Proposition 2.6 that $\mathrm{Const}_{\mathbb{Z}/p} = \mathcal{M}_h$ where $\tilde{h}^* = \tilde{H}^*(-; \mathbb{Z}/p)$. If $H \in \mathcal{F}$ then it contains a non-trivial p-toral subgroup and therefore $\chi(H) = 0 \bmod p$. A stable map $\xi : G/H_+ \to G/D_+$ where $D \in \mathcal{D}$ is a $\mathbb{Z}$-linear combination of (see [28, XIX.3])

$$G/H_+ \xrightarrow{\tau} G/L_+ \xrightarrow{\lambda_+} G/D_+ \tag{1}$$

where τ is the transfer map associated to the inclusion $L \le H$ and λ is a G-map of spaces. Note that $L \in \mathcal{D}$ because $\mathcal{D}$ is convex. Also observe that

$$H^0(BL; \mathbb{Z}/p) \xrightarrow{\mathrm{tr}_L^H} H^0(BH; \mathbb{Z}/p) \xrightarrow{\approx} H^0(BL; \mathbb{Z}/p)$$

is multiplication by $\chi(H/L) = 0 \bmod p$ and the second map is an isomorphism because BL and BH are connected. Therefore $\mathcal{M}_h(\tau) = \mathrm{tr}_L^H = 0$ and consequently $\mathcal{M}_h(-)$ applied to (1) vanishes. It follows that $\mathrm{Const}_{\mathbb{Z}/p}(\xi) = 0$. ∎

5. A topological version of Quillen's Theorem A

5.1. A *topological category* is a category enriched over the category $\mathcal{T}$ of compactly generated topological spaces; see Borceux [6, §§6 and 7.2] and Steenrod [35]. The morphism spaces of a topological category $\mathbf{C}$ are denoted $\mathbf{C}(C, C')$. A functor $F : \mathbf{C} \to \mathcal{T}$ is an assignment of a space $F(C)$ for every object $C \in \mathbf{C}$ and continuous maps $F(C) \times \mathbf{C}(C, C') \to F(C')$ for any objects $C, C' \in \mathbf{C}$ which are compatible with the identities and composition rules (see [6, §6.2]).

The *simplicial replacement* of a functor $F : \mathbf{C} \to \mathcal{T}$ is the simplicial object $\coprod_* F$ in $\mathcal{T}$ where

$$\coprod\nolimits_n F = \coprod_{C_0, \ldots, C_n \in \mathbf{C}} F(C_0) \times \mathbf{C}(C_{n-1}, C_n) \times \cdots \times \mathbf{C}(C_0, C_1).$$

Composition of arrows induces the face maps d_k and d_0 is the structure map $F(C_0) \times \mathbf{C}(C_0, C_1) \to F(C_1)$ of the functor F. Degeneracies are formed by inserting identity morphisms. Compare with [9, p. 337] and [19]. The *homotopy colimit of* F is the geometric realisation of $\coprod_* F$, that is,

$$\operatorname*{hocolim}_{\mathbf{C}} F := |\textstyle\coprod_* F|.$$

Throughout we shall assume that morphism spaces in $\mathbf{C}$ are *cofibrant*, i.e. have the homotopy type of retracts of CW-complexes and the inclusions $\{\mathrm{id}_C\} \subseteq \mathbf{C}(C, C)$ are closed cofibrations for all $C \in \mathbf{C}$. When the values of F are also cofibrant, this construction is homotopy invariant, that is, if $F \to F'$ is a natural transformation such that $F(C) \simeq F'(C)$ for all $C \in \mathbf{C}$, then the homotopy colimits of F and F' are homotopy equivalent. See e.g. Hollender and Vogt [19].

5.2. Fix a compact Lie group G. An *internal G-space category* is a category object in the category of G-spaces; see [5, §8]. More concretely, an internal G-space category $\mathbf{C}$ consists of two G-spaces $\mathbf{C}_0$ (objects) and $\mathbf{C}_1$ (morphisms) together with continuous G-maps $d_1, d_0 : \mathbf{C}_1 \to \mathbf{C}_0$ (d_1 for domain and d_0 for codomain) and $s_0 : \mathbf{C}_0 \to \mathbf{C}_1$ (identities). It is also equipped with a "composition map" $c : \mathbf{C}_2 \to \mathbf{C}_1$ where $\mathbf{C}_2 \subseteq \mathbf{C}_1 \times \mathbf{C}_1$ is the space of composable arrows in $\mathbf{C}$. The maps d_0, d_1, s_0 and c are subject to the obvious associativity and unitality relations [5, §8.1]. Alternatively, we require that upon forgetting the topologies on $\mathbf{C}_0$ and $\mathbf{C}_1$, this structure becomes a small category in the usual sense (see Borceux [5, §8.1.5] using $C = G$). When G is trivial we call $\mathbf{C}$ an *internal space category*. The slogan is that an internal space category is one whose object and morphism sets carry non-trivial topologies.

A *functor* $\Phi : \mathbf{C} \to \mathbf{D}$ between internal G-space categories consists of *continuous* G-maps between the object and morphism spaces of these categories which become a functor between small categories upon forgetting the topologies on $\mathbf{C}$ and $\mathbf{D}$. See [5, §8.1.2].

A *natural transformation* $t : \Phi \Rightarrow \Psi$ is a G-map $t : \mathbf{C}_0 \to \mathbf{D}_1$ with the obvious properties (see [5, §8.1.3]). Equivalently, it is a functor $t : \mathbf{C} \times \{0 \to 1\} \to \mathbf{D}$ such that $t|_{\mathbf{C}\times\{0\}} = \Phi$ and $t|_{\mathbf{C}\times\{1\}} = \Psi$.

The *nerve* of an internal G-space category $\mathbf{C}$ is the simplicial space $\mathrm{Nr}\,\mathbf{C}$ where the space $\mathrm{Nr}_k\,\mathbf{C}$ of k-simplices is the obvious subspace of $\mathbf{C}_0 \times \mathbf{C}_1^k$ of k composable arrows. Explicitly

$$\mathrm{Nr}_k\,\mathbf{C} = \{(C, c_k, \ldots, c_0) : C = d_1(c_0),\ d_0(c_i) = d_1(c_{i+1})\}.$$

Face and degeneracy maps are defined in the usual way (cf. [9, Ch. XI, §2] and [19]).

A functor $\Phi : \mathbf{C} \to \mathbf{D}$ gives rise to an obvious simplicial map between the nerves. A natural transformation $t : \Phi \Rightarrow \Psi$ gives rise to a simplicial map $\mathrm{Nr}\,\mathbf{C} \times \Delta[1] \to \mathrm{Nr}\,\mathbf{D}$, whence a homotopy from $|\mathrm{Nr}\,\Phi|$ to $|\mathrm{Nr}\,\Psi|$.

The nerve of an internal G-space category is rarely Reedy cofibrant (see A.2.1). Hence, the realisation of nerves of internal space categories is not in general homotopically invariant.

5.3. Throughout G is a compact Lie group and $G\mathcal{T}$ is the category of compactly generated G-spaces. Fix a topological category $\mathbf{C}$ and a functor $F : \mathbf{C} \to G\mathcal{T}$. The *transporter category* of F is the internal G-space category $\operatorname{Tr} F$ whose structure is determined by the first three spaces in the simplicial replacement $\coprod_* F$ (5.1). That is, the object and morphism spaces are $\coprod_0 F$ and $\coprod_1 F$, and the category structure is described by

$$\begin{tikzcd} \coprod_0 F & \coprod_1 F \arrow[l, shift left=2, "d_0"'] \arrow[l, shift right=2, "d_1"] & \coprod_2 F \arrow[l, "d_1"'] \\ \end{tikzcd} \quad s_0 : \coprod_0 F \to \coprod_1 F.$$

It is not hard to check that $\operatorname{Tr} F$ is an internal G-space category and that

$$\operatorname{Nr}\ \operatorname{Tr}(F) \approx \coprod\nolimits_* F. \tag{1}$$

In particular (cf. Thomason [37] and Dwyer [13, §2]),

$$\operatorname*{hocolim}_{\mathbf{C}} F \approx |\operatorname{Nr}\operatorname{Tr}(F)|.$$

Fix a functor $j : \mathbf{C} \to \mathbf{D}$ between internal space categories (5.2). The *over category* $(D{\downarrow}j)$ where D is an object in $\mathbf{D}$ is an internal space category whose object space is the subspace of $\mathbf{C}_0 \times \mathbf{D}_1$ consisting of the pairs (C, d) where $d \in \mathbf{D}(D, jC)$. The morphism space is the subspace of $\mathbf{C}_0 \times \mathbf{D}_1 \times \mathbf{C}_1$ consisting of the triples (C, d, c) where (C, d) is an object of $(D{\downarrow}j)$ and c is a morphism $c \in \mathbf{C}(C, C')$. Such a triple is a morphism $(C, d) \to (C', j(c) \circ d)$. Compare with the discrete version [26, §II.6].

It is easily seen that $\operatorname{Nr}_k(D{\downarrow}j)$ can be identified with the subspace of $\mathbf{C}_0 \times \mathbf{D}_1 \times (\mathbf{C}_1)^{\times k}$ of the points

$$\{(C_0, D \xrightarrow{d} jC_0, C_0 \xrightarrow{c_1} \cdots \xrightarrow{c_k} C_k)\}.$$

Quillen's Theorem A [32, Theorem A] asserts that a functor $j : \mathbf{C} \to \mathbf{D}$ of *small* categories induces a weak homotopy equivalence on nerves if for every object $D \in \mathbf{D}$ the nerve of the comma category $(D{\downarrow}j)$ is contractible. His theorem fails when $\mathbf{C}$ and $\mathbf{D}$ are internal space categories.

Example 5.4. Let X be any non-empty connected space distinct from a point. Let X^d denote its underlying set. One may consider X and X^d as internal space categories $\mathbf{X}$ and $\mathbf{X}^d$ whose object and morphism spaces are X and X^d respectively. Thus $\mathbf{X}$ and $\mathbf{X}^d$ have only identity morphisms. The identity functor $j : \mathbf{X}^d \to \mathbf{X}$ gives rise to contractible spaces $|(Y{\downarrow}j)|$ for all objects $Y \in \mathbf{X}$. However $|j|$ is canonically identified with $j : X^d \to X$ which is not an equivalence.

DEFINITION 5.5. Fix a functor $j : \mathbf{C} \to \mathbf{D}$ of internal space categories. The *join* of j is the bisimplicial space $\mathfrak{X}(j)$ where $\mathfrak{X}_{s,t}(j)$ is the subspace of $\mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}) \times \mathbf{D}_1 \times \mathrm{Nr}_t(\mathbf{C})$ consisting of

$$\mathfrak{X}_{s,t}(j) = \{(D_s \xrightarrow{d_s} \cdots \xrightarrow{d_1} D_0,\, D_0 \xrightarrow{d} j(C_0),\, C_0 \xrightarrow{c_1} \cdots \xrightarrow{c_t} C_t)\}.$$

Face and degeneracy maps are defined in the obvious way. For every $s \geq 0$ we obtain a simplicial space $\mathfrak{X}_{s,*}(j)$ and for every $t \geq 0$ a simplicial space $\mathfrak{X}_{*,t}(j)$. These simplicial spaces are augmented (A.2.4) in an obvious way by the projections

$$\pi_v : \mathfrak{X}_{s,t}(j) \to \mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}), \qquad \pi_h : \mathfrak{X}_{s,t}(j) \to \mathrm{Nr}_t(\mathbf{C}).$$

Throughout $D_\bullet$ denotes elements $D_s \to \cdots \to D_0$ in $\mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}})$ and $C_\bullet$ denotes elements $C_0 \to \cdots \to C_t$ in $\mathrm{Nr}_t(\mathbf{C})$. It easily follows from the definitions that for every $D_\bullet$ there is a pullback square

$$\begin{array}{ccc} \mathrm{Nr}_t(D_0{\downarrow}j) & \longrightarrow & \{D_\bullet\} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{incl}} \\ \mathfrak{X}_{s,t}(j) & \xrightarrow{\pi_v} & \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}} \end{array} \tag{5.6}$$

DEFINITION 5.7. A functor of internal space categories $j : \mathbf{C} \to \mathbf{D}$ is called *tame* if the pullback squares (5.6) are homotopy cartesian for all $s, t \geq 0$ and all $D_\bullet \in \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$. That is, $\mathrm{Nr}_t(D_0{\downarrow}j)$ is weakly equivalent, via the natural map, to the homotopy fibre of $\mathfrak{X}_{s,t}(j) \xrightarrow{\pi_v} \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$. We call j *absolutely tame* if π_v are Serre fibrations for all $s, t \geq 0$ and all objects $D_\bullet \in \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$.

The main result of this section is a topological variant of Quillen's Theorem A [32]. The statement of Quillen's result is very clean in the sense that there are no restrictions on the categories and functors it applies to. In light of the theorem below, the reason for this is that every function between discrete sets is a Serre fibration and every simplicial set is Reedy cofibrant (see A.2). In other words the nerve of every small category is Reedy cofibrant and every functor between small categories is tame.

THEOREM 5.8. *Let $j : \mathbf{C} \to \mathbf{D}$ be a tame functor of internal space categories. Assume that*

(a) $\mathrm{Nr}\,\mathbf{C}$ *and* $\mathrm{Nr}\,\mathbf{D}$ *are Reedy cofibrant,*
(b) *for every $D \in \mathbf{D}$ the nerve of $(D{\downarrow}j)$ is Reedy cofibrant and its realisation is contractible.*

Then $|j| : |\mathrm{Nr}\,\mathbf{C}| \to |\mathrm{Nr}\,\mathbf{D}|$ is a homotopy equivalence.

Proof. We follow Quillen's original proof. Note that the functor j induces a commutative diagram of augmented bisimplicial spaces

(1)
$$
\begin{tikzcd}
\mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}) \arrow[d, "\mathrm{id}"'] & \mathfrak{X}_{s,t}(j) \arrow[l, "\pi_v"'] \arrow[r, "\pi_h"] \arrow[d, "j_*"] & \mathrm{Nr}_t\,\mathbf{C} \arrow[d, "\mathrm{Nr}_t\,j"] \\
\mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}) & \mathfrak{X}_{s,t}(1_{\mathbf{D}}) \arrow[l, "\pi_v"'] \arrow[r, "\pi_h"] & \mathrm{Nr}_t\,\mathbf{D}
\end{tikzcd}
$$

We first claim that for every $t \geq 0$ the augmented simplicial spaces

(2)
$$\mathfrak{X}_{*,t}(j) \xrightarrow{\pi_h} \mathrm{Nr}_t\,\mathbf{C} \quad \text{and} \quad \mathfrak{X}_{*,t}(1_{\mathbf{D}}) \xrightarrow{\pi_h} \mathrm{Nr}_t\,\mathbf{D}$$

admit left contractions (A.2.4). Indeed, define $s_{-1} : \mathrm{Nr}_t\,\mathbf{C} \to \mathfrak{X}_{0,t}(j)$ by

$$s_{-1}(C_\bullet) = (jC_0, jC_0 \xrightarrow{\mathrm{id}} jC_0, C_\bullet).$$

For all $r \geq 0$ define $s_{-1} : \mathfrak{X}_{r,t}(j) \to \mathfrak{X}_{r+1,t}(j)$ by

$$s_{-1}(D_\bullet, D_0 \xrightarrow{d} jC_0, C_\bullet) = (D_\bullet \xrightarrow{d} jC_0, jC_0 \xrightarrow{\mathrm{id}} jC_0, C_\bullet).$$

The verification of the continuity of these maps and that for every t they give rise to a left contraction, is straightforward. By replacing j with $1_{\mathbf{D}}$ we obtain a left contraction for the second augmented simplicial space (2).

Now consider the augmented simplicial spaces

(3)
$$\mathfrak{X}_{s,*}(1_{\mathbf{D}}) \xrightarrow{\pi_v} \mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}) \qquad (s \geq 0).$$

They admit left contractions defined by

$$
\begin{aligned}
s_{-1} : \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}} \to \mathfrak{X}_{s,0}(1_{\mathbf{D}}), &\quad s_{-1}(D_\bullet) = (D_\bullet, D_0 \xrightarrow{\mathrm{id}} D_0, D_0), \\
s_{-1} : \mathfrak{X}_{s,t}(1_{\mathbf{D}}) \to \mathfrak{X}_{s,t+1}(1_{\mathbf{D}}), &\quad s_{-1}(D_\bullet, D_0 \xrightarrow{d} D'_0, D'_\bullet) \\
&\qquad\qquad = (D_\bullet, D_0 \xrightarrow{\mathrm{id}} D_0, D_0 \xrightarrow{d} D'_\bullet).
\end{aligned}
$$

We leave it to the reader to verify the continuity of these assignments and the simplicial identities.

For a space X we denote $X' = |\mathrm{Sing}(X)|$. There is a natural weak equivalence $X' \to X$ and X' is always a CW-complex, that is a cofibrant space. By applying this construction to (2) and (3) we obtain augmented simplicial objects of cofibrant spaces with left contractions (A.2.4), hence weak homotopy equivalences

(4)
$$\operatorname*{hocolim}_s \mathfrak{X}_{s,t}(j)' \xrightarrow{\sim} (\mathrm{Nr}_t\,\mathbf{C})' \xrightarrow{\sim} \mathrm{Nr}_t\,\mathbf{C},$$

(5)
$$\operatorname*{hocolim}_s \mathfrak{X}_{s,t}(1_{\mathbf{D}})' \xrightarrow{\sim} (\mathrm{Nr}_t\,\mathbf{D})' \xrightarrow{\sim} \mathrm{Nr}_t\,\mathbf{D},$$

(6)
$$\operatorname*{hocolim}_t \mathfrak{X}_{s,t}(1_{\mathbf{D}})' \xrightarrow{\sim} (\mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}})' \xrightarrow{\sim} \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}.$$

We now consider the augmented simplicial spaces $\mathfrak{X}_{s,*}(j) \xrightarrow{\pi_v} \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$ for all $s \geq 0$. Since j is tame, for every $D_\bullet \in \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$ we have the following commutative ladder of cofibrant spaces whose rows are homotopy fibre sequences over the point $D_\bullet$:

$$\begin{tikzcd}
\mathrm{Nr}_t(D_0{\downarrow}j)' \arrow[r] \arrow[d, "\sim"] & \mathfrak{X}_{s,t}(j)' \arrow[r] \arrow[d, "\sim"] & \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}} \arrow[d, equal] \\
\mathrm{Nr}_t(D_0{\downarrow}j) \arrow[r] & \mathfrak{X}_{s,t}(j) \arrow[r] & \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}
\end{tikzcd}$$

Puppe's Lemma [12, pp. 180] implies that

$$\operatorname*{hocolim}_t \mathrm{Nr}_t(D_0{\downarrow}j)' \to \operatorname*{hocolim}_t \mathfrak{X}_{s,t}(j)' \to \mathrm{Nr}_s\,\mathbf{D}^{\mathrm{op}}$$

is a homotopy fibre sequence over $D_\bullet$. By assumption $\mathrm{Nr}(D_0{\downarrow}j)$ is Reedy cofibrant, therefore there are homotopy equivalences (A.2.3)

$$\operatorname*{hocolim}_t \mathrm{Nr}_t(D_0{\downarrow}j)' \xrightarrow{\simeq} \operatorname*{hocolim}_t \mathrm{Nr}_t(D_0{\downarrow}j) \xrightarrow{\simeq} |\mathrm{Nr}(D_0{\downarrow}j)| \simeq *.$$

Thus the homotopy fibres of

$$\operatorname*{hocolim}_t \mathfrak{X}_{s,t}(j)' \xrightarrow{\pi_v} \mathrm{Nr}_s(\mathbf{D}^{\mathrm{op}}) \tag{7}$$

over every point in the base are contractible, and this map is therefore a homotopy equivalence. Now, (6), (7) and the left hand square of (1) imply that

$$\operatorname*{hocolim}_t \mathfrak{X}_{s,t}(j)' \to \operatorname*{hocolim}_t \mathfrak{X}_{s,t}(1_{\mathbf{D}})'$$

are homotopy equivalences for all $s \geq 0$ and therefore

$$\operatorname*{hocolim}_s \operatorname*{hocolim}_t \mathfrak{X}_{s,t}(j)' \to \operatorname*{hocolim}_s \operatorname*{hocolim}_t \mathfrak{X}_{s,t}(1_{\mathbf{D}})'$$

is a homotopy equivalence. By commuting homotopy colimits there is a homotopy equivalence

$$\operatorname*{hocolim}_t \operatorname*{hocolim}_s \mathfrak{X}_{s,t}(j)' \xrightarrow{\simeq} \operatorname*{hocolim}_t \operatorname*{hocolim}_s \mathfrak{X}_{s,t}(1_{\mathbf{D}})'. \tag{8}$$

The equivalences (4), (5) combined with (8) and the right hand square in (1) show that the left arrow in the following commutative square is a homotopy equivalence.

$$\begin{tikzcd}
\mathrm{hocolim}_t\,\mathrm{Nr}_t\,\mathbf{C} \arrow[r] \arrow[d, "\sim"'] & |\mathrm{Nr}\,\mathbf{C}| \arrow[d, "|j|"] \\
\mathrm{hocolim}_t\,\mathrm{Nr}_t\,\mathbf{D} \arrow[r] & |\mathrm{Nr}\,\mathbf{D}|
\end{tikzcd}$$

The horizontal arrows are homotopy equivalences because $\mathrm{Nr}\,\mathbf{C}$ and $\mathrm{Nr}\,\mathbf{D}$ are Reedy cofibrant. Therefore $|j|$ is a homotopy equivalence. ∎

REMARK 5.9. The category $(D{\downarrow}j)$ in Theorem 5.8 can be replaced with $(j{\downarrow}D)$. To see this, note that $|j|$ is an equivalence if and only if $|j^{\mathrm{op}}|$ is one. The latter is true, by Theorem 5.8, if $\mathrm{Nr}(D{\downarrow}j^{\mathrm{op}})$ are Reedy cofibrant and their realisations are contractible. Note that $(D{\downarrow}j^{\mathrm{op}}) = (j{\downarrow}D)^{\mathrm{op}}$ whose nerve is Reedy cofibrant and contractible if and only if this is the case for $(j{\downarrow}D)$.

We close this section by giving a criterion for a functor between internal G-space categories to satisfy the conditions of Theorem 5.8.

LEMMA 5.10. *Let G be a compact Lie group. Suppose that $f : X \to Y$ is a map of G-spaces such that Y/G is discrete. Then f is a Serre fibration.*

Proof. Since $Y = \coprod_{P \in Y/G} P$ where P is an orbit of G it follows that $f = \coprod_P (X_P \xrightarrow{f_P} P)$ where X_P denotes $f^{-1}(P)$. Bredon [10, II.3.2] shows that the f_P's are fibre bundles, which implies the lemma. ∎

DEFINITION 5.11. Let G be a compact Lie group. A *G-model* is a simplicial G-space $\mathbf{X}$ such that $\mathbf{X}/G$ is a simplicial set, that is, $\mathbf{X}_n/G$ is discrete for all n.

REMARK. The geometric realisation of a G-model $\mathbf{X}$ is a G-CW complex whose spaces of n-cells (2.1) are the spaces $N\mathbf{X}_n$ of non-degenerate simplices of $\mathbf{X}_n$.

There is a natural context in which G-models arise. Consider a functor $\tilde{F} : \mathbf{C} \to G\mathcal{T}$ whose values are transitive G-spaces and $\mathbf{C}$ is a small category. The simplicial replacement $\coprod_* \tilde{F}$ is clearly a G-model because by inspection $\coprod_* \tilde{F}/G = \mathrm{Nr}\,\mathbf{C}$.

COROLLARY 5.12. *Fix a compact Lie group G. If the nerve $\mathrm{Nr}\,\mathbf{D}$ of an internal G-space category $\mathbf{D}$ is a G-model, then it is Reedy cofibrant. A functor $j : \mathbf{C} \to \mathbf{D}$ of internal G-space categories is absolutely tame if $\mathrm{Nr}\,\mathbf{D}$ is a G-model.*

Proof. The first assertion follows from Proposition A.2.2, and the second from Lemma 5.10 which implies that the arrow π_v in the pullback square (5.6) is a Serre fibration. ∎

6. Stable splittings and spectral sequences. In this section we relate G-acyclicity to split exact chain complexes of spectra and to the collapsing of the Bousfield–Kan spectral sequence. The main results of this section are Theorem 6.4 and Proposition 6.5.

A chain complex in an additive category is a sequence of objects

$$\cdots \to A_{n+1} \xrightarrow{\partial_n} A_n \xrightarrow{\partial_{n-1}} A_{n-1} \to \cdots$$

such that $\partial_{n-1} \circ \partial_n = 0$ for all n. We shall be interested in chain complexes of spectra.

DEFINITION 6.1. A chain complex T of spectra is called *split exact* if it is isomorphic in $\bar{h}\mathcal{S}$ to a sum of complexes of the form $\cdots \to 0 \to A \xrightarrow{\simeq} A \to 0 \to \cdots$.

Consider the skeletal filtration $\{X_n\}$ of a G-CW complex X. There are maps $X_n/X_{n-1} \xrightarrow{\partial_n} \Sigma X_{n-1}/X_{n-2}$ associated to the inclusions $X_{n-2} \subseteq X_{n-1} \subseteq X_n$. Clearly $\partial_{n-1} \circ \partial_n$ are null homotopic. Also observe that $X_1/X_0 \xrightarrow{\partial_0} \Sigma X_{0+} \to \Sigma X_+$ where the second arrow is induced by the inclusion $X_0 \subseteq X_1 \subseteq X$, is null homotopic.

DEFINITION 6.2. Let $\{X_n\}$ be the skeletal filtration of a G-CW complex X. Define a chain complex $T^G X$ of G-spectra indexed by integers $n \geq 0$ by

$$\cdots \to \Sigma^n X_n/X_{n-1} \xrightarrow{\Sigma^{-n}\partial_n} \Sigma^{n-1} X_{n-1}/X_{n-2} \to \cdots \to \Sigma^{-1} X_1/X_0 \xrightarrow{\Sigma^{-1}\partial_0} X_{0+}.$$

The chain complex $T^{a,G}X$ indexed by integers $n \geq -1$ is obtained by augmenting $T^G X$ via the inclusion $X_0 \subseteq X$, that is, $T^{a,G}X$ has the form

$$\cdots \to \Sigma^n X_n/X_{n-1} \xrightarrow{\Sigma^{-n}\partial_n} \Sigma^{n-1} X_{n-1}/X_{n-2} \to \cdots \to X_{0+} \to X_+.$$

Define chain complexes of spectra

$$TX = T^G X \wedge_G EG_+ \quad \text{and} \quad T^a X = T^{a,G} X \wedge_G EG_+.$$

In the notation of (2.1), $T_n^G X \simeq \mathfrak{X}_{n+}$, hence $T_n X \simeq \bigvee_{G/H \subseteq \mathfrak{X}_n} BH_+$. Fix a cohomology theory h^*. By the definition of $C^*_G(X; M)$ in 4.11(2) and Definition 2.5,

$$C^*_G(X; \mathcal{M}_h) \approx \tilde{H}^0_G(T^G X; \mathcal{M}_h) \approx \tilde{h}^0 TX.$$

It follows that

$$H^*_G(X; \mathcal{M}_h) \approx H^* \tilde{h}^0 TX. \tag{6.3}$$

The first goal of this section is to prove

THEOREM 6.4. *Let X be a G-CW complex of dimension $d < \infty$. If X is G-acyclic for all p-local coefficient functors $\mathcal{M}_h$* (2.5) *then:*

(i) *After localisation at $S\mathbb{Z}_{(p)}$* (*see* A.3) *the chain complex of spectra $T^a X$ is split exact. In particular*

$$(X_{hG+})_{S\mathbb{Z}_{(p)}} \simeq \sum_{k=0}^{d} (-1)^k \bigvee_{G/H \subseteq \mathfrak{X}_k} (BH_+)_{S\mathbb{Z}_{(p)}}.$$

(ii) *The natural map $X_{hG} \to *_{hG} = BG$ induces an equivalence of $S\mathbb{Z}_{(p)}$-localised spectra $(X_{hG+})_{S\mathbb{Z}_{(p)}} \simeq (BG_+)_{S\mathbb{Z}_{(p)}}$.*

Proof. For convenience, replace $T^a(\mathbf{X})$ with its $S\mathbb{Z}_{(p)}$-localisation $T^a(\mathbf{X})_{S\mathbb{Z}_{(p)}}$. This remains a chain complex of connective spectra by Proposition A.3.1.

(i) If $d = 0$ there is nothing to prove because $X = X_0$. We therefore assume $d > 0$. Consider the Mackey functor $\mathcal{M}_h$ where $h = T_d X$. Since $T_d X$ is $\mathbb{Z}_{(p)}$-local so is $\mathcal{M}_h$ so by hypothesis $H^d_G(X; \mathcal{M}_h) = 0$. Equivalently, in light of (6.3), the homomorphism

$$\tilde{h}^0(T_{d-1}X) = [T_{d-1}X, T_d X] \to [T_d X, T_d X] = \tilde{h}^0(T_d X)$$

is surjective. A preimage of the identity on $T_d X$ is a left inverse for the differential $T_d X \to T_{d-1}X$ in TX. We deduce that

$$T_{d-1}X \simeq T'_{d-1}X \vee T_d X$$

for some $T'_{d-1}X$ which is connective and $S\mathbb{Z}_{(p)}$-local because it is a summand of $T_{d-1}X$.

If $d-1 > 0$ we set $h = T'_{d-1}X$. Once again, $\mathcal{M}_h$ is p-local so $H^{d-1}_G(X; \mathcal{M}_h) = 0$ by hypothesis. Equivalently, in light of (6.3),

$$\tilde{h}^0(T_{d-2}X) = [T_{d-2}X, T'_{d-1}X] \to [T'_{d-1}X, T'_{d-1}X] = \tilde{h}^0(T'_{d-1}X)$$

is surjective. A preimage of the identity on $T'_{d-1}X$ yields a splitting

$$T_{d-2}X \simeq T'_{d-2}X \vee T'_{d-1}X,$$

where $T'_{d-2}X$ is $S\mathbb{Z}_{(p)}$-local and connective, being a retract of one.

We continue in this way and show that $T_n X \simeq T'_n X \vee T'_{n+1}X$ for all $n \geq 0$ where $T'_n X$ are $\mathbb{Z}_{(p)}$-local and connective. In particular $T^a X$ takes the form

$$(1) \qquad \begin{aligned} (X_{hG+})_{S\mathbb{Z}_{(p)}} \leftarrow T'_0 X \vee T'_1 X \leftarrow T'_1 X \vee T'_2 X \leftarrow \cdots \\ \cdots \leftarrow T'_{d-2}X \vee T'_{d-1}X \leftarrow T'_{d-1}X \vee T_d X \leftarrow T_d X. \end{aligned}$$

Consider the filtration $\{X_{ihG}\}$ of X_{hG}. It gives rise to a convergent first-quadrant spectral sequence

$$E^1_{i,j} = \tilde{H}_{i+j}(X_{ihG}/X_{(i-1)hG}; \mathbb{Z}_{(p)}) \;\Rightarrow\; H_{i+j}(X_{hG}; \mathbb{Z}_{(p)}).$$

Theorem A.3.2 shows that $E^1_{*j} = H_j(T_* X; \mathbb{Z}_{(p)})$ and (1) shows that the spectral sequence collapses at its E^2-page to its vertical axis and therefore

$$H_j(X_{hG}; \mathbb{Z}_{(p)}) \approx E^2_{0,j} = H_j(T'_0 X; \mathbb{Z}_{(p)}).$$

It follows that $T'_0 X \to (X_{hG+})_{S\mathbb{Z}_{(p)}}$ is an $H_*(-; \mathbb{Z}_{(p)})$-isomorphism, and therefore an equivalence by Theorem A.3.2. This establishes the splitting of $T^a X$. As a consequence

$$(X_{hG_+})_{S\mathbb{Z}_{(p)}} \simeq \sum_{k=0}^{d} (-1)^k (T_k X)_{S\mathbb{Z}_{(p)}} \simeq \sum_{k=0}^{d} (-1)^k (\mathfrak{X}_{khG_+})_{S\mathbb{Z}_{(p)}}.$$

(ii) For any cohomology theory represented by a $S\mathbb{Z}_{(p)}$-local spectrum h we have

$$[BG_+, h] = \tilde{h}^0(BG_+) = \mathcal{M}_h(*) \approx H^0_G(*; \mathcal{M}_h) \approx H^0_G(X; \mathcal{M}_h) \\ \approx H^0(\tilde{h}^0 T(\mathbf{X})) \approx \tilde{h}^0(T'_0(\mathbf{X})) = [T'_0(\mathbf{X}), h].$$

Therefore $(BG_+)_{S\mathbb{Z}_{(p)}} \simeq T'_0X$. We have already seen in part (i) that $T'_0X \simeq (X_{hG_+})_{S\mathbb{Z}_{(p)}}$. The proof is complete. ∎

Our next goal in this section is to interpret the collapsing of the Bousfield–Kan spectral sequence in terms of G-acyclicity of spaces. Our treatment is inspired by Dwyer's insight [14].

PROPOSITION 6.5. *Consider a functor $\tilde{F} : \mathbf{C} \to G\mathcal{T}$ where $\mathbf{C}$ is a small category and $F(C)$ is an orbit of G for every object $C \in \mathbf{C}$. Let $F : \mathbf{C} \to \mathcal{T}$ denote $\tilde{F}_{hG}$ and consider a cohomology theory h^* represented by a spectrum h. Then the E_2-page of the Bousfield–Kan spectral sequence* [9, Ch. XII.5.8]

$$E_2^{i,j} = \varprojlim_{\mathbf{C}^{\mathrm{op}}}{}^i h^j(F) \;\Rightarrow\; h^{i+j}(\operatorname*{hocolim}_{\mathbf{C}} F) \tag{1}$$

can be identified with (*see* 5.3 *and* 2.5)

$$H^i_G(|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{F}|; \mathcal{M}_{\Sigma^j h}).$$

In particular the Bousfield–Kan spectral sequence of F with respect to h^* collapses at its E_2-page to the vertical axis if and only if $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{F}|$ is G-acyclic for all the Mackey functors $\mathcal{M}_{\Sigma^j h}$.

Proof. Consider the simplicial replacements $\coprod_* F$ and $\coprod_* \tilde{F}$ (see [9, p. 337] and 5.1) and set $X = |\coprod_* \tilde{F}|$. Note that $\coprod_* \tilde{F}$ is a G-model, hence X is a G-CW complex. Clearly $\coprod_* F = (\coprod_* \tilde{F})_{hG}$ and by definition

$$\operatorname*{hocolim}_{\mathbf{C}} F = |\textstyle\coprod_* F| = X_{hG}.$$

The simplicial structure of $\coprod_* F$ induces a filtration on its realisation which gives rise to the Bousfield–Kan spectral sequence (1). Since $X_n/X_{n-1} \simeq S^n \wedge \mathfrak{X}_{n+}$ (see 2.1), the E_1-page of this spectral sequence is

$$E_1^{i,j} = \tilde{h}^{i+j}(X_{ihG}/X_{(i-1)hG}) = \tilde{H}^i_G(X_{ihG}/X_{(i-1)hG}; \mathcal{M}_{\Sigma^j h}).$$

By 4.11(2) we can identify the jth row of the E_1-page with $C^*_G(X; \mathcal{M}_{\Sigma^j h})$. Consequently, the E_2-page of this spectral sequence has the form

$$E_2^{i,j} = H^i C^*_G(X; \mathcal{M}_{\Sigma^j h}) = H^i_G(X; \mathcal{M}_{\Sigma^j h}). \tag{2}$$

The result follows by recalling from 5.3 that $X = |\coprod_* \tilde{F}| = |\mathrm{Nr}\,\mathrm{Tr}\,\tilde{F}|$. ∎

We end this section with a remark on the chain complex TX (6.2) in favourable cases.

DEFINITION 6.6. A G-model $\mathbf{X}$ (5.11) is *combinatorial* if the face of every non-degenerate simplex of $\mathbf{X}/G$ is a non-degenerate simplex. We call $\mathbf{X}$ a *finite* G-model if the simplicial set $\mathbf{X}/G$ has finitely many non-degenerate simplices.

Note that if $\mathbf{X}$ is a finite G-model then $|\mathbf{X}|$ is a finite G-CW complex. Consider now a combinatorial G-model $\mathbf{X}$ and a spectrum h. By identifying the groups $H^i_G(|\mathbf{X}|;\mathcal{M}_{\Sigma^j h}) = H^i\tilde{h}^j T|\mathbf{X}|$ with the Bousfield–Kan spectral sequence [9, Ch. XI.7.3] of the geometric realisation of the simplicial space $\mathbf{X}_{hG}$, we can identify $T|\mathbf{X}|$ with the chain complex of spectra

$$\cdots \to (N\mathbf{X}_n)_{hG_+} \xrightarrow{\sum_{k=0}^n (-1)^k \partial_k} (N\mathbf{X}_{n-1})_{hG_+} \to \cdots$$
$$\cdots \to (N\mathbf{X}_1)_{hG_+} \xrightarrow{\partial_0-\partial_1} (N\mathbf{X}_0)_{hG_+}$$

where ∂_i are the face maps in the simplicial space $\mathbf{X}_{hG}$, and $N\mathbf{X}_n$ are the spaces of non-degenerate n-simplices of $\mathbf{X}$.

Since we shall not need this explicit description of $T|\mathbf{X}|$ in what follows, the details are left to the interested reader.

7. Posets of subgroups. A *topological poset* is a topological space X together with a partial order. It gives rise to an internal space category (5.2) whose object space is X and its morphism space is the obvious subspace of $X \times X$ consisting of the pairs (x_0, x_1) such that $x_0 \le x_1$. The space $\mathrm{Nr}_s X$ (see 5.2) can be identified with the subspace of $X^{\times(s+1)}$ of the $s+1$-tuples $x_0 \le x_1 \le \cdots \le x_s$.

If Y is another topological poset, a functor $f : X \to Y$ is the same as an order preserving continuous map $f : X \to Y$. Continuity on morphism spaces is automatic from the continuity of f. Similarly, a natural transformation T of functors $f \Rightarrow g : X \to Y$ exists if and only if $f(x) \le g(x)$ for every $x \in X$. Continuity of the map $X \xrightarrow{(f,g)} Y \times Y$ which defines T is automatic from the continuity of f and g. We immediately deduce

PROPOSITION 7.1. *Let X be a topological poset. If $x_0 \in X$ is a terminal (resp. initial) object, then $|\mathrm{Nr}\, X|$ is contractible.*

Proof. The constant function $F_{x_0} : X \to X$ is continuous and order preserving, hence a functor. The continuous assignment $x \mapsto (x, x_0)$ provides a natural transformation $\mathrm{Id} \to F_{x_0}$ (resp. $F_{x_0} \to \mathrm{Id}$). There results a homotopy from $1_{|\mathrm{Nr}\, X|}$ to the constant map $|\mathrm{Nr}\, X| \xrightarrow{x_0} |\mathrm{Nr}\, X|$. ∎

DEFINITION 7.2. Fix a topological poset X and an element $x_0 \in X$. The subposet of all $x \in X$ such that $x_0 \le x$ is denoted $[x_0, -)_X$. Similarly, $(x_0, -)_X$ denotes the subposet of the elements $x \in X$ such that $x_0 < x$.

Proposition 7.1 shows that the geometric realisation of the nerve of $[x_0, -)_X$ is contractible. Note that if j is the inclusion of the poset X in Y, then $(y{\downarrow}j)$ is isomorphic to the poset $[y,-)_Y \cap X$ which we shall denote by $[y,-)_X$.

Fix a compact Lie group G. Recall that G acts by conjugation on any collection $\mathcal{H}$ of subgroups of G and we topologise $\mathcal{H}$ as the coproduct of its orbits (i.e. conjugacy classes). We partially order $\mathcal{H}$ by inclusion of subgroups. Since conjugation in G carries inclusions to inclusions, the poset $\mathcal{H}$ becomes an internal G-space category. The points of $\mathrm{Nr}_s\,\mathcal{H}$ are chains of inclusions $H_0 \le \cdots \le H_s$ in $\mathcal{H}$. The isotropy group of such an s-simplex is $\bigcap_i N_G H_i$ and its orbit, i.e. its conjugacy class, is denoted by $(H_0 \le \cdots \le H_s)$.

LEMMA 7.3 (Bredon [10, II.5.7]). *Let H, K be closed subgroups of a compact Lie group G. Then $(G/H)^K/NK$ is a finite set.*

PROPOSITION 7.4. *Let $\mathcal{H}$ be a collection in a compact Lie group G. Then*

(i) $\mathrm{Nr}\,\mathcal{H}$ *is a combinatorial G-model* (6.6) *which is finite if $\mathcal{H}/G$ is finite. The non-degenerate simplices of* $\mathrm{Nr}\,\mathcal{H}$ *have the form $H_0 \lneq H_1 \lneq \cdots \lneq H_n$.*

(ii) *Let P be a subgroup of G fixed by $K \le G$, that is, $K \le NP$. Then the nerves of $[P,-)_{\mathcal{H}^K}$ and $(P,-)_{\mathcal{H}^K}$ are $N_{NP}(K)$-models.*

Proof. (i) We prove by induction on s that $\mathrm{Nr}_s\,\mathcal{H}/G$ is a discrete (resp. finite if $\mathcal{H}$ is finite). The case $s = 0$ follows from the definition of the topology on $\mathcal{H}$. Now, $\mathrm{Nr}_s\,\mathcal{H}$ is a G-subspace of $\mathcal{H} \times \mathrm{Nr}_{s-1}\,\mathcal{H}$. For an orbit (H_0) in $\mathcal{H}$ and an orbit $(H_1 \le \cdots \le H_s)$ in $\mathrm{Nr}_{s-1}\,\mathcal{H}$, the points of $\mathrm{Nr}_s\,\mathcal{H}$ belonging to $(H_0) \times (H_1 \le \cdots \le H_s)$ are in a G-equivariant one-to-one correspondence with a G-subspace of

$$G \times_{NH_0} (H_1 \le \cdots \le H_s)^{H_0} \approx G \times_{NH_0} \Big(G/\bigcap_{i=1}^{s} NH_i\Big)^{H_0}$$

which consists of finitely many orbits by Lemma 7.3. By the induction hypothesis on $\mathrm{Nr}_{s-1}\,\mathcal{H}/G$ it follows that $\mathrm{Nr}_s\,\mathcal{H}/G$ is discrete, being a coproduct of finite sets (resp. it is finite as a finite union of finite sets).

Clearly the non-degenerate simplices in $\mathrm{Nr}\,\mathcal{H}$ are the chains $H_0 < \cdots < H_s$, thus the face of any such s-simplex is non-degenerate. It follows that $\mathrm{Nr}\,\mathcal{H}$ is a combinatorial G-model.

(ii) Set $\mathcal{I} = [P,-)_{\mathcal{H}}$ and $\mathcal{J} = (P,-)_{\mathcal{H}}$. Clearly $\mathcal{I}, \mathcal{J}$ are NP-subposets of $\mathcal{H}^P$. Note that $\mathrm{Nr}\,\mathcal{H}^P$ is an NP-model by Lemma 7.3 and part (i). It immediately follows that $\mathrm{Nr}_s\,\mathcal{I}/NP$ and $\mathrm{Nr}_s\,\mathcal{J}/NP$ are discrete for all $s \ge 0$, that is, $\mathrm{Nr}\,\mathcal{I}$ and $\mathrm{Nr}\,\mathcal{J}$ are NP-models. Note that $K \le NP$ so we can apply Lemma 7.3 to $\mathrm{Nr}\,\mathcal{I}$ and $\mathrm{Nr}\,\mathcal{J}$ and deduce that $\mathrm{Nr}\,\mathcal{I}^K$ and $\mathrm{Nr}\,\mathcal{J}^K$ are $N_{NP}(K)$-models. ∎

DEFINITION 7.5. Fix a collection $\mathcal{H}$ in G. The *subdivision poset* of $\mathcal{H}$ is the topological poset $s(\mathcal{H})$ whose underlying set consists of the chains $H_0 <$

$\cdots < H_k$ of *proper* inclusions in $\mathcal{H}$. We let G act on $s(\mathcal{H})$ by conjugation and topologise it as the coproduct of its orbits. The elements $\mathbf{H}$ of $s(\mathcal{H})$ are called simplices which we view as finite non-empty subsets of $\mathcal{H}$ ordered linearly by inclusion. There is a unique morphism $\mathbf{H} \to \mathbf{H}'$ in $s(\mathcal{H})$ if $\mathbf{H}'$ is a subset of $\mathbf{H}$.

PROPOSITION 7.6. *Fix a collection $\mathcal{H}$ of subgroups of G. Then* $\mathrm{Nr}\, s(\mathcal{H})$ *is a combinatorial G-model* (6.6).

Proof. Set $\mathcal{K} = s(\mathcal{H})$ and note that $\mathcal{K}/G$ is discrete. For natural numbers $n, k \geq 0$ let $U(k, n)$ denote the set of chains $T_n \subseteq \cdots \subseteq T_1$ of non-empty subsets T_i of $\{0, 1, \ldots, k\}$. Given an $\mathbf{H} = \{H_0 < \cdots < H_k\}$ in $\mathcal{K}$, every element $T_n \subseteq \cdots \subseteq T_1$ in $U(k, n)$ gives rise to an element $\mathbf{H}_0 \to \cdots \to \mathbf{H}_n$ in $\mathrm{Nr}_n\, \mathcal{K}$ where $\mathbf{H}_0 = \mathbf{H}$ and $\mathbf{H}_i$ is the obvious subset of $\mathbf{H}$ corresponding to the inclusion of T_i in $\{0, 1, \ldots, k\}$. Conversely, every element $\mathbf{H}_0 \to \cdots \to \mathbf{H}_n$ in $\mathrm{Nr}_n\, \mathcal{K}$ corresponds to a k-simplex $\mathbf{H}_0$ in $s(\mathcal{H})$ and a unique element in $U(k, n)$. The uniqueness follows by considering $o(H_i)$ (see 3.12). We obtain a G-equivariant continuous bijection

$$\varphi : \coprod_k \mathcal{K} \times U(k, n) \to \mathrm{Nr}_n\, \mathcal{K}.$$

To see that φ^{-1} is also continuous we note that $\mathrm{Nr}_n\, \mathcal{K}$ is a subspace of

$$\mathcal{K}^{\times(n+1)} \cong \coprod_{(\mathbf{H}_0), \ldots, (\mathbf{H}_n) \subseteq \mathcal{K}} (\mathbf{H}_0) \times \cdots \times (\mathbf{H}_n).$$

Therefore it suffices to prove that φ^{-1} is continuous on every subspace $(\mathbf{H}_0) \times \cdots \times (\mathbf{H}_n) \cap \mathrm{Nr}_n\, \mathcal{K}$. This map has its image in $\mathcal{K} \times U(k, n)$ and it is induced by the projection to the first factor $\mathcal{K}^{\times(n+1)} \to \mathcal{K}$ and a constant map $\mathcal{K}^{\times(n+1)} \to U(k, n)$, hence it is continuous. Since the orbit space of the domain of φ is discrete, so is $\mathrm{Nr}_n\, \mathcal{K}/G$. It follows that $\mathrm{Nr}\, s(\mathcal{H})$ is a G-model. The verification that it is combinatorial is straightforward. ∎

DEFINITION 7.7. Fix a collection $\mathcal{H}$ and define a functor $\mu : s(\mathcal{H}) \to \mathcal{H}$ by mapping a k-simplex $\mathbf{H}$ in $s(\mathcal{H})$ to its minimal element, that is,

$$\mu : \{H_0 < \cdots < H_k\} \mapsto H_0.$$

We have to show that the assignment μ is continuous and order preserving.

Proof. A morphism $\mathbf{H} \to \mathbf{H}'$ in $s(\mathcal{H})$ exists if an only if $\mathbf{H}' \subseteq \mathbf{H}$, hence $H_0 \leq H_0'$, so μ respects the partial order. To see that μ is a functor we have to show that it is continuous. This is obvious because it is the composition

$$s(\mathcal{H}) \to \coprod_k \mathcal{H}^{\times(k+1)} \xrightarrow{\coprod_k \mathrm{proj}} \coprod_k \mathcal{H} \xrightarrow{\mathrm{fold}} \mathcal{H}$$

where the projections are to the first factor. ∎

LEMMA 7.8. *Any G-equivariant function $G/H \to G/K$ is continuous.*

Proof. This is immediate from Bredon [10, Theorem I.3.3]. ∎

The next result is known for finite groups (e.g. [33]).

PROPOSITION 7.9. *Fix a collection $\mathcal{H}$ in G. Then μ (7.7) induces a G-homotopy equivalence $|\mathrm{Nr}\, s(\mathcal{H})| \to |\mathrm{Nr}\, \mathcal{H}|$.*

Proof.

CLAIM. *For every $L \in \mathcal{H}$ the nerve of $(L{\downarrow}\mu)$ is an NL-model whose geometric realisation is NL-equivariantly contractible.*

Proof. Proposition 7.6 and Lemma 7.3 imply that $\mathrm{Nr}\, \mathcal{H}^L$ and $\mathrm{Nr}\, s(\mathcal{H})^L$ are NL-models. Therefore $\mathrm{Nr}(L{\downarrow}\mu)$ is an NL-model because $(L{\downarrow}\mu)$ is the NL-subposet of $s(\mathcal{H})^L$ consisting of the simplices $\mathbf{H}$ such that $L \le H_0$. In particular the poset $(L{\downarrow}\mu)$ is a coproduct of its NL-orbits. Define functions $\Phi, \Psi : (L{\downarrow}\mu) \to (L{\downarrow}\mu)$ by

$$\Phi : \mathbf{H} \mapsto \{L\} \cup \mathbf{H}, \qquad \Psi : \mathbf{H} \mapsto \{L\}.$$

Both assignments are NL-equivariant and are therefore continuous by Lemma 7.8. They are also order preserving and therefore Φ and Ψ are functors of NL-posets. For any $\mathbf{H} \in (L{\downarrow}\mu)$, the zigzag of inclusions

$$\mathbf{H} \subseteq \{L\} \cup \mathbf{H} \supseteq \{L\} \tag{1}$$

gives rise to natural transformations $\mathrm{Id} \leftarrow \Phi \to \Psi$. There results an NL-equivariant contraction of $|\mathrm{Nr}(L{\downarrow}\mu)|$ because Ψ is constant. $\blacksquare_{\text{Claim}}$

Propositions 7.4(i) and 7.6 imply that $\mathrm{Nr}\, \mathcal{H}$ and $\mathrm{Nr}\, s(\mathcal{H})$ are G-models, hence $|\mathcal{H}|$ and $|s(\mathcal{H})|$ are G-CW complexes. By [38, Ch. II, Proposition 2.7] it suffices to prove that μ induces a homotopy equivalence $|s(\mathcal{H})|^K \to |\mathcal{H}|^K$ for all $K \le G$.

Lemma 7.3 shows that $\mathrm{Nr}\, \mu^K$ is a map of NK-models. In particular μ^K is absolutely tame by Corollary 5.12. Also, Proposition A.2.2 shows that $\mathrm{Nr}\, \mathcal{H}^K$ and $\mathrm{Nr}\, s(\mathcal{H})^K$ are Reedy cofibrant. We now consider $(L{\downarrow}\mu^K)$ for some $L \in \mathcal{H}^K$, that is, $K \le NL$. Lemma 7.3 and the claim above show that $\mathrm{Nr}(L{\downarrow}\mu^K)$ is an $N_{NL}K$-model and in particular it is Reedy cofibrant by Proposition A.2.2. Furthermore, the NL-equivariant contraction (1) of $|(L{\downarrow}\mu)|$ provides, by restriction, a contraction of $|(L{\downarrow}\mu^K)|$. We are now in a position to apply Theorem 5.8 to μ^K and deduce that μ^K induces a homotopy equivalence on nerves. ∎

The orbit space $s(\mathcal{H})/G$ is easily seen to be a poset which was denoted by $\bar{s}(\mathcal{H})$ in 1.4 where the "tautological" functor $\tilde{\delta}_{\mathcal{H}} : \bar{s}(\mathcal{H}) \to G\mathcal{T}$ was defined. The normaliser decomposition is defined by $\delta_{\mathcal{H}} := (\tilde{\delta}_{\mathcal{H}})_{hG}$.

PROPOSITION 7.10. *Let $\mathcal{H}$ be a collection in G. Then $\mathrm{Tr}\, \tilde{\delta}_{\mathcal{H}} = s(\mathcal{H})$.*

Proof. This follows from a straightforward inspection of the definitions of Tr in 5.3 and $s(\mathcal{H})$ in 7.5. ∎

COROLLARY 7.11. *A collection $\mathcal{H}$ is normaliser sharp for a cohomology theory h^* if $|\mathrm{Nr}\,\mathcal{H}|$ is G-acyclic for $\mathcal{M}_{\Sigma^j h}$ for all j (see* 2.5, 4.1*).*

Proof. This follows from Propositions 6.5, 7.9 and 7.10 and Definition 1.6. ∎

8. Elementary abelian p-subgroups. Fix a prime p. In every compact Lie group G we let $\Omega_1(G)$ denote the closed subgroup of G generated by the elements of order p. It is clearly a characteristic subgroup. It is an elementary abelian p-group if G is abelian. If G is p-toral it is also clear that $\Omega_1(G)$ is not trivial.

PROPOSITION 8.1. *The centre of a non-trivial p-toral group G is not trivial. Furthermore $\Omega_1 Z(G)$ is a non-trivial characteristic elementary abelian p-subgroup of G.*

Proof. The second assertion follows from the first by the remarks above. View G as a space on which G acts by conjugation. Smith theory implies that $\chi(ZG) = \chi(G) \equiv 0 \bmod p$, hence $ZG \neq 1$. ∎

PROPOSITION 8.2 ([22, Lemma 6.1]). *In every compact Lie group there are only finitely many conjugacy classes of elementary abelian p-subgroups.*

Recall from A.1.2 that every compact Lie group K contains a maximal normal p-toral subgroup $O_p(K)$. Also recall that the collection of all the non-trivial elementary abelian p-subgroups of G is denoted $\mathcal{E}_p^0(G)$.

PROPOSITION 8.3. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup and let $\mathcal{E}$ denote $\mathcal{E}_p^0(G)$. Then*

(i) $\mathrm{Nr}\,\mathcal{E}$ *is a finite combinatorial G-model* (6.6).
(ii) *Every $H \in \mathrm{Iso}_G(|\mathrm{Nr}\,\mathcal{E}|)$ contains a non-trivial p-toral subgroup.*
(iii) *If $K \leq G$ is such that $O_p(K) \neq 1$ then $|\mathrm{Nr}\,\mathcal{E}|^K$ is contractible.*

Proof. (i), (ii). First, $\mathrm{Nr}\,\mathcal{E}$ is a finite G-model by Propositions 7.4 and 8.2, and the isotropy groups of the points of $|\mathrm{Nr}\,\mathcal{E}|$ have the form $H = NE_0 \cap \cdots \cap NE_k$ for some $E_0 \leq \cdots \leq E_k$ in $\mathcal{E}$. In particular $1 \neq E_0 \leq H$.

(iii). Consider $K \leq G$ such that $O_p(K) \neq 1$. Set $Z = \Omega_1(ZO_p(K))$ and observe that $Z \in \mathcal{E}$ by Proposition 8.1. Furthermore $Z \lhd NK$ because Z is characteristic in K.

Observe that the poset $\mathcal{E}^K$ is topologically a disjoint union of NK-orbits by Lemma 7.3. Note that if $E \in \mathcal{E}^K$ then K normalises E and therefore $C_E(Z) \neq 1$ (see e.g. [13, Proposition 5.2]). The assignments

$$\Phi : E \mapsto C_E(Z), \qquad \Psi : E \mapsto C_E(Z) \cdot Z, \qquad \Xi : E \mapsto Z$$

are order preserving NK-equivariant functions on $\mathcal{E}^K$. In particular they are endofunctors on $\mathcal{E}^K$ by Lemma 7.8. The zigzag of inclusions

$$E \geq C_E(Z) \leq C_E(Z) \cdot Z \geq Z \quad (E \in \mathcal{E}^K)$$

provides natural transformations $\mathrm{Id} \leftarrow \Phi \rightarrow \Psi \leftarrow \Xi$ which connect the identity on $\mathcal{E}^K$ to the constant functor Ξ. It follows that $|\mathcal{E}|^K$ is contractible. This argument appears in Dwyer [14, §8] who attributes it to Quillen. ∎

Point (a) of the next corollary was obtained by J. Słomińska in [34]. Points (b) and (c) are used to prove Theorems D and A. Note that $\mathrm{Nr}(\mathcal{E}_p^0(G))$ is a finite G-model (6.6) by Propositions 7.4 and 8.2, so in particular there are finitely many conjugacy classes of chains $E_0 < \cdots < E_k$ in $\mathcal{E}_p^0(G)$.

Corollary 8.4. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup, and let $\mathcal{E}$ denote $\mathcal{E}_p^0(G)$. Then*

(a) *$|\mathcal{E}|/G$ is contractible.*
(b) *$|\mathcal{E}|$ is G-acyclic for any p-local coefficient functor $\mathcal{M}_h$* (2.5).
(c) *After localisation at the Moore spectrum $S\mathbb{Z}_{(p)}$, there is an equivalence of spectra*

$$BG_+ \simeq \sum_{(E_0<\cdots<E_k)} (-1)^k \big(B(NE_0 \cap \cdots \cap NE_k)\big)_+$$

where the sum is taken over all the conjugacy classes $(E_0 < \cdots < E_k)$ in $\mathcal{E}$.

Proof. This is immediate from Proposition 8.3 and Theorems 4.3 and 6.4. ∎

In the remainder of this section we will give a new proof for the results of Jackowski and McClure in [22]. Fix a compact Lie group G which contains a non-trivial p-toral subgroup and set $\mathcal{E} = \mathcal{E}_p^0(G)$. Recall that $\mathcal{A}_G(\mathcal{E})$ has discrete morphism spaces (1.3) and therefore $\coprod_* \tilde{\alpha}_{\mathcal{E}}$ is a G-model (5.1, 5.11). In particular $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\alpha}_{\mathcal{E}}|$ is a G-CW complex (5.3).

Proposition 8.5. *In the notation above, let X denote $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\alpha}_{\mathcal{E}}|$ and let P be a maximal p-toral subgroup of G. Consider X as a P-space and set $X' = X_{\text{type-}\mathcal{S}_p^0(P)}$ (see 3.9, 3.10). Then X'^Q is N_PQ-equivalent to a point for every $Q \in \mathcal{S}_p^0(P)$.*

Proof. Fix $Q \in \mathcal{S}_p^0(P)$ and observe that $X'^Q = X^Q$. Set

$$H = N_P Q \cdot CQ.$$

This is a subgroup of NQ and it suffices to show that X^Q is H-equivalent to a point.

Set $\mathbf{C} := \mathrm{Tr}\,\tilde{\epsilon}_{\mathcal{E}}$ and consider the internal NQ-space category $\mathbf{C}^Q$. Clearly $X^Q = |\mathrm{Nr}\,\mathbf{C}^Q|$ and note that $\mathrm{Hom}_G(E, G)^Q = \mathrm{Hom}_G(E, CQ)$. Therefore the object and morphism spaces of $\mathbf{C}^Q$ are

$$\coprod_{E\in\mathcal{E}} \mathrm{Hom}_G(E, CQ), \qquad \coprod_{E',E\in\mathcal{E}} \mathrm{Hom}_G(E, CQ)\times \mathrm{Hom}_G(E', E).$$

Lemma 7.3 shows that $\mathrm{Hom}_G(E, CQ)$ has finitely many NQ-orbits. In addition $NQ/Q\cdot CQ < \infty$ by [39, Lemma 5.9.11], therefore $\mathrm{Hom}_G(E, CQ)$ has finitely many CQ-orbits. In particular $\mathrm{Obj}(\mathbf{C}^Q)$ is a coproduct of H-orbits, and more generally $\mathrm{Nr}\,\mathbf{C}^Q$ is an H-model.

Let $\mathbf{D}$ denote the full subcategory of $\mathbf{C}^Q$ whose objects have the form $E \xrightarrow{c_h} CQ$ for $E \le CQ$ and $h \in H$. It is easily seen that the object and morphism spaces of $\mathbf{D}$ are

$$\coprod_{E\in\mathcal{E}\cap CQ} \mathrm{Hom}_H(E, CQ), \qquad \coprod_{E',E\in\mathcal{E}\cap CQ} \mathrm{Hom}_H(E, CQ)\times \mathrm{Hom}_H(E', E).$$

Let $J : \mathbf{D} \to \mathbf{C}^Q$ denote the inclusion of these internal H-space categories. Fix an object $c \in \mathbf{C}^Q$ of the form $E \xrightarrow{c_g} CQ$. For every $E' \in \mathcal{E}\cap CQ$ the set $\mathrm{Hom}_G(E', c_g(E))$ is finite. Hence, there are only finitely many $c_h \in \mathrm{Hom}_H(E', CQ)$ which admit a morphism $c_h \to c$ in $\mathbf{C}^Q$. This shows that the object space of $(J{\downarrow}c)$ is discrete. The morphism spaces of $(J{\downarrow}c)$ are subspaces of the morphism spaces of the objects of $\mathbf{C}^Q$ and they are therefore discrete as well. We have shown that for every $c \in \mathbf{C}^Q$ the category $(J{\downarrow}c)$ is a small category.

Claim 1. *J induces an H-homotopy equivalence* $|\mathrm{Nr}\,\mathbf{D}| \to |\mathrm{Nr}\,\mathbf{C}^Q|$.

Proof. Since $\mathrm{Nr}\,\mathbf{D}$ and $\mathrm{Nr}\,\mathbf{C}^Q$ are H-models, their geometric realisations are H-CW complexes fixed by Q. To prove the equivalence it suffices to show that for every $K/Q \le H/Q$ the functor J^K induces a homotopy equivalence.

Fix $K/Q \le H/Q$ and consider $c \in \mathbf{C}^K$ of the form $E \xrightarrow{c_g} CK$. First, $(J^K{\downarrow}c)$ is a small category because it is a subcategory of $(J{\downarrow}c)$ which we have shown to have discrete object and morphism spaces. In particular, $\mathrm{Nr}(J^K{\downarrow}c)$ is Reedy cofibrant by A.2.2. Clearly $c_g(E) \in \mathcal{E}\cap CQ$ because $CK \le CQ$. By inspection, the object $\mathrm{id}_{c_g(E)} \xrightarrow{c_{g-1}} c$ is terminal in $(J^k{\downarrow}c)$. It follows that $|\mathrm{Nr}(J^K{\downarrow}c)|$ is contractible.

We have noted that $\mathrm{Nr}\,\mathbf{D}$ and $\mathrm{Nr}\,\mathbf{C}^Q$ are H-models and therefore $\mathrm{Nr}\,\mathbf{D}^K$ and $\mathrm{Nr}\,\mathbf{C}^K$ are N_HK-models (7.3) and in particular Reedy cofibrant by A.2.2. Corollary 5.12 also shows that J^K is tame. We can now apply Theorem 5.8 and Remark 5.9 and deduce that J^K induces a homotopy equivalence. $\blacksquare_{\text{Claim 1}}$

Claim 2. *$|\mathbf{D}|$ is H-equivariantly contractible.*

Proof. Recall that N_PQ is a p-toral group (A.1.1) and let E denote $\Omega_1(ZQ)$ as in Proposition 8.1. Note that E is normal in N_PQ and since it is finite, the action of N_PQ on E factors through the finite p-group of components $\overline{N_PQ}$. It follows that $Z = C_E(\overline{N_PQ})$ is a non-trivial elementary abelian p-subgroup of Q which is central in N_PQ. It must also be central in Q, whence $Z \leq CQ$ and $Z \leq ZH$.

We now deduce that every $\varphi \in \mathrm{Hom}_H(E, CQ)$ where $E \leq CQ$ can be unambiguously extended to $E \cdot Z$ by setting it equal to the identity on Z. We obtain H-equivariant maps $\mathrm{Hom}_H(E, CQ) \to \mathrm{Hom}_H(EZ, CQ)$. This, in particular, gives rise to an H-equivariant map $\zeta : \mathrm{Obj}(\mathbf{D}) \to \mathrm{Obj}(\mathbf{D})$ which is also continuous by Lemma 7.8 because $\mathrm{Nr}\,\mathbf{D}$ is an H-model. Similarly, for every $E, E' \leq \mathcal{E} \cap CQ$ we have an H-equivariant bijection

$$\zeta : \mathrm{Hom}_H(E', E) \times \mathrm{Hom}_H(E, CQ) \to \mathrm{Hom}_H(E'{\cdot}Z, E{\cdot}Z) \times \mathrm{Hom}_H(E{\cdot}Z, CQ).$$

We obtain an endofunctor ζ on $\mathbf{D}$. Define $\gamma : \mathbf{D} \to \mathbf{D}$ to be the constant functor

$$\gamma \ : \ (E \xrightarrow{c_g} CQ) \ \mapsto \ (Z \xrightarrow{\mathrm{incl}} CQ)$$

which is clearly H-equivariant. The zigzag of inclusions $E \leq E \cdot Z \geq Z$ gives rise to natural transformations $\mathrm{Id} \leftarrow \zeta \to \gamma$ which provide an H-equivariant contraction of $|\mathrm{Nr}\,\mathbf{D}|$. $\blacksquare_{\text{Claim 2}}$

Claims 1 and 2 show that $X^Q = |\mathrm{Nr}\,\mathbf{C}^Q|$ is H-equivalent to a point. $\blacksquare$

We are now ready to prove the last part of Theorem D.

THEOREM 8.6. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup. The collection $\mathcal{E} = \mathcal{E}_p^0(G)$ is centraliser sharp for any p-local cohomology theory h^*.*

Proof. Fix a maximal p-toral subgroup P of G and let X denote $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\alpha}_{\mathcal{E}}|$. We will prove below that

(1) $\tilde{H}^*(X/G; \mathbb{Z}_{(p)}) = 0$,

(2) X is P-acyclic for all p-local Mackey functors $\widetilde{\mathcal{M}_h}$.

Proposition 4.2 implies that X is G-acyclic for any p-local $\mathcal{M}_h$ and the result follows from Proposition 6.5.

Proof of (1). Proposition 8.2 implies that the category $\mathcal{A}_G(\mathcal{E})$ contains a finite skeletal subcategory $\mathcal{A}_G^{\mathrm{sk}}(\mathcal{E})$ by choosing a representative from every conjugacy class of $E \in \mathcal{E}$. We then have a G-homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{A}_G^{\mathrm{sk}}(\mathcal{E})^{\mathrm{op}}} \tilde{\alpha}_{\mathcal{E}} := X_{\mathrm{sk}} \simeq X.$$

Since $\mathcal{A}_G^{\mathrm{sk}}(\mathcal{E})$ is a finite category, it follows that X_{sk} has finitely many G-cells in every dimension and in particular X_{sk}/G is a CW-complex of finite type. Therefore X/G is equivalent to a CW-complex of finite type so to prove (1) it is enough to prove that $\tilde{H}^*(X/G; \mathbb{Z}/p) = 0$.

Let M be the constant Mackey functor $\mathrm{Const}_{\mathbb{Z}/p}$ on G. Note that
$$H^*(X/G;\mathbb{Z}/p) \approx H^*_G(X;M).$$
Lemma 4.7 and Proposition 4.12 show that there is a commutative ladder

$$\begin{tikzcd}
H^*_G(X;M) \arrow[r, "\mathrm{res}_P^G"] & H^*_P(X;M) \arrow[r, "i^*"] & H^*_P(X';M) \arrow[r] & H^*_G(X;M) \\
H^*_G(*;M) \arrow[u] \arrow[r, "\mathrm{res}_P^G"] & H^*_P(*;M) \arrow[u] \arrow[ur] \arrow[rr, "\mathrm{tr}_P^G"] & & H^*_G(*;M) \arrow[u]
\end{tikzcd}$$

where i is the inclusion of $X' = X_{\text{type-}\mathcal{S}^0_p(P)}$ in X (3.9, 3.10). Furthermore, the composition in both rows is an isomorphism by Proposition 4.9. Since a retract of an isomorphism is an isomorphism we see that it suffices to prove that $H^*_P(*;M) \to H^*_P(X';M)$ is an isomorphism. This follows from Proposition 8.5 and Lemma 3.2(ii) by noting that M is p-constrained and that X' has finite P-orbit type (see 4.10) and that G has a non-trivial p-toral subgroup.

Proof of (2). Note that $\widetilde{\mathcal{M}}_h\downarrow^G_P$ is p-constrained by Lemma 2.10 and vanishes on orbits whose isotropy group is finite of order prime to p. Proposition 2.4 shows that $H^*_P(X;\widetilde{\mathcal{M}}_h) \approx H^*_P(X';\widetilde{\mathcal{M}}_h)$. With the aid of Proposition 8.5 we can apply Lemma 3.2(ii) again to conclude that X' is P-acyclic for $\widetilde{\mathcal{M}}_h$. ■

9. p-radical subgroups. Recall that $S(G)$ and $S_p(G)$ denote the spaces of (p-toral) subgroups of G endowed with the Hausdorff metric (A.1). The subspaces of non-trivial (p-toral) subgroups are denoted $S^0(G)$ and $S^0_p(G)$. The associated collections $\mathcal{S}(G)$ and $\mathcal{S}_p(G)$ have the same underlying set but they are topologised as the coproduct of their G-orbits.

The following is a consequence of tom Dieck [38, Ch. IV, Proposition 3.4]:

LEMMA 9.1. *Let X be a G-space with finitely many orbit types and assume that $H = \lim_{n\to\infty} H_n$ in $S(G)$. Then $X^H = X^{H_n}$ for sufficiently large n.*

PROPOSITION 9.2. *$\mathcal{B}_p(G)/G$ is finite* (1.1). *In particular $\mathcal{B}^0_p(G)/G$ is finite and* $\mathrm{Nr}\,\mathcal{B}^0_p(G)$ *is a finite G-model.*

Proof. Let $B_p(G)$ denote the subspace of $S_p(G)$ of p-radical subgroups. Note that if P is p-radical and $P < P'$ where P' is p-toral, then it is impossible to have $NP \le NP'$. This is proved in [23, Lemma 1.5(i)] under the assumption that $NP/P < \infty$, which is not used.

To complete the proof we follow [23, Proposition 1.6]. If $B_p(G)/G$ is not finite, the compactness of $S_p(G)/G$ implies the existence of a convergent sequence of distinct points (P_k) in $B_p(G)/G$ whose limit $(P) \in S_p(G)/G$ is distinct from all the (P_k)'s. Proposition A.1.4 implies that we may assume that $P_n \lneq P$ and $P_n \to P$ in $S_p(G)$. Now, G/P is a P-space of finite orbit type by 4.10 so $NP/P = (G/P)^P = (G/P)^{P_n}$ for all large n by 9.1. In particular $NP_n \le NP$, which is a contradiction. It follows that $B_p(G)/G < \infty$ and therefore, by Proposition 7.4, $\mathrm{Nr}\,\mathcal{B}_p(G)$ and $\mathrm{Nr}\,\mathcal{B}_p^0(G)$ are finite G-models. ∎

LEMMA 9.3. *Let $\mathcal{H}$ be a collection with $\mathcal{H}/G < \infty$ and let $P_1 \le P_2 \le \cdots$ be a sequence in G which converges to P in $S_p(G)$. Then $[P,-)_{\mathcal{H}} = [P_n,-)_{\mathcal{H}}$ for some n* (7.2).

Proof. First, $\mathcal{H}^P = \mathcal{H}^{P_m}$ for some m by Lemma 9.1. Also note that $[P,-)_{\mathcal{H}}$ is an NP-subspace of $\mathcal{H}^P$, hence a coproduct of finitely many NP-orbits by Lemma 7.3. Similarly, $[P_m,-)_{\mathcal{H}}$ is an NP_m-subspace of $\mathcal{H}^{P_m}$.

Consider an NP-orbit (Q) in $\mathcal{H}^P$ such that $P \not\le Q$. Clearly $P \not\le Q^g$ for any $g \in NP$. We claim that there exists k such that $P_k \not\le Q^g$ for all $g \in NP$. If this is not the case, choose $g_k \in NP$ for every $k \ge 1$ such that $P_k \le Q^{g_k}$. Using the compactness of NP we may assume that $g_k \to g$ and therefore, by Lemma A.1.5,

$$P = \lim_{k\to\infty} P_k \le \lim_{k\to\infty} Q^{g_k} = Q^g,$$

which is a contradiction. Now, $\mathcal{H}^P = \coprod_{i=1}^t (Q_i)$ where Q_i are representatives for the NP-conjugacy classes of the subgroup $Q \in \mathcal{H}^P$. If $t = 0$ then $\mathcal{H}^P$ is empty and consequently $[P,-)_{\mathcal{H}} \subseteq [P_m,-)_{\mathcal{H}} \subseteq \mathcal{H}^{P_m} = \mathcal{H}^P$ are empty.

Now assume that $t \ge 1$. For every $i \le t$, if $P \le Q_i$ set $n_i = m$. If $P \not\le Q_i$, choose some $n_i \ge m$ as above such that $P_{n_i} \not\le Q_i^g$ for all $g \in NP$. Define $n = \max n_i$. Clearly $n \ge m$ so

$$[P,-)_{\mathcal{H}} \subseteq [P_n,-)_{\mathcal{H}} \subseteq \mathcal{H}^{P_n} = \mathcal{H}^P.$$

The inclusion $[P,-)_{\mathcal{H}} \subseteq [P_n,-)_{\mathcal{H}}$ is an equality because we have arranged n so that every NP-orbit (Q_i) in $\mathcal{H}^P$ which is not in $[P,-)_{\mathcal{H}}$ must lie entirely outside of $[P_n,-)_{\mathcal{H}^P} = [P_n,-)_{\mathcal{H}^{P_n}} = [P_n,-)_{\mathcal{H}}$. ∎

The proof of Proposition 9.4 is adapted from [4, Proposition 6.6.5].

PROPOSITION 9.4. *Let $\mathcal{H}$ be a concave subcollection* (3.8) *of $\mathcal{S}_p(G)$ and set $\mathcal{B} = \mathcal{B}_p(G)$. Then the inclusion $j : \mathcal{B} \cap \mathcal{H} \to \mathcal{H}$ induces a G-homotopy equivalence on their realisations.*

Proof. Fix $K \le G$. We have to show that $|\mathcal{B}\cap\mathcal{H}|^K \to |\mathcal{H}|^K$ is a homotopy equivalence.

For $P \in \mathcal{H}^K$ we can identify $(P{\downarrow}j^K)$ with $[P,-)_{\mathcal{B}\cap\mathcal{H}^K}$. Consider

$$\mathcal{C}(K) = \{P \in \mathcal{H}^K : \text{the realisation of } [P,-)_{\mathcal{B}\cap\mathcal{H}^K} \text{ is not contractible}\}.$$

CLAIM 1. *If $\mathcal{C}(K)$ is not empty, it must contain a maximal element.*

Proof. Consider a chain of subgroups $\{P_\lambda\}$ in $\mathcal{C}(K)$ ordered by inclusion. We will find an upper bound P for this chain in $\mathcal{C}(K)$ and apply Zorn's lemma.

Consideration of the order of the P_λ's (3.12) easily yields a cofinal subsequence $P_1 \leq P_2 \leq \cdots$ in this chain. Using the compactness of $S_p(G)$ (Proposition A.1.6) we may assume that this sequence converges to some $P \in S_p(G)$, and therefore (see e.g. [39, p. 108])

$$P = \overline{\bigcup_{n\geq 1} P_n}.$$

It follows that $P \in \mathcal{H}$ because $\mathcal{H}$ is concave in $\mathcal{S}_p(G)$. Furthermore K must normalise P because it normalises the P_n's, that is, $P \in \mathcal{H}^K$. It remains to show that the nerve of $[P,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is not contractible. This follows immediately from Lemma 9.3 and Proposition 9.2 because $K \leq NP_n$ for all n and $K \leq NP$, so for some large n,

$$[P,-)_{\mathcal{B}\cap\mathcal{H}^K} = ([P,-)_{\mathcal{B}\cap\mathcal{H}})^K = ([P_n,-)_{\mathcal{B}\cap\mathcal{H}})^K = [P_n,-)_{\mathcal{B}\cap\mathcal{H}^K}.$$

But $P_n \in \mathcal{C}(K)$ so the nerve of $[P,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is not contractible. $\blacksquare_{\text{Claim 1}}$

Claim 2. *$\mathcal{C}(K)$ cannot contain a maximal element.*

Proof. Assume that P is a maximal element of $\mathcal{C}(K)$. Clearly $P \notin \mathcal{B}$ by Proposition 7.1 because the nerve of the poset $[P,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is not contractible so it cannot contain a minimal element. Thus, $[P,-)_{\mathcal{B}\cap\mathcal{H}^K} = (P,-)_{\mathcal{B}\cap\mathcal{H}^K}$. We shall now obtain a contradiction to the existence of P by proving that the realisation of $(P,-)_{\mathcal{B}\cap\mathcal{H}^K}$ must, in fact, be contractible.

Let ℓ denote the inclusion $(P,-)_{\mathcal{B}\cap\mathcal{H}^K} \subseteq (P,-)_{\mathcal{H}^K}$. The nerves of these topological posets are $N_{NP}(K)$-models by Proposition 7.4. Therefore they are Reedy cofibrant by A.2.2, and Corollary 5.12 also shows that ℓ is tame. If Q is an element in $(P,-)_{\mathcal{H}^K}$ then the realisation of $(Q{\downarrow}\ell) = [Q,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is contractible by the maximality of P in $\mathcal{C}(K)$. Furthermore $[Q,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is an $N_{NQ}(K)$-model by 7.4 and in particular its nerve is Reedy cofibrant. We can now apply Theorem 5.8 to conclude that ℓ induces a weak homotopy equivalence on realisations.

To complete the proof of Claim 2 we now show that $|\mathrm{Nr}(P,-)_{\mathcal{H}^K}|$ is contractible. The nerve of $(P,-)_{\mathcal{H}^K}$ is an $N_{NP}(K)$-model by Proposition 7.4 and in particular this poset is a coproduct of its $N_{NP}K$-orbits. Define self-maps Φ, Ψ, Ξ of $(P,-)_{\mathcal{H}^K}$ which for every $Q \in (P,-)_{\mathcal{H}^K}$ have the effect

$$\Phi : Q \mapsto N_Q P, \qquad \Psi : Q \mapsto N_Q P \cdot O_p(NP), \qquad \Xi : Q \mapsto O_p(NP).$$

The assignments are well defined because (i) $P \lneq N_Q P$ because $P \lneq Q$, (ii) $N_Q P$ normalises $O_p(NP)$ and both are p-toral, (iii) $O_p(NP) \gneq P$ because $P \notin \mathcal{B}$ and (iv) K normalises Q and P hence it normalises $N_Q(P)$ and

$O_p(NP)$. The assignments are also $N_{NP}K$-equivariant and order preserving and therefore, by Proposition 7.8, Φ, Ψ and Ξ are endofunctors of the topological poset $(P,-)_{\mathcal{H}^K}$. The zigzag of inclusions

$$Q \geq N_Q(P) \leq N_Q(P)O_p(NP) \geq O_p(NP), \qquad Q \in (P,-)_{\mathcal{H}^K},$$

is clearly $N_{NP}K$-equivariant and provides a zigzag of natural transformations $\mathrm{Id} \leftarrow \Phi \rightarrow \Psi \leftarrow \Xi$ which connects the identity functor to the constant functor Ξ. This shows that $|\mathrm{Nr}(P,-)_{\mathcal{H}^K}|$ is contractible. $\blacksquare_{\text{Claim 2}}$

The nerves of $\mathcal{B}\cap\mathcal{H}^K$ and $\mathcal{H}^K$ are NK-models by 7.3 and 7.4, hence their nerves are Reedy cofibrant (A.2.2) and j^K is tame by Corollary 5.12. Similarly, for every $P \in \mathcal{H}^K$ the nerve of $(P{\downarrow}j^K) = [P,-)_{\mathcal{B}\cap\mathcal{H}^K}$ is an $N_{NP}(K)$-model by Proposition 7.4, hence its nerve is Reedy cofibrant. Claims 1 and 2 show that $\mathcal{C}(K)$ must be empty, that is $|\mathrm{Nr}(P{\downarrow}j^K)| \simeq *$ for all $P \in \mathcal{H}^K$. Theorem 5.8 shows that j^K induces a homotopy equivalence on realisations. Since this is true for all K, we conclude that $|j|$ is a G-homotopy equivalence. $\blacksquare$

PROPOSITION 9.5. *Let $\mathcal{H}$ be a non-empty concave subcollection of $\mathcal{S}_p(G)$. Then $|\mathcal{H}|^K$ is contractible whenever $O_p(K) \in \mathcal{H}$.*

Proof. Denote $Q = O_p(K)$. Clearly $NK \leq NQ$. The assignments $\Phi : P \mapsto PQ$ and $\Psi : P \mapsto Q$ define functions $\mathcal{H}^K \to \mathcal{H}^K$. They are order preserving and NK-equivariant, and therefore Propositions 7.4 and 7.8 show that these are endofunctors of the topological poset $\mathcal{H}^K$. The NK-equivariant zigzag of inclusions

$$P \leq PQ \geq Q, \qquad P \in \mathcal{H}^K,$$

gives rise to a zigzag of natural transformations $\mathrm{Id} \to \Phi \leftarrow \Psi$ which connect the identity on $\mathcal{H}^K$ to the constant functor Ψ. Therefore $|\mathcal{H}|^K = |\mathcal{H}^K|$ is contractible. $\blacksquare$

COROLLARY 9.6. *Let G be a compact Lie group which contains a non-trivial p-toral subgroup. Let $\mathcal{S}$ denote $\mathcal{S}_p^0(G)$ and let $\mathcal{B}$ denote $\mathcal{B}_p^0(G)$. Then* $\mathrm{Nr}\,\mathcal{B}$ *is a finite G-model and*

(a) $|\mathrm{Nr}\,\mathcal{B}|/G$ *and* $|\mathrm{Nr}\,\mathcal{S}|/G$ *are contractible.*
(b) $|\mathrm{Nr}\,\mathcal{B}|$ *and* $|\mathrm{Nr}\,\mathcal{S}|$ *are G-acyclic for any p-local coefficient functors* $\mathcal{M}_h$ (2.5).
(c) *After localisation at the Moore spectrum $S\mathbb{Z}_{(p)}$, there is an equivalence of spectra*

$$BG_+ \simeq \sum_{(P_0<\cdots<P_k)} (-1)^k \big(B(NP_0 \cap \cdots \cap NP_k)\big)_+$$

where the sum is over all the conjugacy classes of $(P_0 < \cdots < P_k)$ in $\mathcal{B}$.

Proof. Note that $\mathrm{Nr}\,\mathcal{B}$ is a finite G-model by 9.2 and 7.4. Proposition 9.4 shows that $|\mathrm{Nr}\,\mathcal{B}| \to |\mathrm{Nr}\,\mathcal{S}|$ is a G-homotopy equivalence. Proposition 9.5 shows that $|\mathrm{Nr}\,\mathcal{S}|^K \simeq *$ for all $K \le G$ such that $O_p(K) \ne 1$ and therefore also $|\mathrm{Nr}\,\mathcal{B}|^K \simeq *$. The result is immediate from Theorems 4.3 and 6.4 applied to $|\mathrm{Nr}\,\mathcal{B}|$. ∎

Part (a) of this theorem can be improved to generalise Symonds' theorem [36] to compact Lie groups.

Corollary 9.7. *Let $\mathcal{H}$ be a concave non-empty collection in $\mathcal{S}_p(G)$. Then $|\mathrm{Nr}\,\mathcal{H}|/G$ is contractible.*

Proof. Set $\mathcal{B} = \mathcal{B}_p(G)$. The inclusion $\mathcal{B} \cap \mathcal{H} \subseteq \mathcal{H}$ induces a G-homotopy equivalence on realisations by 9.4. Therefore, it suffices to show $|\mathrm{Nr}\,\mathcal{B}\cap\mathcal{H}|/G \simeq *$. Now, $\mathcal{B} \cap \mathcal{H}/G$ is finite by 9.2, so $|\mathcal{B} \cap \mathcal{H}|$ has finite orbit type by Proposition 7.4. Consider a maximal p-toral subgroup Q of $H \in \mathrm{Iso}_G(|\mathcal{B}\cap\mathcal{H}|\cup *)$. Then either $Q = P$ where P is a maximal p-toral subgroup of G or Q is a maximal p-toral subgroup of $NQ_0 \cap \cdots \cap NQ_k$ for some $Q_0 \le \cdots \le Q_k$ in $\mathcal{H}$ so $Q_0 \le Q$. In either case, since $\mathcal{H}$ is concave and non-empty we see that $Q \in \mathcal{H}$. For any subgroup $K/Q \le NQ/Q$ one clearly has $O_p(K) \in \mathcal{H}$ because $Q \le O_p(K)$ and $\mathcal{H}$ is concave. Proposition 9.5 shows that $(|\mathcal{H}|^Q)^{K/Q} = |\mathcal{H}|^K$ is contractible. It follows that $|\mathcal{H}|^Q$ is NQ-equivariantly contractible and therefore so is $|\mathcal{B}\cap\mathcal{H}|^Q$. The key Lemma 3.2 applies with $\mathcal{P} = \mathrm{Syl}_p(\mathrm{Iso}_G(|\mathcal{B}\cap \mathcal{H}|))$, hence $|\mathcal{B} \cap \mathcal{H}|/G \simeq *$. ∎

10. Proof of Theorem C

Lemma 10.1. *Let $\mathbf{C}$ be a topological category whose morphism spaces are all compact manifolds and let $\tilde{F} : \mathbf{C} \to \mathcal{T}$ be a functor whose values are compact manifolds. Then $\mathrm{Nr}\,\mathrm{Tr}\,\tilde{F}$ is Reedy cofibrant.*

Proof. Set $\mathbf{X} = \mathrm{Nr}\,\mathrm{Tr}\,\tilde{F}$ and recall that

$$\mathbf{X}(k) = \coprod_{C_0,\dots,C_k\in\mathbf{C}} \tilde{F}(C_0) \times \prod_{i=1}^{k} \mathbf{C}(C_{i-1}, C_i).$$

The space $D\mathbf{X}(k)$ of degenerate k-simplices of $\mathbf{X}$ is easily seen to be a CW-subcomplex of $\mathbf{X}(k)$ because for every $C \in \mathbf{C}$ the one-point space $\{\mathrm{id}_C\}$ is a subcomplex of $\mathbf{C}(C, C)$. Therefore $D\mathbf{X}(k) \subseteq \mathbf{X}(k)$ is a cofibration. Moreover, $D\mathbf{X}(k)$ is topologically a coproduct of compact spaces indexed by $k + 1$-element subsets of $\mathrm{Obj}(\mathbf{C})$.

Consider the nth latching object $L\mathbf{X}(n)$ which is the direct limit over Γ_n of the spaces $\mathbf{X}(k)$ for $k < n$ (see A.2.1). Since Γ_n is a finite category, one sees by inspection that $L\mathbf{X}(n)$ is topologically a coproduct of compact spaces indexed by the $n+1$-element subsets of $\mathrm{Obj}(\mathbf{C})$. Moreover, the natural maps $\lambda_n : L\mathbf{X}(n) \to D\mathbf{X}(n)$ respect the coproduct decomposition above.

It follows that λ_n is a closed map and since it is bijective by A.2.2, they are homeomorphisms. This shows that $L\mathbf{X}(n) \to \mathbf{X}(n)$ are cofibrations and therefore $\mathbf{X}$ is Reedy cofibrant. ■

We are now ready to prove Theorem C. Recall that a G-map $f : X \to Y$ induces a homotopy equivalence $X_{hG} \to Y_{hG}$ if f is a homotopy equivalence. Observe that for any functor $\tilde{F} : \mathbf{C} \to G\mathcal{T}$ where $\mathbf{C}$ is a topological category and $\tilde{F}(C)$ are G-orbits for all $C \in \mathbf{C}$, we have, by 5.1 and 5.3,

$$\operatorname*{hocolim}_{\mathbf{C}} \tilde{F}_{hG} = |\coprod_*(\tilde{F}_{hG})| = |(\coprod_* \tilde{F})_{hG}| = |\coprod_* \tilde{F}|_{hG}.$$

Our theorem will follow by the definitions of $\beta_{\mathcal{H}}, \alpha_{\mathcal{H}}, \delta_{\mathcal{H}}$ in 1.2–1.4 if we construct natural G-equivariant maps which are (non-equivariant) homotopy equivalences

$$\text{(i) } |\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}}| \to |\mathcal{H}|, \qquad \text{(ii) } |\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\alpha}_{\mathcal{H}}| \to |\mathcal{H}|, \qquad \text{(iii) } |\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\delta}_{\mathcal{H}}| \to |\mathcal{H}|.$$

The map (iii) is obtained from Propositions 7.9 and 7.10. It is a G-homotopy equivalence which is more than we actually require. For the other two maps recall that for every subgroup H and $g \in G$ we denote ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$.

Construction of the map (i). Recall that the object space of $\mathcal{O}_G(\mathcal{H})$ is discrete, hence

$$(10.2) \qquad \begin{aligned} \mathrm{Obj}(\mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}}) &= \coprod_{H \in \mathcal{H}} G/H, \\ \mathrm{Mor}(\mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}}) &= \coprod_{H_0, H_1 \in \mathcal{H}} G/H_0 \times (G/H_1)^{H_0}. \end{aligned}$$

Define a functor $j : \mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}} \to \mathcal{H}$ whose effect on object and morphism spaces is

$$\coprod_{H \in \mathcal{H}} G/H \xrightarrow{gH \mapsto {}^gH} \mathcal{H},$$

$$\coprod_{H_0, H_1 \in \mathcal{H}} G/H_0 \times (G/H_1)^{H_0} \xrightarrow{(gH_0, kH_1) \mapsto ({}^gH_0, {}^{gk}H_1)} \mathcal{H}_1 \subseteq \mathcal{H} \times \mathcal{H}.$$

This is clearly well defined because $kH_1 \in G/H_1^{H_0}$ implies $H_0^k \leq H_1$. The assignments are also G-equivariant. Lemma 7.8 shows that j is continuous on objects, and consequently on morphisms because $(gH_0, kH_1) \mapsto {}^{gk}H_1$ is the composition of j with the evaluation map $G/H_0 \times G/H_1^{H_0} = G/H_0 \times \mathrm{map}_G(G/H_0, G/H_1) \to G/H_1$ is continuous. The assignments also respect identities and must respect compositions because $\mathcal{H}$ is a poset. It follows that j is a functor of internal G-space categories and therefore $|j|$ defines a G-equivariant map $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}}| \to |\mathrm{Nr}\,\mathcal{H}|$. Now $\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\beta}_{\mathcal{H}}$ is Reedy cofibrant by Lemma 10.1 and $\mathrm{Nr}\,\mathcal{H}$ is Reedy cofibrant by Propositions 7.4 and A.2.2.

Corollary 5.12 also shows that j is absolutely tame. We now fix $K \in \mathcal{H}$ and examine $(K{\downarrow}j)$. By inspection

$$(K{\downarrow}j) = (\operatorname{Tr}\tilde{\beta}_{\mathcal{H}})^K = \operatorname{Tr}\tilde{\beta}_{\mathcal{H}}^K.$$

Lemma 10.1 applies to $\tilde{\beta}_{\mathcal{H}}^K$ and shows that $\operatorname{Nr}(K{\downarrow}j)$ is Reedy cofibrant. Let $\mathcal{O}$ denote the category $\mathcal{O}_G(\mathcal{H})$ (see 1.2) and observe that $\tilde{\beta}_{\mathcal{H}}^K$ is the representable functor $\mathcal{O}(G/K,-)$. It follows from 5.3 that

$$|\operatorname{Nr}(K{\downarrow}j)| = |\operatorname{Nr}\operatorname{Tr}\tilde{\beta}_{\mathcal{H}}^K| = \operatorname*{hocolim}_{\mathcal{O}} \mathcal{O}(G/K,-),$$

which is contractible (see e.g. the reduction theorem [19, 4.4]). We are now in a position to apply Theorem 5.8 and deduce that $|j|$ is a G-equivariant map which is a non-equivariant homotopy equivalence $|\operatorname{Nr}\operatorname{Tr}\tilde{\beta}_{\mathcal{H}}| \to |\mathcal{H}|$.

Construction of the map (ii). The object space of $\mathcal{A}_G(\mathcal{H})$ is discrete, therefore the object and morphism spaces of $\operatorname{Tr}(\tilde{\alpha}_{\mathcal{H}})$ are given by

$$(10.3) \qquad \coprod_{H\in\mathcal{H}} \operatorname{Hom}_G(H,G), \qquad \coprod_{H_0,H_1\in\mathcal{H}} \operatorname{Hom}_G(H_1,G)\times\operatorname{Hom}_G(H_0,H_1).$$

Define a functor $j:\operatorname{Tr}\tilde{\alpha}_{\mathcal{H}} \to \mathcal{H}$ whose effect on object and morphism spaces is

$$\coprod_{H\in\mathcal{H}} \operatorname{Hom}_G(H,G) \xrightarrow{c_x \mapsto {}^xH} \mathcal{H},$$

$$\coprod_{H_0,H_1\in\mathcal{H}} \operatorname{Hom}_G(H_1,G)\times\operatorname{Hom}_G(H_0,H_1) \xrightarrow{(c_x,c_g)\mapsto({}^{xg}H_0,{}^xH_1)} \mathcal{H}_1 \subset \mathcal{H}\times\mathcal{H}.$$

These assignments are G-equivariant so Lemma 7.8 shows that the map on object spaces is continuous. Consequently, the map on morphisms is continuous. Furthermore j clearly respects identities and it must respect compositions because $\mathcal{H}$ is a poset. It follows that j is a functor of internal G-space categories and therefore $|j|$ is a G-map.

Note that for any subgroups $J,L\le G$ the space $\operatorname{Hom}_G(J,L)$ is homeomorphic to $N_G(J,L)/C_GJ$, hence it is a compact manifold. Lemma 10.1 implies that $\operatorname{Nr}\operatorname{Tr}\tilde{\alpha}_{\mathcal{H}}$ is Reedy cofibrant. Propositions 7.4 and A.2.2 show that $\operatorname{Nr}\mathcal{H}$ is Reedy cofibrant. Corollary 5.12 shows that j is absolutely tame. We now fix $K\in\mathcal{H}$ and consider $(j{\downarrow}K)$. By inspection

$$\operatorname{Obj}(j{\downarrow}K) = \coprod_{H\in\mathcal{H}} \operatorname{Hom}_G(H,K),$$

$$\operatorname{Mor}(j{\downarrow}K) = \coprod_{H_0,H_1\in\mathcal{H}} \operatorname{Hom}_G(H_1,K)\times\operatorname{Hom}_G(H_0,H_1).$$

Denote $\mathcal{A} := \mathcal{A}_G(\mathcal{H})$ and observe that $(j{\downarrow}K)$ is equal to $\operatorname{Tr}\mathcal{A}(-,K)$ where $\mathcal{A}(-,K)$ is the representable functor $\mathcal{A}^{\mathrm{op}}\to\mathcal{T}$. It follows from Lemma 10.1 that $\operatorname{Nr}(j{\downarrow}K)$ is Reedy cofibrant. Furthermore, from 5.3 and the reduction theorem [19, 4.4] it follows that

$$|\mathrm{Nr}(j{\downarrow}K)| = |\mathrm{Nr}\,\mathrm{Tr}\,\mathcal{A}(-,K)| = \operatorname*{hocolim}_{\mathcal{A}^{\mathrm{op}}} \mathcal{A}(-,K) \simeq *.$$

We can now apply Theorem 5.8 and Remark 5.9 to deduce that $|j|$ induces a G-equivariant map which is a homotopy equivalence $|\mathrm{Nr}\,\mathrm{Tr}\,\tilde{\alpha}_{\mathcal{H}}| \to |\mathcal{H}|$.

A.1. The space of subgroups and p-toral groups. A compact Lie group G is called *p-toral* if the connected component of the identity, denoted G_0, is a torus and $\bar{G} := G/G_0$ is a finite p-group. The following is contained in [24, Lemmas A.1, A.2, A.3]

LEMMA A.1.1. *Every p-toral subgroup of a compact Lie group G can be extended to a maximal p-toral subgroup. All the maximal p-toral subgroups of G are conjugate. Furthermore, if $Q \le P$ is an inclusion of p-toral groups then $N_P(Q)$ is p-toral and it properly contains Q if and only if Q is properly contained in P.*

Note that a quotient of a p-toral group P is p-toral because P_0 is a torus of finite index in P. Furthermore, p-toral groups are closed under extensions, that is, if $1 \to P' \to G \xrightarrow{\pi} P \to 1$ is a short exact sequence where P and P' are p-toral, then G is p-toral. Note that $P'_0 \lhd G_0$ because it is characteristic in P' and that G_0/P'_0 is a subgroup of P_0, hence a torus. It follows from [16, Proposition 2.6] that G_0 is a torus. Finally G/G_0 is an extension of the finite group $P'/G_0 \cap P'$ by the p-group $\overline{P}$, hence G is p-toral.

One can easily make sense of the following definition (see e.g. [3, §3]).

DEFINITION A.1.2. Every compact Lie group G contains a unique maximal normal p-toral subgroup $O_p(G)$.

PROPOSITION A.1.3. *Let H be a subgroup of a p-toral group G. Then H_0 is a torus and H contains a characteristic* abelian *subgroup $O_{p'}(H)$ which contains H_0 and such that $O_{p'}(H)/H_0$ is an abelian group of order prime to p and $H/O_{p'}(H)$ is a finite group of order p-power. If $H \le K \le G$ then $O_{p'}(H) \le O_{p'}(K)$.*

Proof. Clearly H_0 is a subgroup of G_0, whence a torus. Since G_0 is abelian, $G_0 \cap H = H_0 \times \Gamma$ where Γ is a finite abelian group. Clearly $\Gamma = \Gamma_p \oplus \Gamma_{p'}$ where the latter is the subgroup of the elements of order prime to p. Define $O_{p'}(H) = H_0 \times \Gamma_{p'}$. Clearly $O_{p'}(H)/H_0$ is the subgroup of $\bar{H}$ generated by all the elements of order prime to p. Therefore $O_{p'}(H)$ is characteristic in H and $|H/O_{p'}(H)|$ is a p-power. If $H \le K$ then clearly $O_{p'}(H) \le G_0 \cap K$. It follows that $O_{p'}(H)K_0/K_0$ is a quotient of $O_{p'}(H)/H_0$, hence $O_{p'}(H)K_0$ is a subgroup of $O_{p'}(K)$. ■

Let G be a compact Lie group. There is a bi-invariant metric d on G which can be used to equip the set $F(G)$ of all closed non-empty subsets of G with the Hausdorff metric ρ; see e.g. [39, p. 108]. In this way $F(G)$ becomes a compact metric space and moreover ρ is invariant under the left and right action of G by translation. Note that $\rho(U, gU) \leq d(g,1)$ for any $U \in F(G)$.

The subspace $S(G)$ of all the closed subgroups of G is a closed subspace of $F(G)$, hence it is compact. It has an invariant action of G via conjugation. The quotient space $C(G) = S(G)/G$ is a countable compact metric space whose elements are the conjugacy classes of subgroups (H). We recall from [38, Proposition IV.3.3]

Proposition A.1.4. *If $(H_n) \to (K)$ in $C(G)$ then there exists n_0 such that for all $n \geq n_0$ there are $K_n \leq K$ such that $(H_n) = (K_n)$ and furthermore $K_n \to K$ in $S(G)$.*

We also observe

Proposition A.1.5.

(a) *If $g_n \to g$ in G then $H^{g_n} \to H^g$ in $S(G)$.*
(b) *If $H_n \leq K_n$ in $S(G)$ and $H_n \to H$ and $K_n \to K$ then $H \leq K$.*

Proof. Note that $\rho(H, H^x) \leq \rho(H, Hx) + \rho(Hx, x^{-1}Hx) \leq 2d(x,1)$ so if $g_n \to 1$ then $H^{g_n} \to H$. Point (b) follows from [39, p. 108]:

$$H = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j=n}^{\infty} H_j} \leq \bigcap_{n=1}^{\infty} \overline{\bigcup_{j=n}^{\infty} K_j} = K. \blacksquare$$

Let $S_p(G)$ denote the subspace of $S(G)$ of p-toral subgroups.

Proposition A.1.6. *$S_p(G)$ is a compact subspace of $S(G)$.*

Proof. It suffices to show that $S_p(G)$ is closed in $S(G)$. Let P_n be a sequence in $S_p(G)$ which converges to some H in $S(G)$. Fix a maximal p-toral subgroup P of H and note that $(P_n) \to (H)$ in $C(G)$ so A.1.1 and A.1.4 imply that for all large n there exist $g_n \in G$ such that $P_n^{g_n} \leq P$. We may assume that $g_n \to g$ because G is compact. Proposition A.1.5 implies

$$H = \lim_{n\to\infty} P_n \leq \lim_{n\to\infty} P^{g_n^{-1}} = P^{g^{-1}} \leq H^{g^{-1}}.$$

That is, $g \in NH$, hence $H = P$ is p-toral. $\blacksquare$

A.2. Homotopy colimits and simplicial spaces. Let $\mathcal{T}$ denote the category of compactly generated spaces. There is a standard model category structure on $\mathcal{T}$ where fibrations are Serre fibrations and weak equivalences are weak homotopy equivalences [20]. Cofibrant spaces are homotopy equivalent to retracts of CW-complexes.

For a small category $\mathbf{C}$ there is a functor $E\mathbf{C} : \mathbf{C}^{\mathrm{op}} \to \mathcal{T}$ defined by $C \mapsto |(C\downarrow\mathbf{C})|$. Taking coends [26] with $E\mathbf{C}$ gives rise to a functor hocolim :

$\mathcal{T}^{\mathbf{C}} \to \mathcal{T}$ (see e.g. [9, p. 327]),

$$\operatorname*{hocolim}_{\mathbf{C}} F = E\mathbf{C} \otimes_{\mathbf{C}} F \equiv \int^{C \in \mathbf{C}} E\mathbf{C}(C) \times F(C).$$

When the values of $F : \mathbf{C} \to \mathcal{T}$ are cofibrant, $\operatorname{hocolim} F$ agrees up to homotopy with the total derived functor $L\operatorname{colim} F$ (see [20, §1.3.2]). More details can be found in [15] or [18].

A.2.1. The Reedy structure. Let Δ denote the category whose objects are the sets $[n] = \{0, 1, \dots, n\}$ and its morphisms are order preserving functions (cf. [20, Ch. 3]). We denote by $\overleftarrow{\Delta}$ the subcategory consisting of all the surjective functions only. Let Γ_n denote the full subcategory of the comma category $(\overleftarrow{\Delta}^{\mathrm{op}}\downarrow[n])$ of all the objects except the identity $[n] \to [n]$.

A *simplicial space* is a functor $\mathbf{X} : \Delta^{\mathrm{op}} \to \mathcal{T}$. The category of simplicial spaces admits a useful simplicial model category structure known as the "Reedy model structure". The cofibrant objects are described as follows. There is an obvious composition $\Gamma_n \to \Delta^{\mathrm{op}} \xrightarrow{\mathbf{X}} \mathcal{T}$ whose colimit is denoted $L\mathbf{X}(n)$ and is called the nth *latching object* of $\mathbf{X}$. The inclusion $\Gamma_n \subset (\overleftarrow{\Delta}^{\mathrm{op}}\downarrow[n])$ induces a map $L\mathbf{X}(n) \to \mathbf{X}(n)$. A simplicial space $\mathbf{X}$ is Reedy cofibrant if $\mathbf{X}(n)$ are cofibrant and the maps $L\mathbf{X}(n) \to \mathbf{X}(n)$ are cofibrations for all n. See [20, Ch. 5] and [15] for more details.

Here is an important example. It shows that every simplicial set is a Reedy cofibrant simplicial space.

Proposition A.2.2. *Fix a compact Lie group G and a simplicial G-space $\mathbf{X}$ such that $\mathbf{X}(n)/G$ is a discrete set for every n. Then $\mathbf{X}$ is a Reedy cofibrant simplicial space. In fact the natural maps $L\mathbf{X}(n) \to \mathbf{X}(n)$ are homeomorphisms onto the subspaces $D\mathbf{X}(n)$ of degenerate n-simplices of $\mathbf{X}$. Moreover $\mathbf{X}_{hG}$ is a Reedy cofibrant simplicial space.*

Proof. According to [9, Ch. VIII.2.3] for every $x \in \mathbf{X}(n)$ there exists a unique $k \le n$ and a unique non-degenerate simplex $y \in \mathbf{X}(k)$ and a unique surjection $\varphi : [n] \twoheadrightarrow [k]$ such that $x = \mathbf{X}(\varphi)(y)$. It follows that there are natural bijections, where $N\mathbf{X}(n)$ is the subspace of non-degenerate n-simplices of $\mathbf{X}$,

$$D\mathbf{X}(n) \approx \coprod_{[n]\twoheadrightarrow[k],\, k<n} N\mathbf{X}(k) \approx L\mathbf{X}(n).$$

Since G-equivariant bijections $G/H \to G/H'$ are homeomorphisms (see e.g. Lemma 7.8) and since $\mathbf{X}(n)$ is a coproduct of its orbits, we see that $L\mathbf{X}(n) \to \mathbf{X}(n)$ is a homeomorphism onto $D\mathbf{X}(n)$. It follows that it is also a cofibration because $\mathbf{X}(n) = D\mathbf{X}(n) \sqcup N\mathbf{X}(n)$ and $N\mathbf{X}(n)$ is a coproduct of its orbits which are compact manifolds and in particular finite CW-complexes.

The second assertion follows from the fact that $EG \times_G \mathbf{X}(n)$ are clearly CW-complexes and from the fact that $EG\times_G-$ commutes with colimits. This shows that $L\mathbf{X}_{hG}(n) = L\mathbf{X}(n)_{hG} = D\mathbf{X}(n)_{hG}$ and therefore $L\mathbf{X}_{hG}(n) \to \mathbf{X}_{hG}(n)$ is a cofibration. ∎

A.2.3. The geometric realisation $|\mathbf{X}|$ is the coend $\int^{n\in\Delta} \Delta^n \times \mathbf{X}(n)$ where Δ^n is the standard simplex [20, p. 135]. There is a natural map [9, Ch. XI.2.6]

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathbf{X} \to |\mathbf{X}|.$$

It is not in general a homotopy equivalence, unless $\mathbf{X}$ is Reedy cofibrant. The reason is that the standard cosimplicial space $\Delta^\bullet$ is Reedy cofibrant so for every space T the simplicial space $\mathrm{map}(\Delta^\bullet, T)$ is a Reedy fibrant. For more details see [15] and [18].

A.2.4. An *augmentation* of a simplicial space $\mathbf{X}$ is a map $\mathbf{X}(0) \xrightarrow{d_0} Y$ such that the maps $d_0d_0, d_0d_1 : X(1) \to Y$ are equal. We denote Y by $\mathbf{X}(-1)$. A *left contraction* for an augmented simplicial space $\mathbf{X}$ is a collection of maps $s_{-1} : \mathbf{X}(n) \to \mathbf{X}(n+1)$ for all $n \geq -1$ such that the simplicial identities hold, that is,

$$\begin{aligned} &d_0 s_{-1} = \mathrm{id}, \\ &d_i s_{-1} = s_{-1} d_{i-1} \qquad \text{for all } i > 0, \\ &s_i s_{-1} = s_{-1} s_{i-1} \qquad \text{for all } i \geq 0. \end{aligned}$$

When this is the case, and all the spaces $\mathbf{X}(n)$ are cofibrant, the natural map

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \mathbf{X} \to \mathbf{X}(-1)$$

is a homotopy equivalence. See for example [7, §6]. To see this, let Δ^c be the subcategory of Δ whose morphisms $\varphi : [n] \to [k]$ are functions such that $\varphi(0) = 0$. Let Δ^a denote the subcategory of Δ^c where the maps φ are also required to have the property that $\varphi(i) > 0$ if $i > 0$. Note that $[0]$ is an initial object in Δ^a. It is easy to see that an augmented simplicial object is a contravariant functor from Δ^a. An augmented simplicial object with a left contraction is the same as a contravariant functor from Δ^c. It is easy to see that the inclusion of these categories is left cofinal [9, p. 316].

A.3. Bousfield localisation of spectra. Throughout $h\mathcal{S}$ denotes the homotopy category of spectra as defined, e.g. by Boardman. A very readable source is Adams [1]. This is a triangulated category with smash products $A \wedge B$ and function complexes $F(A, B)$ which satisfy the usual adjunctions. A spectrum E gives rise to a homology theory $E_*(-) = [S^0, -\wedge E]_*$ where S^0 denotes the sphere spectrum. It also defines a cohomology theory $E^*(-) = [-, E]_{-*}$.

Recall from [8] that a spectrum A is called *E-acyclic* if $E_*(A) = 0$. A morphism $f : A \to B$ of spectra is an *E-equivalence* if $E_*(f)$ is an isomorphism. A spectrum T is *E-local* if $[f, T]_*$ is an isomorphism for every E-equivalence $f : A \to B$. It easily follows that an E-equivalence of two E-local spectra is an equivalence. Bousfield proves in [8] that every spectrum X is associated with a canonical triangle

$$_EX \to X \xrightarrow{\eta} X_E \xrightarrow{\Sigma} {_EX}$$

where $_EX$ is E-acyclic and X_E is E-local. It is clear that η is an E-equivalence and X_E is called the *E-localization* of X. Bousfield shows that

(i) There is a natural equivalence $(X \vee Y)_E \simeq X_E \vee Y_E$.
(ii) There is a natural equivalence $(\Sigma X)_E \simeq \Sigma(X_E)$.
(iii) $(-)_E$ preserves triangles.

Let G be an abelian group. Let SG denote the Moore spectrum of type $(G, 0)$, that is: (i) $\pi_i SG = 0$ if $i < 0$, (ii) $H_0 SG \approx G$ and (iii) $H_i SG = 0$ if $i > 0$. Let J be a set of primes and $\mathbb{Z}_{(J)}$ be the integers localized at J.

Proposition A.3.1 ([8, Proposition 2.4]). *For every spectrum X there is a natural equivalence $X_{S\mathbb{Z}_{(J)}} \simeq S\mathbb{Z}_{(J)} \wedge X$. Moreover $\pi_* X_{S\mathbb{Z}_{(J)}} = \mathbb{Z}_{(J)} \otimes \pi_* X$ and X is $S\mathbb{Z}_{(J)}$-local if and only if the groups $\pi_* X$ are J-local, namely uniquely $p \notin J$ divisible.*

Theorem A.3.2 ([8, Theorem 3.1]). *If X is a connective spectrum then*

$$X_{H\mathbb{Z}_{(p)}} \simeq X_{S\mathbb{Z}_{(p)}} \simeq X \wedge S\mathbb{Z}_{(p)}.$$

We deduce

Proposition A.3.3. *Consider an $S\mathbb{Z}_{(p)}$-local spectrum E. A map $f : X \to Y$ of connective spectra induces an isomorphism $E^*(Y) \approx E^*(X)$ if f is an $H_*(-; \mathbb{Z}_{(p)})$-isomorphism.*

Proof. Note that $\eta : X \to X_{S\mathbb{Z}_{(p)}}$ and $\eta : Y \to Y_{S\mathbb{Z}_{(p)}}$ are $S\mathbb{Z}_{(p)}$-equivalences. Since f is an $H\mathbb{Z}_{(p)}$-equivalence, it is an $S\mathbb{Z}_{(p)}$-equivalence by A.3.2. Now use the fact that E is $S\mathbb{Z}_{(p)}$-local by A.3.1 and the definition $E^*(-) = [-, E]_*$ to obtain the result. ∎

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References

[1] J. F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, IL, 1974.

[2] J. C. Becker and D. H. Gottlieb, *The transfer map and fiber bundles*, Topology 14 (1975), 1–12.

[3] D. J. Benson, *Fusion in compact Lie groups*, Quart. J. Math. Oxford Ser. (2) 47 (1996), 261–268.

[4] —, *Representations and Cohomology. II. Cohomology of Groups and Modules*, Cambridge Stud. Adv. Math. 31, Cambridge Univ. Press, Cambridge, 1991.

[5] F. Borceux, *Handbook of Categorical Algebra. 1. Basic Category Theory*, Encyclopedia Math. Appl. 50, Cambridge Univ. Press, Cambridge, 1994.

[6] —, *Handbook of Categorical Algebra. 2. Categories and Structures*, Encyclopedia Math. Appl. 51, Cambridge Univ. Press, Cambridge, 1994.

[7] A. K. Bousfield, *Cosimplicial resolutions and homotopy spectral sequences in model categories*, Geom. Topol. 7 (2003), 1001–1053.

[8] —, *The localization of spectra with respect to homology*, Topology 18 (1979), 257–281.

[9] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, Berlin, 1972.

[10] G. E. Bredon, *Introduction to Compact Transformation Groups*, Pure Appl. Math. 46, Academic Press, New York, 1972.

[11] K. U. Bux, *Orbit spaces of subgroup complexes, Morse theory, and a new proof of a conjecture of Webb*, in: Proc. 1999 Topology and Dynamics Conference (Salt Lake City, UT), Topology Proc. 24 (1999), 39–51.

[12] E. Dror Farjoun, *Cellular Spaces, Null Spaces and Homotopy Localization*, Lecture Notes in Math. 1622, Springer, Berlin, 1996.

[13] W. G. Dwyer, *Homology decompositions for classifying spaces of finite groups*, Topology 36 (1997), 783–804.

[14] —, *Sharp homology decompositions for classifying spaces of finite groups*, in: Group Representations: Cohomology, Group Actions and Topology (Seattle, WA, 1996), Proc. Sympos. Pure Math. 63, Amer. Math. Soc., Providence, RI, 1998, 197–220.

[15] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan and J. H. Smith, *Homotopy Limit Functors on Model Categories and Homotopical Categories*, Math. Surveys Monogr. 113, Amer. Math. Soc., 2004.

[16] W. G. Dwyer and C. W. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. London Math. Soc. 30 (1998), 337–364.

[17] J. P. C. Greenlees and J. P. May, *Equivariant stable homotopy theory*, in: Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, 277–323.

[18] P. S. Hirschhorn, *Model Categories and Their Localizations*, Math. Surveys Monogr. 99, Amer. Math. Soc., 2003.

[19] J. Hollender and R. M. Vogt, *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Arch. Math. (Basel) 59 (1992), 115–129.

[20] M. Hovey, *Model Categories*, Math. Surveys Monogr. 63, Amer. Math. Soc., 1999.

[21] S. Illman, *The equivariant triangulation theorem for actions of compact Lie groups*, Math. Ann. 262 (1983), 487–501.

[22] S. Jackowski and J. McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology 31 (1992), 113–132.

[23] S. Jackowski, J. McClure and B. Oliver, *Homotopy classification of self-maps of BG via G-actions. I*, Ann. of Math. 135 (1992), 183–226.

[24] S. Jackowski, J. McClure and B. Oliver, *Homotopy classification of self-maps of BG via G-actions. II*, ibid. (2) 135 (1992), 227–270.

[25] L. G. Lewis Jr., J. P. May, M. Steinberger and J. E. McClure, *Equivariant Stable Homotopy Theory*, Lecture Notes in Math. 1213, Springer, Berlin, 1986.

[26] S. MacLane, *Categories for the Working Mathematician*, 2nd ed., Grad. Texts in Math. 5, Springer, 1998.

[27] J. Martino and S. Priddy, *Minami–Webb type decompositions for compact Lie groups*, in: Recent Progress in Homotopy Theory (Baltimore, MD, 2000), Contemp. Math. 293, Amer. Math. Soc., 233–249.

[28] J. P. May, *Equivariant Homotopy and Cohomology Theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., 1996.

[29] N. Minami, *Hecke algebras and cohomotopical Mackey functors*, Trans. Amer. Math. Soc. 351 (1999), 4481–4513.

[30] R. Oliver, *A transfer for compact Lie group actions*, Proc. Amer. Math. Soc. 97 (1986), 546–548.

[31] R. S. Palais, *The classification of G-spaces*, Mem. Amer. Math. Soc. 36 (1960), 72 pp.

[32] D. Quillen, *Higher algebraic K-theory. I*, in: Algebraic K-theory, I: Higher K-theories (Seattle, WA, 1972), Lecture Notes in Math. 341, Springer, Berlin, 1973, 85–147.

[33] J. Słomińska, *Homotopy colimits on E-I-categories*, in: Algebraic Topology, Poznań, 1989, Lecture Notes in Math. 1474, Springer, 1991, 273–294.

[34] —, *Homotopy decompositions of orbit spaces and the Webb conjecture*, Fund. Math. 169 (2001), 105–137.

[35] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967), 133–152.

[36] P. Symonds, *The orbit space of the p-subgroup complex is contractible*, Comment. Math. Helv. 73 (1998), 400–405.

[37] R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 91–109.

[38] T. tom Dieck, *Transformation Groups*, de Gruyter Stud. Math. 8, de Gruyter, Berlin, 1987.

[39] —, *Transformation Groups and Representation Theory*, Lecture Notes in Math. 766, Springer, Berlin, 1979.

[40] P. J. Webb, *A split exact sequence of Mackey functors*, Comment. Math. Helv. 66 (1991), 34–69.

Assaf Libman
Department of Mathematical Sciences
King's College
University of Aberdeen
Aberdeen AB24 3UE, Scotland, U.K.
E-mail: assaf@maths.abdn.ac.uk

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