C\(^1\)-maps having hyperbolic periodic points

by

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Abstract. We show that the C\(^1\)-interior of the set of maps satisfying the following conditions:

(i) periodic points are hyperbolic,
(ii) singular points belonging to the nonwandering set are sinks,

coincides with the set of Axiom A maps having the no cycle property.

1. Introduction. Let \( M \) be a closed C\(^\infty\)-manifold, \( \| \cdot \| \) be a Riemannian metric on \( M \) and \( \pi : TM \to M \) be the tangent bundle. Let \( C^1(M) \) be the space of \( C^1 \)-differentiable maps from \( M \) into itself endowed with the \( C^1 \)-topology. Then \( C^1(M) \) contains the set \( \text{Diff}^1(M) \) of \( C^1 \)-diffeomorphisms and this subset is open in \( C^1(M) \).

The \( C^1 \)-stability conjecture on \( \text{Diff}^1(M) \) of Palis and Smale was solved by Mañé [12] as follows: if a \( C^1 \)-diffeomorphism \( f \) is structurally stable, then \( f \) satisfies Axiom A and the strong transversality. By using the techniques obtained in proving the conjecture, Palis [18] showed that if there exists a nonempty open subset \( U \) of \( \text{Diff}^1(M) \) such that all periodic points of each \( g \in U \) are hyperbolic, then every diffeomorphism belonging to \( U \) can be approximated by Axiom A diffeomorphisms with no cycles. Next it was checked in [1] that \( U \) consists of Axiom A diffeomorphisms with no cycles. We remark here that the methods of Liao [7] which proved the \( C^1 \)-stability in the 2-dimensional case were also useful in the higher dimensional case (the 2-dimensional case was also proved in Sannami [22]).

In this paper we shall discuss the problem of whether stability of \( C^1 \)-differentiable maps implies Axiom A and no cycles.

Concerning this problem Przytycki proved the following remarkable results: Anosov differentiable maps which are not diffeomorphisms or expand-
ings do not satisfy $C^1$-structural stability [20], and if a differentiable map $f$ satisfies Axiom A and has no singular points in the nonwandering set, then $f$ is $C^1$ $\Omega$-stable if and only if $f$ satisfies strong Axiom A and has no cycles [21]. On the other hand, we know [23] that expanding maps are structurally stable.

In view of these developments, we shall discuss in detail how stability of diffeomorphisms can be adapted to the more complicated situation of $C^1$-maps, and so we shall focus on the noninvertible case (that is, the case of differentiable maps which are not diffeomorphisms).

In order to state our result let us recall a few notations and basic results about $C^1$-maps.

Let $f \in C^1(M)$. For a periodic point $p$ of $f$, denote by $\varrho(f,p)$ the minimal integer $n > 0$ satisfying $f^n(p) = p$. We say that $\varrho(f,p)$ is the period of $p$ for $f$. A periodic point $p$ is called hyperbolic if $D_p f^{\varrho(f,p)} : T_p M \to T_p M$ has no eigenvalues of absolute value one; then $T_p M$ splits into the direct sum $T_p M = E^s(p) \oplus E^u(p)$ of subspaces $E^s(p)$ and $E^u(p)$ such that

$$(1.1) \quad (a) \ D_p f^{\varrho(f,p)}(E^s(p)) \subset E^s(p), \quad D_p f^{\varrho(f,p)}(E^u(p)) = E^u(p),$$

(b) there are $c > 0$ and $0 < \lambda < 1$ such that for $n > 0$,

$$(i) \quad \|D f^n(v)\| \leq c \lambda^n \|v\| \quad (v \in E^s(p)),$$

$$(ii) \quad \|D f^n(v)\| \geq c^{-1} \lambda^{-n} \|v\| \quad (v \in E^u(p)).$$

A hyperbolic periodic point $p$ is said to be a sink (resp. source) if $T_p M = E^s(p)$ (resp. $T_p M = E^u(p)$).

We denote by $\mathbb{M} = \prod_{-\infty}^{\infty} M$ the topological product of $M$’s, and define an injective continuous map $\tilde{f} : \mathbb{M} \to \mathbb{M}$ by

$$\tilde{f}((x_n)) = (f(x_n))$$

for $(x_n) \in \mathbb{M}$. Then $P^0 \circ \tilde{f} = f \circ P^0$ where

$$(1.2) \quad P^0 : \mathbb{M} \to M$$

is the natural projection defined by $P^0((x_n)) = x_0$. For $\Lambda \subset M$ put

$$(1.3) \quad \Lambda_f = \{(x_n) \in \mathbb{M} : x_n \in \Lambda, \ f(x_n) = x_{n+1}, \ n \in \mathbb{Z}\}.$$

Then $\Lambda_f$ is $\tilde{f}$-invariant ($\tilde{f}(\Lambda_f) = \Lambda_f$) and $\tilde{f}|_{\Lambda_f} : \Lambda_f \to \Lambda_f$ is a homeomorphism when $\Lambda_f \neq \emptyset$. Notice that $\Lambda$ is not necessarily $f$-invariant.

We say that $(M_f, \tilde{f})$ is the inverse limit system of $(M, f)$. Notice that if $f : M \to M$ is a diffeomorphism, then the inverse limit system of $(M, f)$ is equal to the original system $(M, f)$.

Let $TM$ be the subspace of $\mathbb{M} \times TM$ defined by

$$TM = \{((\bar{x}, v)) \in \mathbb{M} \times TM : P^0(\bar{x}) = \pi(v)\}.$$
and define a Finsler metric $\| \cdot \|$ on $TM$ by
\[
\| (\tilde{x}, v) \| = \| v \| \quad ( (\tilde{x}, v) \in TM).
\]
Define the projection $\tilde{P}^0 : TM \rightarrow TM$ by
\[
\tilde{P}^0(\tilde{x}, v) = v
\]
for $(\tilde{x}, v) \in TM$. Then $\tilde{P}^0(T_{\tilde{x}}M) = T_{x_0}M$ and the restriction $\tilde{P}^0|T_{\tilde{x}}M : T_{\tilde{x}}M \rightarrow T_{x_0}M$ is a linear isomorphism.

We define a $C^0$-vector bundle
\[
\tilde{\pi} : TM \rightarrow \mathbb{M}
\]
by $\tilde{\pi}(\tilde{x}, v) = \tilde{x}$ for $(\tilde{x}, v) \in TM$, and write $T_{\tilde{x}}M = \tilde{\pi}^{-1}(\tilde{x})$ for $\tilde{x} \in \mathbb{M}$. Let $D\tilde{f} : TM \rightarrow TM$ be defined by
\[
D\tilde{f}(\tilde{x}, v) = (\tilde{f}(\tilde{x}), D_{x_0}f(v)) \quad ((\tilde{x}, v) = ((x_n), v) \in TM),
\]
where $x_0$ is a point in $(x_n)$ and $D_{x_0}f$ is the derivative of $f$ at $x_0$.

We say that a closed $f$-invariant subset $\Lambda$ is hyperbolic if the vector bundle $TM|\Lambda_f = \bigcup_{x \in \Lambda_f} T_{\tilde{x}}M$ splits into the Whitney sum $TM|\Lambda_f = E^s \oplus E^u$ of subbundles $E^s$ and $E^u$ satisfying the following conditions:

(1) $D\tilde{f}(E^s) \subset E^s$, $D\tilde{f}(E^u) = E^u$,
(2) $D\tilde{f}|E^u : E^u \rightarrow E^u$ is injective,
(3) there exist $c > 0$ and $0 < \lambda < 1$ such that for $n \geq 0$,
\[
\|D\tilde{f}^n|E^s\| \leq c\lambda^n, \quad \| (D\tilde{f}|E^u)^{-n} \| \leq c\lambda^n,
\]
where $\|T\|$ denotes the supremum norm of a linear bundle map $T$. It is checked from the techniques in [20, §0 and §1] that

(1) $E^s$ and $E^u$ are $C^0$-vector bundles over $\Lambda_f$,
(2) there exist $0 < \lambda < 1$ and a new norm $\| \cdot \|$ such that
\[
\|D\tilde{f}|E^s\| \leq \lambda, \quad \| (D\tilde{f}|E^u)^{-1} \| \leq \lambda,
\]
(3) if $P^0(\tilde{x}) = P^0(\tilde{y})$ for $\tilde{x}, \tilde{y} \in \Lambda_f$, then $E^s(\tilde{x}) = E^s(\tilde{y})$, but in general $E^u(\tilde{x}) \neq E^u(\tilde{y})$.

Let $f$, in particular, be a $C^1$-map from $M$ onto itself. Then $f$ is called Anosov if $M$ is hyperbolic. An Anosov map $f$ is said to be expanding if $E^u(\tilde{x}) = T_{\tilde{x}}M$ for $\tilde{x} \in M_f$.

For $\tilde{x} = (x_n) \in M_f$ and $\varepsilon > 0$ put
\[
W^s_\varepsilon(\tilde{x}, f) = \{ y \in M : d(x_n, f^n(y)) \leq \varepsilon \text{ for } n \geq 0 \},
\]
\[
W^u_\varepsilon(\tilde{x}, f) = \{ y \in M : \text{there exists } \tilde{y} = (y_n) \in M_f \text{ with } y_0 = y \text{ such that } d(x_n, y_n) \leq \varepsilon \text{ for } n \geq 0 \}.
\]
Then \( W^s_\varepsilon(\bar{x}, f) = W^s_\varepsilon(\bar{y}, f) \) if \( P^0(\bar{x}) = P^0(\bar{y}) \) for \( \bar{x}, \bar{y} \in M_f \), and
\[
W^s_\varepsilon(\bar{x}, f) \subset f^{-1}(W^s_\varepsilon(f(\bar{x}), f)), \quad W^u_\varepsilon(\bar{x}, f) \subset f(W^u_\varepsilon(f^{-1}(\bar{x}), f)).
\]

If \( A \) is hyperbolic, then it follows from [5, Theorem 5.1] that \( \{W^s_\varepsilon(\bar{x}, f)\}_{\bar{x} \in A_f} \) and \( \{W^u_\varepsilon(\bar{x}, f)\}_{\bar{x} \in A_f} \) are continuous families of \( C^1 \)-disks in \( M \) such that
\[
T_{x_0}W^\sigma_\varepsilon(\bar{x}, f) = \overline{F}^0(E^\sigma(\bar{x}))
\]
for \( \bar{x} = (x_n) \in A_f \) and \( \sigma = s, u \). It is easily checked that for \( \bar{x} \in A_f \),
\[
W^s(\bar{x}, f) = \bigcup_{n \geq 0} f^{-n}(W^s_\varepsilon(f^n(\bar{x}), f)), \quad W^u(\bar{x}, f) = \bigcup_{n \geq 0} f^n(W^u_\varepsilon(f^{-n}(\bar{x}), f)),
\]
where
\[
W^s(\bar{x}, f) = \{ y \in M : d(x_n, f^n(y)) \to 0 \text{ as } n \to \infty \}, \quad W^u(\bar{x}, f) = \{ y \in M : \text{there exists } \bar{y} = (y_n) \in M_f \text{ with } y_0 = y \text{ such that } d(x_{-n}, y_{-n}) \to 0 \text{ as } n \to \infty \}.
\]
Notice that \( W^\sigma(\bar{x}, f) \) (\( \sigma = s, u \)) is not an immersed submanifold whenever \( f \) is noninvertible.

A closed \( f \)-invariant set \( A \) is said to be isolated if there is a compact neighborhood \( U \) of \( A \) satisfying \( A_f = U_f \). If, in particular, \( f \) is a diffeomorphism, then \( A_f = U_f \) means \( A = \bigcap_{n=-\infty}^{\infty} f^n(U) \).

If \( A \) is isolated and there is a point \( x \in A \) such that \( \{f^n(x) : n \geq 0\} \) is dense in \( A \), then \( A \) is called a basic set. It follows from [20, Theorem 3.11] and [21, p. 62] that an isolated hyperbolic set \( A \) decomposes into a finite disjoint union \( A = A_1 \cup \ldots \cup A_s \) of basic sets \( A_i \) since the inverse limit system \( \hat{f} \) of \( f \) is an expansive homeomorphism with the shadowing property.

We say that there exists an \( n \)-cycle in \( A \) if there exists \( \{A_{ij} : 1 \leq j \leq n+1\} \) such that

1. \( A_{i_1} = A_{i_{n+1}} \),
2. \( A_{ij} \neq A_{ik} \) (\( 1 \leq j \neq k \leq n \)),
3. \( \{W^s(A_{ij}, f) \setminus A_{ij}\} \cap \{W^u(A_{ij+1}, f) \setminus A_{ij+1}\} \neq \emptyset \) (\( 1 \leq j \leq n \)),

where
\[
W^s(A_i, f) = \bigcup_{\bar{x} \in (A_i)_f} W^s(\bar{x}, f), \quad W^u(A_i, f) = \bigcup_{\bar{x} \in (A_i)_f} W^u(\bar{x}, f).
\]
We say sometimes that \( A_i \) has a homoclinic point when it has a 1-cycle.

The subset
\[
\Omega(f) = \{ x \in M : \text{for any neighborhood } U \text{ of } x \text{ there is } n > 0 \text{ such that } f^n(U) \cap U \neq \emptyset \}
\]
is closed and satisfies $f(\Omega(f)) \subset \Omega(f)$. We say that $\Omega(f)$ is the nonwandering set. Notice that if the set of periodic points, $\text{Per}(f)$, is dense in $\Omega(f)$, then $f(\Omega(f)) = \Omega(f)$. Recall that $f$ satisfies Axiom A if $\text{Per}(f)$ is dense in $\Omega(f)$ and $\Omega(f)$ is hyperbolic. When $f$ satisfies Axiom A, it is easily checked that $\Omega(f)$ is isolated, and so $\Omega(f)$ decomposes into a finite disjoint union of basic sets. We say that an Axiom A differentiable map $f$ has no cycles if there are no cycles in $\Omega(f)$. Define

$$\mathcal{P}(M) = \{ f \in C^1(M) : \text{every periodic point of } f \text{ is hyperbolic} \},$$

$$\mathcal{AN}(M) = \{ f \in C^1(M) : f \text{ satisfies Axiom A and has no cycles} \}.$$

Since $\mathcal{AN}(M)$ is open in $C^1(M)$ [14, Theorem B], we have $\mathcal{AN}(M) \subset \text{int } \mathcal{P}(M)$. Here $\text{int } E$ denotes the interior of $E$.

If $D_x f : T_x M \to T_{f(x)} M$ is not injective, then $x$ is called a singular point for $f$. Denote by $S(f)$ the set of all singular points of $f$. Obviously, $S(f)$ is a closed subset of $M$. Notice that an expanding map has no singular points.

Let $f \in C^1(M)$. Then $f$ is said to be $C^1$-structurally stable if there exists a neighborhood $U(f)$ of $f$ such that for $g \in U(f)$, $g$ is topologically conjugate to $f$. A differentiable map which is $C^1$-structurally stable has no singular points [8, p. 381]. But this is not true for $C^2$-structural stability [2, Theorem 3]. We say that $f$ is $C^1$ $\Omega$-stable if there exists a neighborhood $U(f)$ of $f$ such that $g|\Omega(g)$ is topologically conjugate to $f|\Omega(f)$ for all $g \in U(f)$. Notice that $C^1$-differentiable maps satisfying $C^1$ $\Omega$-stability belong to $\text{int } \mathcal{P}(M)$. This follows from [3, Theorem 1].

Our main theorem is the following:

**Theorem A.** If a $C^1$-map $f$ belonging to $\text{int } \mathcal{P}(M)$ satisfies the condition

$$\Omega(f) \cap S(f) \subset \{ p \in \text{Per}(f) : p \text{ is a sink} \},$$

then $f$ satisfies Axiom A and has no cycles.

The proof of this theorem is based upon results related to stability problems from Mañé [12], Palis [18] and Przytycki [21].

If $f$ satisfies Axiom A and $\Omega(f)$ is the disjoint union $\Omega_1 \cup \Omega_2$ of two closed $f$-invariant sets such that:

(i) $f|\Omega_1$ is injective,

(ii) $\Omega_2$ is contained in the closure of all source periodic points,

then $f$ is said to satisfy strong Axiom A. When $f$ is a diffeomorphism, the notion of strong Axiom A coincides with that of Axiom A.

As an extension of the result of Przytycki [21, Theorem A] stated above we have:

**Corollary B.** If $f \in C^1(M)$ satisfies the assumption of Theorem A, then the following are equivalent:
(1) \( f \) satisfies strong Axiom A and has no cycles,
(2) \( f \) is \( C^1 \) \( \Omega \)-stable.

2. Proof of Theorem A. To show Theorem A we need the following propositions, where \( \text{cl}(E) \) denotes the closure of \( E \).

**Proposition 1.** If \( f \in \text{int} \mathcal{P}(M) \) and \( \{ \Omega(f) \setminus \text{cl}(\text{Per}(f)) \} \cap S(f) = \emptyset \), then \( \Omega(f) = \text{cl}(\text{Per}(f)) \).

This will follow from the techniques used to prove the closing lemma for \( C^1 \)-maps with finite singular points (see Wen [26] and [27, Theorem A]).

Let \( f \in \mathcal{P}(M) \). Then every periodic point \( p \) of \( f \) is hyperbolic. Thus \( p \) satisfies (1.1). We set

\[
(2.1) \quad I_i(f) = \{ p \in \text{Per}(f) : \dim E^s(p) = i \} \quad (0 \leq i \leq \dim M)
\]

where \( E^s(p) \) is as in (1.1), and denote by \( \#E \) the cardinality of \( E \).

**Proposition 2.** Every \( f \in \text{int} \mathcal{P}(M) \) has the following properties:

(a) \( \#I_{\dim M}(f) < \infty \),
(b) \( \text{cl}(I_0(f)) \) is hyperbolic.

Proposition 2(a) was proved in [19, Theorem 4.1] for diffeomorphisms and in [6] for differentiable maps without singular points. We shall give the proof of (a) for the general case. (b) is clear for diffeomorphisms because \( I_0(f) = I_{\dim M}(f^{-1}) \). Unfortunately it is not true that \( \#I_0(f) < \infty \) for the noninvertible case, and so we have to give a proof. To do that, the technique of [12, Theorem I.4] is useful.

We define

\[ \mathcal{F}(M) = \{ f \in \text{int} \mathcal{P}(M) : f \text{ satisfies the assumption of Theorem A} \} \]

and put

\[
(2.2) \quad A(i_0) = \bigcup_{i=0}^{i_0} \text{cl}(I_i(f)) \quad (0 \leq i_0 \leq \dim M).
\]

**Proposition 3.** Let \( f \in \mathcal{F}(M) \) and \( 0 \leq i_0 \leq \dim M - 2 \). If \( A(i_0) \) is hyperbolic and \( A(i_0) \cap \text{cl}(I_{i_0+1}(f)) = \emptyset \), then \( \text{cl}(I_{i_0+1}(f)) \) is hyperbolic.

This will be shown using the methods of [12, p. 167].

**Proposition 4.** Let \( f \in \mathcal{F}(M) \). Then

(a) \( \text{cl}(I_0(f)) \cap \bigcup_{i=1}^{\dim M} \text{cl}(I_i(f)) = \emptyset \),
(b) if \( 1 \leq i_0 \leq \dim M - 2 \) and \( A(i_0) \) is hyperbolic, then \( A(i_0) \cap \text{cl}(I_{i_0+1}(f)) = \emptyset \).

Proposition 4(a) is clear for diffeomorphisms because \( \#I_0(f) < \infty \), but we have to prove it for \( C^1 \)-maps. We shall derive a contradiction by showing
that if (a) is false then \( f \) has homoclinic points. (b) was given in [1, §3] for diffeomorphisms. We shall give the proof of (b) for the class \( \mathcal{F}(M) \) of differentiable maps which contains the diffeomorphisms.

Once Propositions 2–4 are established, we conclude that \( \text{cl}(\text{Per}(f)) \) is hyperbolic when \( f \in \mathcal{F}(M) \).

Indeed, \( \text{cl}(\text{Idim}_M(f)) = \text{Idim}_M(f) \) and \( \text{cl}(I_0(f)) \) are hyperbolic by Proposition 2. From Propositions 3 and 4 it follows that \( \text{cl}(I_i(f)) \) (\( 1 \leq i \leq \dim M - 1 \)) are hyperbolic. Thus \( \text{cl}(\text{Per}(f)) = \bigcup_{i=0}^{\dim M} \text{cl}(I_i(f)) \) is hyperbolic.

Combining this result and Proposition 1 shows that each \( f \in \mathcal{F}(M) \) satisfies Axiom A. Using the techniques of [17, Theorem, p. 221], it is checked that if \( f \in \text{int} \mathcal{P}(M) \) satisfies Axiom A, then \( f \) has no cycles. Therefore Theorem A is proved.

Thus it remains to show Propositions 1–4. We devote the rest of this paper to the proofs.

3. Proof of Proposition 1. We first prepare some auxiliary results.

For \( x \in M \) and \( \xi > 0 \) put \( T_x M(\xi) = \{ v \in T_x M : \| v \| \leq \xi \} \). Then there exists \( \xi > 0 \) such that the exponential map \( \exp_x : T_x M(\xi) \to M \) is a \( C^\infty \)-embedding for all \( x \in M \).

The following Lemmas 3.1 and 3.2 were proved in [3, Lemma 1.1] and [12, Lemma 1.8] for diffeomorphisms. But their proofs can be adapted to the noninvertible case, and so we omit them.

For \( E \subset M \), let \( B_\varepsilon(E) \) denote the closed ball defined by

\[
B_\varepsilon(E) = \{ y \in M : d(x, y) \leq \varepsilon \text{ for some } x \in E \}.
\]

**Lemma 3.1.** Let \( f \in C^1(M) \). For every neighborhood \( \mathcal{U}(f) \) of \( f \) there exist a neighborhood \( \mathcal{U}_1(f) \subset \mathcal{U}(f) \) of \( f \) and \( \varepsilon_1 > 0 \) such that for \( g \in \mathcal{U}_1(f) \), a neighborhood \( U \) of a finite sequence \( \theta = \{ x_1, \ldots, x_N \} \) with \( x_i \neq x_j \) (\( i \neq j \)) and linear maps \( L_i : T_{x_i} M \to T_{g(x_i)} M \) (\( 1 \leq i \leq N \)) with \( \| L_i - D_{x_i} g \| \leq \varepsilon \) there are \( \bar{g} \in \mathcal{U}(f) \) and \( \delta > 0 \) with the following properties:

(a) \( B_{4\delta}(\theta) \subset U \),
(b) \( \bar{g}(x) = g(x) \) (\( x \in \theta \cup \{ M \setminus B_{4\delta}(\theta) \} \)),
(c) \( \bar{g}(x) = \exp_{\bar{g}(x_i)} \circ L_i \circ \exp_{x_i}^{-1} \) (\( x \in B_\delta(x_i) \), \( 1 \leq i \leq N \)).

For \( f \in \mathcal{P}(M) \), \( 0 \leq i \leq \dim M \) and \( n > 0 \) define

\[
\text{Per}^n(f) = \{ p \in \text{Per}(f) : f^n(p) = p \}, \quad I_i^n(f) = I_i(f) \cap \text{Per}^n(f),
\]
where \( I_i(f) \) is defined in (2.1) for \( 0 \leq i \leq \dim M \).

**Lemma 3.2.** Let \( f \in \text{int} \mathcal{P}(M) \) and \( \mathcal{U}(f) \) be a connected open neighborhood of \( f \) contained in \( \text{int} \mathcal{P}(M) \). Then, for all \( g \in \mathcal{U}(f) \), \( 0 \leq i \leq \dim M \) and \( n > 0 \),

\[
\# I_i^n(f) = \# I_i^n(g) < \infty.
\]
Lemma 3.3. If $f \in \text{int} \mathcal{P}(M)$, then $f(\Omega(f)) = \Omega(f)$.

Proof. If $f$ is a diffeomorphism, then the assertion is clear. Thus it suffices to show it for the noninvertible case. It is clear that $f(\Omega(f)) \subset \Omega(f)$. Suppose there is $q \in M$ such that $q \in \Omega(f) \setminus f(\Omega(f))$. Since $q \in \Omega(f)$, there exist sequences $\{x_i\}$ of points and $\{k_i\}$ of positive integers such that

$$d(x_i, q) \leq 1/i \quad \text{and} \quad d(f^{k_i}(x_i), q) \leq 1/i.$$ 

We can suppose that $\{f^{k_i-1}(x_i)\}$ converges to $y$ as $i \to \infty$. Then $y \in f^{-1}(q)$ and so $y \notin \Omega(f)$. Thus there is a neighborhood $U(y)$ of $y$ such that $f^j(U(y)) \cap U(y) = \emptyset$ for $j > 0$. Then for $i > 0$ large enough we have

$$f^{k_i-1}(x_i) \in U(y) \quad \text{and} \quad f^k(x_i) \notin U(y) \quad (0 \leq k < k_i - 1).$$ 

Since $f(\text{cl}(\text{Per}(f))) = \text{cl}(\text{Per}(f))$, we have $q \notin \text{cl}(\text{Per}(f))$. Let $U(q)$ be a neighborhood of $q$ satisfying $U(q) \cap \text{cl}(\text{Per}(f)) = \emptyset$, and let $\mathcal{U}(f)$ be a connected open neighborhood of $f$ contained in $\text{int} \mathcal{P}(M)$. By taking $U(y)$ and $\mathcal{U}(f)$ small enough we can suppose that for all $g \in \mathcal{U}(f)$,

$$g(U(y)) \subset U(q).$$ 

By using Lemma 3.1 we can find $h \in \mathcal{U}(f)$ such that

$$\begin{align*}
(3.1) & \quad f^{k_i-1}(x_i) \in U(y) \quad \text{and} \quad f^k(x_i) \notin U(y) \quad (0 \leq k < k_i - 1). \\
(3.2) & \quad g(U(y)) \subset U(q).
\end{align*}$$ 

(as above, $S(h)$ denotes the set of singular points of $h$). Then there is a neighborhood $V \subset U(y)$ of $y$ such that $h|V : V \to h(V)$ is a diffeomorphism. Thus, for $i > 0$ large enough there is $x_i' \in V$ satisfying $h(x_i') = x_i$. Since $h(y) = q$ and $x_i \to q$ as $i \to \infty$, we have $x_i' \to y$ as $i \to \infty$. Thus, for $i > 0$ large enough we can construct a diffeomorphism $\varphi : M \to M$ such that

$$\varphi(f^{k_i-1}(x_i)) = x_i', \quad \{x \in M : \varphi(x) \neq x\} \subset U(y), \quad g = h \circ \varphi \in \mathcal{U}(f)$$ 

and so

$$g(f^{k_i-1}(x_i)) = x_i.$$

Then

$$g^{k_i}(f^{k_i-1}(x_i)) = f^{k_i-1} \circ g(f^{k_i-1}(x_i)) = f^{k_i-1}(x_i)$$

by (3.1) and (3.3), and $g(U(y)) \subset U(q)$ by (3.2). Thus,

$$g(f^{k_i-1}(x_i)) \in \text{Per}(g) \cap U(q) \neq \emptyset.$$ 

Since $U(y) \cap \text{cl}(\text{Per}(f)) = \emptyset$, we have $f(z) = g(z)$ for $z \in \text{cl}(\text{Per}(f))$. Therefore, $\#\text{Per}^n(f) < \#\text{Per}^n(g)$ for $n = k_i$, which contradicts Lemma 3.2. □

Lemma 3.4. Let $f \in C^1(M)$ and $q \in \Omega(f)$. If $f^{-1}(q') \cap \Omega(f) \neq \emptyset$ for all $q' \in f^{-n}(q) \cap \Omega(f)$ where $n \geq 0$, and if $\bigcup_{k \geq 0} f^{-k}(q) \cap \Omega(f) \cap S(f) = \emptyset$, then for every neighborhood $\mathcal{U}(f)$ of $f$ and every neighborhood $U(q)$ of $q$ there is $g \in \mathcal{U}(f)$ such that
(1) \( \text{Per}(g) \cap U(q) \neq \emptyset \),
(2) \( \{ x \in M : f(x) \neq g(x) \} \subseteq \bigcup_{n>0} f^{-n}(U(q)) \).

Lemma 3.4 easily follows from [27, Theorem A], and so we omit the proof.

**Proof of Proposition 1.** Proposition 1 was proved in [9, Lemma 3.1] for the case when \( f \) is a diffeomorphism. Thus it remains to give the proof for the noninvertible case. Suppose that \( q \in \Omega(f) \setminus \text{cl}(\text{Per}(f)) \). By Lemma 3.3 we have \( f^{-1}(q') \cap \Omega(f) \neq \emptyset \) for all \( q' \in f^{-n}(q) \cap \Omega(f) \) and \( n \geq 0 \). Since \( f(\text{cl}(\text{Per}(f))) = \text{cl}(\text{Per}(f)) \), we have
\[
\{ f^{-n}(q) \cap \Omega(f) \} \cap \text{cl}(\text{Per}(f)) = \emptyset
\]
for \( n \geq 0 \). Thus,
\[
\{ f^{-n}(q) \cap \Omega(f) \} \cap S(f) = \emptyset
\]
for \( n \geq 0 \) because \( \{ \Omega(f) \setminus \text{cl}(\text{Per}(f)) \} \cap S(f) = \emptyset \). Hence the assumptions of Lemma 3.4 were satisfied.

Let \( U(f) \) be a connected open neighborhood of \( f \) contained in \( \text{int} \mathcal{P}(M) \) and \( U(q) \) be a neighborhood of \( q \) satisfying \( U(q) \cap \text{cl}(\text{Per}(f)) = \emptyset \). By Lemma 3.4 there is \( g \in U(f) \) such that \( \text{Per}(g) \cap U(q) \neq \emptyset \) and
\[
\{ z \in M : f(z) \neq g(z) \} \subseteq \bigcup_{n \geq 0} \{ f^{-n}(U(q)) : n \geq 0 \}.
\]
Since \( f(\text{cl}(\text{Per}(f))) = \text{cl}(\text{Per}(f)) \), we have
\[
\bigcup_{n \geq 0} \{ f^{-n}(U(q)) : n \geq 0 \} \cap \text{cl}(\text{Per}(f)) = \emptyset,
\]
and so \( f(z) = g(z) \) for \( z \in \text{cl}(\text{Per}(f)) \). Therefore, \#\( \text{Per}^n(f) \) < \#\( \text{Per}^n(g) \) for some \( n > 0 \), which contradicts Lemma 3.2.

**4. Proof of Proposition 2(a).** Let \( f \in \text{int} \mathcal{P}(M) \). Then it follows from [10, Theorem 4.1] that there exist a neighborhood \( U(f) \subseteq \text{int} \mathcal{P}(M) \) of \( f \) and numbers \( 0 < \lambda_0 < 1 \), \( m_0 > 0 \) and \( \tau_0 > 0 \) such that for all \( g \in U(f) \) the following hold:

(a) for \( \tilde{p} = (p_n) \in \bigcup_{i=1}^{\dim M} I_i(g)_g \) with \( \varrho(g,p_0) = n > \tau_0 \),
\[
(4.1) \prod_{j=0}^{[n/m_0]-1} \| D\tilde{g}^{m_0} \varrho(g,m_0j(\tilde{p})) \| \leq \lambda_0^{[n/m_0]},
\]

(b) for \( \tilde{p} = (p_n) \in \bigcup_{i=0}^{\dim M-1} I_i(g)_g \) with \( \varrho(g,p_0) = n > \tau_0 \),
\[
(4.2) \prod_{j=0}^{[n/m_0]-1} \| (D\tilde{g}^{m_0} \varrho(g,m_0j(\tilde{p})))^{-1} \| \leq \lambda_0^{[n/m_0]},
\]
(c) for \( \tilde{p} = (p_n) \in \bigcup_{i=1}^{\dim M} I_i(g)_g \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Dg^{m_0} | E^u(\tilde{g}^{m_0 j}(\tilde{p})) \| \leq \log \lambda_0,
\]

(d) for \( \tilde{p} = (p_n) \in \bigcup_{i=0}^{\dim M-1} I_i(g)_g \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| (Dg^{m_0} | E^u(\tilde{g}^{m_0 j}(\tilde{p})))^{-1} \| \leq \log \lambda_0,
\]

where \( I_i(g) \) is as in (2.1), \( I_i(g)_g \) is as in (1.3) and \( \lfloor r \rfloor \) denotes the greatest integer not greater than \( r \).

Let \( \varepsilon_1 \) satisfy the conclusion of Lemma 3.1 for \( U_0(f) \) and let \( \lambda_0 < \lambda_1 < 1 \). Choose \( \varepsilon_0 > 0 \) such that \((1 + \varepsilon_0)\lambda_1 < 1 \) and \( \varepsilon_0 < \frac{1}{2}(\varepsilon_1/2)^{m_0} \), and take \( H_1 \geq 1 \) satisfying \( \varepsilon_0 > e^{-H_1} \). Denote by \( N(\lambda_0, \lambda_1) > 0 \) the smallest integer satisfying

\[
N(\lambda_0, \lambda_1) \log(\lambda_1/\lambda_0) > H_1,
\]

and write

\[
C(\lambda_0, \lambda_1) = \frac{\log(\lambda_1/\lambda_0)}{H_1}.
\]

**Lemma 4.1.** Let a sequence \( \{ p(n) : 0 \leq n \leq N - 1 \} \) satisfy the following:

- (i) \( N \geq N(\lambda_0, \lambda_1) \),
- (ii) \( p(n) > 0 \),
- (iii) \( -H_1 \leq \log p(n) \),
- (iv) \( \prod_{n=0}^{N-1} p(n) \leq \lambda_0^N \).

Then there exist an integer \( k \) with \( k > NC(\lambda_0, \lambda_1) \) and a sequence \( 0 \leq n_1 < \ldots < n_k < N - 1 \) such that for \( 1 \leq j \leq k \) and \( n_j < l \leq N - 1 \),

\[
\prod_{n=n_j+1}^{n_l} p(n) \leq \lambda^{l-n_j}_1.
\]

The statement of Lemma 4.1 is a reformulation of the result stated in [19, Lemma, p. 212] and [12, Lemma II.3], and so we omit the proof.

We set

\[
Q = \{ x \in \cl(I_{\dim M}(f)) : \| D_x f^{m_0} \| < \varepsilon_0 \}.
\]

Then there is \( \delta > 0 \) such that

\[
(a) \quad \text{if } d(x, y) \leq 2\delta \ (x \in \cl(I_{\dim M}(f)) \setminus Q, \ y \in M), \text{ then} \quad \| D_y f^{m_0} \| \leq (1 + \varepsilon_0)\| D_x f^{m_0} \|,
\]

\[
(b) \quad \text{if } d(x, y) \leq 2\delta \ (x, y \in \cl(I_{\dim M}(f))), \text{ then} \quad \| D_y f^{m_0} \| - \| D_x f^{m_0} \| \leq \varepsilon_0.
\]
Put $\lambda_2 = (1 + \varepsilon_0)\lambda_1$. Since $M$ is compact, there is $K > 0$ such that for 
$x_1, \ldots, x_K \in M$ with $x_i \neq x_j$ ($i \neq j$) there exist $x_i, x_j$ ($1 \leq i \neq j \leq K$) satisfying $d(x_i, x_j) \leq (1 - \lambda_2)\delta$. Let $N' > 0$ be an integer such that $K \leq N'C(\lambda_0, \lambda_1)$.

To obtain the conclusion of Proposition 2(a) suppose that $\sharp I_{\dim M}(f) = \infty$. Since $\sharp \Per^n(f) < \infty$ for $n > 0$ (by Lemma 3.2), there is a periodic point $p \in I_{\dim M}(f)$ with period $\varrho(f, p)$ satisfying

\[ \varrho(f, p) \geq \max\{\tau_0, m_0N', m_0N(\lambda_0, \lambda_1)\}. \]

Put $N = [\varrho(f, p)/m_0]$. If $q = f^{m_0n}(p) \in Q$ for some $0 \leq n \leq N - 1$, then we can construct a family $\{L_{f^i(q)} : T_{f^i(q)}M \rightarrow T_{f^{i+1}(q)}M\}_{i=0}^{m_0-1}$ of isomorphisms such that

\[ \|L_{f^i(q)} - D_{f^i(q)}f\| \leq \varepsilon_1, \]

\[ \inf\{\|L_{f^i(q)}(v)\| : v \in T_{f^i(q)}M \text{ with } \|v\| = 1\} \geq \varepsilon_1/2. \]

By Lemma 3.1 there is $g \in \mathcal{U}_0(f)$ such that

1. $g(x) = f(x)$ for $x \in \{p, f(p), \ldots, f^{\varrho(f, p)-1}(p)\}$,
2. if $f^{m_0n}(p) \not\in Q$ for $0 \leq n \leq N - 1$, then $D_{f^i(p)}g = D_{f^i(p)}f$ for $m_0n \leq i \leq m_0(n+1) - 1$,
3. if $f^{m_0n}(p) \in Q$ for $0 \leq n \leq N - 1$, then $D_{f^i(p)}g = L_{f^i(p)}f$ for $m_0n \leq i \leq m_0(n+1) - 1$,
4. $D_{f^i(p)}g = D_{f^i(p)}f$ for $Nm_0 \leq i \leq \varrho(f, p) - 1$.

Define a function $p(\cdot) : \{0, 1, \ldots, N-1\} \rightarrow \mathbb{R}$ by

\[ p(n) = \|D_{f^{m_0n}(p)}g^{m_0}\|. \]

Then $-H_1 < \log p(n)$ for $0 \leq n \leq N - 1$. Since $g \in \mathcal{U}_0(f)$, by (4.1) we have

\[ \prod_{n=0}^{N-1} p(n) \leq \lambda_0^N, \]

and so $\{p(n)\}$ satisfies the conditions of Lemma 4.1. Thus there are an integer $k > K$ and a sequence $0 \leq n_1 < \ldots < n_k < N - 1$ such that

\[ \prod_{n=n_j+1}^{l} p(n) \leq \lambda_1^{l-n_j} \quad (1 \leq j \leq k, \ n_j < l \leq N - 1). \]

By the choice of $K$ there are $0 \leq i < j \leq k$ such that

\[ d(g^{m_{0i}}(p), g^{m_{0j}}(p)) \leq (1 - \lambda_2)\delta. \]

By (4.7) and (4.8) it is easily checked that

\[ \begin{align*}
1 & (g^{m_0(n_j-n_i)}B_\delta(g^{m_{0i}}(p))) \text{ is a Lipschitz map and its Lipschitz constant is less than } \lambda_2 < 1, \\
2 & g^{m_0(n_j-n_i)}(B_\delta(g^{m_{0i}}(p))) \subset B_\delta(g^{m_{0i}}(p)).
\end{align*} \]
Thus there is a unique \( z \in B_\delta(g^{m_0}(p)) \) satisfying \( g^{m_0(n_j-n_i)}(z) = z \). Since 
\[ N = \lfloor g(f,p)/m_0 \rfloor \] \( 0 < n_1 < \ldots < n_k < N-1 \), we have \( 0 < m_0(n_j-n_i) < g(f,p) \), and so \( z \neq g^{m_0}(p) \). On the other hand, since 
\[ g^{e(f,p)m_0(n_j-n_i)} : B_\delta(g^{m_0}(p)) \to B_\delta(g^{m_0}(p)) \]
is a contraction, we have \( z = g^{m_0}(p) \), which is a contradiction. Thus \( \#I_{\text{dim}\, M}(f) = \infty \) cannot happen. Therefore Proposition 2(a) is proved.

5. Proof of Key lemma (Lemma 5.1) and Proposition 2(b). Let \( A \) be a closed \( f \)-invariant set. We say that a \( D\hat{f} \)-invariant subbundle \( E \subset TM|A_f \) is contracting if \( D\hat{f}|E \) is contracting, and that \( E \) is expanding if \( D\hat{f}|E \) is expanding.

Let \( f \in \text{int} \mathcal{P}(M) \) and \( I_i(f) \) be as in (2.1). Let \( m_0 \) and \( \lambda_0 \) satisfy (4.1)–(4.4). It follows from [10, Proposition II.1] that \( TM|\text{cl}(I_i(f))_f \) \( 1 \leq i \leq \text{dim}\, M - 1 \) splits into the Whitney sum \( TM|\text{cl}(I_i(f))_f = \tilde{E}_i^s + \tilde{E}_i^u \) of subbundles \( \tilde{E}_i^s \) and \( \tilde{E}_i^u \) such that

\[
\begin{align*}
(a) & \quad D\hat{f}^{m_0}(\tilde{E}_i^s) \subset \tilde{E}_i^s, \quad D\hat{f}^{m_0}(\tilde{E}_i^u) = \tilde{E}_i^u, \\
(b) & \quad D\hat{f}^{m_0}|\tilde{E}_i^u : \tilde{E}_i^u \to \tilde{E}_i^u \text{ is injective,} \\
(c) & \quad \|D\hat{f}^{m_0}|\tilde{E}_i^u(x)\| \cdot \|(D\hat{f}^{m_0}|\tilde{E}_i^u(x))^{-1}\| \leq \lambda_0 \text{ for } \tilde{x} \in \text{cl}(I_i(f))_f.
\end{align*}
\]

It is easily checked from [20, §0 and §1] that for \( 1 \leq i \leq \text{dim}\, M - 1 \),

\[
\begin{align*}
(1) & \quad \tilde{E}_i^s \text{ and } \tilde{E}_i^u \text{ are } C^0\text{-vector bundles over } \text{cl}(I_i(f))_f, \\
(2) & \quad \text{if } \tilde{x} = (x_n), \tilde{y} = (y_n) \in \text{cl}(I_i(f))_f \text{ satisfy } x_0 = y_0, \text{ then } \tilde{E}_i^s(\tilde{x}) = \tilde{E}_i^s(\tilde{y}), \text{ and so we write } \tilde{E}_i^s(x_0) = \tilde{E}_i^u(\tilde{x}) \subset T_{x_0}M \text{ where } \tilde{E}_i^u(\tilde{x}) = \text{cl}(I_i(f))_f \text{ in general),}
(3) & \quad \text{cl}(I_i(f)) \text{ is hyperbolic if and only if } \tilde{E}_i^s \text{ is contracting and } \tilde{E}_i^u \text{ expanding.}
\end{align*}
\]

In the case when \( i = 0 \), \( \text{cl}(I_0(f)) \) is hyperbolic if and only if \( TM|\text{cl}(I_0(f))_f \) is expanding. If \( f \) is a diffeomorphism, then we know [19, Theorem 4.1] that \( \#I_0(f) < \infty \) and \( I_0(f) \) is hyperbolic.

**Lemma 5.1.** Let \( f \in \text{int} \mathcal{P}(M) \). Then

\[
\begin{align*}
(a) & \quad TM|\text{cl}(I_0(f))_f \text{ is expanding,} \\
(b) & \quad \text{if } f \in \mathcal{F}(M) \text{ and } \tilde{E}_i^s \text{ is contracting for some } 1 \leq i \leq \text{dim}\, M - 1, \text{ then } \tilde{E}_i^u \text{ is expanding.}
\end{align*}
\]

If we establish Lemma 5.1, then we obtain Proposition 2(b) from Lemma 5.1(a). The proof of Lemma 5.1(a) is similar to that of (b), and so we omit it. To show (b) we suppose that \( \tilde{E}_i^s \) is contracting and \( \tilde{E}_i^u \) is not expanding for some \( 1 \leq i \leq \text{dim}\, M - 1 \). Then we can find a periodic point
Moreover
the following property
be a compact invariant set
d Let
N
(5
To do that we need the techniques used in proving Theorem
I.4 of [12].
By (5.1)(b) we can define
\( D\tilde{f}^{-m_0}|E_i^{u}: \tilde{E}_i^u \rightarrow \tilde{E}_i^u \)
by \( D\tilde{f}^{-m_0}|E_i^{u}(\tilde{x}) = (D\tilde{f}^{-m_0}|E_i^{u}(\tilde{f}^{-m_0}(\tilde{x})))^{-1} \)
for \( \tilde{x} \in \text{cl}(I_i(f))_f \). We say that
for \( \tilde{x} \in \text{cl}(I_i(f))_f \) and \( n > 0 \) the pair \( (\tilde{x}, \tilde{f}^{m_0n}(\tilde{x})) \) is a \( \gamma \)-string if
\[
\prod_{j=1}^{n} \|D\tilde{f}^{-m_0}|E_i^{u}(\tilde{f}^{m_0j}(\tilde{x}))\| \leq \gamma^n,
\]
and that it is a uniform \( \gamma \)-string if \( (\tilde{f}^{m_0k}(\tilde{x}), \tilde{f}^{m_0n}(\tilde{x})) \) is a \( \gamma \)-string for
\( 0 \leq k < n \). Let us say that for \( 0 \leq N < n \) a pair \( (\tilde{x}, \tilde{f}^{m_0n}(\tilde{x})) \) is an
\((N, \gamma)\)-obstruction if \( (\tilde{x}, \tilde{f}^{m_0k}(\tilde{x})) \) is not a \( \gamma \)-string for \( N \leq k < n \).
Take \( \gamma_i \) \( (0 \leq i \leq 4) \) with
\[
0 < \lambda_0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1.
\]
Let \( N(\gamma_i, \gamma_j) \) and \( C(\gamma_i, \gamma_j) \) \( (0 \leq i < j \leq 4) \) be as in (4.5) and (4.6), and let \( \tilde{d} \) be a compatible metric for the product topological space \( \mathbb{M} \).

**Lemma 5.2.** If \( \tilde{E}_i^u \) is not expanding, then for every \( \varepsilon > 0 \) there exists
a compact invariant set \( \Lambda(\varepsilon) \subset \text{cl}(I_i(f))_f \) such that each \( \tilde{x}^0 \in \Lambda(\varepsilon) \) has
the following property: there exist \( \tilde{x}^1 \in \Lambda(\varepsilon) \cap I_i(f)_f \) arbitrarily near to \( \tilde{x}^0 \),
\( n_1 \geq 0 \) and \( \tilde{y} \in \Lambda(\varepsilon) \) such that
\[
(a) \quad \tilde{d}(\tilde{f}^{m_0n_1}(\tilde{x}^1), \tilde{y}) < \varepsilon/4,
(b) \quad (\tilde{y}, \tilde{f}^{m_0n}(\tilde{y})) \text{ is an } (N(\gamma_3, \gamma_4), \gamma_2)\text{-obstruction for } n > N(\gamma_3, \gamma_4),
(c) \text{ if } n_1 > 0, \text{ then } (\tilde{x}^1, \tilde{f}^{m_0n_1}(\tilde{x}^1)) \text{ is a uniform } \gamma_4\text{-string.}
\]
Moreover \( \Lambda(\varepsilon) \) is the closure of its interior in \( \text{cl}(I_i(f))_f \).

**Lemma 5.2** is checked in the same way as Lemma II.7 of [12], so we omit
the proof.

The following lemma is stated in [12, Lemma II.5].

**Lemma 5.3.** Let \( \tilde{x} \in \text{cl}(I_i(f))_f \) and let \( n, r \) and \( l \) be nonnegative integers
with \( 0 \leq r \leq r + l \leq n \). If \( (\tilde{x}, \tilde{f}^{m_0n}(\tilde{x})) \) is a \( \gamma_0 \)-string containing an
\((N(\gamma_3, \gamma_4), \gamma_2)\)-obstruction \( (\tilde{f}^{m_0r}(\tilde{x}), \tilde{f}^{m_0(r+l)}(\tilde{x})) \) such that
(a) \( n \geq N(\gamma_0, \gamma_4), \)
(b) \( nC(\gamma_0, \gamma_4) > r + l, \)
(c) \( r + l \geq N(\gamma_1, \gamma_2) \) and
(d) \( (r + l)C(\gamma_1, \gamma_2) > r + N(\gamma_3, \gamma_4), \)

then there exists a uniform \( \gamma_4 \)-string \( (\tilde{x}, \tilde{f}^{m_0}(\tilde{x})), r + l \leq m \leq n, \) that is not a \( \gamma_1 \)-string.

Let \( A(\varepsilon) \) be as in Lemma 5.2 and fix \( \tilde{x}^0 \in A(\varepsilon) \). Choose \( \tilde{x}^1 \in A(\varepsilon) \cap I_i(f)_f \), \( \tilde{y} \in A(\varepsilon) \) and \( n_1 > 0 \) as in Lemma 5.2 and take \( N_1 \) with
\[
N_1 > \max\{N(\gamma_3, \gamma_4), N(\gamma_1, \gamma_2)\}.
\]
Since \( A(\varepsilon) \) is the closure of its interior in \( \text{cl}(I_i(f))_f \) and \( (\tilde{y}, \tilde{f}^{m_0}N_1(\tilde{y})) \) is an \( (N(\gamma_3, \gamma_4), \gamma_2) \)-obstruction, there exists \( \tilde{x}^2 \in A(\varepsilon) \cap I_i(f)_f \) such that
\[
\tilde{d}(\tilde{x}^2, \tilde{y}) < \varepsilon/4 \quad \text{and} \quad (\tilde{x}^2, \tilde{f}^{m_0}N_1(\tilde{x}^2)) \text{ is an } (N(\gamma_3, \gamma_4), \gamma_2) \text{-obstruction.}
\]
Since \( \tilde{x}^2 \in I_i(f)_f \) and \( \lambda_0 < \gamma_0 \), we deduce by (4.4) that \( (\tilde{x}^2, \tilde{f}^{m_0}n(\tilde{x}^2)) \) is a \( \gamma_0 \)-string for \( n \) large enough. Thus \( (\tilde{x}^2, \tilde{f}^{m_0}n(\tilde{x}^2)) \) satisfies the conditions of Lemma 5.3 \( (r = 0, l = N_1) \), and so we can choose \( N_1 \leq n_2 \leq n \) such that
\[
(\tilde{x}^2, \tilde{f}^{m_0}n_2(\tilde{x}^2)) \text{ is a uniform } \gamma_4 \text{-string, but not a } \gamma_1 \text{-string.}
\]

Put \( K = \min_{\tilde{z} \in \text{cl}(I_i(f))_f} \|D\tilde{f}^{m_0}|\tilde{E}^u_i(\tilde{x})\| > 0 \) and take \( 0 < k_0 < 1 \) with \( \lambda_0 < k_0^2\gamma_1 \) and \( \gamma_4 < k_0 \). Since \( N_1 \) is large enough and \( n_2 \geq N_1 \), we can suppose that
\[
\gamma_1^{n_2}K^{n_1} \geq (k_0\gamma_1)^{n_2+n_1}.
\]
Continuing in this manner we obtain the following lemma.

**Lemma 5.4.** Suppose that \( \tilde{E}^u_i \) is not expanding. Then for all \( \varepsilon > 0 \) and \( \gamma_1, \gamma_4 \) with \( 0 < \lambda_0 < \gamma_1 < \gamma_4 < 1 \) there exist sequences \( \{\tilde{x}^j\}_{j \geq 1} \subset A(\varepsilon) \) and \( \{n_j\}_{j \geq 1} \) such that
\[
(1) \tilde{d}(\tilde{f}^{m_0}n_j(\tilde{x}^j), \tilde{x}^{j+1}) < \varepsilon/2 \quad (j \geq 1),
(2) \text{if } n_j > 0, \text{ then } (\tilde{x}^j, \tilde{f}^{m_0}n_j(\tilde{x}^j)) \text{ is a uniform } \gamma_4 \text{-string,}
(3) \text{if } j \geq 2 \text{ is even, then } n_j > 0 \text{ and } (\tilde{x}^j, \tilde{f}^{m_0}n_j(\tilde{x}^j)) \text{ is not a } \gamma_1 \text{-string,}
\]
and
\[
\gamma_1^nK^{n_1} \geq (k_0\gamma_1)^{n_2+n_1}.
\]
To show (5.3) we extend the continuous bundles \( \tilde{E}_i^s \) and \( \tilde{E}_i^u \) to a neighborhood of \( \text{cl}(I_i(f))_f \). In the same way as in the proof of [4, Theorem (4.2)] it is checked that there are a closed neighborhood \( V \) of \( \text{cl}(I_i(f))_f \) and a \( C^0 \)-splitting \( TM_f|V = \tilde{E}_i^s \oplus \tilde{E}_i^u \) such that
\[
\begin{align*}
(a) & \tilde{x} \in V \cap \tilde{f}^{m_0}(V), \text{ then } D\tilde{f}^{m_0}(\tilde{E}_i^s(\tilde{x})) = \tilde{E}_i^s(\tilde{f}^{m_0}(\tilde{x})) \quad (\sigma = s, u), \\
(b) & \tilde{E}_i^\sigma|\text{cl}(I_i(f))_f = \tilde{E}_i^\sigma \quad (\sigma = s, u),
\end{align*}
\]
(c) there is $0 < \lambda_0 < \lambda < 1$ such that for $\tilde{x} \in V \cap \tilde{f}^{-m_0}(V)$,

$$\|D\tilde{f}^{m_0}\tilde{E}_{i}^{u}(\tilde{x})\| \cdot \|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{f}^{m_0}(\tilde{x}))\| \leq \lambda.$$ 

Choose $\delta > 0$ such that if $\tilde{x} \in \text{cl}(I_i(f))_f$ and $\tilde{y} = (y_i) \in M_f$ satisfy $\tilde{d}(\tilde{x}, \tilde{y}) < \delta$, then $\tilde{y} \in V$ and

$$k_0\|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{y})\| \leq \|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{x})\| \leq k_0^{-1}\|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{y})\|. \quad (5.4)$$

Let $0 < \varepsilon < \delta$ be sufficiently small. Choose $\{\tilde{x}_j\}_{j \geq 1}$ and $\{n_j\}_{j \geq 1}$ satisfying the assertion of Lemma 5.4 for this $\varepsilon$. Without loss of generality we suppose that $\tilde{d}(\tilde{x}_1, \tilde{f}^{m_0n_k}(\tilde{x}_k)) < \varepsilon/2$ for some large $k > 0$ because $\Lambda(\varepsilon)$ is compact. Then we have to find $\tilde{p} \in \text{Per}(\tilde{f})$ such that

$$\tilde{f}^{m_0n}(\tilde{p}) = \tilde{p}, \quad (5.5)$$

$$\tilde{d}(\tilde{f}^{m_0l}(\tilde{f}^{m_0(n_0+n_1+\ldots+n_{j-1})}(\tilde{p})), \tilde{f}^{m_0l}(\tilde{x}_j)) < \delta \quad (0 \leq l \leq n_j, \ 1 \leq j \leq k),$$

where $n = n_1 + \ldots + n_k$ and $n_0 = 0$.

If (5.5) is established, then the point $\tilde{p}$ meets our requirement. In fact it suffices to see that (5.3) holds for $\tilde{p}$. By (5.4) and Lemma 5.4(2) we have

$$\prod_{l=1}^{n} \|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{f}^{m_0l}(\tilde{p}))\| < k_0^{-m_0}\gamma_4,$$

and so $\tilde{E}_{i}^{u}(\tilde{p}) \subset E^{u}(\tilde{p})$ since $k_0^{-1}\gamma_4 < 1$. Thus $\tilde{p} \in \bigcup_{j=1}^{M-1} I_j(f)_f$. On the other hand, by (5.4), (5.5) and Lemma 5.4(3),

$$\prod_{l=1}^{n} \|D\tilde{f}^{-m_0}\tilde{E}_{i}^{u}(\tilde{f}^{m_0l}(\tilde{p}))\| \geq k_0^{2n}\gamma_1^n > \lambda_0^n.$$ 

Therefore we obtain (5.3).

It remains to show (5.5). To do that we apply the local stable manifold theorem for diffeomorphisms ([5], [23]).

For $\tilde{x} \in M$ and $\xi > 0$ put $T_{\tilde{x}}M(\xi) = \{ (\tilde{x}, v) \in TM : \|v\| \leq \xi \}$. Then $\exp_{\tilde{x}} : T_{\tilde{x}}M(\xi) \to M$ defined by

$$\exp_{\tilde{x}} = \exp_{x_0} \circ \tilde{P}_0 | T_{\tilde{x}}M(\xi)$$

is a $C^\infty$-embedding for small $\xi > 0$ as described in §3. Since $S(f) \cap \text{cl}(I_i(f)) = \emptyset$, there exists $0 < r_0 < \xi$ such that

$$F_{\tilde{x}}^{-1} = (\exp_{\tilde{x}}^{-1} \circ \tilde{f}^{m_0} \circ \exp_{\tilde{f}^{-m_0}})^{-1} | T_{\tilde{x}}M(r_0)$$

is a $C^1$-embedding for $\tilde{x} \in \text{cl}(I_i(f))_f$.

Let $\tilde{x} \in \text{cl}(I_i(f))_f$ and $\tilde{E}_{i}^{\sigma}(\tilde{x})$ be as in (5.1) for $\sigma = s, u$. We put $\tilde{E}_{i}^{\sigma}(\tilde{x}, r) = \tilde{E}_{i}^{\sigma}(\tilde{x}) \cap T_{\tilde{x}}M(r) \ (r > 0)$ and denote by $0$ the zero vector of
We put
\[ \Sigma^b(\bar{x}, r) = \{ \sigma : \tilde{E}^s_i(\bar{x}, r) \to \tilde{E}_i^u(\bar{x}) : \max_{v \in \tilde{E}_i^u(\bar{x}, r)} \| \sigma(v) \| < \infty, \ \text{Lip}(\sigma) \leq 1 \}, \]
and
\[ \Sigma^0(\bar{x}, r) = \{ \sigma \in \Sigma^b(\bar{x}, r) : \| \sigma(o) \| \leq r \}. \]
Here Lip(\sigma) denotes a Lipschitz constant of \( \sigma \). Define \( d'(\sigma, \sigma') = \max_{v \in \tilde{E}_i^u(\bar{x}, r)} \| \sigma(v) - \sigma'(v) \| \) (\( \sigma, \sigma' \in \Sigma^b(\bar{x}, r) \)).

Then \((\Sigma^b(\bar{x}, r), d')\) is a complete metric space and \( \Sigma^0(\bar{x}, r) \) is a closed subset of \( \Sigma^b(\bar{x}, r) \).

Let \( \varepsilon_0 > 0 \) be small enough and choose \( 0 < r_1 \leq r_0 \) satisfying
\[ \text{Lip}((F_{\bar{x}}^{-1} - D_{\bar{x}} \tilde{f}^{-m_0})|T_\bar{x}M(2r_1)) \leq \varepsilon_0 \quad \text{for } \bar{x} \in \text{cl}(I_i(f))_. \]

Since \( \tilde{E}^s_i \) is contracting by the assumption of Lemma 5.1(b), we have
\[ \| D\tilde{f}^{-m_0}|\tilde{E}^s_i \| \leq 1 \]
for some \( \mu < 1 \) (take \( m_0 \) large enough if necessary). Let \( p^\sigma : \tilde{E}^s_i \oplus \tilde{E}^u_i \to \tilde{E}^\sigma_i \) (\( \sigma = s, u \)) be the natural projection. Then it is easily checked that if \( \sigma \in \Sigma^0(\bar{x}, r_1) \), then \( p^s \circ F_{\bar{x}}^{-1} \circ (\text{id}, \sigma) : \tilde{E}^s_i(\bar{x}, r) \to \tilde{E}^s_i(\tilde{f}^{-m_0}(\bar{x})) \) is an embedding such that
\[ p^s \circ F_{\bar{x}}^{-1} \circ (\text{id}, \sigma) : (\tilde{E}^s_i(\bar{x}, r_1)) \supset \tilde{E}^s_i(\tilde{f}^{-m_0}(\bar{x}), r_1(1 - 2\varepsilon_0\mu)/\mu), \]
and so the graph transformation
\[ \Gamma_{\bar{x}}(\sigma) = (p^u \circ F_{\bar{x}}^{-1} \circ (\sigma, \text{id})) \circ [p^s \circ F_{\bar{x}}^{-1} \circ (\sigma, \text{id})]^{-1} |\tilde{E}^u_i(\tilde{f}^{-m_0}(\bar{x}), r_1) \]
is well defined and \( F_{\bar{x}}^{-1}(\text{graph}(\sigma)) \supset \text{graph}(\Gamma_{\bar{x}}(\sigma)) \). Moreover, from (5.1)(c) it follows that for \( \sigma, \sigma' \in \Sigma^0(\bar{x}, r_1) \),
\begin{align*}
(1) \ & \text{Lip}(\Gamma_{\bar{x}}(\sigma)) \leq \frac{\varepsilon_0\mu + \lambda_0}{1 - 2\varepsilon_0\mu} < 1, \\
(2) \ & \| \Gamma_{\bar{x}}(\sigma)(o) \| \leq \{ \| D\tilde{f}^{-m_0}|\tilde{E}^u_i(\bar{x}) \| + \varepsilon_0 \} \frac{\mu}{1 - 2\varepsilon_0\mu} \| \sigma(o) \|, \\
(3) \ & d'(\Gamma_{\bar{x}}(\sigma), \Gamma_{\bar{x}}(\sigma')) \leq \{ \| D\tilde{f}^{-m_0}|\tilde{E}^u_i(\bar{x}) \| + 2\varepsilon_0 \} \frac{\mu}{1 - 2\varepsilon_0\mu} d'(\sigma, \sigma').
\end{align*}

By (1) we have \( \Gamma_{\bar{x}}(\sigma) \subset \Sigma^b(\tilde{f}^{-m_0}(\bar{x}), r_1) \).

We are now in a position to prove (5.5). Let \( \{ \bar{x}_j \}_{j \geq 1} \) and \( \{ n_j \}_{j \geq 1} \) satisfy the conclusion of Lemma 5.4 for \( \varepsilon > 0 \) small enough, and let \( k > 0 \) satisfy \( \tilde{d}(\bar{x}^1, \tilde{f}^{n_k m_0}(\bar{x}^k)) < \varepsilon/2 \). If \( n_j = 0 \) then \( j \) is odd by Lemma 5.4(3), and so \( n_{j+1} > 0 \). Thus we suppose that \( n_j > 0 \), \((\bar{x}^j, \tilde{f}^{n_j m_0}(\bar{x}^j))\) is a uniform \( \gamma_4 \)-string for \( 1 \leq j \leq k \), \( \tilde{d}(\tilde{f}^{n_j m_0}(\bar{x}^j), \bar{x}^{j+1}) < \varepsilon \) for \( 1 \leq j \leq k - 1 \) and \( \tilde{d}(\tilde{f}^{n_k m_0}(\bar{x}^k), \bar{x}^1) < \varepsilon \). To avoid complication we show (5.5) for the case when \( k = 1 \).
Choose \( \varepsilon_0 > 0 \) with \( e^{\varepsilon_0} \gamma_4 < 1 \) and suppose
\[
2\varepsilon_0 < (e^{\varepsilon_0} - 1) \inf_{x \in \text{cl}(I_i(f))} \| D_x \tilde{f}^{-m_0} \|
\]
because \( \varepsilon_0 \) is small enough. We put \( \Gamma = \Gamma_{\tilde{f}^{m_0}(\bar{x}_1)} \circ \ldots \circ \Gamma_{\tilde{f}^{n_{1}m_0}(\bar{x}_1)} \). By applying inductively the above estimates (1)–(3), we find that for \( \sigma, \sigma' \in \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \),
\[
(1') \quad \tilde{\Gamma}(\sigma) \subset \Sigma^0(\bar{x}_1, r_1),
\]
\[
(2') \quad \| \tilde{\Gamma}(\sigma)(o) \| \leq \left\{ \prod_{j=1}^{n_1} \left\{ \| D \tilde{f}^{-m_0} | \tilde{E}_i^u(\tilde{f}^{jm_0}(\bar{x}_1)) \| + \varepsilon_0 \right\} \frac{\mu}{1 - 2\varepsilon_0 \mu} \right\} \| \sigma(o) \|
\]
\[
\leq \left( e^{\varepsilon_0} \frac{\mu}{1 - 2\varepsilon_0 \mu} \right)^{n_1} \left\{ \prod_{j=1}^{n_1} \| D \tilde{f}^{-m_0} | \tilde{E}_i^u(\tilde{f}^{jm_0}(\bar{x}_1)) \| \right\} \| \sigma(o) \|
\]
\[
\leq \left( e^{\varepsilon_0} \gamma_4 \frac{\mu}{1 - 2\varepsilon_0 \mu} \right)^{n_1} \| \sigma(o) \|,
\]
\[
(3') \quad d'(\tilde{\Gamma}(\sigma), \tilde{\Gamma}(\sigma')) \leq \left( e^{\varepsilon_0} \gamma_4 \frac{\mu}{1 - 2\varepsilon_0 \mu} \right)^{n_1} d'(\sigma, \sigma').
\]

Let \( \sigma \in \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \). Since \( \tilde{E}_i^\sigma \) is continuous (\( \sigma = s, u \)) and \( \tilde{d}(\bar{x}_1, \tilde{f}^{n_{1}m_0}(\bar{x}_1)) < \varepsilon \), by (1) and (2') there is a unique \( \bar{\sigma} \in \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \) such that
\[
\text{graph}(\bar{\sigma}) \subset \exp_{\tilde{f}^{n_{1}m_0}(\bar{x}_1)}^{-1} \circ \text{exp}_{\bar{x}_1} \circ \tilde{\Gamma}(\sigma),
\]
and so we can define \( \Gamma^0 : \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \rightarrow \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \) by \( \Gamma^0(\sigma) = \bar{\sigma} \). From (3') it follows that \( \Gamma^0 \) is a contracting map, and thus it has a unique fixed point \( \sigma_0 \in \Sigma^0(\tilde{f}^{n_{1}m_0}(\bar{x}_1), r_1) \). Then \( \tilde{f}^{n_{1}m_0}(\text{graph}(\sigma_0)) \subset \text{graph}(\sigma_0) \). By Brouwer’s theorem there is \( p \in \text{graph}(\sigma_0) \) such that \( \tilde{f}^{n_{1}m_0}(p) = p \). Put \( \tilde{p} = \ldots, p, f(p), \ldots, \tilde{f}^{n_{1}m_0-1}(p), p, \ldots \in \text{Per}(\tilde{f}) \). Then it is easily checked that \( \tilde{p} \) meets our requirement.

### 6. Proof of Proposition 3. To show Proposition 3 we need properties of Borel probability measures used in [12, §1 and §3]. Denote by \( \mathcal{M}(X) \) the set of all Borel probability measures on a compact metric space \( X \). Let \( f : X \rightarrow X \) be a continuous map and \( \Lambda \) be a closed \( f \)-invariant set. We denote by \( \mathcal{M}(f|\Lambda) \) the set of all \( f \)-invariant measures belonging to \( \mathcal{M}(\Lambda) \) and by \( \mathcal{M}_e(f|\Lambda) \) that of all ergodic \( f \)-invariant measures.

Let \( f \in \text{int} \mathcal{P}(M) \) and \( I_i(f) \) be as in (2.1). Let \( m_0 \) and \( \lambda_0 \) be numbers satisfying (4.1)–(4.4), (5.1) and (5.2).

**Lemma 6.1.** Let \( f \in \mathcal{F}(M) \) and \( 0 \leq i_0 \leq \dim M - 2 \) be as in Proposition 3, and \( \Lambda(i_0) \) be as in (2.2). If \( \mu \in \mathcal{M}(f^m_0|\text{cl}(I_{i_0+1}(f))) \) satisfies
then \( \mu(\Lambda(i_0)) > 0 \).

This result was proved in [12, Theorem I.6] for diffeomorphisms. For the noninvertible case we can apply the method given in [12].

**Lemma 6.2.** Let \( f \) and \( i_0 \) be as in Lemma 6.1. Suppose that \( \mu \in \mathcal{M}_e(f^{m_0}|\overline{cl(I_{i_0+1}(f))}) \). Then, given a neighborhood \( V \) of \( \mu \) in \( \mathcal{M}(X) \) and a compact set \( D \) disjoint from the support of \( \mu \), there exist a \( C^1 \)-map \( g \), arbitrarily \( C^1 \)-close to \( f \) and coinciding with \( f \) on \( D \), and a periodic orbit \( \tilde{\rho} = (p_n) \) of \( g \) with period \( mm_0 \) such that

1. \( \mu_0 = m^{-1} \sum_{n=0}^{m-1} \delta_{p_{nm_0}} \in V \),
2. \( p_{nm_0} \not\in D \) for \( n \in \mathbb{Z} \),

where \( \delta_x \) is the point measure supported at \( x \).

Lemma 6.2 was obtained in [12, Theorem III.1] by using the ergodic closing lemma proved in [10, Theorem A] for diffeomorphisms and in [13, Theorem, p. 173] for \( C^1 \)-maps without singular points. However the proof in [13] can be adapted to our case. Thus we omit the proof of Lemma 6.2.

**Proof of Lemma 6.1.** The proof is very similar to that of [12, Theorem I.6]. Let \( \mu \in \mathcal{M}(f^{m_0}|\overline{cl(I_{i_0+1}(f))}) \) satisfy (6.1). We first check the case when \( \mu \) is ergodic.

Let \( W \subset \mathbb{M} \) be a small neighborhood of \( \overline{cl(I_{i_0+1}(f))} \). Choose an open neighborhood \( W_0 \) of \( \overline{cl(I_{i_0+1}(f))} \) such that if \( \tilde{x} = (x_n) \in \mathbb{M} \) satisfies \( x_{nm_0} \in W_0 \) for \( n \in \mathbb{Z} \), then \( \tilde{x} \in W \). By Lemma 6.2 there are \( g \), arbitrarily \( C^1 \)-near \( f \), and a periodic orbit \( \tilde{\rho} = (p_n) \) of \( g \) with period \( mm_0 \) such that \( \mu_0 = m^{-1} \sum_{n=0}^{m-1} \delta_{p_{nm_0}} \) is close to \( \mu \) in \( \mathcal{M}(M) \) and \( p_{nm_0} \not\in M \setminus W_0 \) for \( n \in \mathbb{Z} \). Then \( \mu_0 \) concentrates on \( W_0 \).

Since \( g \) is \( C^1 \)-near \( f \), we can suppose \( g \in \mathcal{P}(M) \). As in (2.1) and (2.2) define

\[
I_i(g) = \{ q \in \text{Per}(g) : \dim E^s(q) = i \}, \quad \Lambda'(i) = \bigcup_{k=0}^{i} \overline{cl(I_k(g))}
\]

for \( 0 \leq i \leq \dim M \). Since \( p_0 \) is a periodic point of \( g \) with period \( mm_0 \), it is hyperbolic and so the tangent space \( T_p M \) splits as in (1.1). If we prove

\[
\dim E^s(p_0) \leq i_0,
\]

then \( \mu_0(\Lambda'(i_0)) = 1 \). Since \( \Lambda'(i_0) \) and \( \mu_0 \) converge to \( \Lambda(i_0) \) and \( \mu \) respectively as \( g \to f \), we have \( \mu(\Lambda(i_0)) = 1 \). Lemma 6.1 proved.

Thus it is enough to show (6.2). To do that we use a continuous splitting

\[
TM|W = \hat{E}^s_{i_0+1} \oplus \hat{E}^u_{i_0+1}
\]
that is an extension of the splitting
\[ T\mathcal{M}|\text{cl}(I_{i_0+1}(f))f = \tilde{E}_{i_0+1}^s \oplus \tilde{E}_{i_0+1}^u \]
as in (5.1) (cf. [4, Lemma 4.4]). Let \( g \) be close to \( f \). Then we know that \( W(\tilde{g}) = \bigcap_{n \in \mathbb{Z}} \tilde{g}^n(W) \) has a \( D\tilde{g}^{m_0} \)-invariant splitting \( T\mathcal{M}|W(\tilde{g}) = \tilde{E}_{s}^g \oplus \tilde{E}_{u}^g \) such that \( \tilde{E}_{s}^g(\tilde{x}) \) is close to \( \tilde{E}_{i_0+1}^s(\tilde{x}) \) for \( \tilde{x} \in W(\tilde{g}), \sigma = s,u \) (cf. [5, §2]). If \( \tilde{x} = (x_n), \tilde{y} = (y_n) \in W(\tilde{g}) \) satisfy \( x_0 = y_0 \), then \( \tilde{E}_{s}^g(\tilde{x}) = \tilde{E}_{s}^g(\tilde{y}) \), and so we write \( \tilde{E}_{s}^g(x_0) = \mathcal{P}^0(\tilde{E}_{s}^g(\tilde{x})) (\subset T_{x_0}M) \). Notice that \( \tilde{E}_{s}^u(\tilde{x}) \neq \tilde{E}_{s}^u(\tilde{y}) \) generally.

Define a number \( \lambda > 0 \) by
\[
\log \lambda = \int_{\text{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}\|	ilde{E}_{i_0+1}^s\| d\mu > \log \lambda_0.
\]
Take \( \lambda_i \ (i = 1,2) \) with \( 0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda \). Since \( \tilde{E}_{s}^g \) \((\sigma = s,u)\) and \( \mu_0 \) are close to \( \tilde{E}_{s}^g \) \( \text{and} \mu \) respectively, by (5.1) and (6.1) we can suppose that for \( \tilde{x} \in W(\tilde{g}) \),
\[
\|D\tilde{g}^{m_0}\tilde{E}_{s}^g(\tilde{x})\| \cdot \|(D\tilde{g}^{m_0}\tilde{E}_{u}^g(\tilde{x}))^{-1}\| \leq \lambda_1
\]
and
\[
\int_{W_0} \log \|Dg^{m_0}\tilde{E}_{s}^g\| d\mu_0 \geq \log \lambda_2.
\]
Since \( pnm_0 \in W_0 \) for \( n \in \mathbb{Z} \), we have \( \tilde{p} \in W(\tilde{g}) \). Thus, by (6.4) and the definition of \( \mu_0 \),
\[
\prod_{j=0}^{m-1} \|D\tilde{g}^{m_0}\tilde{E}_{s}^g(\tilde{g}^{m_0j}(\tilde{p}))\| \geq \lambda_2^m.
\]
From (6.3) it follows that
\[
\|D\tilde{g}^{-m_0}\tilde{E}_{u}^g(\tilde{p})\| \leq \prod_{i=0}^{m-1} \|D\tilde{g}^{-m_0}\tilde{E}_{u}^g(\tilde{g}^{m_0i}(\tilde{p}))\|
\leq \prod_{i=0}^{m-1} \lambda_1 \|D\tilde{g}^{m_0}\tilde{E}_{s}^g(\tilde{g}^{m_0i}(\tilde{p}))\|^{-1}
\leq (\lambda_1/\lambda_2)^m < 1,
\]
and so \( \mathcal{P}^0(\tilde{E}_{s}^u(\tilde{p})) \subset E^u(p_0) \) where \( \mathcal{P}^0 \) is defined as in (1.4). This implies that \( \dim E^s(\tilde{p}) \leq i_0 + 1 \).

If \( \dim E^s(p_0) = i_0 + 1 \), then we have \( \dim \tilde{E}_{s}^u(\tilde{p}) = \dim E^u(p_0) \), and so \( \mathcal{P}^0(\tilde{E}_{s}^u(\tilde{p})) = E^u(p_0) \). Thus it is easily checked that \( \tilde{E}_{s}^u(p_0) = E^s(p_0) \) since \( p_0 \) is hyperbolic and \( \tilde{E}_{s}^u(p_0) \) is \( D_{p_0}g^{m_0} \)-invariant.
By (6.5),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Dg^{m_0} | E^s(p_{m_0j}) \| = \lim_{n \to \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \| Dg^{m_0} | \tilde{E}_g^{s}(g^{m_0j}(\tilde{p})) \| \geq \log \lambda_2,
\]
which contradicts (4.3). Therefore, \( \dim E^s(p_0) \leq i_0 \).

If \( \mu \) is not ergodic, then by using the ergodic decomposition theorem we can check that \( \mu(\Lambda(i_0)) > 0 \) (cf. for the proof, see [12, Theorem I.6]).

**Lemma 6.3.** Let \( f \in \mathcal{P}(M) \) and \( 0 \leq i_0 \leq \dim M - 2 \). If
\[
\int_{\text{cl}(I_{i_0+1}(f))} \log \| Df^{m_0} | \tilde{E}_g^{s} \|\, d\mu < 0
\]
for \( \mu \in \mathcal{M}_e(f^{m_0}|\text{cl}(I_{i_0+1}(f))) \), then \( \tilde{E}_g^{s} \) is contracting.

The proof of Lemma 6.3 is very similar to that of [12, Lemma I.5], and so we omit it.

Lemmas 6.1, 6.3 and 5.1(b) yield Proposition 3 as follows: suppose that \( \Lambda(i_0) \cap \text{cl}(I_{i_0+1}(f)) = \emptyset \). Then \( \mu(\Lambda(i_0)) = 0 \) for \( \mu \in \mathcal{M}_e(f^{m_0}|\text{cl}(I_{i_0+1}(f))) \), and by Lemma 6.1,
\[
\int_{\text{cl}(I_{i_0+1}(f))} \log \| Df^{m_0} | \tilde{E}_g^{s} \|\, d\mu \leq \log \lambda_0 < 0
\]
for \( \mu \in \mathcal{M}(f^{m_0}|\text{cl}(I_{i_0+1}(f))) \). Therefore \( \text{cl}(I_{i_0+1}(f)) \) is hyperbolic by Lemmas 6.3 and 5.1(b). The proof of Proposition 3 is complete.

**7. Proof of Proposition 4(a).** Before starting the proof we notice that if \( f \) is a diffeomorphism, then the inverse limit system of \( (M, f) \) equals the original system \( (M, f) \), and thus all the results for the inverse limit system can be transferred to the original system.

To show Proposition 4(a) we prepare the following two lemmas.

**Lemma 7.1.** Let \( f \in \text{int} \mathcal{P}(M) \) and let \( \Lambda(i_0) \) be as in (2.2) for \( f \) and \( 0 \leq i_0 \leq \dim M - 2 \). Let \( \mathcal{U}(f) \subset \text{int} \mathcal{P}(M) \) be a connected neighborhood of \( f \). Suppose that \( \Lambda(i_0) \) is hyperbolic and \( g \in \mathcal{U}(f) \) satisfies \( g = f \) in a neighborhood of \( \Lambda(i_0) \). Then \( g \) has no cycles in \( \Lambda(i_0) \).

For the proof of Lemma 7.1 we need the following:

**Lemma 7.2.** Let \( g \in C^1(M) \) and \( p \in M \) be a hyperbolic fixed point of \( g \). Suppose that there is \( \tilde{x} = (x_n) \in M_g \) satisfying the following:
(1) \( d(x_n, p) \to 0, \ d(x_{-n}, p) \to 0 \ (n \to \infty) \),

(2) \( D_{x_{-n}} g^{2n}(T_{x_{-n}} W^u_\varepsilon(\hat{p}, g)) + T_{x_n} W^s_\varepsilon(\hat{p}, g) = T_{x_n} M \) for \( n > 0 \) large enough, where \( \hat{p} = (\ldots, p, p, p, \ldots) \in \text{Per}(g)_g \) and \( W^s_\varepsilon(\hat{p}, g) \) (\( \sigma = s, u \)) is as in (1.5).

Then for every neighborhood \( U(x_0) \) of \( x_0 \) there is a hyperbolic periodic point \( q \) such that \( \dim E^s(q) = \dim E^s(p) \), where \( E^s(p) \) is the subspace of \( T_p M \) as in (1.1).

Lemma 7.2 was proved in [24] and [16, Appendix] for diffeomorphisms and extended in [25, Theorem 4.2] to differentiable maps.

**Proof of Lemma 7.1.** Let \( \mathcal{U}(f) \) and \( g \in \mathcal{U}(f) \) satisfy the assumptions of Lemma 7.1. Notice that \( \Lambda(i_0) \) is an isolated hyperbolic set of \( g \). Suppose that \( \Lambda(i_0) \) has a cycle for \( g \). By using the techniques described in [17, Theorem, p. 221] there exist \( h \in \mathcal{U}(f) \), \( p \in \Lambda(i_0) \) and \( \hat{x} = (x_n) \in M_h \) satisfying the assumptions (1) and (2) of Lemma 7.2 and \( h = g = f \) on some neighborhood of \( \Lambda(i_0) \). Then it follows from Lemma 7.2 that \( \#I^n_i(f) < \#I^n_i(h) \) for some \( 0 \leq i \leq i_0 \) and \( n > 0 \). This contradicts Lemma 3.2. ■

Let \( f \in \mathcal{F}(M) \). Since \( \text{cl}(I_0(f)) \) is hyperbolic by Proposition 2(b), it is isolated and can be written as a finite disjoint union \( \text{cl}(I_0(f)) = \Lambda_1 \cup \ldots \cup \Lambda_s \) of basic sets \( \Lambda_i \). Since \( TM[\text{cl}(I_0(f))]_f \) is expanding, there exist \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) such that for \( 1 \leq a \leq s \),

\[
(7.1) \quad 
\begin{align*}
(\text{i}) & \ W^u_{3\varepsilon}(\hat{x}, f) = B_{3\varepsilon}(x_0) \ (\hat{x} = (x_n) \in (\Lambda_a)_f), \\
(\text{ii}) & \text{if } \hat{x} = (x_n) \in (\Lambda_a)_f \text{ and } y \in B_{3\varepsilon}(x_0), \text{ then there is a unique point } y_{-1} \in B_{3\varepsilon}(x_{-1}) \text{ such that } f(y_{-1}) = y, \\
(\text{iii}) & \ d(x, y) \leq \lambda d(f(x), f(y)) \ (x, y \in B_{3\varepsilon}(\Lambda_a)), \\
(\text{iv}) & \ (\Lambda_a)_f = \{\hat{y} = (y_n) \in M_f : y_n \in B_{3\varepsilon}(\Lambda_a), \ n \geq 0\}.
\end{align*}
\]

Choose \( 0 < \delta_1 < \varepsilon - \varepsilon \lambda \) such that if \( d(x, y) \leq \delta_1 \ (x, y \in M) \) then \( d(f(x), f(y)) \leq \varepsilon \). It is easily checked that for every connected neighborhood \( \mathcal{U}(f) \) of \( f \) contained in \( \text{int} \mathcal{P}(M) \) there is \( 0 < \delta_2 < \delta_1 \) such that if \( d(x, z) \leq \delta_2 \ (x, z \in M) \) then we can construct a diffeomorphism \( \varphi : M \to M \) satisfying

\[
(7.2) \quad 
\begin{align*}
(\text{i}) & \ \varphi(z) = x, \\
(\text{ii}) & \ \{y \in M : \varphi(y) \neq y\} \subset B_{\delta_1}(z), \\
(\text{iii}) & \ f \circ \varphi \in \mathcal{U}(f).
\end{align*}
\]

From the properties of differentiable maps belonging to \( \mathcal{F}(M) \) we have

\[ \{\text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f)\} \cap S(f) = \emptyset. \]
Since \( \#I_{\dim M}(f) < \infty \) by Proposition 2(a), \( \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \) is closed. Thus there is 0 < \( \delta_3 < \delta_2 \) such that if \( x, y \in M \) satisfy \( d(x, y) \leq \delta_3 \), then for every point \( x_{-1} \in f^{-1}(x) \) with \( x_{-1} \in \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \) there exists a unique \( y_{-1} \in f^{-1}(y) \) satisfying \( d(x_{-1}, y_{-1}) \leq \delta_2 \).

If Proposition 4(a) is false, we have

\[
\Lambda_a \cap \text{cl}(I_i(f)) \neq \emptyset
\]

for some 1 \( \leq a \leq s \) and 0 < \( i \leq \dim M \). Proposition 2 ensures that \( \text{cl}(I_0(f)) \setminus I_{\dim M}(f) = \emptyset \), and so \( i \neq \dim M \). Choose \( x \in \Lambda_a, p \in I_i(f) \) with \( d(x, p) \leq \delta_3 \) and a periodic point \( \tilde{p} \in I_i(f) \) with \( p_0 = p \). By (7.1)(iv) there is 0 < \( n < g(p, f) \) such that \( p_{-j} \in B_{2\varepsilon}(\Lambda_a) \) (0 < \( j \leq n - 1 \)) and \( p_{-n} \notin B_{2\varepsilon}(\Lambda_a) \). Then for 0 \( \leq j \leq n - 1 \) there is \( x_{-j} \in \Lambda_a \) such that

\[
(f(x_{-j}) = x_{-j+1} \quad \text{and} \quad d(x_{-j}, p_{-j}) \leq \delta_3.
\]

Indeed, there is a unique \( x_{-1} \in f^{-1}(x) \) such that \( d(x_{-1}, p_{-1}) \leq \delta_2 < \varepsilon \) because \( d(x, p) \leq \delta_3 \) and \( p_{-1} \in \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \). Obviously, \( x_{-1} \in B_{3\varepsilon}(\Lambda_a) \) since \( p_{-1} \in B_{2\varepsilon}(\Lambda_a) \). By (7.1)(i) we have \( x_{-1} \in W^s(\Lambda_a, f) \cap W^u(\Lambda_a, f) \). Since \( \Lambda_a \) has no cycles by Lemma 7.1, we have \( x_{-1} \in \Lambda_a \). By (7.1)(ii), (iii),

\[
d(x_{-1}, p_{-1}) \leq \lambda d(x, p) \leq \delta_3.
\]

Continuing in this fashion we obtain (7.3).

Since \( d(x_{-(n-1)}, p_{-(n-1)}) \leq \delta_3 \) (by (7.3)) and \( p_{-n} \in \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \), we can find a unique point

\[
x_{-n} \in f^{-1}(x_{-(n-1)}) \subset f^{-1}(\Lambda_a)
\]

such that \( d(x_{-n}, p_{-n}) \leq \delta_2 \). By (7.2) there is a diffeomorphism \( \varphi : M \to M \) such that

(i) \( \varphi(p_{-n}) = x_{-n} \),
(ii) \( \{y \in M : \varphi(y) \neq y\} \subset B_{\delta_1}(p_{-n}) \),
(iii) \( f \circ \varphi \in \mathcal{U}(f) \).

For simplicity we write \( g = f \circ \varphi \). Obviously

\[
g(y) = f(y) \quad (y \in M \setminus B_{\delta_1}(p_{-n})),
\]

\[
g(p_{-n}) = f \circ \varphi(p_{-n}) = f(x_{-n}) = x_{-(n-1)} \in \Lambda_a.
\]

Since \( p_{-n} \notin \Lambda_a \) and \( g^i(p_{-n}) \in \Lambda_a \) for \( i > 0 \), we have

\[
p_{-n} \in W^s(\Lambda_a, g) \setminus \Lambda_a.
\]

If we establish that

\[
p_{-n} \in W^u(\Lambda_a, g) \setminus \Lambda_a,
\]
then $A_a$ has a 1-cycle, that is, $p_{-n} \in \{W^u(A_a, g) \setminus A_a\} \cap \{W^u(A_a, g) \setminus A_a\}$. This contradicts Lemma 7.1. Hence for $1 \leq a \leq s$,

$$\dim M \geq \sum_{i=1}^d \dim \cl(I_i(f)) = 0.$$ 

This shows Proposition 4(a).

Thus it only remains to prove (7.4). Since $p_{-n} \notin B_{2\varepsilon}(A_a)$ and $p_0 = p_{g(p,f)} \in B_{\delta^3}(A_a) \subset B_{2\varepsilon}(A_a)$, there is $n + 1 \leq m \leq g(p,f)$ such that $p_{-j} \notin B_{2\varepsilon}(A_a)$ for $n \leq j \leq m - 1$, and $p_{-m} \in B_{2\varepsilon}(A_a)$. Then $d(p_{-n}, p_{-j}) > \delta_1$ for $n + 1 \leq j \leq m$.

Indeed, if there is $n + 1 \leq j \leq m$ such that $d(p_{-n}, p_{-j}) \leq \delta_1$, then

$$d(p_{-(n-1)}, p_{-(j-1)}) = d(f(p_{-n}), f(p_{-j})) \leq \varepsilon.$$ 

Since $p_{-(n-1)} \in B_{\delta^3}(A_a)$ by (7.3), we have $p_{-(j-1)} \in B_{2\varepsilon}(A_a)$, which contradicts the choice of $m$.

Thus $g^j(p_{-m}) \notin B_{\delta^1}(p_{-n})$ for $0 \leq j \leq m - n - 1$, and so

$$g^{m-n}(p_{-m}) = p_{-n}.$$ 

Since $p_{-m} \in B_{2\varepsilon}(A_a)$, by (7.1)(i)-(iii) there is $\tilde{q} \in M_f$ with $q_0 = p_{-m}$ such that

$$d(q_{-j}, A_a) \leq \lambda^j d(p_{-m}, A_a) \leq 2\varepsilon \lambda^j \leq 2\varepsilon \lambda \quad (j \geq 1)$$

where $d(q, A) = \min_{x \in A} d(q, x)$ for $q \in M$ and a closed subset $A$. Then

$$d(q_{-j}, p_{-n}) \geq d(p_{-n}, A_a) - d(q_{-j}, A_a) > 2(\varepsilon - \varepsilon \lambda) > \delta_1,$$

and so $q_{-j} \notin B_{\delta^1}(p_{-n})$ ($j \geq 1$). Put

$$p'_j = \begin{cases} 
  g^j(p_{-n}) & \text{if } j \geq 0, \\
  g^{m-n+j}(p_{-m}) & \text{if } -m + n \leq j \leq -1, \\
  q_{m-n+j} & \text{if } j \leq -m + n - 1.
\end{cases}$$

Then $(p'_j) \in M_g$ and $d(p'_{-j}, A_a) \to 0$ as $j \to \infty$. This implies that $p_{-n} = p'_0 \in W^u(A_a, g)$, and (7.4) holds since $p_{-n} \notin A_a$.

8. Proof of Proposition 4(b). Let $f \in \mathcal{F}(M)$ and $A(i_0)$ be as in the statement of Proposition 4(b). Then $A(i_0)$ is hyperbolic and isolated by Lemma 7.1. Thus $A(i_0)$ splits into a union $A_1 \cup \ldots \cup A_s$ of basic sets. Fix $\varepsilon_0 > 0$. For $1 \leq a \leq s$ we define

$$V^+_a = \bigcup \{W^s_{\varepsilon_0}(\tilde{x}, f) : \tilde{x} \in (A_a)_f\}, \quad V^-_a = \bigcup \{W^u_{\varepsilon_0}(\tilde{x}, f) : \tilde{x} \in (A_a)_f\}.$$ 

Fix $0 < r_0 < 1$ and $0 < \delta_0 < 1$. For $n \geq 0$ define

$$r_{n+1} = r_n^{1+\delta_0}, \quad V(r_n, A_a) = \{x \in M : d(x, V^+_a) \leq r_n, \ d(x, V^-_a) \leq r_n\}.$$ 

Then $V(r_n, A_a) \setminus A_a$ since $r_n \to 0$ as $n \to \infty$. 

\[\text{C}^1\text{-maps having hyperbolic periodic points}\]
Let $m \geq 0$ be an integer and $\xi = (x_0, x_{-1}, \ldots, x_{-m})$ be a finite sequence in $M$. We say that $\xi$ is a string if

$$f(x_{-j}) = x_{-j+1} \quad \text{for } 1 \leq j \leq m.$$ 

Notice that the notion of string described here is different from that of $\gamma$-string introduced at the beginning of §5. For convenience of notation we make no distinction between a string $\xi$ and a set $\{x_0, x_{-1}, \ldots, x_{-m}\}$.

Let $\xi = (x_0, \ldots, x_{-m})$ and $\eta = (y_0, \ldots, y_{-n})$ be strings ($0 \leq n \leq m$). Then $\eta$ is said to be a substring of $\xi$ if there is $0 \leq j \leq m - n$ such that $x_{-j-l} = y_{-l}$ for $0 \leq l \leq n$ (Figure 1(a)). If, in particular, $m = n$, then we have $\eta = \xi$.

![Fig. 1](image)

Let $\sigma$ be a substring of $\xi = (x_0, x_{-1}, \ldots, x_{-m})$ written as

$$\sigma = (x_{-l}, x_{-l-1}, \ldots, x_{-t+1}, x_{-t})$$

for some $0 < l \leq t < m$. If $\sigma$ satisfies

(a) $\sigma \subset V(r_0, \Lambda_a)$,
(b) $\sigma \cap V(r_{-n}, \Lambda_a) \neq \emptyset$,
(c) $x_{-t+1}, x_{-t-1} \not\in V(r_0, \Lambda_a)$,

then we say that $\sigma$ is a $(\xi, n; a)$-string. If $x_{-j} \in V(r_0, \Lambda_a)$ for some $1 \leq j \leq m - 1$, then there is a $(\xi, 0; a)$-string containing $x_{-j}$ if and only if $x_{-j_1}, x_{-j_2} \not\in V(r_0, \Lambda_a)$ for some $j_1$ and $j_2$ with $0 \leq j_1 < j < j_2 \leq m$.

For $(\xi, 0; a)$-strings

$$\sigma_1 = (x_{-l_1}, x_{-l_1-1}, \ldots, x_{-t_1+1}, x_{-t_1})$$
$$\sigma_2 = (x_{-l_2}, x_{-l_2-1}, \ldots, x_{-t_2+1}, x_{-t_2})$$

we introduce an order by

$$\sigma_1 < \sigma_2 \quad \text{if} \quad t_1 < l_2$$

(Figure 1(b)).
Since \( \{ \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \} \cap S(f) = \emptyset \) and \( \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \) is closed by Proposition 2(a), we can choose a compact neighborhood \( U_0 \) of \( \text{cl}(\text{Per}(f)) \setminus I_{\dim M}(f) \) satisfying \( U_0 \cap S(f) = \emptyset \). Hereafter \( U_0 \) is fixed.

Suppose that a string \( \xi \) contained in \( U_0 \) has the property that

(C) there exist \( (\xi, n + 1; a) \)-strings \( \sigma_1 \) and \( \sigma_2 \) with \( \sigma_1 < \sigma_2 \) satisfying

\[
\sigma \cap V(r_n, \Lambda_a) = \emptyset \quad \text{for every } (\xi, 0; a) \text{-string } \sigma \text{ with } \sigma_1 < \sigma < \sigma_2.
\]

If \( n \) is large enough, by using the condition (C) we can show ([11] and [15, Theorem A, p. 57] that there exists \( g C^1 \)-near \( f \) such that \( g = f \) in a neighborhood of \( \Lambda_a \) and \( \Lambda_a \) has a 1-cycle. However this is inconsistent with Lemma 7.1.

Thus (C) cannot happen when a string \( \xi \) satisfies \( \xi \subset U_0 \) and \( n \) is large enough.

To show Proposition 4(b) we derive a contradiction by proving that if

\[
\Lambda_a \cap \text{cl}(I_{t_0 + 1}(f)) \neq \emptyset
\]

for some \( 1 \leq a \leq s \), then there exists a string \( \xi \) satisfying the condition (C) for \( n > 0 \) large enough. To do that we prepare auxiliary results.

Since \( \Lambda_a \) \( (1 \leq a \leq s) \) has no homoclinic points by Lemma 7.1, we have the following:

**Lemma 8.1** [15, Proposition 4]. Let \( \{ \xi^k \} \) be a sequence of strings with \( \xi^k \subset U_0 \). Suppose that

1. if \( \xi^k = (x_0^k, x_1^k, \ldots, x_{m_k - 1}^k, x_m^k) \), then \( m_k \not\rightarrow \infty \) as \( k \rightarrow \infty \),
2. \( \mu_k = m_k^{-1} \sum_{i=1}^{m_k} \delta_{x_{i-1}^k} \) converges to \( \mu \in \mathcal{M}(f) \),
3. \( \mu(\Lambda_a) > 0 \) for some \( 1 \leq a \leq s \).

Then for \( N, K > 0 \) there exist integers \( n \geq N, k \geq K \) and a \( (\xi^k, n + 1; a) \)-string \( \sigma_1 \) such that \( \sigma \cap V(r_n, \Lambda_a) = \emptyset \) for every \( (\xi^k, 0; a) \)-string \( \sigma \neq \sigma_1 \).

Let \( \xi \) be a string and for \( 1 \leq a \leq s \) define

\[
N_a(\xi) = \min \{ n \geq 0 : \xi \cap V(r_{n+1}, \Lambda_a) = \emptyset \}.
\]

If a string \( \xi = (x_0, x_1, \ldots, x_m) \) satisfies

1. \( N_a(\xi) > 0 \),
2. \( x_0, x_m \not\in V(r_0, \Lambda_a) \),

then there exists a \( (\xi, N_a(\xi); a) \)-string.

**Lemma 8.2.** Let \( \xi^k = (x_{0}^{k}, \ldots, x_{m_{k}}^{k}) \) and \( \eta^k = (y_{0}^{k}, \ldots, y_{n_{k}}^{k}) \) be strings with \( \xi^k, \eta^k \subset U_0 \) for \( k > 0 \). Suppose that

1. \( x_{0}^{k}, x_{m_{k}}^{k}, y_{0}^{k}, y_{n_{k}}^{k} \not\in \bigcup_{c=1}^{s} V(r_0, \Lambda_c) \) for \( k > 0 \),
2. \( \eta^k \) is a substring of \( \xi^k \) for \( k > 0 \),
3. \( m_k \not\rightarrow \infty \) and \( n_k \not\rightarrow \infty \) as \( k \rightarrow \infty \).
(4) \( \mu_k^0 = m_k^{-1} \sum_{i=1}^{m_k} \delta_{x_i} \) converges to \( \mu^0 \) and \( \mu_k^1 = n_k^{-1} \sum_{i=1}^{n_k} \delta_{y_i} \) converges to \( \mu^1 \).

If there are \( 1 \leq a, b \leq s \) and \( L \geq 0 \) such that \( \mu^0(A_a) > 0 \) and

\[
\limsup_{k \to \infty} (N_a(\xi^k) - N_b(\eta^k)) \leq L,
\]

then \( \mu^1(A_b) > 0 \).

For the proof of Lemma 8.2 we need the following two lemmas:

**Lemma 8.3** [15, Proposition 1]. There exist \( 0 < \gamma < \lambda < 1 \) such that for \( 1 \leq a \leq s \) and \( x \in V(r_0, A_a) \),

1. \( \gamma d(f(x), V_a^+) \leq d(x, V_a^+) \),
2. \( d(x, V_a^+) \leq \lambda d(f(x), V_a^+) \),
3. there is \( y \in f^{-1}(x) \) such that \( \gamma d(y, V_a^-) \leq d(f(y), V_a^-) = d(x, V_a^-) \),
4. \( d(f(x), V_a^-) \leq \lambda d(x, V_a^-) \).

Let \( 0 < \gamma < \lambda < 1 \) be as in Lemma 8.3 and set

\[
C_{1,n} = \frac{\log r_n}{2 \log \gamma} \quad \text{and} \quad C_{2,n} = 2(1 + \delta_0) \frac{\log r_n}{\log \lambda}
\]

for \( n \geq 0 \).

**Lemma 8.4.** Let \( \xi \) be a string with \( \xi \subset U_0 \). For \( n \) large enough there is \( N_n > n \) such that for every \( (\xi, 0; a) \)-string \( \sigma \),

1. if \( \sigma \) is a \( (\xi, i; a) \)-string for some \( i \geq N_n \), then
   \[
   \#\{\sigma \cap V(r_n, A_a)\} \geq C_{1,n}(1 + \delta_0)^{i-n},
   \]
2. if \( \sigma \) is not a \( (\xi, i+1; a) \)-string for some \( i \geq N_n \), then
   \[
   \#\{\sigma \cap V(r_n, A_a)\} \leq C_{2,n}(1 + \delta_0)^{i-n}.
   \]

**Proof.** (1) follows easily from [15, Lemma 5(b)].

To obtain (2) it is enough to show that (2) holds when \( \sigma \cap V(r_n, A_a) \neq \emptyset \).

Let \( \xi = (x_0, \ldots, x_m) \) and \( \sigma = (x_{-k_1}, \ldots, x_{-k_2}) \). Then \( 0 < k_1 < k_2 < m \).

Since \( \sigma \cap V(r_n, A_a) \neq \emptyset \), there is \( k_1 < t \leq k_2 \) satisfying \( x_{-t} \in \sigma \cap V(r_n, A_a) \).

Choose the smallest integers \( 0 < l_1 < t \) and \( 0 < l_2 < m - t \) such that \( x_{-t+l_1+1} \notin V(r_n, A_a) \) and \( x_{-t-l_2-1} \notin V(r_n, A_a) \). Then

\[
(8.4) \quad \#\{\sigma \cap V(r_n, A_a)\} = l_1 + l_2.
\]

Indeed, since \( d(x_{-t+l_1+1}, V_a^-) \leq \lambda^{l_1+1} d(x_{-t}, V_a^-) \leq r_n \) by Lemma 8.3(4), we have

\[
d(x_{-t+l_1+1}, V_a^+) > r_n.
\]
By Lemma 8.3(2), for \( k_1 \leq j \leq t - l_1 - 1 \),

\[
d(x_{-j}, V_{a}^{+}) \geq (1/\lambda)^{(t - l_1 - 1) - j} d(x_{-(t - l_1 - 1 - j)}, V_{a}^{+})
\]

\[
\geq d(x_{-t + l_1 + 1}, V_{a}^{+}) > r_n.
\]

This implies that

\[
x_{-j} \not\in V(r_n, \Lambda_a) \quad (k_1 \leq j \leq t - l_1 - 1).
\]

Suppose that there is \( j_1 \) with \( t + l_2 + 1 < j_1 < k_2 \) such that \( x_{-j_1} \in V(r_n, \Lambda_a) \). Then we can find \( j_2 \) with \( t + l_2 + 1 \leq j_2 < j_1 \leq k_2 \) such that \( x_{-j_2} \not\in V(r_n, \Lambda_a) \). Thus,

\[
d(x_t, V_{a}^{+}) \geq (1/\lambda)^{-t + j_2} d(x_{-j_2}, V_{a}^{+}) > d(x_{-j_2}, V_{a}^{+}) > r_n,
\]

which contradicts \( x_{-t} \in V(r_n, \Lambda_a) \). That is, \( x_{-j} \not\in V(r_n, \Lambda_a) \) for \( t + l_2 + 1 \leq j \leq k_2 \). Therefore we have (8.4).

From [15, Lemma 5(a)] we have the inequality

\[
l_1 + l_2 \leq C_{2,n}(1 + \delta_0)^{i-n}.
\]

Therefore we have (2) by (8.4).

**Proof of Lemma 8.2.** Let \( \{\xi^k\}, \{\mu_k^0\} \) and \( \mu^0 \) be as in Lemma 8.2. Since \( \mu^0(\Lambda_a) > 0 \) and \( \text{int} \, V(r_n, \Lambda_a) \searrow \Lambda_a \) \((n \to \infty)\), we have

\[
0 < \mu^0(\Lambda_a) = \lim_{n \to \infty} \mu^0(\text{int} \, V(r_n, \Lambda_a))
\]

\[
\leq \lim_{n \to \infty} \lim_{k \to \infty} \mu^0_k(\text{int} \, V(r_n, \Lambda_a))
\]

\[
= \lim_{n \to \infty} \lim_{k \to \infty} \frac{1}{m_k} \sum_{i=1}^{m_k} \delta_{x_{-i}}(V(r_n, \Lambda_a))
\]

\[
= \lim_{n \to \infty} \lim_{k \to \infty} \frac{\#(\xi^k \cap V(r_n, \Lambda_a))}{m_k}.
\]

Thus,

\[
N_a(\xi^k) \to \infty \quad (k \to \infty),
\]

where \( N_a(\xi^k) \) is defined in (8.2). Without loss of generality we suppose that \( N_a(\xi^k) > 0 \) for \( k > 0 \). Then, by (1) of Lemma 8.2, \( \xi^k \) satisfies (8.3) and so there is a \( (\xi^k, N_a(\xi^k); a) \)-string, say \( \sigma^k(a) \), for \( k > 0 \).

First we prove that

\[
\lim_{n \to \infty} \lim_{k \to \infty} \sup \frac{\#(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k} > 0.
\]

To see this write

\[
\sigma^k(a) = (x_{-l_k}^k, \ldots, x_{-s_k}^k) \quad (0 < l_k < s_k < m_k)
\]

for \( k > 0 \). Then we have the two sequences \( \{l_k\} \) and \( \{m_k - s_k\} \).
If, in particular, \( \{l_k\} \) and \( \{m_k - s_k\} \) are bounded, then by (8.5) we have (8.7) as follows:

\[
0 < \lim_{n \to \infty} \liminf_{k \to \infty} \frac{\#(\xi^k \cap V(r_n, A_a))}{m_k}
\leq \lim_{n \to \infty} \limsup_{k \to \infty} \frac{\#(\xi^k \cap V(r_n, A_a))}{m_k}
\leq \lim_{n \to \infty} \limsup_{k \to \infty} \frac{l_k + \#(\sigma^k(a) \cap V(r_n, A_a)) + (m_k - s_k)}{m_k}
= \lim_{n \to \infty} \limsup_{k \to \infty} \frac{\#(\sigma^k(a) \cap V(r_n, A_a))}{m_k}.
\]

To conclude (8.7) for the case when either \( \{l_k\} \) or \( \{m_k - s_k\} \) is unbounded we divide the proof into the following three cases:

1. both \( \{l_k\} \) and \( \{m_k - s_k\} \) are unbounded,
2. \( \{l_k\} \) is unbounded, and \( \{m_k - s_k\} \) is bounded,
3. \( \{l_k\} \) is bounded, and \( \{m_k - s_k\} \) is unbounded.

Case (1): Suppose that \( \{l_k\} \) and \( \{m_k - s_k\} \) are increasing sequences, and put

\[
\xi^k_+ = (x^k_0, x^k_1, \ldots, x^k_{-l_k+1}), \quad \mu^+_k = \frac{1}{l_k - 1} \sum_{j=1}^{l_k-1} \delta_{x^k_{-j}},
\]

\[
\xi^k_- = (x^k_{-s_k-1}, x^k_{-s_k-2}, \ldots, x^k_{-m_k}), \quad \mu^-_k = \frac{1}{m_k - s_k - 1} \sum_{j=1}^{m_k-s_k-1} \delta_{x^k_{-s_k-1-j}}
\]

for \( k > 0 \). Then \( \xi^k_+, \xi^k_- \subset \xi^k \subset U_0 \) for \( k > 0 \). Since \( \xi^k = \xi^k_+ \cup \sigma^k(a) \cup \xi^k_- \), we have

\[
\frac{\#(\xi^k \cap V(r_n, A_a))}{m_k}
= \frac{1}{m_k} \{\#(\xi^k_+ \cap V(r_n, A_a)) + \#(\sigma^k(a) \cap V(r_n, A_a)) + \#(\xi^k_- \cap V(r_n, A_a))\}
= \frac{1}{m_k} \sum_{j=1}^{l_k-1} \delta_{x^k_{-j}}(V(r_n, A_a)) + \frac{\#(\sigma^k(a) \cap V(r_n, A_a))}{m_k}
+ \frac{1}{m_k} \sum_{j=1}^{m_k-s_k-1} \delta_{x^k_{-s_k-1-j}}(V(r_n, A_a))
\]
Thus for every \((8.8)\) by \((8.9)\),

\[
\text{respectively, by (8.5),}
\]

\[
\text{which yields}
\]

\[
\text{To obtain (8.7) it suffices to show that}
\]

\[
\mu^+(A_a) = 0 \text{ and } \mu^-(A_a) = 0.
\]

Suppose that \(\mu^+(A_a) > 0\). Since \(\{\xi^+_k\}, \{\mu^+_k\}\) and \(\mu^+\) satisfy the assumptions (1)–(3) of Lemma 8.1, for \(N > 0\) large enough there exist \(n > N\), \(k > 0\) and a \((\xi^+_k, n + 1; a)\)-string \(\overline{\sigma}\) such that

\[
(8.8) \quad \sigma \cap V(r_n, A_a) = \emptyset
\]

for every \((\xi^+_k, 0; a)\)-string \(\sigma \neq \overline{\sigma}\). Since \(\overline{\sigma} \subset \xi^+_k \subset \xi^k\), \(\overline{\sigma}\) is a \((\xi^k, n+1; a)\)-string. Thus

\[
\overline{\sigma} \subset V(r_0, A_a), \quad \emptyset \neq \overline{\sigma} \cap V(r_{n+1}, A_a) \subset \xi^k \cap V(r_{n+1}, A_a),
\]

which yields \(N_a(\xi^k) \geq n+1\). Since \(\sigma^k(a)\) is a \((\xi^k, N_a(\xi^k); a)\)-string, we have

\[
(8.9) \quad \sigma^k(a) \cap V(r_{n+1}, A_a) \neq \emptyset.
\]

Define a string as

\[
\overline{\xi}^k = (x_0^k, a_{-1}^k, \ldots, x_{l_k}^k, \ldots, x_{s_k}^k, \ldots, x_{s_k-1}^k).
\]

Then \(\xi^k_k \subset \overline{\xi}^k\). Since \(\overline{\sigma}\) is a \((\xi^k_k, n + 1; a)\)-string, it is a \((\overline{\xi}^k, n + 1; a)\)-string. By \((8.9)\), \(\sigma^k(a)\) is a \((\overline{\xi}^k, n + 1; a)\)-string. Thus, by \((8.8)\) we have

\[
\sigma \cap V(r_n, A_a) = \emptyset
\]

for every \((\overline{\xi}_k^k; 0; a)\)-string \(\sigma\) with \(\overline{\sigma} < \sigma < \sigma^k(a)\). This implies the condition (C). Since \(n\) is large enough, we have a contradiction, and so \(\mu^+(A_a) > 0\) cannot happen. Similarly we have \(\mu^-(A_a) = 0\). Therefore \((8.7)\) holds in case (1).

In a similar way we obtain \((8.7)\) for cases (2) and (3).

To complete the proof of Lemma 8.2 let \(\{\eta^k\}\) be as in the statement of the lemma. Since \(\limsup_{k \to \infty} (N_a(\xi^k) - N_b(\eta^k)) \leq L\), we have

\[
(8.10) \quad N_b(\eta^k) \geq N_a(\xi^k) - L
\]

for \(k\) large enough, and so \(\lim_{k \to \infty} N_b(\eta^k) = \infty\) by \((8.6)\). Without loss of generality we suppose that \(N_b(\eta^k) > 0\) for \(k > 0\). Then \(\eta^k\) satisfies \((8.3)\) by
(1) of Lemma 8.2. Thus we can choose a \((\eta^k, N_b(\eta^k); b)\)-string, say \(\tau^k(b)\), for \(k > 0\). For \(\eta^k\) define

\[\mu^1_k = \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y^k_i}\]

and suppose that \(\mu^1_k \to \mu^1\) as \(k \to \infty\). Then

\[
\mu^1(A_b) \geq \lim_{n \to \infty} \limsup_{k \to \infty} \mu^1_k(V(r_n, A_b))
\]

\[
= \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y^k_i}(V(r_n, A_b))
\]

\[
= \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{n_k} \#(\eta^k \cap V(r_n, A_b))
\]

\[
\geq \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{n_k} \#(\tau^k(b) \cap V(r_n, A_b)).
\]

For \(n\) large enough let \(N_n\) be as in Lemma 8.4. Since \(N_a(\xi^k) \to \infty\) and \(N_b(\eta^k) \to \infty\) as \(k \to \infty\), we have \(N_a(\xi^k) \geq N_n\) and \(N_b(\eta^k) \geq N_n\) for \(k\) large enough. Then, by Lemma 8.4(1),

\[
\#(\tau^k(b) \cap V(r_n, A_b)) \geq C_{1,n}(1 + \delta_0)^{N_b(\eta^k) - n}.
\]

On the other hand, by Lemma 8.4(2) and (8.2),

\[
\#(\sigma^k(a) \cap V(r_n, A_a)) \leq C_{2,n}(1 + \delta_0)^{N_a(\xi^k) - n}.
\]

Since \(\eta^k = (y^k_0, \ldots, y^k_{n_k})\) is a substring of \(\xi^k = (x^k_0, \ldots, x^k_{m_k})\), we have \(n_k \leq m_k\) for \(k > 0\). Therefore, by (8.10),

\[
\frac{1}{n_k} \#(\tau^k(b) \cap V(r_n, A_b)) \geq \frac{1}{n_k} C_{1,n}(1 + \delta_0)^{N_b(\eta^k) - n}
\]

\[
\geq \frac{1}{m_k} C_{1,n}(1 + \delta_0)^{N_a(\xi^k) - n - L - n}
\]

\[
\geq \frac{C_{1,n}}{C_{2,n}} (1 + \delta_0)^{-L} \frac{\#(\sigma^k(a) \cap V(r_n, A_a))}{m_k}
\]

Since

\[0 < C_{1,n}/C_{2,n} = \frac{\log \lambda}{4(1 + \delta_0) \log \gamma} < 1 \quad (n \geq 0),\]

by using (8.7), (8.11) and (8.12) we have the conclusion of Lemma 8.2:

\[
\mu^1(A_b) \geq \lim_{n \to \infty} \limsup_{k \to \infty} \frac{C_{1,n}(1 + \delta_0)^{-L} \#(\sigma^k(a) \cap V(r_n, A_a))}{m_k} > 0. \quad \blacksquare
\]

**Lemma 8.5** [1, (3.15)]. Suppose that there exist \(\overline{\pi}, \overline{N} > 0\) such that for strings \(\xi = (x_0, \ldots, x_{-m})\) and \(\eta = (y_0, \ldots, y_{-m'})\) there are a \((\xi, 0; a)\)-string \(\sigma = (x_{-l}, \ldots, x_{-s})\) and an integer \(t\) with \(l \leq t \leq s\) such that
(1) $x_{-t} \in V(r_N, A_a)$ for some $N \geq \overline{N}$,
(2) $d(x_{-t+j}, y_{-t+j}) \leq r_0/2$ ($0 \leq j \leq t-l$) for some $t'$ with $t-l \leq t' \leq m'$.
Then there is $t' - t + l \leq t_0 \leq t'$ such that
(a) $y_{-t_0} \in V(r_{N-\overline{N}}, A_a)$,
(b) $y_{-j} \in V(r_0, A_a)$ ($t_0 \leq j \leq t'$).

Proof. The proof given in [1] was only done for diffeomorphisms. For completeness we give the full proof.

Since $V(r_n, A_a) \setminus A_a$ as $n \to \infty$, there is a sufficiently large integer $\overline{N} > 0$ satisfying

$$V(r_n, A_a) \subset U(A_a, r_0/2) \quad \text{for } n \geq \overline{N}.$$  

For a string $\xi$ let $\sigma = (x_{-t}, \ldots, x_{-t}, \ldots, x_{-s})$ be a $(\xi, 0; a)$-string satisfying the condition (1) of the lemma. Since $x_{-t} \in V(r_0, A_a)$, by Lemma 8.3(4) we have

$$d(x_{-t+1}, V_a^-) \leq d(x_{-t}, V_a^-) \leq r_0.$$  

By the definition of a $(\xi, 0; a)$-string we have $x_{-t+1} \notin V(r_0, A_a)$, and so

$$d(x_{-t+1}, V_a^+) > r_0.$$  

Thus there is $\hat{t}$ with $l \leq \hat{t} \leq t$ such that

$$d(x_{-t+j}, V_a^+) \leq r_0/2 \quad \text{for } 0 \leq j \leq t - \hat{t}, \quad d(x_{-\hat{t}+1}, V_a^+) > r_0/2.$$  

Since $x_{-t} \in V(r_N, A_a)$, by Lemma 8.3(1),

$$r_0^{(1+\delta_0)^N} = r_N \geq d(x_{-t}, V_a^+) \geq \gamma^{t-\hat{t}+1} d(f^{t-\hat{t}+1}(x_{-t}), V_a^+)$$  

$$= \gamma^{t-\hat{t}+1} d(x_{-\hat{t}+1}, V_a^+) > \gamma^{t-\hat{t}+1} r_0/2,$$

and so

$$t - \hat{t} + 1 > (\log r_0/\log \gamma)(1 + \delta_0)^N - \log(r_0/2)/\log \gamma$$

$$= \frac{\log r_0 - \log(r_0/2)/(1 + \delta_0)^N}{\log \gamma} (1 + \delta_0)^N.$$  

Since $\overline{N}$ is large enough and $N \geq \overline{N}$, we can suppose that

$$t - \hat{t} > \frac{\log r_0}{2 \log \gamma} (1 + \delta_0)^N$$  

(notice that $\overline{N}$ is independent of $\xi$ and $\sigma$).

Let $\eta = (y_0, \ldots, y_{-m'})$ be a string satisfying the condition (2) of Lemma 8.5. Since $x_{-t+j} \in V(r_0, A_a)$ for $0 \leq j \leq t - l$ and $x_{-t} \in U(A_a, r_0/2)$, by
Lemma 8.3(4) we have
\[ d(x_{-t+j}, V_{a}^{-}) \leq \lambda^j d(x_{-t}, V_{a}^{-}) \leq \lambda^j d(x_{-t}, A_a) \leq r_0/2 \quad (0 \leq j \leq t - l), \]
and so, for \(0 \leq j \leq t - \hat{t},\)
\[ d(y_{-t'+j}, V_{a}^{-}) \leq d(y_{-t'+j}, x_{-t+j}) + d(x_{-t+j}, V_{a}^{-}) \leq r_0. \]
On the other hand, by (8.13),
\[ d(y_{-t'+j}, V_{a}^{+}) \leq d(y_{-t'+j}, x_{-t+j}) + d(x_{-t+j}, V_{a}^{+}) \leq r_0 \quad (0 \leq j \leq t - \hat{t}). \]
Thus we have
\[ (8.15) \quad y_{-t'+j} \in V(r_0, A_a) \quad (0 \leq j \leq t - \hat{t}). \]
Put \(t_0 = t' - [(t - \hat{t})/2].\) We show that \(t_0\) satisfies assertions (a) and (b) of Lemma 8.5. Since
\[ t' \geq t_0 \geq t' - (t - \hat{t}) \geq t' - t + l, \]
(b) follows from (8.15).
To see (a) put
\[ \bar{n} = \left\lfloor \frac{\log C_1' - \log C_2'}{\log(1 + \delta_0)} \right\rfloor + 1 \]
where
\[ C_1' = \frac{2 \log r_0}{\log \lambda} \quad \text{and} \quad C_2' = \frac{\log r_0}{2 \log r_0}. \]
Then \(y_{-t_0} \in V(r_{N-\bar{n}}, A_a).\) Indeed, put \(j_0 = [(t - \hat{t})/2].\) Then by (8.15) and Lemma 8.3 we have
\[ (8.16) \quad r_0 \geq d(y_{-t'}, V_{a}^{-}) \geq \lambda^{-j_0} d(y_{-t'+j_0}, V_{a}^{-}) = \lambda^{-j_0} d(y_{-t_0}, V_{a}^{-}), \]
\[ r_0 \geq d(y_{-t'+2j_0}, V_{a}^{+}) \geq \lambda^{-j_0} d(y_{-t'+j_0}, V_{a}^{+}) = \lambda^{-j_0} d(y_{-t_0}, V_{a}^{+}). \]
Suppose that \(y_{-t_0} \not\in V(r_n, A_a)\) for \(n = N - \bar{n}.\) Then
\[ \text{either} \quad d(y_{-t_0}, V_{a}^{+}) > r_n, \quad \text{or} \quad d(y_{-t_0}, V_{a}^{-}) > r_n. \]
In any case, by (8.16) we have \(r_0 > \lambda^{-j_0} r_n = \lambda^{-j_0} r_0^{(1+\delta_0)^n},\) and so
\[ j_0 < (\log r_0/\log \lambda)((1 + \delta_0)^n - 1). \]
Then
\[ t - \hat{t} \leq 2(j_0 + 1) < 2\left(\frac{\log r_0/\log \lambda}{(1 + \delta_0)^n - 1}\right) + 2 \leq C_1'(1 + \delta_0)^n. \]
By (8.14) we have \(C_2'(1 + \delta_0)^N \leq t - \hat{t} < C_1'(1 + \delta_0)^n,\) and so
\[ N - n < \frac{\log C_1' - \log C_2'}{\log(1 + \delta_0)} < \bar{n} = N - n. \]
This is a contradiction. Therefore Lemma 8.5(a) holds. ■
Proof of Proposition 4(b). Let $f \in \mathcal{F}(M)$. As mentioned before $\Lambda(i_0) = \bigcup_{i=1}^{i_0} \text{cl}(I_i(f))$ and $\Lambda(i_0)$ splits into a union

$$\Lambda(i_0) = \Lambda_1 \cup \ldots \cup \Lambda_s$$

of basic sets $\Lambda_i$.

Our aim is to conclude that $\Lambda(i_0) \cap \text{cl}(I_{i_0+1}(f)) = \emptyset$. Suppose that

(*)

$$\Lambda_a \cap \text{cl}(I_{i_0+1}(f)) \neq \emptyset$$

for some $1 \leq a \leq s$. Then there is a sequence $\{p^k\} \subset I_{i_0+1}(f)$ of periodic points such that $d(p^k, \Lambda_a) \to 0$ as $k \to \infty$. Let $m_k = g(p^k, f)$ be the period of $p^k$ for $k > 0$. Since $\Lambda(i_0) \cap I_{i_0+1}(f) = \emptyset$, the sequence $\{m_k : k > 0\}$ tends to infinity as $k \to \infty$. Notice that $m_0$ is not a member of $\{m_k : k > 0\}$. In fact, $m_0$ is the integer satisfying (4.1)--(4.4).

For simplicity we suppose that $p^k \in V(r_0, \Lambda_a)$ for $k > 0$. Since $\Lambda_a$ is isolated and $p^k \notin \Lambda_a$, for $k > 0$ we put

$$t_k = \min \{0 < t < m_k : f^t(p^k) \notin V(r_0, \Lambda_a)\}.$$ 

Choose a periodic orbit

$$\tilde{q}^k = (q^k_j) \in I_{i_0+1}(f)$$

with $q^k_0 = f^{t_k}(p^k)$ for $k > 0$ (Figure 2). Then $q^k_{-t_k} = p^k \in V(r_0, \Lambda_a)$ and $q^k_m \notin V(r_0, \Lambda_a)$. Define a sequence of strings

$$\xi^k = (q^k_0, q^k_{-1}, \ldots, q^k_{-m_k+1}, q^k_{-m_k})$$

for $k > 0$. Then each $\xi^k$ consists of a periodic orbit and

$$\xi^k \subset I_{i_0+1}(f) \subset U_0$$

where $U_0$ is the compact neighborhood defined before the condition (C).

For $k > 0$ we put

$$N(\xi^k) = \max \{N_b(\xi^k) : 1 \leq b \leq s\},$$

where $N_b(\xi^k)$ is defined in (8.2). For some $1 \leq b \leq s$ we can find a sequence $k'$ of integers such that $N(\xi^{k'}) = N_b(\xi^{k'})$. To simplify the notations suppose that for $k > 0$,

$$N_a(\xi^k) = N(\xi^k), \quad q^k_{-t_k} = p^k \in V(r_{N_a(\xi^k)}, \Lambda_a).$$

Since $d(p^k, \Lambda_a) \to 0$, we have

$$N(\xi^k) \to \infty$$

as $k \to \infty$. Thus we can suppose that $N(\xi^k)$ is large enough for $k > 0$ and $\{N(\xi^k)\}$ is an increasing sequence.

Since $q^k_{-t_k} \in V(r_0, \Lambda_a)$ and $q^k_{-m_k} \notin V(r_0, \Lambda_a)$, we put

$$s_k = \min \{t_k \leq s < m_k : q^k_{-s-1} \notin V(r_0, \Lambda_a)\} \quad (k > 0).$$
Combining the definitions of $t_k$ and $s_k$, for $k > 0$ we have

$$q^k_t \in V(r_0, A_a) \quad (1 \leq t \leq s_k).$$

Since $q^k_0, q^k_{-s_k-1} \notin V(r_0, A_a)$ and $0 < t_k \leq s_k$ for $k > 0$, by (8.20) we find that

$$(8.22) \quad \sigma^k = (q^k_{-1}, q^k_{-2}, \ldots, q^k_{-s_k+1}, q^k_{-s_k})$$

is a $(\xi^k, N(\xi^k); a)$-string (Figure 2). Notice that $\# \sigma^k \to \infty$ as $k \to \infty$ by (8.21) and Lemma 8.4(1), and so $s_k \to \infty$ as $k \to \infty$.

With the above preparations we shall deduce Proposition 4(b) through the nine claims below.

**Claim 1.** For $k > 0$ there is $s_k < j < m_k$ such that

$$q^k_{-j} \in V(r_{N(\xi^k)-1}, A_a).$$

**Proof.** If this is false, then there is $k > 0$ such that $\sigma \cap V(r_{N(\xi^k)-1}, A_a) = \emptyset$ for every $(\xi^k; 0; a)$-string $\sigma$ with $\sigma^k < \sigma$. Let $\zeta^k$ be a string

$$\zeta^k = (q^k_0, \ldots, q^k_{-m_k} = q^k_0, q^k_{-m_k-1}, \ldots, q^k_{-2m_k}),$$

and let

$$\tau^k = (q^k_{-m_k-1}, q^k_{-m_k-2}, \ldots, q^k_{-m_k-s_k})$$

be a $(\xi^k, N(\xi^k); a)$-string. Then $\tau^k \subset \zeta^k \subset U_0$ and $\xi^k$ is a substring of $\zeta^k$. Obviously, $\sigma^k$ and $\tau^k$ are $(\xi^k, N(\xi^k); a)$-strings, and $\sigma \cap V(r_{N(\xi^k)-1}, A_a) = \emptyset$ for every $(\xi^k; 0; a)$-string $\sigma$ with $\sigma^k < \sigma < \tau^k$. Therefore we have the condition (C). Since $N(\xi^k)$ is large enough, we have a contradiction as before. Thus we have Claim 1. \[\blacksquare\]
Fix an integer \( u \geq 1 \), and choose \( K_0(u) > 0 \) large enough satisfying \( N(\xi^k) > u \) for \( k \geq K_0(u) \). Since \( V(r_N(\xi^k)_{-1}, \Lambda_a) \subset V(r_N(\xi^k)_{-u}, \Lambda_a) \) for \( k \geq K_0(u) \), we put

\[
(8.23) \quad t_k(u) = \min\{s_k < j < m_k : q_j^k \in V(r_N(\xi^k)_{-u}, \Lambda_a)\} \quad (k \geq K_0(u)).
\]

This is well defined by Claim 1, and so choose a \((\xi^k, N(\xi^k) - u; a)\)-string

\[
(8.24) \quad \sigma_k(u) = (q_{-m_k(u)}, \ldots, q_{-t_k(u)}, \ldots, q_{-s_k(u)}) \quad (\subset V(r_0, \Lambda_a))
\]

for \( k \geq K_0(u) \) (Figure 2).

**Claim 2.** Under the above notations, for \( k \geq K_0(u) \) we have

1. \( s_k + 1 < m_k(u) < t_k(u) < s_k(u) < m_k \),
2. \( s_k(u) - m_k(u) \rightarrow \infty \) as \( k \rightarrow \infty \),
3. \( \{q_{-s_k-1}, q_{-s_k-2}; \ldots, q_{-m_k(u)+1}\} \cap V(r_N(\xi^k)_{-u}, \Lambda_a) = \emptyset \),
4. \( \sigma_k(u) \cap V(r_N(\xi^k)_{-u+1}, \Lambda_a) = \emptyset \).

**Proof.** (1) and (3) are clear. By (8.21) and Lemma 8.4(1) we have \( \|\sigma^k = s_k(u) - m_k(u) + 1 \rightarrow \infty \) as \( k \rightarrow \infty \), and so we have (2). If (4) is false, then \( \sigma^k(u) \) is a \((\xi^k, N(\xi^k) - u+1; a)\)-string and \( \sigma^k \) is a \((\xi^k, N(\xi^k) - u+1; a)\)-string.

By the definition of \( t_k(u) \) we deduce that \( \sigma \cap V(r_N(\xi^k)_{-u}, \Lambda_a) = \emptyset \) for every \((\xi^k, 0; a)\)-string \( \sigma \) with \( \sigma^k < \sigma < \sigma^k(u) \). This implies the condition (C). Since \( K_0(u) \) is large enough, \( N(\xi^k) - u \) is large enough for \( k \geq K_0(u) \). Thus we have a contradiction, and (4) is proved. \( \blacksquare \)

Let \( \lambda_0 \) and \( m_0 \) be the numbers described at the beginning of §4 and let the splitting \( TM|\text{cl}(I_{i_0+1}(f)) = E^{s}_{i_0+1} \oplus E^{u}_{i_0+1} \) be as in (5.1). For simplicity write \( E = E^{s}_{i_0+1} \) and \( F = E^{u}_{i_0+1} \). Then

\[
(8.25) \quad \|D\tilde{f}^{m_0}|E(\tilde{x})\| : \|D\tilde{f}^{-m_0}|F(\tilde{f}^{m_0}(\tilde{x}))\| \leq \lambda_0
\]

for \( \tilde{x} \in \text{cl}(I_{i_0+1}(f)) \). Let \( P^0 \) and \( \overline{P}^0 \) be as in (1.2) and (1.4). As mentioned in (5.2)(2) we have \( \overline{P}^0(E(\tilde{x})) = \overline{P}^0(E(\tilde{y})) \) when \( \tilde{x}, \tilde{y} \in \text{cl}(I_{i_0+1}(f)) \) satisfy \( P^0(\tilde{x}) = \overline{P}^0(\tilde{y}) \). Thus we write

\[
E(x_0) = \overline{P}^0(E(\tilde{x})) \subset T_{x_0}M
\]

for \( \tilde{x} = (x_i) \in \text{cl}(I_{i_0+1}(f)) \), and then \( \|Df|E(x_0)\| = \|D\tilde{f}|E(\tilde{x})\| \).

**Claim 3.** For \( \varepsilon > 0 \) there exist continuous families

\[
\{Z^s_\varepsilon(\tilde{x}, f^{m_0}) : \tilde{x} \in \text{cl}(I_{i_0+1}(f))\} \quad \text{and} \quad \{Z^u_\varepsilon(\tilde{x}, f^{m_0}) : \tilde{x} \in \text{cl}(I_{i_0+1}(f))\}
\]

of \( C^1\)-disks on \( M \) such that

(a) for \( \tilde{x} = (x_i) \in \text{cl}(I_{i_0+1}(f)) \) and \( \sigma = s, u \),

\[
x_0 \in Z^\sigma_\varepsilon(\tilde{x}, f^{m_0}) \subset B_\varepsilon(x_0),
\]
(b) for \( \tilde{x} = (x_i) \in \text{cl}(I_{i_0+1}(f))_f \),
\[
T_{x_0} Z^{s}_e(\tilde{x}, f^{m_0}) = E(x_0) \quad \text{and} \quad T_{\tilde{x}_0} Z^{u}_e(\tilde{x}, f^{m_0}) = \overline{P^0}(F(\tilde{x})) ,
\]
(c) there is \( 0 < \varepsilon' \leq \varepsilon \) such that
\[
f^{m_0}(Z^{\sigma}_e(\tilde{x}, f^{m_0})) \subset Z^{\sigma}_e(\tilde{f}^{m_0}(\tilde{x}), f^{m_0})
\]
for \( \tilde{x} \in \text{cl}(I_{i_0+1}(f))_f \) and \( \sigma = s, u \),
(d) there is \( \delta = \delta(\varepsilon) > 0 \) such that if \( \hat{d}(\tilde{x}, \tilde{y}) \leq \delta \) \( (\tilde{x}, \tilde{y} \in \text{cl}(I_{i_0+1}(f))_f) \) then
\[
Z^s_e(\tilde{x}, f^{m_0}) \cap Z^u_e(\tilde{y}, f^{m_0})
\]
is a one-point set and the intersection is transversal.

Proof. This follows from [9, Proposition 2.3] and [5, Theorem 5.1].

Fix \( \gamma_0 \) with \( \lambda_0 < \gamma_0 < 1 \). Then we have:

**Claim 4.** For fixed \( u \geq 1 \) let \( K_0(u) \) be as above. Then there exists \( K_1(u) > K_0(u) \) such that for \( k \geq K_1(u) \) there is \( l \) with \( 0 < l \leq \lfloor s_k(u)/m_0 \rfloor \) such that for \( 0 \leq r < l \),
\[
\prod_{t=r+1}^{l} \| Df^{m_0} | E(q^k_{m_0 t}) \| \leq \gamma_0^{l-r} .
\]

Proof. If the claim is false, then for some \( u \geq 1 \) there exist infinitely many \( k \geq K_0(u) \) such that
\[
(8.26) \quad \prod_{t=1}^{l} \| Df^{m_0} | E(q^k_{m_0 t}) \| > \gamma_0^l
\]
for \( l \) with \( 0 < l \leq \lfloor s_k(u)/m_0 \rfloor \). Without loss of generality we suppose that (8.26) holds for \( k > 0 \).

Define the Borel probability measures \( \mu_k \) by
\[
\mu_k = \frac{1}{\lfloor s_k(u)/m_0 \rfloor} \sum_{j=1}^{\lfloor s_k(u)/m_0 \rfloor} \delta_{q^k_{m_0 j}}.
\]
Then \( \mu_k \) converges to \( \mu \) belonging to \( \mathcal{M}(f^{m_0}|\text{cl}(I_{i_0+1}(f))) \) (take a subsequence if necessary). Since, by (8.26),
\[
\int_{\text{cl}(I_{i_0+1}(f))} \log \| Df^{m_0} | E \| \, d \mu = \lim_{k \to \infty} \int_{\text{cl}(I_{i_0+1}(f))} \log \| Df^{m_0} | E \| \, d \mu_k
\]
\[ = \lim_{k \to \infty} \frac{1}{[s_k(u)/m_0]} \log \prod_{j=1}^{[s_k(u)/m_0]} \|Df^{m_0}E(q_{m_0}^{j})\| \]
\[ \geq \log \gamma_0 > \log \lambda_0, \]
by Lemma 6.1 we have
\[ \mu(A(i_0)) > 0. \]

Let \( \xi^k (k > 0) \) be the sequence of strings in (8.18). For \( k > 0 \) define a substring of \( \xi^k \) as
\[ \tilde{\xi}^k = (q_0^k, q_{-1}^k, \ldots, q_{-s_k(u)}^k, q_{-s_k(u)-1}^k), \]
for \( \tilde{\xi}^k \) define
\[ \overline{\mu}_k = \frac{1}{s_k(u) + 1} \sum_{j=1}^{s_k(u)+1} \delta_{q_{-j}^k}, \]
and put
\[ V_n = \bigcup_{b=1}^{s} V(r_n, A_b) \quad (n \geq 0). \]
Since \( \overline{\mu}_k \) converges to \( \overline{\mu} \) and \( V_n \setminus A(i_0) \) as \( n \to \infty \), we have
\[ \mu(A(i_0)) \]
\[ \geq \lim_{n \to \infty} \limsup_{k \to \infty} \overline{\mu}_k(V_n) = \lim_{n \to \infty} \limsup_{k \to \infty} \frac{\#(\tilde{\xi}^k \cap V_n)}{s_k(u) + 1} \]
\[ \geq \lim_{n \to \infty} \limsup_{k \to \infty} \frac{1}{s_k(u) + 1} \#((q_{m_0}^k, q_{-2m_0}^k, \ldots, q_{-m_0}^k) \cap V_n) \]
\[ = \lim_{n \to \infty} \limsup_{k \to \infty} \frac{[s_k(u)/m_0]}{s_k(u) + 1} \mu_k(V_n) \]
\[ \geq \frac{1}{m_0} \lim_{n \to \infty} \liminf_{k \to \infty} \mu_k(\text{int } V_n) \geq \frac{1}{m_0} \mu(A(i_0)) > 0, \]
from which
\[ \overline{\mu}(A_b) > 0 \]
for some \( 1 \leq b \leq s. \)

For \( k > 0 \) define
\[ \hat{\xi}^k = (q_{-s_k-1}^k, q_{-s_k-2}^k, \ldots, q_{-s_k(u)-1}^k) \quad (\subset \tilde{\xi}^k), \]
\[ \hat{\mu}_k = \frac{1}{s_k(u) - s_k} \sum_{j=1}^{s_k(u)-s_k} \delta_{q_{-s_k-1-j}}. \]
By Claim 2(1), (2) we have
\[ s_k(u) - s_k = \{ s_k(u) - m_k(u) \} + \{ m_k(u) - s_k \} \to \infty \]
as \( k \to \infty \), and so we suppose that \( \hat{\mu}_k \) converges to \( \hat{\mu} \). By (8.23) and Claim 2(1),
\[ q^{k}_{-t_k(u)} \in V(r_{N(\xi^k) - u}, A_a) \cap \tilde{\xi}^k \]
for \( k \geq K_0(u) \), and so
\[ N_a(\tilde{\xi}^k) \geq N(\xi^k) - u \quad (k \geq K_0(u)). \]
Since \( N(\xi^k) \geq N_b(\xi^k) \geq N_b(\hat{\xi}^k) \) by (8.20), we have
\[ N_b(\hat{\xi}^k) - N_a(\hat{\xi}^k) \leq N(\xi^k) - (N(\xi^k) - u) = u. \]
Since \( \hat{\xi}^k \) is a substring of \( \tilde{\xi}^k \) and \( \hat{\xi}^k \subset U_0 \), \( \hat{\xi}^k \) and \( \tilde{\xi}^k \) satisfy the conditions (1)–(4) of Lemma 8.2. Thus by (8.28), (8.29) and Lemma 8.2 we have
\[ \hat{\mu}(A_a) > 0. \]
Hence \( \{ \hat{\xi}^k \} \), \( \{ \hat{\mu}_k \} \) and \( \hat{\mu} \) satisfy the conditions (1)–(3) of Lemma 8.1, and so there exist a sufficiently large \( n > 0 \), \( k \geq K_0(u) \) and a \((\hat{\xi}^k, n + 1; a)\)-string \( \hat{\sigma}_1 \) such that
\[ \sigma \cap V(r_n, A_a) = \emptyset \]
for every \((\hat{\xi}^k, 0; a)\)-string \( \sigma \neq \hat{\sigma}_1 \).
Since
\[ \xi^k \cap V(r_{n+1}, A_a) \supset \hat{\xi}^k \cap V(r_{n+1}, A_a) \supset \hat{\sigma}_1 \cap V(r_{n+1}, A_a) \neq \emptyset, \]
by (8.2) and (8.19) we have
\[ N(\xi^k) = N_a(\xi^k) \geq n + 1. \]
Thus the \((\xi^k, N(\xi^k); a)\)-string \( \sigma^k \) of (8.22) contains a \((\xi^k, n + 1; a)\)-string. Since \( \hat{\xi}^k \) is a substring of \( \xi^k \), \( \hat{\sigma}_1 \) is a \((\xi^k, n + 1; a)\)-string. If \( \sigma \) is a \((\xi^k, 0; a)\)-string with \( \sigma^k < \sigma < \hat{\sigma}_1 \), then \( \sigma \) is a \((\xi^k, 0; a)\)-string with \( \sigma \neq \hat{\sigma}_1 \). Thus, by (8.30) we have \( \sigma \cap V(r_n, A_a) = \emptyset \). This yields the condition (C). Since \( n \) is large enough, we have a contradiction as before. Claim 4 is proved. ■

For fixed \( u \geq 1 \) let \( s_k(u) \) be an integer satisfying (1) and (2) of Claim 2 for \( u \geq K_0(u) \), and let \( K_1(u) \) be as in Claim 4. For \( k \geq K_1(u) \) define
\[ l_k(u) = \max \left\{ 0 < l \leq \left[ \frac{s_k(u)}{m_0} \right] : 0 \leq r < l, \right. \]
\[ \prod_{t=r+1}^{l} \| D f^{m_0} | E(q_{-m_0 t}) \| \leq \gamma_0^{l-r} \} \].
Then for $k \geq K_1(u)$ and $0 < i \leq l_k(u)$ we have

$$\|Df^{m_0i}|E(q^k_{-m_0l_k(u)})\| \leq \prod_{t=l_k(u)-i+1}^{l_k(u)} \|Df^{m_0}|E(q^k_{-m_0t})\| \leq \gamma_0^i < 1.$$ 

Since

$$E(q^k_{-m_0l_k(u)}) = T_{q^k_{-m_0l_k(u)}} Z^s(\tilde{f}^{-m_0l_k(u)}(\tilde{q}^k), f^{m_0})$$

by Claim 3(b), for $\varepsilon > 0$ small enough we have

$$d(f^{m_0i}(y), q^k_{-m_0l_k(u)+m_0i}) \leq \varepsilon \quad (0 \leq i \leq l_k(u))$$

for $y \in Z^s(\tilde{f}^{-m_0l_k(u)}(\tilde{q}^k), f^{m_0})$. Therefore, if $\theta > 0$ is sufficiently small compared with $\varepsilon$, then

$$(8.32)\quad d(f^j(y), q^k_{-m_0l_k(u)+j}) \leq \varepsilon \quad (0 \leq j \leq m_0l_k(u))$$

for $y \in Z^s(\tilde{f}^{-m_0l_k(u)}(\tilde{q}^k), f^{m_0})$ (subset $Z^s(\tilde{f}^{-m_0l_k(u)}(\tilde{q}^k), f^{m_0})$). Notice that $\theta$ does not depend on $k$ and $u$.

Since $\dim E(x) = i_0 + 1$ for $x \in \text{cl}(I_{i_0+1}(f))$ and $A(i_0)$ is hyperbolic, by taking $\varepsilon_0 > 0$ small enough we have

$$\|Df^{m_0}|E(x)\| > 1 \quad (x \in U_{2\varepsilon_0}(A(i_0)) \cap \text{cl}(I_{i_0+1}(f))).$$

Here $U_{\varepsilon}(G) = \{y \in M : d(G, y) < \varepsilon\}$ for a closed set $G$. Since $r_0$ is arbitrary in (8.1), we can assume that $0 < r_0 < \varepsilon_0$. Thus,

$$V_0 = \bigcup_{b=1}^{s} V(r_0, A_b) \subset U_{\varepsilon_0}(A(i_0)).$$

The choice of $l_k(u)$ ensures that $\|Df^{m_0}|E(q^k_{-m_0l_k(u)})\| \leq \gamma_0 < 1$, and so

$$(8.33)\quad q^k_{-m_0l_k(u)} \notin U_{2\varepsilon_0}(A(i_0)) \supset V(r_0, A_a) \quad (k \geq K_1(u)).$$

By (8.24) and Claim 2(1), for $k \geq K_1(u)$ we have

$$(8.34)\quad s_k + 1 \leq m_0l_k(u) < m_k(u) < s_k(u),$$

and so by Claim 2(2),

$$(8.35)\quad s_k(u) - m_0l_k(u) = \{s_k(u) - m_k(u)\} + \{m_k(u) - m_0l_k(u)\} \to \infty$$

as $k \to \infty$. Thus $\{s_k(u)/m_0 - l_k(u)\}$ is unbounded.

For simplicity suppose that $[s_k(u)/m_0] - l_k(u) \geq 0$ for $k \geq K_1(u)$.

CLAIM 5. Under the above notations, for fixed $u \geq 1$ we have

$$\prod_{t=l_k(u)+1}^{r} \|Df^{m_0}|E(q^k_{-m_0t})\| \geq \gamma_0^{r-l_k(u)}$$

for $k \geq K_1(u)$ and $r$ with $l_k(u) < r \leq [s_k(u)/m_0]$. 


Proof. If this is false, then there are \( k \geq K_1(u) \) and \( l_k(u) < s \leq [s_k(u)/m_0] \) such that

\[
\prod_{t=l_k(u)+1}^{r} \| Df^{m_0} | E(q_{-m_0t})^k \| \geq \gamma_0^{r-l_k(u)} \quad (l_k(u) < r < s),
\]

\[
\prod_{t=l_k(u)+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| < \gamma_0^{s-l_k(u)},
\]

and for \( l_k(u) < r < s \),

\[
(8.36) \quad \prod_{t=r+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| = \prod_{t=l_k(u)+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| \frac{\prod_{t=l_k(u)+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \|}{\prod_{t=l_k(u)+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \|} < \frac{\gamma_0^{s-l_k(u)}}{\gamma_0^{r-l_k(u)}} = \gamma_0^{s-r}.
\]

Since, for \( 0 \leq r < l_k(u) \),

\[
(8.37) \quad \prod_{t=r+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| = \prod_{t=r+1}^{l_k(u)} \| Df^{m_0} | E(q_{-m_0t})^k \| \cdot \prod_{t=l_k(u)+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| < \gamma_0^{l_k(u)-r} \cdot \gamma_0^{s-l_k(u)} = \gamma_0^{s-r},
\]

from (8.36) and (8.37) we have

\[
\prod_{t=r+1}^{s} \| Df^{m_0} | E(q_{-m_0t})^k \| \leq \gamma_0^{s-r}
\]

for \( 0 \leq r < s \), which contradicts the choice of \( l_k(u) \). Therefore Claim 5 holds.

By Claim 5 and (8.25) we have

\[
\| D\tilde{f}^{m_0i} | F(\tilde{f}^{m_0l_k(u)}(q)) \| \leq \prod_{t=l_k(u)}^{l_k(u)+i-1} \| D\tilde{f}^{m_0} | F(\tilde{f}^{m_0t}(q)) \| \leq \prod_{t=l_k(u)+1}^{l_k(u)+i} \lambda_0 \| Df^{m_0} | E(q_{-m_0t})^k \|^{-1} \leq (\lambda_0 \gamma_0^{-1})^i < 1
\]

for \( k \geq K_1(u) \) and \( 0 < i \leq [s_k(u)/m_0] - l_k(u) \). Thus the following statement is easily checked from (b) and (c) of Claim 3: for every \( \epsilon > 0 \) we can take a
small number $0 < \theta < \varepsilon$ such that if $y \in Z^u_{\theta}(\tilde{f}^{-m_0l_k(u)}(\tilde{q}^k), f^{m_0})$, then there is a string $(y_0, \ldots, y_{-s_k(u)})$ with $y_0 = y$ satisfying
\begin{equation}
(8.38) \quad d(y_j, q_{-m_0l_k(u)-j}^k) \leq \varepsilon \quad (0 \leq j \leq s_k(u))
\end{equation}
where
\begin{equation}
(8.39) \quad s_k(u) = s_k(u) + 1 - m_0l_k(u).
\end{equation}

For fixed $u \geq 1$ let $K_1(u)$ be as in Claim 5. For $k \geq K_1(u)$ define
\begin{equation}
\xi_1^k(u) = (q_{-m_0l_k(u)}, q_{-m_0l_k(u)}^k, \ldots, q_{-m_0l_k(u)}^{k+2}, q_{-m_0l_k(u)+1})
\end{equation}
where $m_k(u)$ is as in Claim 2.

CLAIM 6. For every $v \geq 1$ there is $K(u, v) \geq K_1(u)$ such that for $k \geq K(u, v)$,
\begin{equation}
\xi_1^k(u) \cap V(r_{N(\xi^k)}-u-v, A_a) = \emptyset,
\end{equation}
where $N(\xi^k)$ is as in (8.19).

Proof. Suppose that this is false. Then there is $v \geq 1$ such that for infinitely many $k$ with $k \geq K_1(u)$,
\begin{equation}
(8.40) \quad \xi_1^k(u) \cap V(r_{N(\xi^k)}-u-v, A_a) \neq \emptyset.
\end{equation}
Without loss of generality we suppose that (8.40) holds for $k \geq K_1(u)$.

It is clear that $\{\xi_1^k(u)\} \subset U_0$. Since $q_{-m_0l_k(u)}, q_{-m_0l_k(u)+1} \notin V(r_0, A_a)$ for $k \geq K_1(u)$, $\xi_1^k(u)$ contains a $(\xi_1^k(u), N(\xi^k) - u - v; a)$-string, and so by (8.21) and Lemma 8.4(1),
\begin{equation}
m_k(u) - 1 - m_0l_k(u) \geq \#\{\xi_1^k(u) \cap V(r_0, A_a)\} \to \infty
\end{equation}
as $k \to \infty$. For $\xi_1^k(u)$ we define
\begin{equation}
\mu_k^1 = \frac{1}{(m_k(u) - 1) - m_0l_k(u)} \sum_{j=1}^{(m_k(u)-1)-m_0l_k(u)} \delta_{q_{-m_0l_k(u)-j}^k}
\end{equation}
and let $\mu^1$ be an accumulation point of $\mu_k^1$. If we establish that
\begin{equation}
(8.41) \quad \mu^1(A_a) > 0,
\end{equation}
then $\{\xi_1^k(u)\}$ satisfies the conditions (1)–(3) of Lemma 8.1. Thus there are sufficiently large integers $n$, $k$ and a $(\xi_1^k(u), n+1; a)$-string $\sigma_1$ such that for every $(\xi_1^k(u), 0; a)$-string $\sigma \neq \sigma_1$,
\begin{equation}
\sigma \cap V(r_n, A_a) = \emptyset.
\end{equation}
Since this implies the condition (C) for $n$ large enough, we have a contradiction.
Thus it is enough to show (8.41) to obtain Claim 6. For \( k \geq K_1(u) \) define the Borel probability measures \( \nu_k \) by

\[
\nu_k = \frac{1}{[s_k(u)/m_0]} \sum_{j=1}^{[s_k(u)/m_0]} \delta_{q_{m_0l_k(u)-m_0j}^k}.
\]

Then \( \nu_k \) converges to \( \nu \in \mathcal{M}(f^{m_0}|\text{cl}(I_{i_0+1}(f))) \) (take a subsequence if necessary). Since

\[
\int_{\text{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\| \, d\nu = \lim_{k \to \infty} \int_{\text{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\| \, d\nu_k
\]

\[
\geq \log \gamma_0 > \log \lambda_0
\]

by Claim 5, we find that \( \nu(A(i_0)) > 0 \) by Lemma 6.1.

For \( k \geq K_1(u) \) define a string

\[
\zeta^k_1(u) = (q_{-m_0l_k(u)+1}^k, \ldots, q_{-m_0l_k(u)}^k, \ldots, q_{-s_k(u)}^k, q_{-s_k(u)-1}^k).
\]

Then \( \xi^k_1(u) \subset \zeta^k_1(u) \) since \( m_k(u) < s_k(u) \). For \( \zeta^k_1(u) \) we define

\[
\nu^1_k = \frac{1}{s_k(u)+1-m_0l_k(u)} \sum_{j=1}^{s_k(u)+1-m_0l_k(u)} \delta_{q_{m_0l_k(u)-j}^k}.
\]

Then \( \nu^1_k \) converges to \( \nu^1 \in \mathcal{M}(f) \) by (8.35). By the same calculation as in (8.27) we have

\[
\nu^1(A(i_0)) \geq \frac{1}{m_0} \nu(A(i_0)) > 0,
\]

and so

\[
\nu^1(A_b) > 0
\]

for some \( 1 \leq b \leq s \).

Since \( \zeta^k_1(u) \) is a substring of \( \xi^k \), we have \( N_b(\xi^k) \geq N_b(\zeta^k_1(u)) \), and so by (8.19),

\[
N(\xi^k) \geq N_b(\zeta^k_1(u)) \quad (k \geq K_1(u)).
\]

Thus, by (8.40),

\[
N_b(\zeta^k_1(u)) - N_a(\zeta^k_1(u)) \leq N(\xi^k) - (N(\xi^k) - u - v) = u + v.
\]

Since \( \zeta^k_1(u) \) is a substring of \( \zeta^k(u) \), by (8.42) and Lemma 8.2 we have \( \mu^1(A_a) > 0 \). Thus (8.41) was proved.

**Claim 7.** Let \( u_1 \) and \( u_2 \) be integers with \( 1 \leq u_2 < u_1 \), and let \( K(u_1,1) \) and \( K(u_2,1) \) be as in Claim 6. Then, for \( k \geq \max\{K(u_1,1), K(u_2,1)\}, \)

\[
m_0l_k(u_1) < m_0l_k(u_2).
\]

**Proof.** Since \( m_0l_k(u_1) < s_k(u_1) \) by (8.34), it is enough to show that \( s_k(u_1) < m_0l_k(u_2) \). Otherwise \( s_k(u_1) \geq m_0l_k(u_2) \) for some \( k \geq \max\{K(u_1,1), \)
$K(u_2, 1)$. By (8.24) we have

$$q^k_{-t} \in V(r_0, A_a) \quad (m_k(u_1) \leq t \leq s_k(u_1)).$$

Since $q_{-m_0l_k(u_2)}^k \not\in V(r_0, A_a)$ by (8.33), we have

$$m_0l_k(u_2) < m_k(u_1),$$

and so by Claim 2(3),

$$q^k_{-t} \not\in V(r_{N(\xi_k) - u_1}, A_a) \supset V(r_{N(\xi_k) - u_2 - 1}, A_a)$$

for $s_k + 1 \leq t \leq m_0l_k(u_2)$. Combining this result and Claim 6, we have

$$q^k_{-t} \not\in V(r_{N(\xi_k) - u_2 - 1}, A_a)$$

for $s_k + 1 \leq t \leq m_k(u_2) - 1$. Then

$$\sigma \cap V(r_{N(\xi_k) - u_2 - 1}, A_a) = \emptyset$$

for every $(\xi_k, 0; a)$-string $\sigma$ with $\sigma^k < \sigma < \sigma^k(u_2)$. Since $\sigma^k$ and $\sigma^k(u_2)$ are $(\xi_k, N(\xi_k) - u_2; a)$-strings, the condition (C) holds. Since $N(\xi_k)$ is large enough, we have a contradiction. Thus we have Claim 7. ■

Let $\gamma$ be as in Lemma 8.3, and $\overline{\pi}$ and $\overline{N}$ be as in Lemma 8.5. Let $r_0$ be a sufficiently small positive number as in (8.1). Choose $\varepsilon > 0$ such that

$$\varepsilon < \min\{(1 - \lambda)r_0, \gamma r_0/3\},$$

and take a small number $\delta > 0$ satisfying (8.32) and (8.38). Let $\delta = \delta(\theta) > 0$ be as in Claim 3(d). Since $M$ is compact, there is $v_0 > 0$ such that

$$\max\{d(\tilde{x}, \tilde{y}) : \tilde{x}, \tilde{y} \in \mathbb{M}\}/v_0 \leq \delta.$$
Then

\[(8.47) \quad (i) \ 0 \leq w_2 < u_1 \leq v_0(2n + 1), \]

\[(ii) \ 2\tilde{n} + 1 \leq u_1 - w_2 \leq v_0(2n + 1), \]

\[(iii) \ \tilde{d}(\tilde{f}^{-m_0l_k(u_1)}(\tilde{q}^k), \tilde{f}^{-m_0l_k(u_2)}(\tilde{q}^k)) \leq \delta. \]

By Claim 3(d),

\[Z_\theta^s(\tilde{f}^{-m_0l_k(u_1)}(\tilde{q}^k), f^{m_0}) \cap Z_\theta^u(\tilde{f}^{-m_0l_k(u_2)}(\tilde{q}^k), f^{m_0})\]

is one point; denote it by \(z\).

**Claim 8.** Let \(\overline{\sigma}_k(u_2)\) be as in (8.39) and \(l_k(u)\) be as in (8.31). For the above point \(z\) there is a string

\[\eta = (z_1, z_0, z_1, \ldots, z_{m_0l_k(u_1)}, \ldots, z_{m_0l_k(u_1)} - \overline{\sigma}_k(u_2), z_{m_0l_k(u_1)} - \overline{\sigma}_k(u_2) - 1)\]

such that

\[(i) \ z_{-m_0l_k(u_1)} = z, \]

\[(ii) \ d(z_{-j}, q^k_{-j}) \leq \varepsilon \ (0 \leq j \leq m_0l_k(u_1)), \]

\[(iii) \ d(z_{-m_0l_k(u_1)-j}, q^k_{-m_0l_k(u_2)-j}) \leq \varepsilon \ (0 \leq j \leq \overline{\sigma}_k(u_2)), \]

\[(iv) \ \text{either} \ z_1 \not\in V(r_0, \Lambda_a), \ or \ z_0 \not\in V(r_0, \Lambda_a), \]

\[(v) \ \text{either} \ z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2) \not\in V(r_0, \Lambda_a), \ or \ z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2) - 1 \not\in V(r_0, \Lambda_a). \]

**Proof.** For \(-1 \leq j \leq m_0l_k(u_1)\) put

\[z_{-j} = f^{m_0l_k(u_1)-j}(z).\]

Since \(z \in Z_\theta^u(\tilde{f}^{-m_0l_k(u_2)}(\tilde{q}^k), f^{m_0})\), by (8.38) we can take a string

\[(z_{-m_0l_k(u_1)}), \ldots, z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2))\]

with \(z_{-m_0l_k(u_1)} = z\) to satisfy (iii). Let \(z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2) - 1\) be an arbitrary point belonging to the inverse image of \(z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2)\). Then \(\eta = (z_1, \ldots, z_{-m_0l_k(u_1)} - \overline{\sigma}_k(u_2) - 1)\) is a string.

Clearly (i) holds. Since \(z \in Z_\theta^u(\tilde{f}^{-m_0l_k(u_1)}(\tilde{q}^k), f^{m_0})\), by (8.32) we see that \(\eta\) satisfies (ii).

It remains to show (iv) and (v). Since \(q^k_0 \not\in V(r_0, \Lambda_a)\), we can check that

\[d(q^k_0, V_a^+) > r_0.\]

If \(z_0 \in V(r_0, \Lambda_a)\), then by (ii) we have

\[d(z_0, V_a^+) \geq d(q^k_0, V_a^+) - d(z_0, q^k_0) > r_0 - \varepsilon.\]

By Lemma 8.3(2) and (8.43),

\[d(z_1, V_a^+) = d(f(z_0), V_a^+) \geq \frac{1}{\lambda} d(z_0, V_a^+) > \frac{1}{\lambda} (r_0 - \varepsilon) > r_0,\]

and so \(z_1 \not\in V(r_0, \Lambda_a)\). Thus (iv) is proved. Similarly we can check (v). Therefore Claim 8 holds. ■
Hereafter let $K$ be as in (8.44) and $k$ be an integer so large that $k \geq K$. Since $q_{m_0 l_k(u_1)} \notin U_{2\delta_0}(A(i_0))$ by (8.33), it follows from Claim 8(ii) that

\[ z_{-m_0 l_k(u_1)} \notin V(r_0, A_a). \]  

Let $\sigma^k = (q_{-1}, \ldots, q_{-t_k}, \ldots, q_{-s_k})$ be the $(\xi^k, N(\xi^k); a)$-string of (8.22). Then, by (8.20) and (8.45),

\[ q_{-t_k} \in \sigma^k \cap V(r_{N(\xi^k)}, A_a) \quad \text{and} \quad N(\xi^k) \geq \overline{N}. \]

By (8.34) we have $t_k < s_k < m_0 l_k(u_1)$, and so by Claim 8(ii) and (8.43),

\[ d(q_{-t_k+j}, z_{-t_k+j}) \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t_k - 1). \]

Thus we have the conditions (1) and (2) of Lemma 8.5, and so there is $1 \leq t_1 \leq t_k$ such that

\[ z_{-t_1} \in V(r_{N(\xi^k)-\overline{\pi}}, A_a) \quad \text{and} \quad z_{-j} \in V(r_0, A_a) \quad (t_1 \leq j \leq t_k). \]

Since $0 \leq t_1 \leq t_k \leq m_0 l_k(u_1)$ and $z_{-m_0 l_k(u_1)} \notin V(r_0, A_a)$, by (8.43) and Claim 8(iv) there exists an $(\eta, 0; a)$-string $\sigma_1$ containing $z_{-t_1}$.

Let $u_2$ be as in (8.46). For $u_2$ let $\sigma^k(u_2) = (q_{-m_k(u_2)}, \ldots, q_{-t_k(u_2)}, \ldots, q_{-s_k(u_2)})$ be the $(\xi^k, N(\xi^k) - u_2; a)$-string defined as in (8.24). By (8.23) and (8.45) we have

\[ q_{-t_k(u_2)} \in V(r_{N(\xi^k) - u_2}, A_a), \quad N(\xi^k) - u_2 \geq N(\xi^k) - v_0(2\overline{\pi} + 1) \geq \overline{N}. \]

For $0 \leq j \leq t_k(u_2) - m_k(u_2)$, by (8.34) and Claim 2(1) we have

\[ 0 < -j + t_k(u_2) - m_0 l_k(u_2) < \overline{\pi}(u_2), \]

and so by Claim 8(iii) and (8.43),

\[ d(q_{-t_k(u_2)+j}, z_{-t_k(u_2)+w(k)+j}) \]
\[ = d(q_{-m_0 l_k(u_2) - \{ -j + t_k(u_2) - m_0 l_k(u_2) \}}, z_{-m_0 l_k(u_1) - \{ -j + t_k(u_2) - m_0 l_k(u_2) \}}) \]
\[ \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t_k(u_2) - m_k(u_2)) \]

where

\[ w(k) = m_0 l_k(u_2) - m_0 l_k(u_1). \]

Thus we have the conditions (1) and (2) of Lemma 8.5, and so there is $t_2$ with $m_k(u_2) - w(k) \leq t_2 \leq t_k(u_2) - w(k)$ such that

\[ z_{-t_2} \in V(r_{N(\xi^k) - u_2 - \overline{\pi}}, A_a), \]
\[ z_{-j} \in V(r_0, A_a) \quad (t_2 \leq j \leq t_k(u_2) - w(k)). \]

Since $m_0 l_k(u_1) < m_k(u_2) - w(k) \leq t_2$ and $z_{-m_0 l_k(u_1)} \notin V(r_0, A_a)$, by (8.43) and Claim 8(v) there exists an $(\eta, 0; a)$-string $\sigma_2$ containing $z_{-t_2}$.

Since, by the choice of $t_1$ and $t_2$,

\[ t_1 \leq t_k < m_0 l_k(u_1) < m_k(u_2) - w(k) \leq t_2, \]
we have $\sigma_1 \neq \sigma_2$. By (8.47)(ii),
\[ N(\xi^k) - \bar{\eta} \geq N(\xi^k) - u_2 - \bar{\eta} \geq N(\xi^k) - u_1 + \bar{\eta} + 1, \]
and so by (8.49) and (8.50),
\[ z_{-t_1}, z_{-t_2} \in V(r_{N(\xi^k) - u_1 + \bar{\eta} + 1}, A_a). \]
Therefore $\sigma_1$ and $\sigma_2$ are $(\eta, N(\xi^k) - u_1 + \bar{\eta} + 1, A_a)$-strings and $\sigma_1 \neq \sigma_2$.

**Claim 9.** Let $\sigma_1$ and $\sigma_2$ be as above. For every $(\eta, 0; a)$-string $\sigma$ with $\sigma_1 < \sigma < \sigma_2$ we have
\[ \sigma \cap V(r_{N(\xi^k) - u_1 + \bar{\eta}}, A_a) = \emptyset \]
for $k \geq K$.

If we establish Claim 9, then the condition (C) holds for $N(\xi^k) - u_1 + \bar{\eta}$ ($\geq N(\xi^k) - v_0(2\bar{\eta} + 1)$) large enough. This implies the existence of a 1-cycle for $A_a$, which is inconsistent with Lemma 7.1. This contradiction has been derived through the nine claims under the assumption given in (*).

Therefore the assumption $A_a \cap cl(I_{i+1}(f)) \neq \emptyset$ is invalid, which yields Proposition 4(b). To finish the proof it thus suffices to check that Claim 9 is true.

**Proof of Claim 9.** If Claim 9 is false, then there is an $(\eta, 0; a)$-string $\sigma$ such that $\sigma_1 < \sigma < \sigma_2$ and
\[ \sigma \cap V(r_{N(\xi^k) - u_1 + \bar{\eta}}, A_a) \neq \emptyset \]
for some $k \geq K$. Write
\[ \sigma = (z_{-l}, \ldots, z_{-s}) \quad (\subset V(r_0, A_a)) \]
for some $l, s$ with $l \leq s$. Choose $l \leq t \leq s$ such that
\[ z_{-t} \in \sigma \cap V(r_{N(\xi^k) - u_1 + \bar{\eta}}, A_a). \]
Since $\sigma_1 < \sigma < \sigma_2$, we have $t_1 < l \leq t \leq s < t_2$. Since $z_{-m_0l_k(u_1)} \notin V(r_0, A_0)$ by (8.48), we have two cases to consider:

(a) $s < m_0l_k(u_1)$,  \quad (b) $m_0l_k(u_1) < l$.

**Case (a):** By Claim 8(ii), (8.45) and (8.51) we have
\[ z_{-t} \in V(r_{N(\xi^k) - u_1 + \bar{\eta}}, A_a), \]
\[ N(\xi^k) - u_1 + \bar{\eta} \geq N(\xi^k) - v_0(2\bar{\eta} + 1) \geq \bar{N}, \]
\[ d(z_{-t+j}, q_{-t+j}^k) \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t - l), \]
and so $\sigma$ satisfies the conditions (1) and (2) of Lemma 8.5. Replacing $\xi$ by $\eta$ and $\eta$ by $\xi^k$ in Lemma 8.5, we can take $l \leq t_3 \leq t$ such that
\[ q_{-t_3}^k \in V(r_{N(\xi^k) - u_1}, A_a), \quad q_{-j}^k \in V(r_0, A_a) \quad (t_3 \leq j \leq t). \]
By (8.34),
\[ t_3 \leq t \leq s < m_0 l_k(u_1) \leq m_k(u_1) - 1, \]
and so by Claim 2(3),
\[ q_{t_3}^k \in V(r_N(\xi^k) - u_1, A_a) \cap \{q_{-1}^k, \ldots, q_{-m_k(u_1)+1}^k\} \subset \sigma^k = \{q_{-1}^k, \ldots, q_{-s_k}^k\}. \]
Since \( q_{-s_k}^k \notin V(r_0, A_a) \), by (8.52) we have
\[ (8.53) \quad q_{-j}^k \in \sigma^k \subset V(r_0, A_a) \quad (1 \leq j \leq t). \]
Since \( \sigma = (z_{-1}, \ldots, z_{-s}) \) is an \((\eta, 0; a)\)-string, we have
\[ z_{-t} \in V(r_0, A_a) \quad \text{and} \quad z_{-t+1} \notin V(r_0, A_a). \]
Thus it is easily checked by using Lemma 8.3 that \( d(z_{-t+1}, V_{a^+}) > r_0 \). Thus,
\[ (8.54) \quad d(z_{-t}, V_{a^+}) \geq \gamma d(f(z_{-t}), V_{a^+}) = \gamma d(z_{-t+1}, V_{a^+}) > \gamma r_0. \]
Since \( t_1 < l \leq t \) and \( f^{l-t_1}(q_{-t_1}^k) = q_{-t_1}^k \), by (8.53) and Lemma 8.3(2) we have
\[ d(q_{-t_1}^k, V_{a^+}) \leq \lambda^{l-t_1} d(f^{l-t_1}(q_{-t_1}^k), V_{a^+}) < d(q_{-t_1}^k, V_{a^+}). \]
Thus, by Claim 8(ii), (8.49) and (8.54),
\[ 0 < d(q_{-t_1}^k, V_{a^+}) - d(q_{-l}^k, V_{a^+}) \leq (d(q_{-t_1}^k, z_{-t_1}) + d(z_{-t_1}, V_{a^+})) - (d(z_{-l}, V_{a^+}) - d(z_{-l}, q_{-l}^k)) \leq (\varepsilon + r_N(\xi^k) - \pi) - (\gamma r_0 - \varepsilon) < r_N(\xi^k) - \pi - \gamma r_0 / 3 < 0. \]
This is a contradiction.

Case (b): By Claim 8(iii), (8.45) and (8.51) we have
\[ z_{-t} \in V(r_N(\xi^k) - u_1 + \pi, A_a), \]
\[ N(\xi^k) - u_1 + \pi \geq N(\xi^k) - v_0(2\pi + 1) \geq N, \]
\[ d(z_{-t+j}, q_{-l-w(k)+j}^k) \leq \varepsilon < r_0 / 2 \quad (0 \leq j \leq t - l), \]
and so \( \sigma \) satisfies the conditions (1) and (2) of Lemma 8.5. Thus,
\[ (8.55) \quad q_{-t_4}^k \in V(r_N(\xi^k) - u_1, A_a), \quad q_{-j}^k \in V(r_0, A_a) \quad (t_4 \leq j \leq t + w(k)) \]
for some \( t_4 \) with \( l + w(k) \leq t_4 \leq t + w(k) \). Since \( u_1 - u_2 \leq v_0(2\pi + 1) \) by (8.47)(ii), it follows that
\[ N(\xi^k) - u_1 \geq N(\xi^k) - u_2 - v_0(2\pi + 1), \]
and so by (8.55) and Claim 6,
\[ q_{-t_4}^k \in V(r_N(\xi^k) - u_2 - v_0(2\pi + 1)) \cap \{q_{m_k(u_2)}, \ldots, q_{s_k(u_2)}\} \subset \sigma^k(u_2) = \{q_{-m_k(u_2)}, \ldots, q_{-s_k(u_2)}\} \]
because \( k \geq K \geq K(u_2, v_0(2\pi + 1)) \) by (8.44). Since \( q_{-m_k(u_2)+1}^k \notin V(r_0, A_a) \), by (8.55) we have
\[ q_{-j}^k \in \sigma^k(u_2) \subset V(r_0, A_a) \quad (t + w(k) \leq j \leq s_k(u_2)). \]
Since \( \sigma \) and \( \sigma_2 \) contain \( z_{-t} \) and \( z_{-t_2} \) respectively and satisfy \( \sigma < \sigma_2 \), there is \( t < t'_2 < t_2 \) such that

\[
z_{-t'_2} \in V(r_0, A_a) \quad \text{and} \quad z_{-t'_2+1} \notin V(r_0, A_a),
\]

and so by Claim 8(iii) and (8.50),

\[
0 < d(q^k_{-t-w(k)}, V^+_a) - d(q^k_{-t'_2-w(k)}, V^+_a) \\
\leq (d(q^k_{-t-w(k)}, z_t) + d(z_{-t}, V^+_a)) - (d(z_{-t'_2}, V^+_a) - d(z_{-t'_2}, q^k_{-t'_2-w(k)})) \\
\leq (\varepsilon + r_N(\xi^k - u_1 + \pi)) - (\gamma r_0 - \varepsilon) < r_N(\xi^k - u_1 + \pi) - \gamma r_0 / 3 < 0.
\]

This is a contradiction. Therefore we have Claim 9.

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**References**


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