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C^1-maps having hyperbolic periodic points

by

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Abstract. We show that the C^1-interior of the set of maps satisfying the following conditions:

(i) periodic points are hyperbolic,
(ii) singular points belonging to the nonwandering set are sinks,

coincides with the set of Axiom A maps having the no cycle property.

1. Introduction. Let M be a closed C^∞-manifold, $\|\cdot\|$ be a Riemannian metric on M and $\pi : TM \to M$ be the tangent bundle. Let $C^1(M)$ be the space of C^1-differentiable maps from M into itself endowed with the C^1-topology. Then $C^1(M)$ contains the set $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms and this subset is open in $C^1(M)$.

The C^1-stability conjecture on $\mathrm{Diff}^1(M)$ of Palis and Smale was solved by Mañé [12] as follows: if a C^1-diffeomorphism f is structurally stable, then f satisfies Axiom A and the strong transversality. By using the techniques obtained in proving the conjecture, Palis [18] showed that if there exists a nonempty open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ such that all periodic points of each $g \in \mathcal{U}$ are hyperbolic, then every diffeomorphism belonging to $\mathcal{U}$ can be approximated by Axiom A diffeomorphisms with no cycles. Next it was checked in [1] that $\mathcal{U}$ consists of Axiom A diffeomorphisms with no cycles. We remark here that the methods of Liao [7] which proved the C^1-stability in the 2-dimensional case were also useful in the higher dimensional case (the 2-dimensional case was also proved in Sannami [22]).

In this paper we shall discuss the problem of whether stability of C^1-differentiable maps implies Axiom A and no cycles.

Concerning this problem Przytycki proved the following remarkable results: Anosov differentiable maps which are not diffeomorphisms or expand-

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ings do not satisfy C^1-structural stability [20], and if a differentiable map f satisfies Axiom A and has no singular points in the nonwandering set, then f is C^1 Ω-stable if and only if f satisfies strong Axiom A and has no cycles [21]. On the other hand, we know [23] that expanding maps are structurally stable.

In view of these developments, we shall discuss in detail how stability of diffeomorphisms can be adapted to the more complicated situation of C^1-maps, and so we shall focus on the noninvertible case (that is, the case of differentiable maps which are not diffeomorphisms).

In order to state our result let us recall a few notations and basic results about C^1-maps.

Let $f \in C^1(M)$. For a periodic point p of f, denote by $\varrho(f,p)$ the minimal integer $n > 0$ satisfying $f^n(p) = p$. We say that $\varrho(f,p)$ is the *period* of p for f. A periodic point p is called *hyperbolic* if $D_p f^{\varrho(f,p)} : T_pM \to T_pM$ has no eigenvalues of absolute value one; then T_pM splits into the direct sum $T_pM = E^{\mathrm{s}}(p) \oplus E^{\mathrm{u}}(p)$ of subspaces $E^{\mathrm{s}}(p)$ and $E^{\mathrm{u}}(p)$ such that

(1.1) (a) $D_p f^{\varrho(f,p)}(E^{\mathrm{s}}(p)) \subset E^{\mathrm{s}}(p),\ D_p f^{\varrho(f,p)}(E^{\mathrm{u}}(p)) = E^{\mathrm{u}}(p)$,

(b) there are $c > 0$ and $0 < \lambda < 1$ such that for $n > 0$,

(i) $\|Df^n(v)\| \le c\lambda^n \|v\|$ $(v \in E^{\mathrm{s}}(p))$,

(ii) $\|Df^n(v)\| \ge c^{-1}\lambda^{-n} \|v\|$ $(v \in E^{\mathrm{u}}(p))$.

A hyperbolic periodic point p is said to be a *sink* (resp. *source*) if $T_pM = E^{\mathrm{s}}(p)$ (resp. $T_pM = E^{\mathrm{u}}(p)$).

We denote by $\mathbb{M} = \prod_{-\infty}^{\infty} M$ the topological product of M's, and define an injective continuous map $\widetilde{f} : \mathbb{M} \to \mathbb{M}$ by

$$\widetilde{f}((x_n)) = (f(x_n))$$

for $(x_n) \in \mathbb{M}$. Then $P^0 \circ \widetilde{f} = f \circ P^0$ where

$$P^0 : \mathbb{M} \to M \tag{1.2}$$

is the natural projection defined by $P^0((x_n)) = x_0$. For $\Lambda \subset M$ put

$$\Lambda_f = \{(x_n) \in \mathbb{M} : x_n \in \Lambda,\ f(x_n) = x_{n+1},\ n \in \mathbb{Z}\}. \tag{1.3}$$

Then Λ_f is $\widetilde{f}$-invariant ($\widetilde{f}(\Lambda_f) = \Lambda_f$) and $\widetilde{f}|\Lambda_f : \Lambda_f \to \Lambda_f$ is a homeomorphism when $\Lambda_f \neq \emptyset$. Notice that Λ is not necessarily f-invariant.

We say that $(M_f, \widetilde{f})$ is the *inverse limit system* of (M, f). Notice that if $f : M \to M$ is a diffeomorphism, then the inverse limit system of (M, f) is equal to the original system (M, f).

Let $T\mathbb{M}$ be the subspace of $\mathbb{M} \times TM$ defined by

$$T\mathbb{M} = \{(\widetilde{x}, v) \in \mathbb{M} \times TM : P^0(\widetilde{x}) = \pi(v)\}$$

and define a Finsler metric $\|\cdot\|$ on $T\mathbb{M}$ by

$$\|(\widetilde{x}, v)\| = \|v\| \quad ((\widetilde{x}, v) \in T\mathbb{M}).$$

Define the projection $\overline{P}^0 : T\mathbb{M} \to TM$ by

$$\overline{P}^0(\widetilde{x}, v) = v \tag{1.4}$$

for $(\widetilde{x}, v) \in T\mathbb{M}$. Then $\overline{P}^0(T_{\widetilde{x}}\mathbb{M}) = T_{x_0}M$ and the restriction $\overline{P}^0|T_{\widetilde{x}}\mathbb{M} : T_{\widetilde{x}}\mathbb{M} \to T_{x_0}M$ is a linear isomorphism.

We define a C^0-vector bundle

$$\widetilde{\pi} : T\mathbb{M} \to \mathbb{M}$$

by $\widetilde{\pi}(\widetilde{x}, v) = \widetilde{x}$ for $(\widetilde{x}, v) \in T\mathbb{M}$, and write $T_{\widetilde{x}}\mathbb{M} = \widetilde{\pi}^{-1}(\widetilde{x})$ for $\widetilde{x} \in \mathbb{M}$. Let $D\widetilde{f} : T\mathbb{M} \to T\mathbb{M}$ be defined by

$$D\widetilde{f}(\widetilde{x}, v) = (\widetilde{f}(\widetilde{x}), D_{x_0}f(v)) \quad ((\widetilde{x}, v) = ((x_n), v) \in T\mathbb{M}),$$

where x_0 is a point in (x_n) and $D_{x_0}f$ is the derivative of f at x_0.

We say that a closed f-invariant subset Λ is *hyperbolic* if the vector bundle $T\mathbb{M}|\Lambda_f = \bigcup_{\widetilde{x}\in\Lambda_f} T_{\widetilde{x}}\mathbb{M}$ splits into the Whitney sum $T\mathbb{M}|\Lambda_f = E^{\mathrm{s}} \oplus E^{\mathrm{u}}$ of subbundles E^{s} and E^{u} satisfying the following conditions:

(a) $D\widetilde{f}(E^{\mathrm{s}}) \subset E^{\mathrm{s}}$, $D\widetilde{f}(E^{\mathrm{u}}) = E^{\mathrm{u}}$,
(b) $D\widetilde{f}|E^{\mathrm{u}} : E^{\mathrm{u}} \to E^{\mathrm{u}}$ is injective,
(c) there exist $c > 0$ and $0 < \lambda < 1$ such that for $n \geq 0$,

$$\|D\widetilde{f}^n|E^{\mathrm{s}}\| \leq c\lambda^n, \quad \|(D\widetilde{f}|E^{\mathrm{u}})^{-n}\| \leq c\lambda^n,$$

where $\|T\|$ denotes the supremum norm of a linear bundle map T. It is checked from the techniques in [20, §0 and §1] that

(1) E^{s} and E^{u} are C^0-vector bundles over Λ_f,
(2) there exist $0 < \lambda < 1$ and a new norm $\|\cdot\|$ such that

$$\|D\widetilde{f}|E^{\mathrm{s}}\| \leq \lambda, \quad \|(D\widetilde{f}|E^{\mathrm{u}})^{-1}\| \leq \lambda,$$

(3) if $P^0(\widetilde{x}) = P^0(\widetilde{y})$ for $\widetilde{x}, \widetilde{y} \in \Lambda_f$, then $E^{\mathrm{s}}(\widetilde{x}) = E^{\mathrm{s}}(\widetilde{y})$, but in general $E^{\mathrm{u}}(\widetilde{x}) \neq E^{\mathrm{u}}(\widetilde{y})$.

Let f, in particular, be a C^1-map from M onto itself. Then f is called *Anosov* if M is hyperbolic. An Anosov map f is said to be *expanding* if $E^{\mathrm{u}}(\widetilde{x}) = T_{\widetilde{x}}M$ for $\widetilde{x} \in M_f$.

For $\widetilde{x} = (x_n) \in M_f$ and $\varepsilon > 0$ put

$$\begin{aligned} W^{\mathrm{s}}_\varepsilon(\widetilde{x}, f) &= \{y \in M : d(x_n, f^n(y)) \leq \varepsilon \text{ for } n \geq 0\}, \\ W^{\mathrm{u}}_\varepsilon(\widetilde{x}, f) &= \{y \in M : \text{there exists } \widetilde{y} = (y_n) \in M_f \text{ with } y_0 = y \\ &\qquad \text{such that } d(x_{-n}, y_{-n}) \leq \varepsilon \text{ for } n \geq 0\}. \end{aligned} \tag{1.5}$$

Then $W^s_\varepsilon(\widetilde{x}, f) = W^s_\varepsilon(\widetilde{y}, f)$ if $P^0(\widetilde{x}) = P^0(\widetilde{y})$ for $\widetilde{x}, \widetilde{y} \in M_f$, and

$$W^s_\varepsilon(\widetilde{x}, f) \subset f^{-1}(W^s_\varepsilon(\widetilde{f}(\widetilde{x}), f)), \qquad W^u_\varepsilon(\widetilde{x}, f) \subset f(W^u_\varepsilon(\widetilde{f}^{-1}(\widetilde{x}), f)).$$

If Λ is hyperbolic, then it follows from [5, Theorem 5.1] that $\{W^s_\varepsilon(\widetilde{x}, f)\}_{\widetilde{x}\in\Lambda_f}$ and $\{W^u_\varepsilon(\widetilde{x}, f)\}_{\widetilde{x}\in\Lambda_f}$ are continuous families of C^1-disks in M such that

$$T_{x_0} W^\sigma_\varepsilon(\widetilde{x}, f) = \overline{P}^0(E^\sigma(\widetilde{x}))$$

for $\widetilde{x} = (x_n) \in \Lambda_f$ and $\sigma = \mathrm{s}, \mathrm{u}$. It is easily checked that for $\widetilde{x} \in \Lambda_f$,

$$W^s(\widetilde{x}, f) = \bigcup_{n\geq 0} f^{-n}(W^s_\varepsilon(\widetilde{f}^n(\widetilde{x}), f)),$$
$$W^u(\widetilde{x}, f) = \bigcup_{n\geq 0} f^{n}(W^u_\varepsilon(\widetilde{f}^{-n}(\widetilde{x}), f)),$$

where

$$W^s(\widetilde{x}, f) = \{y \in M : d(x_n, f^n(y)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(\widetilde{x}, f) = \{y \in M : \text{there exists } \widetilde{y} = (y_n) \in M_f \text{ with } y_0 = y \text{ such that } d(x_{-n}, y_{-n}) \to 0 \text{ as } n \to \infty\}.$$

Notice that $W^\sigma(\widetilde{x}, f)$ ($\sigma = \mathrm{s}, \mathrm{u}$) is not an immersed submanifold whenever f is noninvertible.

A closed f-invariant set Λ is said to be *isolated* if there is a compact neighborhood U of Λ satisfying $\Lambda_f = U_f$. If, in particular, f is a diffeomorphism, then $\Lambda_f = U_f$ means $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(U)$.

If Λ is isolated and there is a point $x \in \Lambda$ such that $\{f^n(x) : n \geq 0\}$ is dense in Λ, then Λ is called a *basic set*. It follows from [20, Theorem 3.11] and [21, p. 62] that an isolated hyperbolic set Λ decomposes into a finite disjoint union $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_s$ of basic sets Λ_i since the inverse limit system $\widetilde{f}$ of f is an expansive homeomorphism with the shadowing property.

We say that there exists an *n-cycle* in Λ if there exists $\{\Lambda_{i_j} : 1 \leq j \leq n+1\}$ such that

(1) $\Lambda_{i_1} = \Lambda_{i_{n+1}}$,
(2) $\Lambda_{i_j} \neq \Lambda_{i_k}$ $(1 \leq j \neq k \leq n)$,
(3) $\{W^s(\Lambda_{i_j}, f) \setminus \Lambda_{i_j}\} \cap \{W^u(\Lambda_{i_{j+1}}, f) \setminus \Lambda_{i_{j+1}}\} \neq \emptyset$ $(1 \leq j \leq n)$,

where

$$W^s(\Lambda_i, f) = \bigcup_{\widetilde{x}\in(\Lambda_i)_f} W^s(\widetilde{x}, f), \qquad W^u(\Lambda_i, f) = \bigcup_{\widetilde{x}\in(\Lambda_i)_f} W^u(\widetilde{x}, f).$$

We say sometimes that Λ_i has a *homoclinic point* when it has a 1-cycle.

The subset

$$\Omega(f) = \{x \in M : \text{for any neighborhood } U \text{ of } x \text{ there is } n > 0 \text{ such that } f^n(U) \cap U \neq \emptyset\}$$

is closed and satisfies $f(\Omega(f)) \subset \Omega(f)$. We say that $\Omega(f)$ is the *nonwandering set*. Notice that if the set of periodic points, $\mathrm{Per}(f)$, is dense in $\Omega(f)$, then $f(\Omega(f)) = \Omega(f)$. Recall that f satisfies *Axiom A* if $\mathrm{Per}(f)$ is dense in $\Omega(f)$ and $\Omega(f)$ is hyperbolic. When f satisfies Axiom A, it is easily checked that $\Omega(f)$ is isolated, and so $\Omega(f)$ decomposes into a finite disjoint union of basic sets. We say that an Axiom A differentiable map f *has no cycles* if there are no cycles in $\Omega(f)$. Define

$$\mathcal{P}(M) = \{f \in C^1(M) : \text{every periodic point of } f \text{ is hyperbolic}\},$$
$$\mathcal{AN}(M) = \{f \in C^1(M) : f \text{ satisfies Axiom A and has no cycles}\}.$$

Since $\mathcal{AN}(M)$ is open in $C^1(M)$ [14, Theorem B], we have $\mathcal{AN}(M) \subset \operatorname{int} \mathcal{P}(M)$. Here $\operatorname{int} E$ denotes the interior of E.

If $D_x f : T_x M \to T_{f(x)} M$ is not injective, then x is called a *singular point* for f. Denote by $S(f)$ the set of all singular points of f. Obviously, $S(f)$ is a closed subset of M. Notice that an expanding map has no singular points.

Let $f \in C^1(M)$. Then f is said to be C^1-*structurally stable* if there exists a neighborhood $\mathcal{U}(f)$ of f such that for $g \in \mathcal{U}(f)$, g is topologically conjugate to f. A differentiable map which is C^1-structurally stable has no singular points [8, p. 381]. But this is not true for C^2-structural stability [2, Theorem 3]. We say that f is C^1 Ω-*stable* if there exists a neighborhood $\mathcal{U}(f)$ of f such that $g|\Omega(g)$ is topologically conjugate to $f|\Omega(f)$ for all $g \in \mathcal{U}(f)$. Notice that C^1-differentiable maps satisfying C^1 Ω-stability belong to $\operatorname{int} \mathcal{P}(M)$. This follows from [3, Theorem 1].

Our main theorem is the following:

THEOREM A. *If a C^1-map f belonging to* $\operatorname{int} \mathcal{P}(M)$ *satisfies the condition*

$$\Omega(f) \cap S(f) \subset \{p \in \mathrm{Per}(f) : p \textit{ is a sink}\},$$

then f satisfies Axiom A and has no cycles.

The proof of this theorem is based upon results related to stability problems from Mañé [12], Palis [18] and Przytycki [21].

If f satisfies Axiom A and $\Omega(f)$ is the disjoint union $\Omega_1 \cup \Omega_2$ of two closed f-invariant sets such that:

(i) $f|\Omega_1$ is injective,
(ii) Ω_2 is contained in the closure of all source periodic points,

then f is said to satisfy *strong Axiom A*. When f is a diffeomorphism, the notion of strong Axiom A coincides with that of Axiom A.

As an extension of the result of Przytycki [21, Theorem A] stated above we have:

COROLLARY B. *If $f \in C^1(M)$ satisfies the assumption of Theorem* A, *then the following are equivalent*:

(1) *f satisfies strong Axiom A and has no cycles*,
(2) *f is C^1 Ω-stable.*

2. Proof of Theorem A. To show Theorem A we need the following propositions, where $\mathrm{cl}(E)$ denotes the closure of E.

PROPOSITION 1. *If $f \in \operatorname{int}\mathcal{P}(M)$ and $\{\Omega(f) \setminus \mathrm{cl}(\mathrm{Per}(f))\} \cap S(f) = \emptyset$, then $\Omega(f) = \mathrm{cl}(\mathrm{Per}(f))$.*

This will follow from the techniques used to prove the closing lemma for C^1-maps with finite singular points (see Wen [26] and [27, Theorem A]).

Let $f \in \mathcal{P}(M)$. Then every periodic point p of f is hyperbolic. Thus p satisfies (1.1). We set

$$I_i(f) = \{p \in \mathrm{Per}(f) : \dim E^{\mathrm{s}}(p) = i\} \quad (0 \le i \le \dim M) \tag{2.1}$$

where $E^{\mathrm{s}}(p)$ is as in (1.1), and denote by $\sharp E$ the cardinality of E.

PROPOSITION 2. *Every $f \in \operatorname{int}\mathcal{P}(M)$ has the following properties*:

(a) $\sharp I_{\dim M}(f) < \infty$,
(b) $\mathrm{cl}(I_0(f))$ *is hyperbolic.*

Proposition 2(a) was proved in [19, Theorem 4.1] for diffeomorphisms and in [6] for differentiable maps without singular points. We shall give the proof of (a) for the general case. (b) is clear for diffeomorphisms because $I_0(f) = I_{\dim M}(f^{-1})$. Unfortunately it is not true that $\sharp I_0(f) < \infty$ for the noninvertible case, and so we have to give a proof. To do that, the technique of [12, Theorem I.4] is useful.

We define

$$\mathcal{F}(M) = \{f \in \operatorname{int}\mathcal{P}(M) : f \text{ satisfies the assumption of Theorem A}\}$$

and put

$$\Lambda(i_0) = \bigcup_{i=0}^{i_0} \mathrm{cl}(I_i(f)) \quad (0 \le i_0 \le \dim M). \tag{2.2}$$

PROPOSITION 3. *Let $f \in \mathcal{F}(M)$ and $0 \le i_0 \le \dim M - 2$. If $\Lambda(i_0)$ is hyperbolic and $\Lambda(i_0) \cap \mathrm{cl}(I_{i_0+1}(f)) = \emptyset$, then $\mathrm{cl}(I_{i_0+1}(f))$ is hyperbolic.*

This will be shown using the methods of [12, p. 167].

PROPOSITION 4. *Let $f \in \mathcal{F}(M)$. Then*

(a) $\mathrm{cl}(I_0(f)) \cap \bigcup_{i=1}^{\dim M} \mathrm{cl}(I_i(f)) = \emptyset$,
(b) *if $1 \le i_0 \le \dim M - 2$ and $\Lambda(i_0)$ is hyperbolic, then $\Lambda(i_0) \cap \mathrm{cl}(I_{i_0+1}(f)) = \emptyset$.*

Proposition 4(a) is clear for diffeomorphisms because $\sharp I_0(f) < \infty$, but we have to prove it for C^1-maps. We shall derive a contradiction by showing that if (a) is false then f has homoclinic points. (b) was given in [1, §3] for diffeomorphisms. We shall give the proof of (b) for the class $\mathcal{F}(M)$ of differentiable maps which contains the diffeomorphisms.

Once Propositions 2–4 are established, we conclude that $\mathrm{cl}(\mathrm{Per}(f))$ is hyperbolic when $f \in \mathcal{F}(M)$.

Indeed, $\mathrm{cl}(I_{\dim M}(f)) = I_{\dim M}(f)$ and $\mathrm{cl}(I_0(f))$ are hyperbolic by Proposition 2. From Propositions 3 and 4 it follows that $\mathrm{cl}(I_i(f))$ $(1 \le i \le \dim M - 1)$ are hyperbolic. Thus $\mathrm{cl}(\mathrm{Per}(f)) = \bigcup_{i=0}^{\dim M} \mathrm{cl}(I_i(f))$ is hyperbolic.

Combining this result and Proposition 1 shows that each $f \in \mathcal{F}(M)$ satisfies Axiom A. Using the techniques of [17, Theorem, p. 221], it is checked that if $f \in \operatorname{int} \mathcal{P}(M)$ satisfies Axiom A, then f has no cycles. Therefore Theorem A is proved.

Thus it remains to show Propositions 1–4. We devote the rest of this paper to the proofs.

3. Proof of Proposition 1. We first prepare some auxiliary results.

For $x \in M$ and $\xi > 0$ put $T_xM(\xi) = \{v \in T_xM : \|v\| \le \xi\}$. Then there exists $\xi > 0$ such that the exponential map $\exp_x : T_xM(\xi) \to M$ is a C^∞-embedding for all $x \in M$.

The following Lemmas 3.1 and 3.2 were proved in [3, Lemma 1.1] and [12, Lemma 1.8] for diffeomorphisms. But their proofs can be adapted to the noninvertible case, and so we omit them.

For $E \subset M$, let $B_\varepsilon(E)$ denote the closed ball defined by

$$B_\varepsilon(E) = \{y \in M : d(x, y) \le \varepsilon \text{ for some } x \in E\}.$$

LEMMA 3.1. *Let $f \in C^1(M)$. For every neighborhood $\mathcal{U}(f)$ of f there exist a neighborhood $\mathcal{U}_1(f) \subset \mathcal{U}(f)$ of f and $\varepsilon_1 > 0$ such that for $g \in \mathcal{U}_1(f)$, a neighborhood U of a finite sequence $\theta = \{x_1, \dots, x_N\}$ with $x_i \ne x_j$ $(i \ne j)$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ $(1 \le i \le N)$ with $\|L_i - D_{x_i}g\| \le \varepsilon$ there are $\overline{g} \in \mathcal{U}(f)$ and $\delta > 0$ with the following properties*:

(a) $B_{4\delta}(\theta) \subset U$,
(b) $\overline{g}(x) = g(x)$ $(x \in \theta \cup \{M \setminus B_{4\delta}(\theta)\})$,
(c) $\overline{g}(x) = \exp_{\overline{g}(x_i)} \circ L_i \circ \exp_{x_i}^{-1}(x)$ $(x \in B_\delta(x_i),\ 1 \le i \le N)$.

For $f \in \mathcal{P}(M)$, $0 \le i \le \dim M$ and $n > 0$ define

$$\mathrm{Per}^n(f) = \{p \in \mathrm{Per}(f) : f^n(p) = p\}, \qquad I_i^n(f) = I_i(f) \cap \mathrm{Per}^n(f),$$

where $I_i(f)$ is defined in (2.1) for $0 \le i \le \dim M$.

LEMMA 3.2. *Let $f \in \operatorname{int} \mathcal{P}(M)$ and $\mathcal{U}(f)$ be a connected open neighborhood of f contained in* $\operatorname{int}\mathcal{P}(M)$. *Then, for all $g \in \mathcal{U}(f)$, $0 \le i \le \dim M$ and $n > 0$,*

$$\sharp I_i^n(f) = \sharp I_i^n(g) < \infty.$$

LEMMA 3.3. *If $f \in \operatorname{int}\mathcal{P}(M)$, then $f(\Omega(f)) = \Omega(f)$.*

Proof. If f is a diffeomorphism, then the assertion is clear. Thus it suffices to show it for the noninvertible case. It is clear that $f(\Omega(f)) \subset \Omega(f)$. Suppose there is $q \in M$ such that $q \in \Omega(f) \setminus f(\Omega(f))$. Since $q \in \Omega(f)$, there exist sequences $\{x_i\}$ of points and $\{k_i\}$ of positive integers such that

$$d(x_i, q) \le 1/i \quad \text{and} \quad d(f^{k_i}(x_i), q) \le 1/i.$$

We can suppose that $\{f^{k_i-1}(x_i)\}$ converges to y as $i \to \infty$. Then $y \in f^{-1}(q)$ and so $y \notin \Omega(f)$. Thus there is a neighborhood $U(y)$ of y such that $f^j(U(y)) \cap U(y) = \emptyset$ for $j > 0$. Then for $i > 0$ large enough we have

$$f^{k_i-1}(x_i) \in U(y) \quad \text{and} \quad f^k(x_i) \notin U(y) \quad (0 \le k < k_i - 1). \tag{3.1}$$

Since $f(\mathrm{cl}(\mathrm{Per}(f))) = \mathrm{cl}(\mathrm{Per}(f))$, we have $q \notin \mathrm{cl}(\mathrm{Per}(f))$. Let $U(q)$ be a neighborhood of q satisfying $U(q) \cap \mathrm{cl}(\mathrm{Per}(f)) = \emptyset$, and let $\mathcal{U}(f)$ be a connected open neighborhood of f contained in $\operatorname{int}\mathcal{P}(M)$. By taking $U(y)$ and $\mathcal{U}(f)$ small enough we can suppose that for all $g \in \mathcal{U}(f)$,

$$g(U(y)) \subset U(q). \tag{3.2}$$

By using Lemma 3.1 we can find $h \in \mathcal{U}(f)$ such that

$$\begin{aligned} &\text{(i) } y \notin S(h), \\ &\text{(ii) } f(z) = h(z) \ (z \in \{y\} \cup \{M \setminus U(y)\}) \end{aligned} \tag{3.3}$$

(as above, $S(h)$ denotes the set of singular points of h). Then there is a neighborhood $V \subset U(y)$ of y such that $h|V : V \to h(V)$ is a diffeomorphism. Thus, for $i > 0$ large enough there is $x_i' \in V$ satisfying $h(x_i') = x_i$. Since $h(y) = q$ and $x_i \to q$ as $i \to \infty$, we have $x_i' \to y$ as $i \to \infty$. Thus, for $i > 0$ large enough we can construct a diffeomorphism $\varphi : M \to M$ such that

$$\varphi(f^{k_i-1}(x_i)) = x_i', \qquad \{x \in M : \varphi(x) \ne x\} \subset U(y), \qquad g = h \circ \varphi \in \mathcal{U}(f)$$

and so

$$g(f^{k_i-1}(x_i)) = x_i.$$

Then

$$g^{k_i}(f^{k_i-1}(x_i)) = f^{k_i-1} \circ g(f^{k_i-1}(x_i)) = f^{k_i-1}(x_i)$$

by (3.1) and (3.3), and $g(U(y)) \subset U(q)$ by (3.2). Thus,

$$g(f^{k_i-1}(x_i)) \in \mathrm{Per}(g) \cap U(q) \ne \emptyset.$$

Since $U(y) \cap \mathrm{cl}(\mathrm{Per}(f)) = \emptyset$, we have $f(z) = g(z)$ for $z \in \mathrm{cl}(\mathrm{Per}(f))$. Therefore, $\sharp\,\mathrm{Per}^n(f) < \sharp\,\mathrm{Per}^n(g)$ for $n = k_i$, which contradicts Lemma 3.2. ∎

LEMMA 3.4. *Let $f \in C^1(M)$ and $q \in \Omega(f)$. If $f^{-1}(q') \cap \Omega(f) \ne \emptyset$ for all $q' \in f^{-n}(q) \cap \Omega(f)$ where $n \ge 0$, and if $\{\bigcup_{k\ge 0} f^{-k}(q) \cap \Omega(f)\} \cap S(f) = \emptyset$, then for every neighborhood $\mathcal{U}(f)$ of f and every neighborhood $U(q)$ of q there is $g \in \mathcal{U}(f)$ such that*

(1) $\mathrm{Per}(g) \cap U(q) \neq \emptyset$,
(2) $\{x \in M : f(x) \neq g(x)\} \subset \bigcup_{n>0} f^{-n}(U(q))$.

Lemma 3.4 easily follows from [27, Theorem A], and so we omit the proof.

Proof of Proposition 1. Proposition 1 was proved in [9, Lemma 3.1] for the case when f is a diffeomorphism. Thus it remains to give the proof for the noninvertible case. Suppose that $q \in \Omega(f) \setminus \mathrm{cl}(\mathrm{Per}(f))$. By Lemma 3.3 we have $f^{-1}(q') \cap \Omega(f) \neq \emptyset$ for all $q' \in f^{-n}(q) \cap \Omega(f)$ and $n \geq 0$. Since $f(\mathrm{cl}(\mathrm{Per}(f))) = \mathrm{cl}(\mathrm{Per}(f))$, we have

$$\{f^{-n}(q) \cap \Omega(f)\} \cap \mathrm{cl}(\mathrm{Per}(f)) = \emptyset$$

for $n \geq 0$. Thus,

$$\{f^{-n}(q) \cap \Omega(f)\} \cap S(f) = \emptyset$$

for $n \geq 0$ because $\{\Omega(f) \setminus \mathrm{cl}(\mathrm{Per}(f))\} \cap S(f) = \emptyset$. Hence the assumptions of Lemma 3.4 were satisfied.

Let $\mathcal{U}(f)$ be a connected open neighborhood of f contained in $\mathrm{int}\,\mathcal{P}(M)$ and $U(q)$ be a neighborhood of q satisfying $U(q) \cap \mathrm{cl}(\mathrm{Per}(f)) = \emptyset$. By Lemma 3.4 there is $g \in \mathcal{U}(f)$ such that $\mathrm{Per}(g) \cap U(q) \neq \emptyset$ and

$$\{z \in M : f(z) \neq g(z)\} \subset \bigcup \{f^{-n}(U(q)) : n \geq 0\}.$$

Since $f(\mathrm{cl}(\mathrm{Per}(f))) = \mathrm{cl}(\mathrm{Per}(f))$, we have

$$\bigcup \{f^{-n}(U(q)) : n \geq 0\} \cap \mathrm{cl}(\mathrm{Per}(f)) = \emptyset,$$

and so $f(z) = g(z)$ for $z \in \mathrm{cl}(\mathrm{Per}(f))$. Therefore, $\sharp\,\mathrm{Per}^n(f) < \sharp\,\mathrm{Per}^n(g)$ for some $n > 0$, which contradicts Lemma 3.2.

4. Proof of Proposition 2(a). Let $f \in \mathrm{int}\,\mathcal{P}(M)$. Then it follows from [10, Theorem 4.1] that there exist a neighborhood $\mathcal{U}(f) \subset \mathrm{int}\,\mathcal{P}(M)$ of f and numbers $0 < \lambda_0 < 1$, $m_0 > 0$ and $\tau_0 > 0$ such that for all $g \in \mathcal{U}(f)$ the following hold:

(a) for $\widetilde{p} = (p_n) \in \bigcup_{i=1}^{\dim M} I_i(g)_g$ with $\varrho(g, p_0) = n > \tau_0$,

$$\prod_{j=0}^{[n/m_0]-1} \|D\widetilde{g}^{m_0}|E^{\mathrm{s}}(\widetilde{g}^{m_0 j}(\widetilde{p}))\| \leq \lambda_0^{[n/m_0]}, \tag{4.1}$$

(b) for $\widetilde{p} = (p_n) \in \bigcup_{i=0}^{\dim M-1} I_i(g)_g$ with $\varrho(g, p_0) = n > \tau_0$,

$$\prod_{j=0}^{[n/m_0]-1} \|(D\widetilde{g}^{m_0}|E^{\mathrm{u}}(\widetilde{g}^{m_0 j}(\widetilde{p})))^{-1}\| \leq \lambda_0^{[n/m_0]}, \tag{4.2}$$

(c) for $\widetilde{p} = (p_n) \in \bigcup_{i=1}^{\dim M} I_i(g)_g$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|D\widetilde{g}^{m_0}|E^{\mathrm{s}}(\widetilde{g}^{m_0 j}(\widetilde{p}))\| \le \log \lambda_0, \tag{4.3}$$

(d) for $\widetilde{p} = (p_n) \in \bigcup_{i=0}^{\dim M-1} I_i(g)_g$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(D\widetilde{g}^{m_0}|E^{\mathrm{u}}(\widetilde{g}^{m_0 j}(\widetilde{p})))^{-1}\| \le \log \lambda_0, \tag{4.4}$$

where $I_i(g)$ is as in (2.1), $I_i(g)_g$ is as in (1.3) and $[r]$ denotes the greatest integer not greater than r.

Let ε_1 satisfy the conclusion of Lemma 3.1 for $\mathcal{U}_0(f)$ and let $\lambda_0 < \lambda_1 < 1$. Choose $\varepsilon_0 > 0$ such that $(1+\varepsilon_0)\lambda_1 < 1$ and $\varepsilon_0 < \frac{1}{2}(\varepsilon_1/2)^{m_0}$, and take $H_1 \ge 1$ satisfying $\varepsilon_0 > e^{-H_1}$. Denote by $N(\lambda_0, \lambda_1) > 0$ the smallest integer satisfying

$$N(\lambda_0, \lambda_1) \log(\lambda_1/\lambda_0) > H_1, \tag{4.5}$$

and write

$$C(\lambda_0, \lambda_1) = \frac{\log(\lambda_1/\lambda_0)}{H_1}. \tag{4.6}$$

LEMMA 4.1. *Let a sequence* $\{p(n) : 0 \le n \le N-1\}$ *satisfy the following*:

(i) $N \ge N(\lambda_0, \lambda_1)$,
(ii) $p(n) > 0$,
(iii) $-H_1 \le \log p(n)$,
(iv) $\prod_{n=0}^{N-1} p(n) \le \lambda_0^N$.

Then there exist an integer k *with* $k > NC(\lambda_0, \lambda_1)$ *and a sequence* $0 \le n_1 < \ldots < n_k < N-1$ *such that for* $1 \le j \le k$ *and* $n_j < l \le N-1$,

$$\prod_{n=n_j+1}^{l} p(n) \le \lambda_1^{l-n_j}.$$

The statement of Lemma 4.1 is a reformulation of the result stated in [19, Lemma, p. 212] and [12, Lemma II.3], and so we omit the proof.

We set

$$Q = \{x \in \mathrm{cl}(I_{\dim M}(f)) : \|D_x f^{m_0}\| < \varepsilon_0\}.$$

Then there is $\delta > 0$ such that

(4.7) (a) if $d(x,y) \le 2\delta$ $(x \in \mathrm{cl}(I_{\dim M}(f)) \setminus Q,\ y \in M)$, then

$$\|D_y f^{m_0}\| \le (1+\varepsilon_0)\|D_x f^{m_0}\|,$$

(b) if $d(x,y) \le 2\delta$ $(x, y \in \mathrm{cl}(I_{\dim M}(f)))$, then

$$\big|\,\|D_y f^{m_0}\| - \|D_x f^{m_0}\|\,\big| \le \varepsilon_0.$$

Put $\lambda_2 = (1+\varepsilon_0)\lambda_1$. Since M is compact, there is $K > 0$ such that for $\{x_1, \ldots, x_K\} \subset M$ with $x_i \neq x_j$ $(i \neq j)$ there exist x_i, x_j $(1 \leq i \neq j \leq K)$ satisfying $d(x_i, x_j) \leq (1-\lambda_2)\delta$. Let $N' > 0$ be an integer such that $K \leq N'C(\lambda_0, \lambda_1)$.

To obtain the conclusion of Proposition 2(a) suppose that $\sharp I_{\dim M}(f) = \infty$. Since $\sharp\,\mathrm{Per}^n(f) < \infty$ for $n > 0$ (by Lemma 3.2), there is a periodic point $p \in I_{\dim M}(f)$ with period $\varrho(f,p)$ satisfying

$$\varrho(f,p) \geq \max\{\tau_0, m_0N', m_0N(\lambda_0, \lambda_1)\}.$$

Put $N = [\varrho(f,p)/m_0]$. If $q = f^{m_0n}(p) \in Q$ for some $0 \leq n \leq N-1$, then we can construct a family $\{L_{f^i(q)} : T_{f^i(q)}M \to T_{f^{i+1}(q)}M\}_{i=0}^{m_0-1}$ of isomorphisms such that

$$\|L_{f^i(q)} - D_{f^i(q)}f\| \leq \varepsilon_1,$$
$$\inf\{\|L_{f^i(q)}(v)\| : v \in T_{f^i(q)}M \text{ with } \|v\| = 1\} \geq \varepsilon_1/2.$$

By Lemma 3.1 there is $g \in \mathcal{U}_0(f)$ such that

(1) $g(x) = f(x)$ for $x \in \{p, f(p), \ldots, f^{\varrho(f,p)-1}(p)\}$,

(2) if $f^{m_0n}(p) \notin Q$ for $0 \leq n \leq N-1$, then $D_{f^i(p)}g = D_{f^i(p)}f$ for $m_0n \leq i \leq m_0(n+1)-1$,

(3) if $f^{m_0n}(p) \in Q$ for $0 \leq n \leq N-1$, then $D_{f^i(p)}g = L_{f^i(p)}$ for $m_0n \leq i \leq m_0(n+1)-1$,

(4) $D_{f^i(p)}g = D_{f^i(p)}f$ for $Nm_0 \leq i \leq \varrho(f,p)-1$.

Define a function $p(\cdot) : \{0, 1, \ldots, N-1\} \to \mathbb{R}$ by

$$p(n) = \|D_{f^{m_0n}(p)}g^{m_0}\|.$$

Then $-H_1 < \log p(n)$ for $0 \leq n \leq N-1$. Since $g \in \mathcal{U}_0(f)$, by (4.1) we have

$$\prod_{n=0}^{N-1} p(n) \leq \lambda_0^N,$$

and so $\{p(n)\}$ satisfies the conditions of Lemma 4.1. Thus there are an integer $k > K$ and a sequence $0 \leq n_1 < \ldots < n_k < N-1$ such that

$$\prod_{n=n_j+1}^{l} p(n) \leq \lambda_1^{l-n_j} \quad (1 \leq j \leq k,\ n_j < l \leq N-1). \tag{4.8}$$

By the choice of K there are $0 \leq i < j \leq k$ such that

$$d(g^{m_0n_i}(p), g^{m_0n_j}(p)) \leq (1-\lambda_2)\delta.$$

By (4.7) and (4.8) it is easily checked that

(4.9) (1) $g^{m_0(n_j-n_i)}|B_\delta(g^{m_0n_i}(p))$ is a Lipschitz map and its Lipschitz constant is less than $\lambda_2 < 1$,

(2) $g^{m_0(n_j-n_i)}(B_\delta(g^{m_0n_i}(p))) \subset B_\delta(g^{m_0n_i}(p))$.

Thus there is a unique $z \in B_\delta(g^{m_0 n_i}(p))$ satisfying $g^{m_0(n_j-n_i)}(z) = z$. Since $N = [\varrho(f,p)/m_0]$ and $0 < n_1 < \ldots < n_k < N-1$, we have $0 < m_0(n_j-n_i) < \varrho(f,p)$, and so $z \neq g^{m_0 n_i}(p)$. On the other hand, since

$$g^{\varrho(f,p)m_0(n_j-n_i)} : B_\delta(g^{m_0 n_i}(p)) \to B_\delta(g^{m_0 n_i}(p))$$

is a contraction, we have $z = g^{m_0 n_i}(p)$, which is a contradiction. Thus $\sharp I_{\dim M}(f) = \infty$ cannot happen. Therefore Proposition 2(a) is proved.

5. Proof of Key lemma (Lemma 5.1) and Proposition 2(b). Let Λ be a closed f-invariant set. We say that a $D\widetilde{f}$-invariant subbundle $E \subset T\mathbb{M}|\Lambda_f$ is *contracting* if $D\widetilde{f}|E$ is contracting, and that E is *expanding* if $D\widetilde{f}|E$ is expanding.

Let $f \in \operatorname{int}\mathcal{P}(M)$ and $I_i(f)$ be as in (2.1). Let m_0 and λ_0 satisfy (4.1)–(4.4). It follows from [10, Proposition II.1] that $T\mathbb{M}|\mathrm{cl}(I_i(f))_f$ $(1 \leq i \leq \dim M - 1)$ splits into the Whitney sum $T\mathbb{M}|\mathrm{cl}(I_i(f))_f = \widetilde{E}_i^{\mathrm{s}} \oplus \widetilde{E}_i^{\mathrm{u}}$ of subbundles $\widetilde{E}_i^{\mathrm{s}}$ and $\widetilde{E}_i^{\mathrm{u}}$ such that

(5.1) (a) $D\widetilde{f}^{m_0}(\widetilde{E}_i^{\mathrm{s}}) \subset \widetilde{E}_i^{\mathrm{s}},\ D\widetilde{f}^{m_0}(\widetilde{E}_i^{\mathrm{u}}) = \widetilde{E}_i^{\mathrm{u}}$,
(b) $D\widetilde{f}^{m_0}|\widetilde{E}_i^{\mathrm{u}} : \widetilde{E}_i^{\mathrm{u}} \to \widetilde{E}_i^{\mathrm{u}}$ is injective,
(c) $\|D\widetilde{f}^{m_0}|\widetilde{E}_i^{\mathrm{s}}(\widetilde{x})\| \cdot \|(D\widetilde{f}^{m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{x}))^{-1}\| \leq \lambda_0$ for $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$.

It is easily checked from [20, §0 and §1] that for $1 \leq i \leq \dim M - 1$,

(5.2) (1) $\widetilde{E}_i^{\mathrm{s}}$ and $\widetilde{E}_i^{\mathrm{u}}$ are C^0-vector bundles over $\mathrm{cl}(I_i(f))_f$,
(2) if $\widetilde{x} = (x_n), \widetilde{y} = (y_n) \in \mathrm{cl}(I_i(f))_f$ satisfy $x_0 = y_0$, then $\widetilde{E}_i^{\mathrm{s}}(\widetilde{x}) = \widetilde{E}_i^{\mathrm{s}}(\widetilde{y})$, and so we write $\widetilde{E}_i^{\mathrm{s}}(x_0) = \overline{P}^0(\widetilde{E}_i^{\mathrm{s}}(\widetilde{x}))(\subset T_{x_0}M)$ where $\overline{P}^0$ is defined as in (1.4) (notice that $\widetilde{E}_i^{\mathrm{u}}(\widetilde{x}) \neq \widetilde{E}_i^{\mathrm{u}}(\widetilde{y})$ in general),
(3) $\mathrm{cl}(I_i(f))$ is hyperbolic if and only if $\widetilde{E}_i^{\mathrm{s}}$ is contracting and $\widetilde{E}_i^{\mathrm{u}}$ expanding.

In the case when $i = 0$, $\mathrm{cl}(I_0(f))$ is hyperbolic if and only if $T\mathbb{M}|\mathrm{cl}(I_0(f))_f$ is expanding. If f is a diffeomorphism, then we know [19, Theorem 4.1] that $\sharp I_0(f) < \infty$ and $I_0(f)$ is hyperbolic.

Lemma 5.1. *Let $f \in \operatorname{int}\mathcal{P}(M)$. Then*

(a) $T\mathbb{M}|\mathrm{cl}(I_0(f))_f$ *is expanding*,

(b) *if $f \in \mathcal{F}(M)$ and $\widetilde{E}_i^{\mathrm{s}}$ is contracting for some $1 \leq i \leq \dim M - 1$, then $\widetilde{E}_i^{\mathrm{u}}$ is expanding.*

If we establish Lemma 5.1, then we obtain Proposition 2(b) from Lemma 5.1(a). The proof of Lemma 5.1(a) is similar to that of (b), and so we omit it. To show (b) we suppose that $\widetilde{E}_i^{\mathrm{s}}$ is contracting and $\widetilde{E}_i^{\mathrm{u}}$ is not expanding for some $1 \leq i \leq \dim M - 1$. Then we can find a periodic point

$\widetilde{p} \in \bigcup_{i=0}^{\dim M-1} I_i(f)_f$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(D\widetilde{f}^{m_0}|E^{\mathrm{u}}(\widetilde{f}^{m_0 j}(\widetilde{p})))^{-1}\| > \log \lambda_0, \tag{5.3}$$

which in fact contradicts (4.4). Thus it remains to find a periodic point satisfying (5.3). To do that we need the techniques used in proving Theorem I.4 of [12].

By (5.1)(b) we can define

$$D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}} : \widetilde{E}_i^{\mathrm{u}} \to \widetilde{E}_i^{\mathrm{u}}$$

by $D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{x}) = (D\widetilde{f}^{m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{f}^{-m_0}(\widetilde{x})))^{-1}$ for $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$. We say that for $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$ and $n > 0$ the pair $(\widetilde{x}, \widetilde{f}^{m_0 n}(\widetilde{x}))$ is a γ-*string* if

$$\prod_{j=1}^{n} \|D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{f}^{m_0 j}(\widetilde{x}))\| \le \gamma^n,$$

and that it is a *uniform* γ-*string* if $(\widetilde{f}^{m_0 k}(\widetilde{x}), \widetilde{f}^{m_0 n}(\widetilde{x}))$ is a γ-string for $0 \le k < n$. Let us say that for $0 \le N < n$ a pair $(\widetilde{x}, \widetilde{f}^{m_0 n}(\widetilde{x}))$ is an (N,γ)-*obstruction* if $(\widetilde{x}, \widetilde{f}^{m_0 k}(\widetilde{x}))$ is not a γ-string for $N \le k < n$.

Take γ_i $(0 \le i \le 4)$ with

$$0 < \lambda_0 < \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < 1.$$

Let $N(\gamma_i, \gamma_j)$ and $C(\gamma_i, \gamma_j)$ $(0 \le i < j \le 4)$ be as in (4.5) and (4.6), and let $\widetilde{d}$ be a compatible metric for the product topological space $\mathbb{M}$.

Lemma 5.2. *If $\widetilde{E}_i^{\mathrm{u}}$ is not expanding, then for every $\varepsilon > 0$ there exists a compact invariant set $\Lambda(\varepsilon) \subset \mathrm{cl}(I_i(f))_f$ such that each $\widetilde{x}^0 \in \Lambda(\varepsilon)$ has the following property: there exist $\widetilde{x}^1 \in \Lambda(\varepsilon) \cap I_i(f)_f$ arbitrarily near to $\widetilde{x}^0$, $n_1 \ge 0$ and $\widetilde{y} \in \Lambda(\varepsilon)$ such that*

(a) $\widetilde{d}(\widetilde{f}^{m_0 n_1}(\widetilde{x}^1), \widetilde{y}) < \varepsilon/4$,
(b) $(\widetilde{y}, \widetilde{f}^{m_0 n}(\widetilde{y}))$ *is an* $(N(\gamma_3,\gamma_4),\gamma_2)$-*obstruction for* $n > N(\gamma_3,\gamma_4)$,
(c) *if* $n_1 > 0$, *then* $(\widetilde{x}^1, \widetilde{f}^{m_0 n_1}(\widetilde{x}^1))$ *is a uniform* γ_4-*string.*

Moreover $\Lambda(\varepsilon)$ is the closure of its interior in $\mathrm{cl}(I_i(f))_f$.

Lemma 5.2 is checked in the same way as Lemma II.7 of [12], so we omit the proof.

The following lemma is stated in [12, Lemma II.5].

Lemma 5.3. *Let $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$ and let n, r and l be nonnegative integers with $0 \le r \le r+l \le n$. If $(\widetilde{x}, \widetilde{f}^{m_0 n}(\widetilde{x}))$ is a γ_0-string containing an $(N(\gamma_3,\gamma_4),\gamma_2)$-obstruction $(\widetilde{f}^{m_0 r}(\widetilde{x}), \widetilde{f}^{m_0 (r+l)}(\widetilde{x}))$ such that*

(a) $n \geq N(\gamma_0, \gamma_4)$,
(b) $nC(\gamma_0, \gamma_4) > r + l$,
(c) $r + l \geq N(\gamma_1, \gamma_2)$ *and*
(d) $(r + l)C(\gamma_1, \gamma_2) > r + N(\gamma_3, \gamma_4)$,

then there exists a uniform γ_4*-string* $(\widetilde{x}, \widetilde{f}^{m_0 m}(\widetilde{x}))$, $r + l \leq m \leq n$, *that is not a* γ_1*-string.*

Let $\Lambda(\varepsilon)$ be as in Lemma 5.2 and fix $\widetilde{x}^0 \in \Lambda(\varepsilon)$. Choose $\widetilde{x}^1 \in \Lambda(\varepsilon) \cap I_i(f)_f$, $\widetilde{y} \in \Lambda(\varepsilon)$ and $n_1 > 0$ as in Lemma 5.2 and take N_1 with

$$N_1 > \max\{N(\gamma_3, \gamma_4), N(\gamma_1, \gamma_2)\}.$$

Since $\Lambda(\varepsilon)$ is the closure of its interior in $\mathrm{cl}(I_i(f))_f$ and $(\widetilde{y}, \widetilde{f}^{m_0 N_1}(\widetilde{y}))$ is an $(N(\gamma_3, \gamma_4), \gamma_2)$-obstruction, there exists $\widetilde{x}^2 \in \Lambda(\varepsilon) \cap I_i(f)_f$ such that $\widetilde{d}(\widetilde{x}^2, \widetilde{y}) < \varepsilon/4$ and $(\widetilde{x}^2, \widetilde{f}^{m_0 N_1}(\widetilde{x}^2))$ is an $(N(\gamma_3, \gamma_4), \gamma_2)$-obstruction. Since $\widetilde{x}^2 \in I_i(f)_f$ and $\lambda_0 < \gamma_0$, we deduce by (4.4) that $(\widetilde{x}^2, \widetilde{f}^{m_0 n}(\widetilde{x}^2))$ is a γ_0-string for n large enough. Thus $(\widetilde{x}^2, \widetilde{f}^{m_0 n}(\widetilde{x}^2))$ satisfies the conditions of Lemma 5.3 ($r = 0$, $l = N_1$), and so we can choose $N_1 \leq n_2 \leq n$ such that $(\widetilde{x}^2, \widetilde{f}^{m_0 n_2}(\widetilde{x}^2))$ is a uniform γ_4-string, but not a γ_1-string.

Put $K = \min_{\widetilde{x} \in \mathrm{cl}(I_i(f))_f} \|D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{x})\| > 0$ and take $0 < k_0 < 1$ with $\lambda_0 < k_0^2 \gamma_1$ and $\gamma_4 < k_0$. Since N_1 is large enough and $n_2 \geq N_1$, we can suppose that

$$\gamma_1^{n_2} K^{n_1} \geq (k_0 \gamma_1)^{n_2 + n_1}.$$

Continuing in this manner we obtain the following lemma.

LEMMA 5.4. *Suppose that* $\widetilde{E}_i^{\mathrm{u}}$ *is not expanding. Then for all* $\varepsilon > 0$ *and* γ_1, γ_4 *with* $0 < \lambda_0 < \gamma_1 < \gamma_4 < 1$ *there exist sequences* $\{\widetilde{x}^j\}_{j \geq 1} \subset \Lambda(\varepsilon)$ *and* $\{n_j\}_{j \geq 1}$ *such that*

(1) $\widetilde{d}(\widetilde{f}^{m_0 n_j}(\widetilde{x}^j), \widetilde{x}^{j+1}) < \varepsilon/2$ $(j \geq 1)$,

(2) *if* $n_j > 0$, *then* $(\widetilde{x}^j, \widetilde{f}^{m_0 n_j}(\widetilde{x}^j))$ *is a uniform* γ_4*-string*,

(3) *if* $j \geq 2$ *is even, then* $n_j > 0$ *and* $(\widetilde{x}^j, \widetilde{f}^{m_0 n_j}(\widetilde{x}^j))$ *is not a* γ_1*-string, and*

$$\gamma_1^{n_j} K^{n_{j-1}} \geq (k_0 \gamma_1)^{n_j + n_{j-1}}.$$

To show (5.3) we extend the continuous bundles $\widetilde{E}_i^{\mathrm{s}}$ and $\widetilde{E}_i^{\mathrm{u}}$ to a neighborhood of $\mathrm{cl}(I_i(f))_f$. In the same way as in the proof of [4, Theorem (4.2)] it is checked that there are a closed neighborhood V of $\mathrm{cl}(I_i(f))_f$ and a C^0-splitting $TM_f|V = \widehat{E}_i^{\mathrm{s}} \oplus \widehat{E}_i^{\mathrm{u}}$ such that

(a) if $\widetilde{x} \in V \cap \widetilde{f}^{-m_0}(V)$, then $D\widetilde{f}^{m_0}(\widehat{E}_i^{\sigma}(\widetilde{x})) = \widehat{E}_i^{\sigma}(\widetilde{f}^{m_0}(\widetilde{x}))$ $(\sigma = \mathrm{s}, \mathrm{u})$,
(b) $\widehat{E}_i^{\sigma}|\mathrm{cl}(I_i(f))_f = \widetilde{E}_i^{\sigma}$ $(\sigma = \mathrm{s}, \mathrm{u})$,

(c) there is $0 < \lambda_0 < \lambda < 1$ such that for $\widetilde{x} \in V \cap \widetilde{f}^{-m_0}(V)$,

$$\|D\widetilde{f}^{m_0}|\widehat{E}_i^{\mathrm{s}}(\widetilde{x})\| \cdot \|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{f}^{m_0}(\widetilde{x}))\| \leq \lambda.$$

Choose $\delta > 0$ such that if $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$ and $\widetilde{y} = (y_i) \in M_f$ satisfy $\widetilde{d}(\widetilde{x}, \widetilde{y}) < \delta$, then $\widetilde{y} \in V$ and

$$(5.4) \qquad k_0\|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{y})\| \leq \|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{x})\| \leq k_0^{-1}\|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{y})\|.$$

Let $0 < \varepsilon < \delta$ be sufficiently small. Choose $\{\widetilde{x}^j\}_{j\geq 1}$ and $\{n_j\}_{j\geq 1}$ satisfying the assertion of Lemma 5.4 for this ε. Without loss of generality we suppose that $\widetilde{d}(\widetilde{x}^1, \widetilde{f}^{m_0 n_k}(\widetilde{x}^k)) < \varepsilon/2$ for some large $k > 0$ because $\Lambda(\varepsilon)$ is compact. Then we have to find $\widetilde{p} \in \mathrm{Per}(\widetilde{f})$ such that

$$(5.5) \quad \begin{aligned} &\widetilde{f}^{m_0 n}(\widetilde{p}) = \widetilde{p}, \\ &\widetilde{d}(\widetilde{f}^{m_0 l}(\widetilde{f}^{m_0(n_0+n_1+\ldots+n_{j-1})}(\widetilde{p})), \widetilde{f}^{m_0 l}(\widetilde{x}^j)) < \delta \ (0 \leq l \leq n_j,\ 1 \leq j \leq k), \end{aligned}$$

where $n = n_1 + \ldots + n_k$ and $n_0 = 0$.

If (5.5) is established, then the point $\widetilde{p}$ meets our requirement. In fact it suffices to see that (5.3) holds for $\widetilde{p}$. By (5.4) and Lemma 5.4(2) we have

$$\prod_{l=1}^{n} \|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{f}^{m_0 l}(\widetilde{p}))\| < k_0^{-m_0}\gamma_4^{m_0},$$

and so $\widehat{E}_i^{\mathrm{u}}(\widetilde{p}) \subset E^{\mathrm{u}}(\widetilde{p})$ since $k_0^{-1}\gamma_4 < 1$. Thus $\widetilde{p} \in \bigcup_{j=i}^{\dim M-1} I_j(f)_f$. On the other hand, by (5.4), (5.5) and Lemma 5.4(3),

$$\prod_{l=1}^{n} \|D\widetilde{f}^{-m_0}|\widehat{E}_i^{\mathrm{u}}(\widetilde{f}^{m_0 l}(\widetilde{p}))\| \geq k_0^{2n}\gamma_1^n > \lambda_0^n.$$

Therefore we obtain (5.3).

It remains to show (5.5). To do that we apply the local stable manifold theorem for diffeomorphisms ([5], [23]).

For $\widetilde{x} \in \mathbb{M}$ and $\xi > 0$ put $T_{\widetilde{x}}\mathbb{M}(\xi) = \{(\widetilde{x}, v) \in T\mathbb{M} : \|v\| \leq \xi\}$. Then $\exp_{\widetilde{x}} : T_{\widetilde{x}}\mathbb{M}(\xi) \to M$ defined by

$$\exp_{\widetilde{x}} = \exp_{x_0} \circ \overline{P}^0|T_{\widetilde{x}}\mathbb{M}(\xi)$$

is a C^∞-embedding for small $\xi > 0$ as described in §3. Since $S(f) \cap \mathrm{cl}(I_i(f)) = \emptyset$, there exists $0 < r_0 < \xi$ such that

$$F_{\widetilde{x}}^{-1} = (\exp_{\widetilde{x}}^{-1} \circ f^{m_0} \circ \exp_{\widetilde{f}^{-m_0}\widetilde{x}})^{-1}|T_{\widetilde{x}}\mathbb{M}(r_0)$$

is a C^1-embedding for $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$.

Let $\widetilde{x} \in \mathrm{cl}(I_i(f))_f$ and $\widetilde{E}_i^{\sigma}(\widetilde{x})$ be as in (5.1) for $\sigma = \mathrm{s}, \mathrm{u}$. We put $\widetilde{E}_i^{\mathrm{s}}(\widetilde{x}, r) = \widetilde{E}_i^{\mathrm{s}}(\widetilde{x}) \cap T_{\widetilde{x}}\mathbb{M}(r)$ $(r > 0)$ and denote by o the zero vector of

$T_{\widetilde{x}}\mathbb{M}$. We put

$$\Sigma^{\mathrm{b}}(\widetilde{x},r)=\{\sigma:\widetilde{E}^{\mathrm{s}}_i(\widetilde{x},r)\to\widetilde{E}^{\mathrm{u}}_i(\widetilde{x}):\max_{v\in\widetilde{E}^{\mathrm{s}}_i(\widetilde{x},r)}\|\sigma(v)\|<\infty,\ \mathrm{Lip}(\sigma)\le 1\},$$

and

$$\Sigma^0(\widetilde{x},r)=\{\sigma\in\Sigma^b(\widetilde{x},r):\|\sigma(o)\|\le r\}.$$

Here $\mathrm{Lip}(\sigma)$ denotes a Lipschitz constant of σ. Define

$$d'(\sigma,\sigma')=\max_{v\in\widetilde{E}^{\mathrm{s}}_i(\widetilde{x},r)}\|\sigma(v)-\sigma'(v)\|\qquad(\sigma,\sigma'\in\Sigma^b(\widetilde{x},r)).$$

Then $(\Sigma^b(\widetilde{x},r),d')$ is a complete metric space and $\Sigma^0(\widetilde{x},r)$ is a closed subset of $\Sigma^b(\widetilde{x},r)$.

Let $\varepsilon_0>0$ be small enough and choose $0<r_1\le r_0$ satisfying

$$\mathrm{Lip}((F^{-1}_{\widetilde{x}}-D_{\widetilde{x}}\widetilde{f}^{-m_0})|T_{\widetilde{x}}\mathbb{M}(2r_1))\le\varepsilon_0\qquad\text{for }\widetilde{x}\in\mathrm{cl}(I_i(f))_f.$$

Since $\widetilde{E}^{\mathrm{s}}_i$ is contracting by the assumption of Lemma 5.1(b), we have

$$\|D\widetilde{f}^{m_0}|\widetilde{E}^{\mathrm{s}}_i\|\le\mu<1$$

for some $\mu<1$ (take m_0 large enough if necessary). Let $p^\sigma:\widetilde{E}^{\mathrm{s}}_i\oplus\widetilde{E}^{\mathrm{u}}_i\to\widetilde{E}^\sigma_i$ $(\sigma=\mathrm{s},\mathrm{u})$ be the natural projection. Then it is easily checked that if $\sigma\in\Sigma^0(\widetilde{x},r_1)$, then $p^{\mathrm{s}}\circ F^{-1}_{\widetilde{x}}\circ(\mathrm{id},\sigma):\widetilde{E}^{\mathrm{s}}_i(\widetilde{x},r)\to\widetilde{E}^{\mathrm{s}}_i(\widetilde{f}^{-m_0}(\widetilde{x}))$ is an embedding such that

$$p^{\mathrm{s}}\circ F^{-1}_{\widetilde{x}}\circ(\mathrm{id},\sigma)(\widetilde{E}^{\mathrm{s}}_i(\widetilde{x},r_1))\supset\widetilde{E}^{\mathrm{s}}_i(\widetilde{f}^{-m_0}(\widetilde{x}),r_1(1-2\varepsilon_0\mu)/\mu),$$

and so the graph transformation

$$\Gamma_{\widetilde{x}}(\sigma)=(p^{\mathrm{u}}\circ F^{-1}_{\widetilde{x}}\circ(\sigma,\mathrm{id}))\circ[p^{\mathrm{s}}\circ F^{-1}_{\widetilde{x}}\circ(\sigma,\mathrm{id})]^{-1}|\widetilde{E}^{\mathrm{s}}_i(\widetilde{f}^{-m_0}(\widetilde{x}),r_1)$$

is well defined and $F^{-1}_{\widetilde{x}}(\mathrm{graph}(\sigma))\supset\mathrm{graph}(\Gamma_{\widetilde{x}}(\sigma))$. Moreover, from (5.1)(c) it follows that for $\sigma,\sigma'\in\Sigma^0(\widetilde{x},r_1)$,

(1) $\mathrm{Lip}(\Gamma_{\widetilde{x}}(\sigma))\le\dfrac{\varepsilon_0\mu+\lambda_0}{1-2\varepsilon_0\mu}<1,$

(2) $\|\Gamma_{\widetilde{x}}(\sigma)(o)\|\le\{\|D\widetilde{f}^{-m_0}|\widetilde{E}^{\mathrm{u}}_i(\widetilde{x})\|+\varepsilon_0\}\dfrac{\mu}{1-2\varepsilon_0\mu}\|\sigma(o)\|,$

(3) $d'(\Gamma_{\widetilde{x}}(\sigma),\Gamma_{\widetilde{x}}(\sigma'))\le\{\|D\widetilde{f}^{-m_0}|\widetilde{E}^{\mathrm{u}}_i(\widetilde{x})\|+2\varepsilon_0\}\dfrac{\mu}{1-2\varepsilon_0\mu}d'(\sigma,\sigma').$

By (1) we have $\Gamma_{\widetilde{x}}(\sigma)\subset\Sigma^{\mathrm{b}}(\widetilde{f}^{-m_0}(\widetilde{x}),r_1)$.

We are now in a position to prove (5.5). Let $\{\widetilde{x}^j\}_{j\ge1}$ and $\{n_j\}_{j\ge1}$ satisfy the conclusion of Lemma 5.4 for $\varepsilon>0$ small enough, and let $k>0$ satisfy $\widetilde{d}(\widetilde{x}^1,\widetilde{f}^{n_km_0}(\widetilde{x}^k))<\varepsilon/2$. If $n_j=0$ then j is odd by Lemma 5.4(3), and so $n_{j+1}>0$. Thus we suppose that $n_j>0$, $(\widetilde{x}^j,\widetilde{f}^{n_jm_0}(\widetilde{x}^j))$ is a uniform γ_4-string for $1\le j\le k$, $\widetilde{d}(\widetilde{f}^{n_jm_0}(\widetilde{x}^j),\widetilde{x}^{j+1})<\varepsilon$ for $1\le j\le k-1$ and $\widetilde{d}(\widetilde{f}^{n_km_0}(\widetilde{x}^k),\widetilde{x}^1)<\varepsilon$. To avoid complication we show (5.5) for the case when $k=1$.

Choose $\varepsilon_0' > 0$ with $e^{\varepsilon_0'}\gamma_4 < 1$ and suppose

$$2\varepsilon_0 < (e^{\varepsilon_0'} - 1) \inf_{\widetilde{x} \in \mathrm{cl}(I_i(f))_f} \|D_{\widetilde{x}} \widetilde{f}^{-m_0}\|$$

because ε_0 is small enough. We put $\overline{\Gamma} = \Gamma_{\widetilde{f}^{m_0}(\widetilde{x}^1)} \circ \ldots \circ \Gamma_{\widetilde{f}^{n_1 m_0}(\widetilde{x}^1)}$. By applying inductively the above estimates (1)–(3), we find that for $\sigma, \sigma' \in \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1)$,

(1′) $\overline{\Gamma}(\sigma) \subset \Sigma^0(\widetilde{x}^1, r_1)$,

(2′)
$$\|\overline{\Gamma}(\sigma)(o)\| \le \Big\{ \prod_{j=1}^{n_1} \{\|D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{f}^{jm_0}(\widetilde{x}^1))\| + \varepsilon_0\} \frac{\mu}{1 - 2\varepsilon_0\mu} \Big\} \|\sigma(o)\|$$
$$\le \Big(e^{\varepsilon_0'} \frac{\mu}{1 - 2\varepsilon_0\mu}\Big)^{n_1} \Big\{ \prod_{j=1}^{n_1} \|D\widetilde{f}^{-m_0}|\widetilde{E}_i^{\mathrm{u}}(\widetilde{f}^{jm_0}(\widetilde{x}^1))\| \Big\} \|\sigma(o)\|$$
$$\le \Big(e^{\varepsilon_0'}\gamma_4 \frac{\mu}{1 - 2\varepsilon_0\mu}\Big)^{n_1} \|\sigma(o)\|,$$

(3′)
$$d'(\overline{\Gamma}(\sigma), \overline{\Gamma}(\sigma')) \le \Big(e^{\varepsilon_0'}\gamma_4 \frac{\mu}{1 - 2\varepsilon_0\mu}\Big)^{n_1} d'(\sigma, \sigma').$$

Let $\sigma \in \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1)$. Since $\widetilde{E}_i^\sigma$ is continuous ($\sigma = \mathrm{s}, \mathrm{u}$) and $\widetilde{d}(\widetilde{x}^1, \widetilde{f}^{n_k m_0}(\widetilde{x}^1)) < \varepsilon$, by (1) and (2′) there is a unique $\overline{\sigma} \in \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1)$ such that

$$\mathrm{graph}(\overline{\sigma}) \subset \exp^{-1}_{\widetilde{f}^{n_1 m_0}(\widetilde{x}^1)} \circ \exp_{\widetilde{x}^1} \circ \overline{\Gamma}(\sigma),$$

and so we can define $\Gamma^0 : \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1) \to \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1)$ by $\Gamma^0(\sigma) = \overline{\sigma}$. From (3′) it follows that Γ^0 is a contracting map, and thus it has a unique fixed point $\sigma_0 \in \Sigma^0(\widetilde{f}^{n_1 m_0}(\widetilde{x}^1), r_1)$. Then $f^{n_1 m_0}(\mathrm{graph}(\sigma_0)) \subset \mathrm{graph}(\sigma_0)$. By Brouwer's theorem there is $p \in \mathrm{graph}(\sigma_0)$ such that $f^{n_1 m_0}(p) = p$. Put $\widetilde{p} = (\ldots, p, f(p), \ldots, f^{n_1 m_0 - 1}(p), p, \ldots) \in \mathrm{Per}(\widetilde{f})$. Then it is easily checked that $\widetilde{p}$ meets our requirement.

6. Proof of Proposition 3. To show Proposition 3 we need properties of Borel probability measures used in [12, §1 and §3]. Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on a compact metric space X. Let $f : X \to X$ be a continuous map and Λ be a closed f-invariant set. We denote by $\mathcal{M}(f|\Lambda)$ the set of all f-invariant measures belonging to $\mathcal{M}(\Lambda)$ and by $\mathcal{M}_{\mathrm{e}}(f|\Lambda)$ that of all ergodic f-invariant measures.

Let $f \in \mathrm{int}\,\mathcal{P}(M)$ and $I_i(f)$ be as in (2.1). Let m_0 and λ_0 be numbers satisfying (4.1)–(4.4), (5.1) and (5.2).

Lemma 6.1. *Let $f \in \mathcal{F}(M)$ and $0 \le i_0 \le \dim M - 2$ be as in Proposition* 3, *and $\Lambda(i_0)$ be as in* (2.2). *If $\mu \in \mathcal{M}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$ satisfies*

$$\int\limits_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|\widetilde{E}^{\mathrm{s}}_{i_0+1}\|\, d\mu > \log \lambda_0, \tag{6.1}$$

then $\mu(\Lambda(i_0)) > 0$.

This result was proved in [12, Theorem I.6] for diffeomorphisms. For the noninvertible case we can apply the method given in [12].

LEMMA 6.2. *Let* f *and* i_0 *be as in Lemma* 6.1. *Suppose that* $\mu \in \mathcal{M}_{\mathrm{e}}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$. *Then, given a neighborhood* V *of* μ *in* $\mathcal{M}(X)$ *and a compact set* D *disjoint from the support of* μ, *there exist a* C^1*-map* g, *arbitrarily* C^1 *close to* f *and coinciding with* f *on* D, *and a periodic orbit* $\widetilde{p} = (p_n)$ *of* g *with period* mm_0 *such that*

(a) $\mu_0 = m^{-1}\sum_{n=0}^{m-1} \delta_{p_{nm_0}} \in V$,
(b) $p_{nm_0} \notin D$ *for* $n \in \mathbb{Z}$,

where δ_x *is the point measure supported at* x.

Lemma 6.2 was obtained in [12, Theorem III.1] by using the ergodic closing lemma proved in [10, Theorem A] for diffeomorphisms and in [13, Theorem, p. 173] for C^1-maps without singular points. However the proof in [13] can be adapted to our case. Thus we omit the proof of Lemma 6.2.

Proof of Lemma 6.1. The proof is very similar to that of [12, Theorem I.6]. Let $\mu \in \mathcal{M}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$ satisfy (6.1). We first check the case when μ is ergodic.

Let $W \subset \mathbb{M}$ be a small neighborhood of $\mathrm{cl}(I_{i_0+1}(f))_f$. Choose an open neighborhood W_0 of $\mathrm{cl}(I_{i_0+1}(f))$ such that if $\widetilde{x} = (x_n) \in \mathbb{M}$ satisfies $x_{nm_0} \in W_0$ for $n \in \mathbb{Z}$, then $\widetilde{x} \in W$. By Lemma 6.2 there are g, arbitrarily C^1-near f, and a periodic orbit $\widetilde{p} = (p_n)$ of g with period mm_0 such that $\mu_0 = m^{-1}\sum_{n=0}^{m-1} \delta_{p_{nm_0}}$ is close to μ in $\mathcal{M}(M)$ and $p_{nm_0} \notin M \setminus W_0$ for $n \in \mathbb{Z}$. Then μ_0 concentrates on W_0.

Since g is C^1-near f, we can suppose $g \in \mathcal{P}(M)$. As in (2.1) and (2.2) define

$$I_i(g) = \{q \in \mathrm{Per}(g) : \dim E^{\mathrm{s}}(q) = i\}, \qquad \Lambda'(i) = \bigcup_{k=0}^{i} \mathrm{cl}(I_k(g))$$

for $0 \le i \le \dim M$. Since p_0 is a periodic point of g with period mm_0, it is hyperbolic and so the tangent space T_pM splits as in (1.1). If we prove

$$\dim E^{\mathrm{s}}(p_0) \le i_0, \tag{6.2}$$

then $\mu_0(\Lambda'(i_0)) = 1$. Since $\Lambda'(i_0)$ and μ_0 converge to $\Lambda(i_0)$ and μ respectively as $g \to f$, we have $\mu(\Lambda(i_0)) = 1$. Lemma 6.1 proved.

Thus it is enough to show (6.2). To do that we use a continuous splitting

$$T\mathbb{M}|W = \widehat{E}^{\mathrm{s}}_{i_0+1} \oplus \widehat{E}^{\mathrm{u}}_{i_0+1}$$

that is an extension of the splitting

$$T\mathbb{M}|\mathrm{cl}(I_{i_0+1}(f))_f = \widetilde{E}^{\mathrm{s}}_{i_0+1} \oplus \widetilde{E}^{\mathrm{u}}_{i_0+1}$$

as in (5.1) (cf. [4, Lemma 4.4]). Let g be close to f. Then we know that $W(\widetilde{g}) = \bigcap_{n\in\mathbb{Z}} \widetilde{g}^n(W)$ has a $D\widetilde{g}^{m_0}$-invariant splitting $T\mathbb{M}|W(\widetilde{g}) = \widehat{E}^{\mathrm{s}}_g \oplus \widehat{E}^{\mathrm{u}}_g$ such that $\widehat{E}^{\sigma}_g(\widetilde{x})$ is close to $\widehat{E}^{\sigma}_{i_0+1}(\widetilde{x})$ for $\widetilde{x} \in W(\widetilde{g})$, $\sigma = \mathrm{s}, \mathrm{u}$ (cf. [5, §2]). If $\widetilde{x} = (x_n), \widetilde{y} = (y_n) \in W(\widetilde{g})$ satisfy $x_0 = y_0$, then $\widehat{E}^{\mathrm{s}}_g(\widetilde{x}) = \widehat{E}^{\mathrm{s}}_g(\widetilde{y})$, and so we write $\widehat{E}^{\mathrm{s}}_g(x_0) = \overline{P}^0(\widehat{E}^{\mathrm{s}}_g(\widetilde{x}))$ $(\subset T_{x_0}M)$. Notice that $\widehat{E}^{\mathrm{u}}_g(\widetilde{x}) \neq \widehat{E}^{\mathrm{u}}_g(\widetilde{y})$ generally.

Define a number $\lambda > 0$ by

$$\log\lambda = \int_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|\widetilde{E}^{\mathrm{s}}_{i_0+1}\|\, d\mu > \log\lambda_0.$$

Take λ_i $(i = 1, 2)$ with $0 < \lambda_0 < \lambda_1 < \lambda_2 < \lambda$. Since $\widehat{E}^{\sigma}_g$ $(\sigma = \mathrm{s}, \mathrm{u})$ and μ_0 are close to $\widehat{E}^{\sigma}_{i_0+1}$ and μ respectively, by (5.1) and (6.1) we can suppose that for $\widetilde{x} \in W(\widetilde{g})$,

$$\|D\widetilde{g}^{m_0}|\widehat{E}^{\mathrm{s}}_g(\widetilde{x})\| \cdot \|(D\widetilde{g}^{m_0}|\widehat{E}^{\mathrm{u}}_g(\widetilde{x}))^{-1}\| \le \lambda_1 \tag{6.3}$$

and

$$\int_{W_0} \log\|Dg^{m_0}|\widehat{E}^{\mathrm{s}}_g\|\, d\mu_0 \ge \log\lambda_2. \tag{6.4}$$

Since $p_{nm_0} \in W_0$ for $n \in \mathbb{Z}$, we have $\widetilde{p} \in W(\widetilde{g})$. Thus, by (6.4) and the definition of μ_0,

$$\prod_{j=0}^{m-1} \|D\widetilde{g}^{m_0}|\widehat{E}^{\mathrm{s}}_g(\widetilde{g}^{m_0 j}(\widetilde{p}))\| \ge \lambda_2^m. \tag{6.5}$$

From (6.3) it follows that

$$\begin{aligned}\|D\widetilde{g}^{-mm_0}|\widehat{E}^{\mathrm{u}}_g(\widetilde{p})\| &\le \prod_{i=0}^{m-1} \|D\widetilde{g}^{-m_0}|\widehat{E}^{\mathrm{u}}_g(\widetilde{g}^{m_0 i}(\widetilde{p}))\| \\ &\le \prod_{i=0}^{m-1} \lambda_1\|D\widetilde{g}^{m_0}|\widehat{E}^{\mathrm{s}}_g(\widetilde{g}^{m_0 i}(\widetilde{p}))\|^{-1} \\ &\le (\lambda_1/\lambda_2)^m < 1,\end{aligned}$$

and so $\overline{P}^0(\widehat{E}^{\mathrm{u}}_g(\widetilde{p})) \subset E^{\mathrm{u}}(p_0)$ where $\overline{P}^0$ is defined as in (1.4). This implies that $\dim E^{\mathrm{s}}(p) \le i_0 + 1$.

If $\dim E^{\mathrm{s}}(p_0) = i_0 + 1$, then we have $\dim \widehat{E}^{\mathrm{u}}_g(\widetilde{p}) = \dim E^{\mathrm{u}}(p_0)$, and so $\overline{P}^0(\widehat{E}^{\mathrm{u}}_g(\widetilde{p})) = E^{\mathrm{u}}(p_0)$. Thus it is easily checked that $\widehat{E}^{\mathrm{s}}_g(p_0) = E^{\mathrm{s}}(p_0)$ since p_0 is hyperbolic and $\widehat{E}^{\mathrm{s}}_g(p_0)$ is $D_{p_0}g^{mm_0}$-invariant.

By (6.5),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Dg^{m_0}|E^{\mathrm{s}}(p_{m_0 j}))\|$$
$$=\lim_{n\to\infty}\frac{1}{n}\log\prod_{j=0}^{n-1}\|D\widetilde{g}^{m_0}|\widehat{E}^{\mathrm{s}}_g(\widetilde{g}^{m_0 j}(\widetilde{p}))\|\ge\log\lambda_2,$$

which contradicts (4.3). Therefore, $\dim E^{\mathrm{s}}(p_0)\le i_0$.

If μ is not ergodic, then by using the ergodic decomposition theorem we can check that $\mu(\Lambda(i_0))>0$ (cf. for the proof, see [12, Theorem I.6]). ∎

Lemma 6.3. *Let $f\in\operatorname{int}\mathcal{P}(M)$ and $0\le i_0\le\dim M-2$. If*

$$\int_{\mathrm{cl}(I_{i_0+1}(f))}\log\|Df^{m_0}|\widetilde{E}^{\mathrm{s}}_{i_0+1}\|\,d\mu<0$$

for $\mu\in\mathcal{M}_{\mathrm{e}}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$, then $\widetilde{E}^{\mathrm{s}}_{i_0+1}$ is contracting.

The proof of Lemma 6.3 is very similar to that of [12, Lemma I.5], and so we omit it.

Lemmas 6.1, 6.3 and 5.1(b) yield Proposition 3 as follows: suppose that $\Lambda(i_0)\cap\mathrm{cl}(I_{i_0+1}(f))=\emptyset$. Then $\mu(\Lambda(i_0))=0$ for $\mu\in\mathcal{M}_{\mathrm{e}}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$, and by Lemma 6.1,

$$\int_{\mathrm{cl}(I_{i_0+1}(f))}\log\|Df^{m_0}|\widetilde{E}^{\mathrm{s}}_{i_0+1}\|\,d\mu\le\log\lambda_0<0$$

for $\mu\in\mathcal{M}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$. Therefore $\mathrm{cl}(I_{i_0+1}(f))$ is hyperbolic by Lemmas 6.3 and 5.1(b). The proof of Proposition 3 is complete.

7. Proof of Proposition 4(a). Before starting the proof we notice that if f is a diffeomorphism, then the inverse limit system of (M,f) equals the original system (M,f), and thus all the results for the inverse limit system can be transferred to the original system.

To show Proposition 4(a) we prepare the following two lemmas.

Lemma 7.1. *Let $f\in\operatorname{int}\mathcal{P}(M)$ and let $\Lambda(i_0)$ be as in* (2.2) *for f and $0\le i_0\le\dim M-2$. Let $\mathcal{U}(f)\subset\operatorname{int}\mathcal{P}(M)$ be a connected neighborhood of f. Suppose that $\Lambda(i_0)$ is hyperbolic and $g\in\mathcal{U}(f)$ satisfies $g=f$ in a neighborhood of $\Lambda(i_0)$. Then g has no cycles in $\Lambda(i_0)$.*

For the proof of Lemma 7.1 we need the following:

Lemma 7.2. *Let $g\in C^1(M)$ and $p\in M$ be a hyperbolic fixed point of g. Suppose that there is $\widetilde{x}=(x_n)\in M_g$ satisfying the following*:

(1) $d(x_n, p) \to 0,\ d(x_{-n}, p) \to 0\ (n \to \infty)$,

(2) $D_{x_{-n}} g^{2n}(T_{x_{-n}} W^{\mathrm{u}}_{\varepsilon}(\widetilde{p}, g)) + T_{x_n} W^{\mathrm{s}}_{\varepsilon}(\widetilde{p}, g) = T_{x_n} M$ *for* $n > 0$ *large enough, where* $\widetilde{p} = (\ldots, p, p, p, \ldots) \in \mathrm{Per}(g)_g$ *and* $W^{\sigma}_{\varepsilon}(\widetilde{p}, g)$ $(\sigma = \mathrm{s}, \mathrm{u})$ *is as in* (1.5).

Then for every neighborhood $U(x_0)$ *of* x_0 *there is a hyperbolic periodic point* q *such that* $\dim E^{\mathrm{s}}(q) = \dim E^{\mathrm{s}}(p)$, *where* $E^{\mathrm{s}}(p)$ *is the subspace of* $T_p M$ *as in* (1.1).

Lemma 7.2 was proved in [24] and [16, Appendix] for diffeomorphisms and extended in [25, Theorem 4.2] to differentiable maps.

Proof of Lemma 7.1. Let $\mathcal{U}(f)$ and $g \in \mathcal{U}(f)$ satisfy the assumptions of Lemma 7.1. Notice that $\Lambda(i_0)$ is an isolated hyperbolic set of g. Suppose that $\Lambda(i_0)$ has a cycle for g. By using the techniques described in [17, Theorem, p. 221] there exist $h \in \mathcal{U}(f)$, $p \in \Lambda(i_0)$ and $\widetilde{x} = (x_n) \in M_h$ satisfying the assumptions (1) and (2) of Lemma 7.2 and $h = g = f$ on some neighborhood of $\Lambda(i_0)$. Then it follows from Lemma 7.2 that $\sharp I^n_i(f) < \sharp I^n_i(h)$ for some $0 \le i \le i_0$ and $n > 0$. This contradicts Lemma 3.2. ∎

Let $f \in \mathcal{F}(M)$. Since $\mathrm{cl}(I_0(f))$ is hyperbolic by Proposition 2(b), it is isolated and can be written as a finite disjoint union $\mathrm{cl}(I_0(f)) = \Lambda_1 \cup \ldots \cup \Lambda_s$ of basic sets Λ_i. Since $T\mathbb{M}|\mathrm{cl}(I_0(f))_f$ is expanding, there exist $\varepsilon > 0$ and $0 < \lambda < 1$ such that for $1 \le a \le s$,

(7.1)
(i) $W^{\mathrm{u}}_{3\varepsilon}(\widetilde{x}, f) = B_{3\varepsilon}(x_0)\ (\widetilde{x} = (x_n) \in (\Lambda_a)_f)$,
(ii) if $\widetilde{x} = (x_n) \in (\Lambda_a)_f$ and $y \in B_{3\varepsilon}(x_0)$, then there is a unique point $y_{-1} \in B_{3\varepsilon}(x_{-1})$ such that $f(y_{-1}) = y$,
(iii) $d(x, y) \le \lambda d(f(x), f(y))\ (x, y \in B_{3\varepsilon}(\Lambda_a))$,
(iv) $(\Lambda_a)_f = \{\widetilde{y} = (y_n) \in M_f : y_n \in B_{3\varepsilon}(\Lambda_a),\ n \ge 0\}$.

Choose $0 < \delta_1 < \varepsilon - \varepsilon\lambda$ such that if $d(x, y) \le \delta_1$ $(x, y \in M)$ then $d(f(x), f(y)) \le \varepsilon$. It is easily checked that for every connected neighborhood $\mathcal{U}(f)$ of f contained in $\mathrm{int}\,\mathcal{P}(M)$ there is $0 < \delta_2 < \delta_1$ such that if $d(x, z) \le \delta_2$ $(x, z \in M)$ then we can construct a diffeomorphism $\varphi : M \to M$ satisfying

(7.2)
(i) $\varphi(z) = x$,
(ii) $\{y \in M : \varphi(y) \ne y\} \subset B_{\delta_1}(z)$,
(iii) $f \circ \varphi \in \mathcal{U}(f)$.

From the properties of differentiable maps belonging to $\mathcal{F}(M)$ we have

$$\{\mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)\} \cap S(f) = \emptyset.$$

Since $\sharp I_{\dim M}(f) < \infty$ by Proposition 2(a), $\mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$ is closed. Thus there is $0 < \delta_3 < \delta_2$ such that if $x, y \in M$ satisfy $d(x, y) \le \delta_3$, then for every point $x_{-1} \in f^{-1}(x)$ with $x_{-1} \in \mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$ there exists a unique $y_{-1} \in f^{-1}(y)$ satisfying $d(x_{-1}, y_{-1}) \le \delta_2$.

If Proposition 4(a) is false, we have

$$\Lambda_a \cap \mathrm{cl}(I_i(f)) \neq \emptyset$$

for some $1 \le a \le s$ and $0 < i \le \dim M$. Proposition 2 ensures that $\mathrm{cl}(I_0(f)) \setminus I_{\dim M}(f) = \emptyset$, and so $i \neq \dim M$. Choose $x \in \Lambda_a$, $p \in I_i(f)$ with $d(x, p) \le \delta_3$ and a periodic point $\widetilde{p} \in I_i(f)_f$ with $p_0 = p$. By (7.1)(iv) there is $0 < n < \varrho(p, f)$ such that $p_{-j} \in B_{2\varepsilon}(\Lambda_a)$ $(0 \le j \le n-1)$ and $p_{-n} \notin B_{2\varepsilon}(\Lambda_a)$. Then for $0 \le j \le n-1$ there is $x_{-j} \in \Lambda_a$ such that

$$f(x_{-j}) = x_{-j+1} \quad \text{and} \quad d(x_{-j}, p_{-j}) \le \delta_3. \tag{7.3}$$

Indeed, there is a unique $x_{-1} \in f^{-1}(x)$ such that $d(x_{-1}, p_{-1}) \le \delta_2 < \varepsilon$ because $d(x, p) \le \delta_3$ and $p_{-1} \in \mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$. Obviously, $x_{-1} \in B_{3\varepsilon}(\Lambda_a)$ since $p_{-1} \in B_{2\varepsilon}(\Lambda_a)$. By (7.1)(i) we have $x_{-1} \in W^{\mathrm{s}}(\Lambda_a, f) \cap W^{\mathrm{u}}(\Lambda_a, f)$. Since Λ_a has no cycles by Lemma 7.1, we have $x_{-1} \in \Lambda_a$. By (7.1)(ii), (iii),

$$d(x_{-1}, p_{-1}) \le \lambda d(x, p) \le \delta_3.$$

Continuing in this fashion we obtain (7.3).

Since $d(x_{-(n-1)}, p_{-(n-1)}) \le \delta_3$ (by (7.3)) and $p_{-n} \in \mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$, we can find a unique point

$$x_{-n} \in f^{-1}(x_{-(n-1)}) \subset f^{-1}(\Lambda_a)$$

such that $d(x_{-n}, p_{-n}) \le \delta_2$. By (7.2) there is a diffeomorphism $\varphi : M \to M$ such that

(i) $\varphi(p_{-n}) = x_{-n}$,
(ii) $\{y \in M : \varphi(y) \neq y\} \subset B_{\delta_1}(p_{-n})$,
(iii) $f \circ \varphi \in \mathcal{U}(f)$.

For simplicity we write $g = f \circ \varphi$. Obviously

$$g(y) = f(y) \quad (y \in M \setminus B_{\delta_1}(p_{-n})),$$
$$g(p_{-n}) = f \circ \varphi(p_{-n}) = f(x_{-n}) = x_{-(n-1)} \in \Lambda_a.$$

Since $p_{-n} \notin \Lambda_a$ and $g^i(p_{-n}) \in \Lambda_a$ for $i > 0$, we have

$$p_{-n} \in W^{\mathrm{s}}(\Lambda_a, g) \setminus \Lambda_a.$$

If we establish that

$$p_{-n} \in W^{\mathrm{u}}(\Lambda_a, g) \setminus \Lambda_a, \tag{7.4}$$

then Λ_a has a 1-cycle, that is, $p_{-n} \in \{W^{\mathrm{s}}(\Lambda_a, g) \setminus \Lambda_a\} \cap \{W^{\mathrm{u}}(\Lambda_a, g) \setminus \Lambda_a\}$. This contradicts Lemma 7.1. Hence for $1 \le a \le s$,

$$\Lambda_a \cap \bigcup_{i=1}^{\dim M} \mathrm{cl}(I_i(f)) = \emptyset.$$

This shows Proposition 4(a).

Thus it only remains to prove (7.4). Since $p_{-n} \notin B_{2\varepsilon}(\Lambda_a)$ and $p_0 = p_{\varrho(p,f)} \in B_{\delta_3}(\Lambda_a) \subset B_{2\varepsilon}(\Lambda_a)$, there is $n+1 \le m \le \varrho(p,f)$ such that $p_{-j} \notin B_{2\varepsilon}(\Lambda_a)$ for $n \le j \le m-1$, and $p_{-m} \in B_{2\varepsilon}(\Lambda_a)$. Then $d(p_{-n}, p_{-j}) > \delta_1$ for $n+1 \le j \le m$.

Indeed, if there is $n+1 \le j \le m$ such that $d(p_{-n}, p_{-j}) \le \delta_1$, then

$$d(p_{-(n-1)}, p_{-(j-1)}) = d(f(p_{-n}), f(p_{-j})) \le \varepsilon.$$

Since $p_{-(n-1)} \in B_{\delta_3}(\Lambda_a)$ by (7.3), we have $p_{-(j-1)} \in B_{2\varepsilon}(\Lambda_a)$, which contradicts the choice of m.

Thus $g^j(p_{-m}) \notin B_{\delta_1}(p_{-n})$ for $0 \le j \le m-n-1$, and so

$$g^{m-n}(p_{-m}) = p_{-n}.$$

Since $p_{-m} \in B_{2\varepsilon}(\Lambda_a)$, by (7.1)(i)–(iii) there is $\widetilde{q} \in M_f$ with $q_0 = p_{-m}$ such that

$$d(q_{-j}, \Lambda_a) \le \lambda^j d(p_{-m}, \Lambda_a) \le 2\varepsilon\lambda^j \le 2\varepsilon\lambda \quad (j \ge 1)$$

where $d(q, \Lambda) = \min_{x\in\Lambda} d(q,x)$ for $q \in M$ and a closed subset Λ. Then

$$d(q_{-j}, p_{-n}) \ge d(p_{-n}, \Lambda_a) - d(q_{-j}, \Lambda_a) > 2(\varepsilon - \varepsilon\lambda) > \delta_1,$$

and so $q_{-j} \notin B_{\delta_1}(p_{-n})$ $(j \ge 1)$. Put

$$p'_j = \begin{cases} g^j(p_{-n}) & \text{if } j \ge 0, \\ g^{m-n+j}(p_{-m}) & \text{if } -m+n \le j \le -1, \\ q_{m-n+j} & \text{if } j \le -m+n-1. \end{cases}$$

Then $(p'_j) \in M_g$ and $d(p'_{-j}, \Lambda_a) \to 0$ as $j \to \infty$. This implies that $p_{-n} = p'_0 \in W^{\mathrm{u}}(\Lambda_a, g)$, and (7.4) holds since $p_{-n} \notin \Lambda_a$.

8. Proof of Proposition 4(b). Let $f \in \mathcal{F}(M)$ and $\Lambda(i_0)$ be as in the statement of Proposition 4(b). Then $\Lambda(i_0)$ is hyperbolic and isolated by Lemma 7.1. Thus $\Lambda(i_0)$ splits into a union $\Lambda_1 \cup \ldots \cup \Lambda_s$ of basic sets. Fix $\varepsilon_0 > 0$. For $1 \le a \le s$ we define

$$V_a^+ = \bigcup\{W^{\mathrm{s}}_{\varepsilon_0}(\widetilde{x}, f) : \widetilde{x} \in (\Lambda_a)_f\}, \qquad V_a^- = \bigcup\{W^{\mathrm{u}}_{\varepsilon_0}(\widetilde{x}, f) : \widetilde{x} \in (\Lambda_a)_f\}.$$

Fix $0 < r_0 < 1$ and $0 < \delta_0 < 1$. For $n \ge 0$ define

$$\begin{aligned} &r_{n+1} = r_n^{1+\delta_0}, \\ &V(r_n, \Lambda_a) = \{x \in M : d(x, V_a^+) \le r_n,\ d(x, V_a^-) \le r_n\}. \end{aligned} \tag{8.1}$$

Then $V(r_n, \Lambda_a) \searrow \Lambda_a$ since $r_n \searrow 0$ as $n \to \infty$.

Let $m \geq 0$ be an integer and $\xi = (x_0, x_{-1}, \dots, x_{-m})$ be a finite sequence in M. We say that ξ is a *string* if

$$f(x_{-j}) = x_{-j+1} \quad \text{for} \ \ 1 \leq j \leq m.$$

Notice that the notion of string described here is different from that of γ-string introduced at the beginning of §5. For convenience of notation we make no distinction between a string ξ and a set $\{x_0, x_{-1}, \dots, x_{-m}\}$.

Let $\xi = (x_0, \dots, x_{-m})$ and $\eta = (y_0, \dots, y_{-n})$ be strings $(0 \leq n \leq m)$. Then η is said to be a *substring* of ξ if there is $0 \leq j \leq m - n$ such that $x_{-j-l} = y_{-l}$ for $0 \leq l \leq n$ (Figure 1(a)). If, in particular, $m = n$, then we have $\eta = \xi$.

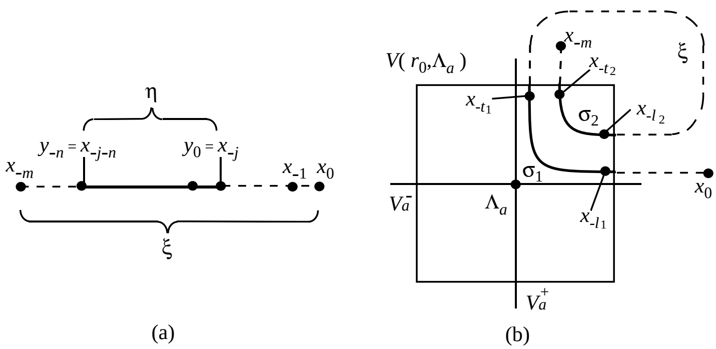

Fig. 1

Let σ be a substring of $\xi = (x_0, x_{-1}, \dots, x_{-m})$ written as

$$\sigma = (x_{-l}, x_{-l-1}, \dots, x_{-t+1}, x_{-t})$$

for some $0 < l \leq t < m$. If σ satisfies

(a) $\sigma \subset V(r_0, \Lambda_a)$,
(b) $\sigma \cap V(r_n, \Lambda_a) \neq \emptyset$,
(c) $x_{-l+1}, x_{-t-1} \notin V(r_0, \Lambda_a)$,

then we say that σ is a $(\xi, n; a)$-*string*. If $x_{-j} \in V(r_0, \Lambda_a)$ for some $1 \leq j \leq m - 1$, then there is a $(\xi, 0; a)$-string containing x_{-j} if and only if $x_{-j_1}, x_{-j_2} \notin V(r_0, \Lambda_a)$ for some j_1 and j_2 with $0 \leq j_1 < j < j_2 \leq m$.

For $(\xi, 0; a)$-strings

$$\sigma_1 = (x_{-l_1}, x_{-l_1-1}, \dots, x_{-t_1+1}, x_{-t_1}),$$
$$\sigma_2 = (x_{-l_2}, x_{-l_2-1}, \dots, x_{-t_2+1}, x_{-t_2}),$$

we introduce an order by

$$\sigma_1 < \sigma_2 \quad \text{if} \quad t_1 < l_2$$

(Figure 1(b)).

Since $\{\mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)\} \cap S(f) = \emptyset$ and $\mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$ is closed by Proposition 2(a), we can choose a compact neighborhood U_0 of $\mathrm{cl}(\mathrm{Per}(f)) \setminus I_{\dim M}(f)$ satisfying $U_0 \cap S(f) = \emptyset$. Hereafter U_0 is fixed.

Suppose that a string ξ contained in U_0 has the property that

(C) there exist $(\xi, n+1; a)$-strings σ_1 and σ_2 with $\sigma_1 < \sigma_2$ satisfying $\sigma \cap V(r_n, \Lambda_a) = \emptyset$ for every $(\xi, 0; a)$-string σ with $\sigma_1 < \sigma < \sigma_2$.

If n is large enough, by using the condition (C) we can show ([11] and [15, Theorem A, p. 57] that there exists g C^1-near f such that $g = f$ in a neighborhood of Λ_a and Λ_a has a 1-cycle. However this is inconsistent with Lemma 7.1.

Thus (C) cannot happen when a string ξ satisfies $\xi \subset U_0$ and n is large enough.

To show Proposition 4(b) we derive a contradiction by proving that if

$$\Lambda_a \cap \mathrm{cl}(I_{i_0+1}(f)) \neq \emptyset$$

for some $1 \le a \le s$, then there exists a string ξ satisfying the condition (C) for $n > 0$ large enough. To do that we prepare auxiliary results.

Since Λ_a $(1 \le a \le s)$ has no homoclinic points by Lemma 7.1, we have the following:

Lemma 8.1 [15, Proposition 4]. *Let $\{\xi^k\}$ be a sequence of strings with $\xi^k \subset U_0$. Suppose that*

(1) *if $\xi^k = (x_0^k, x_{-1}^k, \ldots, x_{-m_k+1}^k, x_{-m_k}^k)$, then $m_k \nearrow \infty$ as $k \to \infty$,*
(2) *$\mu_k = m_k^{-1} \sum_{i=1}^{m_k} \delta_{x_{-i}^k}$ converges to $\mu \in \mathcal{M}(f)$,*
(3) *$\mu(\Lambda_a) > 0$ for some $1 \le a \le s$.*

Then for $N, K > 0$ there exist integers $n \ge N$, $k \ge K$ and a $(\xi^k, n+1; a)$-string σ_1 such that $\sigma \cap V(r_n, \Lambda_a) = \emptyset$ for every $(\xi^k, 0; a)$-string $\sigma \neq \sigma_1$.

Let ξ be a string and for $1 \le a \le s$ define

$$N_a(\xi) = \min\{n \ge 0 : \xi \cap V(r_{n+1}, \Lambda_a) = \emptyset\}. \tag{8.2}$$

If a string $\xi = (x_0, x_{-1}, \ldots, x_{-m})$ satisfies

(8.3) (1) $N_a(\xi) > 0$,
(2) $x_0, x_{-m} \notin V(r_0, \Lambda_a)$,

then there exists a $(\xi, N_a(\xi); a)$-string.

Lemma 8.2. *Let $\xi^k = (x_0^k, \ldots, x_{-m_k}^k)$ and $\eta^k = (y_0^k, \ldots, y_{-n_k}^k)$ be strings with $\xi^k, \eta^k \subset U_0$ for $k > 0$. Suppose that*

(1) *x_0^k, $x_{-m_k}^k$, y_0^k, $y_{-n_k}^k \notin \bigcup_{c=1}^s V(r_0, \Lambda_c)$ for $k > 0$,*
(2) *η^k is a substring of ξ^k for $k > 0$,*
(3) *$m_k \nearrow \infty$ and $n_k \nearrow \infty$ as $k \to \infty$,*

(4) $\mu_k^0 = m_k^{-1}\sum_{i=1}^{m_k}\delta_{x_{-i}^k}$ *converges to* μ^0 *and* $\mu_k^1 = n_k^{-1}\sum_{i=1}^{n_k}\delta_{y_{-i}^k}$ *converges to* μ^1.

If there are $1 \le a, b \le s$ *and* $L \ge 0$ *such that* $\mu^0(\Lambda_a) > 0$ *and*

$$\limsup_{k\to\infty}(N_a(\xi^k) - N_b(\eta^k)) \le L,$$

then $\mu^1(\Lambda_b) > 0$.

For the proof of Lemma 8.2 we need the following two lemmas:

LEMMA 8.3 [15, Proposition 1]. *There exist* $0 < \gamma < \lambda < 1$ *such that for* $1 \le a \le s$ *and* $x \in V(r_0, \Lambda_a)$,

(1) $\gamma d(f(x), V_a^+) \le d(x, V_a^+)$,
(2) $d(x, V_a^+) \le \lambda d(f(x), V_a^+)$,
(3) *there is* $y \in f^{-1}(x)$ *such that* $\gamma d(y, V_a^-) \le d(f(y), V_a^-) = d(x, V_a^-)$,
(4) $d(f(x), V_a^-) \le \lambda d(x, V_a^-)$.

Let $0 < \gamma < \lambda < 1$ be as in Lemma 8.3 and set

$$C_{1,n} = \frac{\log r_n}{2\log\gamma} \quad \text{and} \quad C_{2,n} = 2\frac{(1+\delta_0)\log r_n}{\log\lambda}$$

for $n \ge 0$.

LEMMA 8.4. *Let* ξ *be a string with* $\xi \subset U_0$. *For* n *large enough there is* $N_n > n$ *such that for every* $(\xi, 0; a)$*-string* σ,

(1) *if* σ *is a* $(\xi, i; a)$*-string for some* $i \ge N_n$, *then*

$$\sharp\{\sigma \cap V(r_n, \Lambda_a)\} \ge C_{1,n}(1+\delta_0)^{i-n},$$

(2) *if* σ *is not a* $(\xi, i+1; a)$*-string for some* $i \ge N_n$, *then*

$$\sharp\{\sigma \cap V(r_n, \Lambda_a)\} \le C_{2,n}(1+\delta_0)^{i-n}.$$

Proof. (1) follows easily from [15, Lemma 5(b)].

To obtain (2) it is enough to show that (2) holds when $\sigma \cap V(r_n, \Lambda_a) \ne \emptyset$. Let $\xi = (x_0, \dots, x_{-m})$ and $\sigma = (x_{-k_1}, \dots, x_{-k_2})$. Then $0 < k_1 < k_2 < m$. Since $\sigma \cap V(r_n, \Lambda_a) \ne \emptyset$, there is $k_1 < t \le k_2$ satisfying $x_{-t} \in \sigma \cap V(r_n, \Lambda_a)$. Choose the smallest integers $0 < l_1 < t$ and $0 < l_2 < m - t$ such that $x_{-t+l_1+1} \notin V(r_n, \Lambda_a)$ and $x_{-t-l_2-1} \notin V(r_n, \Lambda_a)$. Then

$$\sharp\{\sigma \cap V(r_n, \Lambda_a)\} = l_1 + l_2. \tag{8.4}$$

Indeed, since $d(x_{-t+l_1+1}, V_a^-) \le \lambda^{l_1+1} d(x_{-t}, V_a^-) \le r_n$ by Lemma 8.3(4), we have

$$d(x_{-t+l_1+1}, V_a^+) > r_n.$$

By Lemma 8.3(2), for $k_1 \le j \le t - l_1 - 1$,

$$
\begin{aligned}
d(x_{-j}, V_a^+) &\ge (1/\lambda)^{(t-l_1-1)-j} d(x_{-j-(t-l_1-1-j)}, V_a^+) \\
&\ge d(x_{-t+l_1+1}, V_a^+) > r_n.
\end{aligned}
$$

This implies that

$$x_{-j} \notin V(r_n, \Lambda_a) \quad (k_1 \le j \le t - l_1 - 1).$$

Suppose that there is j_1 with $t + l_2 + 1 < j_1 \le k_2$ such that $x_{-j_1} \in V(r_n, \Lambda_a)$. Then we can find j_2 with $t + l_2 + 1 \le j_2 < j_1 \le k_2$ such that $x_{-j_2} \notin V(r_n, \Lambda_a)$. Thus,

$$d(x_t, V_a^+) \ge (1/\lambda)^{-t+j_2} d(x_{-j_2}, V_a^+) > d(x_{-j_2}, V_a^+) > r_n,$$

which contradicts $x_{-t} \in V(r_n, \Lambda_a)$. That is, $x_{-j} \notin V(r_n, \Lambda_a)$ for $t + l_2 + 1 \le j \le k_2$. Therefore we have (8.4).

From [15, Lemma 5(a)] we have the inequality

$$l_1 + l_2 \le C_{2,n}(1 + \delta_0)^{i-n}.$$

Therefore we have (2) by (8.4).

Proof of Lemma 8.2. Let $\{\xi^k\}$, $\{\mu_k^0\}$ and μ^0 be as in Lemma 8.2. Since $\mu^0(\Lambda_a) > 0$ and $\operatorname{int} V(r_n, \Lambda_a) \searrow \Lambda_a$ $(n \to \infty)$, we have

$$
\begin{aligned}
0 < \mu^0(\Lambda_a) &= \lim_{n\to\infty} \mu^0(\operatorname{int} V(r_n, \Lambda_a)) \\
&\le \lim_{n\to\infty} \liminf_{k\to\infty} \mu_k^0(\operatorname{int} V(r_n, \Lambda_a)) \\
&= \lim_{n\to\infty} \liminf_{k\to\infty} \frac{1}{m_k} \sum_{i=1}^{m_k} \delta_{x_{-i}^k}(V(r_n, \Lambda_a)) \\
&= \lim_{n\to\infty} \liminf_{k\to\infty} \frac{\sharp(\xi^k \cap V(r_n, \Lambda_a))}{m_k}.
\end{aligned} \tag{8.5}
$$

Thus,

$$N_a(\xi^k) \to \infty \quad (k \to \infty), \tag{8.6}$$

where $N_a(\xi^k)$ is defined in (8.2). Without loss of generality we suppose that $N_a(\xi^k) > 0$ for $k > 0$. Then, by (1) of Lemma 8.2, ξ^k satisfies (8.3) and so there is a $(\xi^k, N_a(\xi^k); a)$-string, say $\sigma^k(a)$, for $k > 0$.

First we prove that

$$\lim_{n\to\infty} \limsup_{k\to\infty} \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k} > 0. \tag{8.7}$$

To see this write

$$\sigma^k(a) = (x_{-l_k}^k, \dots, x_{-s_k}^k) \quad (0 < l_k < s_k < m_k)$$

for $k > 0$. Then we have the two sequences $\{l_k\}$ and $\{m_k - s_k\}$.

If, in particular, $\{l_k\}$ and $\{m_k - s_k\}$ are bounded, then by (8.5) we have (8.7) as follows:

$$\begin{aligned}
0 &< \lim_{n\to\infty} \liminf_{k\to\infty} \frac{\sharp(\xi^k \cap V(r_n, \Lambda_a))}{m_k} \\
&\le \lim_{n\to\infty} \limsup_{k\to\infty} \frac{\sharp(\xi^k \cap V(r_n, \Lambda_a))}{m_k} \\
&\le \lim_{n\to\infty} \limsup_{k\to\infty} \frac{l_k + \sharp(\sigma^k(a) \cap V(r_n, \Lambda_a)) + (m_k - s_k)}{m_k} \\
&= \lim_{n\to\infty} \limsup_{k\to\infty} \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k}.
\end{aligned}$$

To conclude (8.7) for the case when either $\{l_k\}$ or $\{m_k - s_k\}$ is unbounded we divide the proof into the following three cases:

(1) both $\{l_k\}$ and $\{m_k - s_k\}$ are unbounded,

(2) $\{l_k\}$ is unbounded, and $\{m_k - s_k\}$ is bounded,

(3) $\{l_k\}$ is bounded, and $\{m_k - s_k\}$ is unbounded.

Case (1): Suppose that $\{l_k\}$ and $\{m_k - s_k\}$ are increasing sequences, and put

$$\xi_+^k = (x_0^k, x_{-1}^k, \dots, x_{-l_k+1}^k), \qquad \mu_k^+ = \frac{1}{l_k - 1} \sum_{j=1}^{l_k - 1} \delta_{x_{-j}^k},$$

$$\xi_-^k = (x_{-s_k-1}^k, x_{-s_k-2}^k, \dots, x_{-m_k}^k), \quad \mu_k^- = \frac{1}{m_k - s_k - 1} \sum_{j=1}^{m_k - s_k - 1} \delta_{x_{-s_k-1-j}^k}$$

for $k > 0$. Then $\xi_+^k, \xi_-^k \subset \xi^k \subset U_0$ for $k > 0$. Since $\xi^k = \xi_+^k \cup \sigma^k(a) \cup \xi_-^k$, we have

$$\begin{aligned}
&\frac{\sharp(\xi^k \cap V(r_n, \Lambda_a))}{m_k} \\
&\quad = \frac{1}{m_k}\{\sharp(\xi_+^k \cap V(r_n, \Lambda_a)) + \sharp(\sigma^k(a) \cap V(r_n, \Lambda_a)) + \sharp(\xi_-^k \cap V(r_n, \Lambda_a))\} \\
&\quad = \frac{1}{m_k} \sum_{j=1}^{l_k - 1} \delta_{x_{-j}^k}(V(r_n, \Lambda_a)) + \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k} \\
&\qquad + \frac{1}{m_k} \sum_{j=1}^{m_k - s_k - 1} \delta_{x_{-s_k-1-j}^k}(V(r_n, \Lambda_a))
\end{aligned}$$

$$= \frac{l_k - 1}{m_k}\mu_k^+(V(r_n, \Lambda_a)) + \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k}$$
$$+ \frac{m_k - s_k - 1}{m_k}\mu_k^-(V(r_n, \Lambda_a))$$
$$< \mu_k^+(V(r_n, \Lambda_a)) + \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k} + \mu_k^-(V(r_n, \Lambda_a)).$$

Since μ_k^+ and μ_k^- converge to f-invariant probability measures μ^+ and μ^- respectively, by (8.5),

$$0 < \mu^+(\Lambda_a) + \lim_{n\to\infty} \limsup_{k\to\infty} \frac{\sharp(\sigma^k(a) \cap V(r_n, \Lambda_a))}{m_k} + \mu^-(\Lambda_a).$$

To obtain (8.7) it suffices to show that $\mu^+(\Lambda_a) = 0$ and $\mu^-(\Lambda_a) = 0$.

Suppose that $\mu^+(\Lambda_a) > 0$. Since $\{\xi_+^k\}$, $\{\mu_k^+\}$ and μ^+ satisfy the assumptions (1)–(3) of Lemma 8.1, for $N > 0$ large enough there exist $n > N$, $k > 0$ and a $(\xi_+^k, n+1; a)$-string $\overline{\sigma}$ such that

$$\sigma \cap V(r_n, \Lambda_a) = \emptyset \tag{8.8}$$

for every $(\xi_+^k, 0; a)$-string $\sigma \neq \overline{\sigma}$. Since $\overline{\sigma} \subset \xi_+^k \subset \xi^k$, $\overline{\sigma}$ is a $(\xi^k, n+1; a)$-string. Thus

$$\overline{\sigma} \subset V(r_0, \Lambda_a), \qquad \emptyset \neq \overline{\sigma} \cap V(r_{n+1}, \Lambda_a) \subset \xi^k \cap V(r_{n+1}, \Lambda_a),$$

which yields $N_a(\xi^k) \geq n+1$. Since $\sigma^k(a)$ is a $(\xi^k, N_a(\xi^k); a)$-string, we have

$$\sigma^k(a) \cap V(r_{n+1}, \Lambda_a) \neq \emptyset. \tag{8.9}$$

Define a string as

$$\overline{\xi}^k = (x_0^k, x_{-1}^k, \ldots, x_{-l_k+1}^k, \ldots, x_{-s_k}^k, x_{-s_k-1}^k).$$

Then $\xi_+^k \subset \overline{\xi}^k$. Since $\overline{\sigma}$ is a $(\xi_+^k, n+1; a)$-string, it is a $(\overline{\xi}^k, n+1; a)$-string. By (8.9), $\sigma^k(a)$ is a $(\overline{\xi}^k, n+1; a)$-string. Thus, by (8.8) we have

$$\sigma \cap V(r_n, \Lambda_a) = \emptyset$$

for every $(\overline{\xi}^k, 0; a)$-string σ with $\overline{\sigma} < \sigma < \sigma^k(a)$. This implies the condition (C). Since n is large enough, we have a contradiction, and so $\mu^+(\Lambda_a) > 0$ cannot happen. Similarly we have $\mu^-(\Lambda_a) = 0$. Therefore (8.7) holds in case (1).

In a similar way we obtain (8.7) for cases (2) and (3).

To complete the proof of Lemma 8.2 let $\{\eta^k\}$ be as in the statement of the lemma. Since $\limsup_{k\to\infty}(N_a(\xi^k) - N_b(\eta^k)) \leq L$, we have

$$N_b(\eta^k) \geq N_a(\xi^k) - L \tag{8.10}$$

for k large enough, and so $\lim_{k\to\infty} N_b(\eta^k) = \infty$ by (8.6). Without loss of generality we suppose that $N_b(\eta^k) > 0$ for $k > 0$. Then η^k satisfies (8.3) by

(1) of Lemma 8.2. Thus we can choose a $(\eta^k, N_b(\eta^k); b)$-string, say $\tau^k(b)$, for $k > 0$. For η^k define

$$\mu_k^1 = \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y^k_{-i}}$$

and suppose that $\mu_k^1 \to \mu^1$ as $k \to \infty$. Then

$$\begin{aligned}
\mu^1(\varLambda_b) &\geq \lim_{n\to\infty} \limsup_{k\to\infty} \mu_k^1(V(r_n, \varLambda_b)) \\
&= \lim_{n\to\infty} \limsup_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \delta_{y^k_{-i}}(V(r_n, \varLambda_b)) \\
&= \lim_{n\to\infty} \limsup_{k\to\infty} \frac{1}{n_k} \sharp(\eta^k \cap V(r_n, \varLambda_b)) \\
&\geq \lim_{n\to\infty} \limsup_{k\to\infty} \frac{1}{n_k} \sharp(\tau^k(b) \cap V(r_n, \varLambda_b)).
\end{aligned} \tag{8.11}$$

For n large enough let N_n be as in Lemma 8.4. Since $N_a(\xi^k) \to \infty$ and $N_b(\eta^k) \to \infty$ as $k \to \infty$, we have $N_a(\xi^k) \geq N_n$ and $N_b(\eta^k) \geq N_n$ for k large enough. Then, by Lemma 8.4(1),

$$\sharp(\tau^k(b) \cap V(r_n, \varLambda_b)) \geq C_{1,n}(1+\delta_0)^{N_b(\eta^k)-n}.$$

On the other hand, by Lemma 8.4(2) and (8.2),

$$\sharp(\sigma^k(a) \cap V(r_n, \varLambda_a)) \leq C_{2,n}(1+\delta_0)^{N_a(\xi^k)-n}.$$

Since $\eta^k = (y_0^k, \ldots, y_{-n_k}^k)$ is a substring of $\xi^k = (x_0^k, \ldots, x_{-m_k}^k)$, we have $n_k \leq m_k$ for $k > 0$. Therefore, by (8.10),

$$\begin{aligned}
\frac{1}{n_k} \sharp(\tau^k(b) \cap V(r_n, \varLambda_b)) &\geq \frac{1}{n_k} C_{1,n}(1+\delta_0)^{N_b(\eta^k)-n} \\
&\geq \frac{1}{m_k} C_{1,n}(1+\delta_0)^{N_a(\xi^k)-L-n} \\
&\geq \frac{C_{1,n}}{C_{2,n}}(1+\delta_0)^{-L} \frac{\sharp(\sigma^k(a) \cap V(r_n, \varLambda_a))}{m_k}.
\end{aligned} \tag{8.12}$$

Since

$$0 < C_{1,n}/C_{2,n} = (\log\lambda)/(4(1+\delta_0)\log\gamma) < 1 \quad (n \geq 0),$$

by using (8.7), (8.11) and (8.12) we have the conclusion of Lemma 8.2:

$$\mu^1(\varLambda_b) \geq \lim_{n\to\infty} \limsup_{k\to\infty} \frac{C_{1,n}}{C_{2,n}}(1+\delta_0)^{-L} \frac{\sharp(\sigma^k(a) \cap V(r_n, \varLambda_a))}{m_k} > 0. \quad \blacksquare$$

LEMMA 8.5 [1, (3.15)]. *Suppose that there exist $\overline{n}, \overline{N} > 0$ such that for strings $\xi = (x_0, \ldots, x_{-m})$ and $\eta = (y_0, \ldots, y_{-m'})$ there are a $(\xi, 0; a)$-string $\sigma = (x_{-l}, \ldots, x_{-s})$ and an integer t with $l \leq t \leq s$ such that*

(1) $x_{-t} \in V(r_N, \Lambda_a)$ *for some* $N \geq \overline{N}$,

(2) $d(x_{-t+j}, y_{-t'+j}) \leq r_0/2$ $(0 \leq j \leq t-l)$ *for some* t' *with* $t-l \leq t' \leq m'$.

Then there is $t' - t + l \leq t_0 \leq t'$ *such that*

(a) $y_{-t_0} \in V(r_{N-\overline{n}}, \Lambda_a)$,

(b) $y_{-j} \in V(r_0, \Lambda_a)$ $(t_0 \leq j \leq t')$.

Proof. The proof given in [1] was only done for diffeomorphisms. For completeness we give the full proof.

Since $V(r_n, \Lambda_a) \searrow \Lambda_a$ as $n \to \infty$, there is a sufficiently large integer $\overline{N} > 0$ satisfying

$$V(r_n, \Lambda_a) \subset U(\Lambda_a, r_0/2) \quad \text{for } n \geq \overline{N}.$$

For a string ξ let $\sigma = (x_{-l}, \dots, x_{-t}, \dots, x_{-s})$ be a $(\xi, 0; a)$-string satisfying the condition (1) of the lemma. Since $x_{-l} \in V(r_0, \Lambda_a)$, by Lemma 8.3(4) we have

$$d(x_{-l+1}, V_a^-) \leq d(x_{-l}, V_a^-) \leq r_0.$$

By the definition of a $(\xi, 0; a)$-string we have $x_{-l+1} \notin V(r_0, \Lambda_a)$, and so

$$d(x_{-l+1}, V_a^+) > r_0.$$

Since $x_{-t} \in V(r_N, \Lambda_a) \subset U(\Lambda_a, r_0/2)$, we have

$$d(x_{-t}, V_a^+) < d(x_{-t}, \Lambda_a) \leq r_0/2.$$

Thus there is $\widehat{t}$ with $l \leq \widehat{t} \leq t$ such that

$$d(x_{-t+j}, V_a^+) \leq r_0/2 \quad \text{for } 0 \leq j \leq t - \widehat{t}, \qquad d(x_{-\widehat{t}+1}, V_a^+) > r_0/2. \tag{8.13}$$

Since $x_{-t} \in V(r_N, \Lambda_a)$, by Lemma 8.3(1),

$$\begin{aligned} r_0^{(1+\delta_0)^N} = r_N &\geq d(x_{-t}, V_a^+) \geq \gamma^{t-\widehat{t}+1} d(f^{t-\widehat{t}+1}(x_{-t}), V_a^+) \\ &= \gamma^{t-\widehat{t}+1} d(x_{-\widehat{t}+1}, V_a^+) > \gamma^{t-\widehat{t}+1} r_0/2, \end{aligned}$$

and so

$$\begin{aligned} t - \widehat{t} + 1 &> (\log r_0/\log \gamma)(1+\delta_0)^N - \log(r_0/2)/\log \gamma \\ &= \frac{\log r_0 - \log(r_0/2)/(1+\delta_0)^N}{\log \gamma}(1+\delta_0)^N. \end{aligned}$$

Since $\overline{N}$ is large enough and $N \geq \overline{N}$, we can suppose that

$$t - \widehat{t} > \frac{\log r_0}{2 \log \gamma}(1+\delta_0)^N \tag{8.14}$$

(notice that $\overline{N}$ is independent of ξ and σ).

Let $\eta = (y_0, \dots, y_{-m'})$ be a string satisfying the condition (2) of Lemma 8.5. Since $x_{-t+j} \in V(r_0, \Lambda_a)$ for $0 \leq j \leq t - l$ and $x_{-t} \in U(\Lambda_a, r_0/2)$, by

Lemma 8.3(4) we have

$$d(x_{-t+j}, V_a^-) \le \lambda^j d(x_{-t}, V_a^-) \le \lambda^j d(x_{-t}, \Lambda_a) \le r_0/2 \quad (0 \le j \le t-l),$$

and so, for $0 \le j \le t - \widehat{t}$,

$$d(y_{-t'+j}, V_a^-) \le d(y_{-t'+j}, x_{-t+j}) + d(x_{-t+j}, V_a^-) \le r_0.$$

On the other hand, by (8.13),

$$d(y_{-t'+j}, V_a^+) \le d(y_{-t'+j}, x_{-t+j}) + d(x_{-t+j}, V_a^+) \le r_0 \quad (0 \le j \le t - \widehat{t}).$$

Thus we have

$$y_{-t'+j} \in V(r_0, \Lambda_a) \quad (0 \le j \le t - \widehat{t}). \tag{8.15}$$

Put $t_0 = t' - [(t-\widehat{t})/2]$. We show that t_0 satisfies assertions (a) and (b) of Lemma 8.5. Since

$$t' \ge t_0 \ge t' - (t - \widehat{t}) \ge t' - t + l,$$

(b) follows from (8.15).

To see (a) put

$$\overline{n} = \left[\frac{\log C_1' - \log C_2'}{\log(1+\delta_0)}\right] + 1$$

where

$$C_1' = \frac{2\log r_0}{\log \lambda} \quad \text{and} \quad C_2' = \frac{\log r_0}{2\log \gamma}.$$

Then $y_{-t_0} \in V(r_{N-\overline{n}}, \Lambda_a)$. Indeed, put $j_0 = [(t-\widehat{t})/2]$. Then by (8.15) and Lemma 8.3 we have

$$\begin{aligned} r_0 &\ge d(y_{-t'}, V_a^-) \ge \lambda^{-j_0} d(y_{-t'+j_0}, V_a^-) = \lambda^{-j_0} d(y_{-t_0}, V_a^-), \\ r_0 &\ge d(y_{-t'+2j_0}, V_a^+) \ge \lambda^{-j_0} d(y_{-t'+j_0}, V_a^+) = \lambda^{-j_0} d(y_{-t_0}, V_a^+). \end{aligned} \tag{8.16}$$

Suppose that $y_{-t_0} \notin V(r_n, \Lambda_a)$ for $n = N - \overline{n}$. Then

$$\text{either} \quad d(y_{-t_0}, V_a^+) > r_n, \quad \text{or} \quad d(y_{-t_0}, V_a^-) > r_n.$$

In any case, by (8.16) we have $r_0 > \lambda^{-j_0} r_n = \lambda^{-j_0} r_0^{(1+\delta_0)^n}$, and so

$$j_0 < (\log r_0/\log\lambda)\{(1+\delta_0)^n - 1\}.$$

Then

$$t - \widehat{t} \le 2(j_0+1) < 2(\log r_0/\log\lambda)\{(1+\delta_0)^n - 1\} + 2 \le C_1'(1+\delta_0)^n.$$

By (8.14) we have $C_2'(1+\delta_0)^N < t - \widehat{t} < C_1'(1+\delta_0)^n$, and so

$$N - n < \frac{\log C_1' - \log C_2'}{\log(1+\delta_0)} < \overline{n} = N - n.$$

This is a contradiction. Therefore Lemma 8.5(a) holds. ∎

Proof of Proposition 4(b). Let $f \in \mathcal{F}(M)$. As mentioned before $\Lambda(i_0) = \bigcup_{i=1}^{i_0} \mathrm{cl}(I_i(f))$ and $\Lambda(i_0)$ splits into a union

$$\Lambda(i_0) = \Lambda_1 \cup \ldots \cup \Lambda_s$$

of basic sets Λ_i.

Our aim is to conclude that $\Lambda(i_0) \cap \mathrm{cl}(I_{i_0+1}(f)) = \emptyset$. Suppose that

$$(*) \qquad \Lambda_a \cap \mathrm{cl}(I_{i_0+1}(f)) \neq \emptyset$$

for some $1 \leq a \leq s$. Then there is a sequence $\{p^k\} \subset I_{i_0+1}(f)$ of periodic points such that $d(p^k, \Lambda_a) \to 0$ as $k \to \infty$. Let $m_k = \varrho(p^k, f)$ be the period of p^k for $k > 0$. Since $\Lambda(i_0) \cap I_{i_0+1}(f) = \emptyset$, the sequence $\{m_k : k > 0\}$ tends to infinity as $k \to \infty$. Notice that m_0 is not a member of $\{m_k : k > 0\}$. In fact, m_0 is the integer satisfying (4.1)–(4.4).

For simplicity we suppose that $p^k \in V(r_0, \Lambda_a)$ for $k > 0$. Since Λ_a is isolated and $p^k \notin \Lambda_a$, for $k > 0$ we put

$$t_k = \min\{0 < t < m_k : f^t(p^k) \notin V(r_0, \Lambda_a)\}.$$

Choose a periodic orbit

$$\widetilde{q}^k = (q_j^k) \in I_{i_0+1}(f)_f \tag{8.17}$$

with $q_0^k = f^{t_k}(p^k)$ for $k > 0$ (Figure 2). Then $q_{-t_k}^k = p^k \in V(r_0, \Lambda_a)$ and $q_0^k = q_{-m_k}^k \notin V(r_0, \Lambda_a)$. Define a sequence of strings

$$\xi^k = (q_0^k, q_{-1}^k, \ldots, q_{-m_k+1}^k, q_{-m_k}^k) \tag{8.18}$$

for $k > 0$. Then each ξ^k consists of a periodic orbit and

$$\xi^k \subset I_{i_0+1}(f) \subset U_0$$

where U_0 is the compact neighborhood defined before the condition (C).

For $k > 0$ we put

$$N(\xi^k) = \max\{N_b(\xi^k) : 1 \leq b \leq s\}, \tag{8.19}$$

where $N_b(\xi^k)$ is defined in (8.2). For some $1 \leq b \leq s$ we can find a sequence k' of integers such that $N(\xi^{k'}) = N_b(\xi^{k'})$. To simplify the notations suppose that for $k > 0$,

$$N_a(\xi^k) = N(\xi^k), \qquad q_{-t_k}^k = p^k \in V(r_{N_a(\xi^k)}, \Lambda_a). \tag{8.20}$$

Since $d(p^k, \Lambda_a) \to 0$, we have

$$N(\xi^k) \to \infty \tag{8.21}$$

as $k \to \infty$. Thus we can suppose that $N(\xi^k)$ is large enough for $k > 0$ and $\{N(\xi^k)\}$ is an increasing sequence.

Since $q_{-t_k}^k \in V(r_0, \Lambda_a)$ and $q_{-m_k}^k \notin V(r_0, \Lambda_a)$, we put

$$s_k = \min\{t_k \leq s < m_k : q_{-s-1}^k \notin V(r_0, \Lambda_a)\} \quad (k > 0).$$

Combining the definitions of t_k and s_k, for $k > 0$ we have

$$q^k_{-t} \in V(r_0, \Lambda_a) \quad (1 \le t \le s_k).$$

Since $q^k_0, q^k_{-s_k-1} \notin V(r_0, \Lambda_a)$ and $0 < t_k \le s_k$ for $k > 0$, by (8.20) we find that

$$\sigma^k = (q^k_{-1}, q^k_{-2}, \dots, q^k_{-s_k+1}, q^k_{-s_k}) \tag{8.22}$$

is a $(\xi^k, N(\xi^k); a)$-string (Figure 2). Notice that $\sharp\sigma^k \to \infty$ as $k \to \infty$ by (8.21) and Lemma 8.4(1), and so $s_k \to \infty$ as $k \to \infty$.

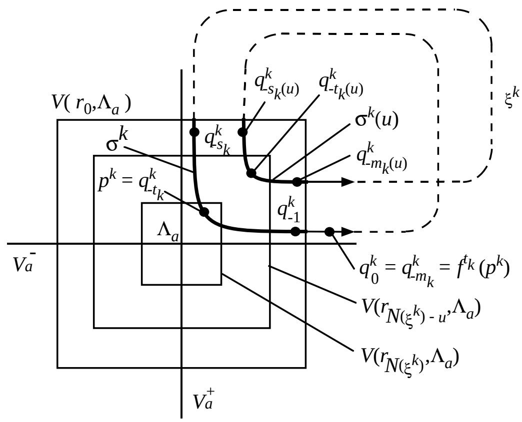

Fig. 2

With the above preparations we shall deduce Proposition 4(b) through the nine claims below.

Claim 1. *For $k > 0$ there is $s_k < j < m_k$ such that*

$$q^k_{-j} \in V(r_{N(\xi^k)-1}, \Lambda_a).$$

Proof. If this is false, then there is $k > 0$ such that $\sigma \cap V(r_{N(\xi^k)-1}, \Lambda_a) = \emptyset$ for every $(\xi^k, 0; a)$-string σ with $\sigma^k < \sigma$. Let ζ^k be a string

$$\zeta^k = (q^k_0, \dots, q^k_{-m_k} = q^k_0, q^k_{-m_k-1}, \dots, q^k_{-2m_k}),$$

and let

$$\tau^k = (q^k_{-m_k-1}, q^k_{-m_k-2}, \dots, q^k_{-m_k-s_k})$$

be a $(\zeta^k, N(\xi^k); a)$-string. Then $\tau^k \subset \zeta^k \subset U_0$ and ξ^k is a substring of ζ^k. Obviously, σ^k and τ^k are $(\zeta^k, N(\xi^k); a)$-strings, and $\sigma \cap V(r_{N(\xi^k)-1}, \Lambda_a) = \emptyset$ for every $(\xi^k, 0; a)$-string σ with $\sigma^k < \sigma < \tau^k$. Therefore we have the condition (C). Since $N(\xi^k)$ is large enough, we have a contradiction as before. Thus we have Claim 1. ∎

Fix an integer $u \geq 1$, and choose $K_0(u) > 0$ large enough satisfying $N(\xi^k) > u$ for $k \geq K_0(u)$. Since $V(r_{N(\xi^k)-1}, \Lambda_a) \subset V(r_{N(\xi^k)-u}, \Lambda_a)$ for $k \geq K_0(u)$, we put

$$(8.23) \qquad t_k(u) = \min\{s_k < j < m_k : q^k_{-j} \in V(r_{N(\xi^k)-u}, \Lambda_a)\} \quad (k \geq K_0(u)).$$

This is well defined by Claim 1, and so choose a $(\xi^k, N(\xi^k) - u; a)$-string

$$(8.24) \qquad \sigma^k(u) = (q^k_{-m_k(u)}, \ldots, q^k_{-t_k(u)}, \ldots, q^k_{-s_k(u)}) \quad (\subset V(r_0, \Lambda_a))$$

for $k \geq K_0(u)$ (Figure 2).

Claim 2. *Under the above notations, for $k \geq K_0(u)$ we have*

(1) $s_k + 1 < m_k(u) < t_k(u) < s_k(u) < m_k$,
(2) $s_k(u) - m_k(u) \to \infty$ *as* $k \to \infty$,
(3) $\{q^k_{-s_k-1}, q^k_{-s_k-2}, \ldots, q^k_{-m_k(u)+1}\} \cap V(r_{N(\xi^k)-u}, \Lambda_a) = \emptyset$,
(4) $\sigma^k(u) \cap V(r_{N(\xi^k)-u+1}, \Lambda_a) = \emptyset$.

Proof. (1) and (3) are clear. By (8.21) and Lemma 8.4(1) we have $\sharp\sigma^k = s_k(u) - m_k(u) + 1 \to \infty$ as $k \to \infty$, and so we have (2). If (4) is false, then $\sigma^k(u)$ is a $(\xi^k, N(\xi^k)-u+1; a)$-string and σ^k is a $(\xi^k, N(\xi^k)-u+1; a)$-string. By the definition of $t_k(u)$ we deduce that $\sigma \cap V(r_{N(\xi^k)-u}, \Lambda_a) = \emptyset$ for every $(\xi^k, 0; a)$-string σ with $\sigma^k < \sigma < \sigma^k(u)$. This implies the condition (C). Since $K_0(u)$ is large enough, $N(\xi^k) - u$ is large enough for $k \geq K_0(u)$. Thus we have a contradiction, and (4) is proved. ∎

Let λ_0 and m_0 be the numbers described at the beginning of §4 and let the splitting $T\mathbb{M}|\mathrm{cl}(I_{i_0+1}(f)) = \widetilde{E}^{\mathrm{s}}_{i_0+1} \oplus \widetilde{E}^{\mathrm{u}}_{i_0+1}$ be as in (5.1). For simplicity write $E = \widetilde{E}^{\mathrm{s}}_{i_0+1}$ and $F = \widetilde{E}^{\mathrm{u}}_{i_0+1}$. Then

$$(8.25) \qquad \|D\widetilde{f}^{m_0}|E(\widetilde{x})\| \cdot \|D\widetilde{f}^{-m_0}|F(\widetilde{f}^{m_0}(\widetilde{x}))\| \leq \lambda_0$$

for $\widetilde{x} \in \mathrm{cl}(I_{i_0+1}(f))_f$. Let P^0 and $\overline{P}^0$ be as in (1.2) and (1.4). As mentioned in (5.2)(2) we have $\overline{P}^0(E(\widetilde{x})) = \overline{P}^0(E(\widetilde{y}))$ when $\widetilde{x}, \widetilde{y} \in \mathrm{cl}(I_{i_0+1}(f))_f$ satisfy $P^0(\widetilde{x}) = P^0(\widetilde{y})$. Thus we write

$$E(x_0) = \overline{P}^0(E(\widetilde{x})) \subset T_{x_0}M$$

for $\widetilde{x} = (x_i) \in \mathrm{cl}(I_{i_0+1}(f))_f$, and then $\|Df|E(x_0)\| = \|D\widetilde{f}|E(\widetilde{x})\|$.

Claim 3. *For $\varepsilon > 0$ there exist continuous families*

$$\{Z^{\mathrm{s}}_\varepsilon(\widetilde{x}, f^{m_0}) : \widetilde{x} \in \mathrm{cl}(I_{i_0+1}(f))_f\} \quad \textit{and} \quad \{Z^{\mathrm{u}}_\varepsilon(\widetilde{x}, f^{m_0}) : \widetilde{x} \in \mathrm{cl}(I_{i_0+1}(f))_f\}$$

of C^1-disks on M such that

(a) *for* $\widetilde{x} = (x_i) \in \mathrm{cl}(I_{i_0+1}(f))_f$ *and* $\sigma = \mathrm{s}, \mathrm{u}$,

$$x_0 \in Z^\sigma_\varepsilon(\widetilde{x}, f^{m_0}) \subset B_\varepsilon(x_0),$$

(b) *for* $\widetilde{x} = (x_i) \in \mathrm{cl}(I_{i_0+1}(f))_f$,

$$T_{x_0} Z^{\mathrm{s}}_{\varepsilon}(\widetilde{x}, f^{m_0}) = E(x_0) \quad \text{and} \quad T_{x_0} Z^{\mathrm{u}}_{\varepsilon}(\widetilde{x}, f^{m_0}) = \overline{P}^0(F(\widetilde{x})),$$

(c) *there is* $0 < \varepsilon' \le \varepsilon$ *such that*

$$f^{m_0}(Z^{\sigma}_{\varepsilon'}(\widetilde{x}, f^{m_0})) \subset Z^{\sigma}_{\varepsilon}(\widetilde{f}^{m_0}(\widetilde{x}), f^{m_0})$$

for $\widetilde{x} \in \mathrm{cl}(I_{i_0+1}(f))_f$ *and* $\sigma = \mathrm{s}, \mathrm{u}$,

(d) *there is* $\delta = \delta(\varepsilon) > 0$ *such that if* $\widetilde{d}(\widetilde{x}, \widetilde{y}) \le \delta$ $(\widetilde{x}, \widetilde{y} \in \mathrm{cl}(I_{i_0+1}(f))_f)$ *then*

$$Z^{\mathrm{s}}_{\varepsilon}(\widetilde{x}, f^{m_0}) \cap Z^{\mathrm{u}}_{\varepsilon}(\widetilde{y}, f^{m_0})$$

is a one-point set and the intersection is transversal.

Proof. This follows from [9, Proposition 2.3] and [5, Theorem 5.1]. ∎

Fix γ_0 with $\lambda_0 < \gamma_0 < 1$. Then we have:

CLAIM 4. *For fixed* $u \ge 1$ *let* $K_0(u)$ *be as above. Then there exists* $K_1(u) > K_0(u)$ *such that for* $k \ge K_1(u)$ *there is* l *with* $0 < l \le [s_k(u)/m_0]$ *such that for* $0 \le r < l$,

$$\prod_{t=r+1}^{l} \|Df^{m_0}|E(q^k_{-m_0 t})\| \le \gamma_0^{l-r}.$$

Proof. If the claim is false, then for some $u \ge 1$ there exist infinitely many $k \ge K_0(u)$ such that

$$\prod_{t=1}^{l} \|Df^{m_0}|E(q^k_{-m_0 t})\| > \gamma_0^{l} \tag{8.26}$$

for l with $0 < l \le [s_k(u)/m_0]$. Without loss of generality we suppose that (8.26) holds for $k > 0$.

Define the Borel probability measures μ_k by

$$\mu_k = \frac{1}{[s_k(u)/m_0]} \sum_{j=1}^{[s_k(u)/m_0]} \delta_{q^k_{-m_0 j}}.$$

Then μ_k converges to μ belonging to $\mathcal{M}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$ (take a subsequence if necessary). Since, by (8.26),

$$\int\limits_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\|\, d\mu = \lim_{k\to\infty} \int\limits_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\|\, d\mu_k$$

$$= \lim_{k\to\infty} \frac{1}{[s_k(u)/m_0]} \log \prod_{j=1}^{[s_k(u)/m_0]} \|Df^{m_0}|E(q^k_{-m_0 j})\|$$

$$\geq \log\gamma_0 > \log\lambda_0,$$

by Lemma 6.1 we have

$$\mu(\Lambda(i_0)) > 0.$$

Let ξ^k $(k > 0)$ be the sequence of strings in (8.18). For $k > 0$ define a substring of ξ^k as

$$\overline{\xi}^k = (q^k_0, q^k_{-1}, \ldots, q^k_{-s_k(u)}, q^k_{-s_k(u)-1}),$$

for $\overline{\xi}^k$ define

$$\overline{\mu}_k = \frac{1}{s_k(u)+1} \sum_{j=1}^{s_k(u)+1} \delta_{q^k_{-j}},$$

and put

$$V_n = \bigcup_{b=1}^{s} V(r_n, \Lambda_b) \quad (n \geq 0).$$

Since $\overline{\mu}_k$ converges to $\overline{\mu}$ and $V_n \searrow \Lambda(i_0)$ as $n \to \infty$, we have

(8.27) $\quad \overline{\mu}(\Lambda(i_0))$

$$\geq \lim_{n\to\infty} \limsup_{k\to\infty} \overline{\mu}_k(V_n) = \lim_{n\to\infty} \limsup_{k\to\infty} \frac{\sharp\{\overline{\xi}^k \cap V_n\}}{s_k(u)+1}$$

$$\geq \lim_{n\to\infty} \limsup_{k\to\infty} \frac{1}{s_k(u)+1} \sharp\{(q^k_{-m_0}, q^k_{-2m_0}, \ldots, q^k_{-[s_k(u)/m_0]m_0}) \cap V_n\}$$

$$= \lim_{n\to\infty} \limsup_{k\to\infty} \frac{[s_k(u)/m_0]}{s_k(u)+1} \mu_k(V_n)$$

$$\geq \frac{1}{m_0} \lim_{n\to\infty} \liminf_{k\to\infty} \mu_k(\operatorname{int} V_n) \geq \frac{1}{m_0} \mu(\Lambda(i_0)) > 0,$$

from which

(8.28) $$\overline{\mu}(\Lambda_b) > 0$$

for some $1 \leq b \leq s$.

For $k > 0$ define

$$\widehat{\xi}^k = (q^k_{-s_k-1}, q^k_{-s_k-2}, \ldots, q^k_{-s_k(u)-1}) \quad (\subset \overline{\xi}^k),$$

$$\widehat{\mu}_k = \frac{1}{s_k(u)-s_k} \sum_{j=1}^{s_k(u)-s_k} \delta_{q_{-s_k-1-j}}.$$

By Claim 2(1), (2) we have

$$s_k(u) - s_k = \{s_k(u) - m_k(u)\} + \{m_k(u) - s_k\} \to \infty$$

as $k \to \infty$, and so we suppose that $\widehat{\mu}_k$ converges to $\widehat{\mu}$. By (8.23) and Claim 2(1),

$$q^k_{-t_k(u)} \in V(r_{N(\xi^k)-u}, \Lambda_a) \cap \widehat{\xi}^k$$

for $k \geq K_0(u)$, and so

$$N_a(\widehat{\xi}^k) \geq N(\xi^k) - u \quad (k \geq K_0(u)).$$

Since $N(\xi^k) \geq N_b(\xi^k) \geq N_b(\overline{\xi}^k)$ by (8.20), we have

$$N_b(\overline{\xi}^k) - N_a(\widehat{\xi}^k) \leq N(\xi^k) - (N(\xi^k) - u) = u. \tag{8.29}$$

Since $\widehat{\xi}^k$ is a substring of $\overline{\xi}^k$ and $\widehat{\xi}^k \subset U_0$, $\widehat{\xi}^k$ and $\overline{\xi}^k$ satisfy the conditions (1)–(4) of Lemma 8.2. Thus by (8.28), (8.29) and Lemma 8.2 we have

$$\widehat{\mu}(\Lambda_a) > 0.$$

Hence $\{\widehat{\xi}^k\}$, $\{\widehat{\mu}_k\}$ and $\widehat{\mu}$ satisfy the conditions (1)–(3) of Lemma 8.1, and so there exist a sufficiently large $n > 0$, $k \geq K_0(u)$ and a $(\widehat{\xi}^k, n+1; a)$-string $\widehat{\sigma}_1$ such that

$$\sigma \cap V(r_n, \Lambda_a) = \emptyset \tag{8.30}$$

for every $(\widehat{\xi}^k, 0; a)$-string $\sigma \neq \widehat{\sigma}_1$.

Since

$$\xi^k \cap V(r_{n+1}, \Lambda_a) \supset \widehat{\xi}^k \cap V(r_{n+1}, \Lambda_a) \supset \widehat{\sigma}_1 \cap V(r_{n+1}, \Lambda_a) \neq \emptyset,$$

by (8.2) and (8.19) we have

$$N(\xi^k) = N_a(\xi^k) \geq n + 1.$$

Thus the $(\xi^k, N(\xi^k); a)$-string σ^k of (8.22) contains a $(\xi^k, n+1; a)$-string. Since $\widehat{\xi}^k$ is a substring of ξ^k, $\widehat{\sigma}_1$ is a $(\xi^k, n+1; a)$-string. If σ is a $(\xi^k, 0; a)$-string with $\sigma^k < \sigma < \widehat{\sigma}_1$, then σ is a $(\widehat{\xi}^k, 0; a)$-string with $\sigma \neq \widehat{\sigma}_1$. Thus, by (8.30) we have $\sigma \cap V(r_n, \Lambda_a) = \emptyset$. This yields the condition (C). Since n is large enough, we have a contradiction as before. Claim 4 is proved. ∎

For fixed $u \geq 1$ let $s_k(u)$ be an integer satisfying (1) and (2) of Claim 2 for $u \geq K_0(u)$, and let $K_1(u)$ be as in Claim 4. For $k \geq K_1(u)$ define

$$l_k(u) = \max\left\{0 < l \leq \left[\frac{s_k(u)}{m_0}\right] : \text{for } 0 \leq r < l, \prod_{t=r+1}^{l} \|Df^{m_0}|E(q^k_{-m_0 t})\| \leq \gamma_0^{l-r}\right\}. \tag{8.31}$$

Then for $k \geq K_1(u)$ and $0 < i \leq l_k(u)$ we have

$$\|Df^{m_0 i}|E(q^k_{-m_0 l_k(u)})\| \leq \prod_{t=l_k(u)-i+1}^{l_k(u)} \|Df^{m_0}|E(q^k_{-m_0 t})\| \leq \gamma_0^i < 1.$$

Since

$$E(q^k_{-m_0 l_k(u)}) = T_{q^k_{-m_0 l_k(u)}} Z^{\mathrm{s}}_{\varepsilon}(\widetilde{f}^{-m_0 l_k(u)}(\widetilde{q}^k), f^{m_0})$$

by Claim 3(b), for $\varepsilon > 0$ small enough we have

$$d(f^{m_0 i}(y), q^k_{-m_0 l_k(u)+m_0 i}) \leq \varepsilon \quad (0 \leq i \leq l_k(u))$$

for $y \in Z^{\mathrm{s}}_{\varepsilon}(\widetilde{f}^{-m_0 l_k(u)}(\widetilde{q}^k), f^{m_0})$. Therefore, if $\theta > 0$ is sufficiently small compared with ε, then

$$d(f^j(y), q^k_{-m_0 l_k(u)+j}) \leq \varepsilon \quad (0 \leq j \leq m_0 l_k(u)) \tag{8.32}$$

for $y \in Z^{\mathrm{s}}_{\theta}(\widetilde{f}^{-m_0 l_k(u)}(\widetilde{q}^k), f^{m_0})$ $(\subset Z^{\mathrm{s}}_{\varepsilon}(\widetilde{f}^{-m_0 l_k(u)}(\widetilde{q}^k), f^{m_0}))$. Notice that θ does not depend on k and u.

Since $\dim E(x) = i_0 + 1$ for $x \in \mathrm{cl}(I_{i_0+1}(f))$ and $\Lambda(i_0)$ is hyperbolic, by taking $\varepsilon_0 > 0$ small enough we have

$$\|Df^{m_0}|E(x)\| > 1 \quad (x \in U_{2\varepsilon_0}(\Lambda(i_0)) \cap \mathrm{cl}(I_{i_0+1}(f))).$$

Here $U_{\varepsilon}(G) = \{y \in M : d(G, y) < \varepsilon\}$ for a closed set G. Since r_0 is arbitrary in (8.1), we can assume that $0 < r_0 < \varepsilon_0$. Thus,

$$V_0 = \bigcup_{b=1}^{\mathrm{s}} V(r_0, \Lambda_b) \subset U_{\varepsilon_0}(\Lambda(i_0)).$$

The choice of $l_k(u)$ ensures that $\|Df^{m_0}|E(q^k_{-m_0 l_k(u)})\| \leq \gamma_0 < 1$, and so

$$q^k_{-m_0 l_k(u)} \notin U_{2\varepsilon_0}(\Lambda(i_0)) \supset V(r_0, \Lambda_a) \quad (k \geq K_1(u)). \tag{8.33}$$

By (8.24) and Claim 2(1), for $k \geq K_1(u)$ we have

$$s_k + 1 \leq m_0 l_k(u) < m_k(u) < s_k(u), \tag{8.34}$$

and so by Claim 2(2),

$$s_k(u) - m_0 l_k(u) = \{s_k(u) - m_k(u)\} + \{m_k(u) - m_0 l_k(u)\} \to \infty \tag{8.35}$$

as $k \to \infty$. Thus $\{[s_k(u)/m_0] - l_k(u)\}$ is unbounded.

For simplicity suppose that $[s_k(u)/m_0] - l_k(u) \geq 0$ for $k \geq K_1(u)$.

Claim 5. *Under the above notations, for fixed $u \geq 1$ we have*

$$\prod_{t=l_k(u)+1}^{r} \|Df^{m_0}|E(q^k_{-m_0 t})\| \geq \gamma_0^{r-l_k(u)}$$

for $k \geq K_1(u)$ and r with $l_k(u) < r \leq [s_k(u)/m_0]$.

Proof. If this is false, then there are $k \geq K_1(u)$ and $l_k(u) < s \leq [s_k(u)/m_0]$ such that

$$\prod_{t=l_k(u)+1}^{r} \|Df^{m_0}|E(q^k_{-m_0t})\| \geq \gamma_0^{r-l_k(u)} \qquad (l_k(u) < r < s),$$

$$\prod_{t=l_k(u)+1}^{s} \|Df^{m_0}|E(q^k_{-m_0t})\| < \gamma_0^{s-l_k(u)},$$

and for $l_k(u) < r < s$,

$$\prod_{t=r+1}^{s} \|Df^{m_0}|E(q^k_{-m_0t})\| = \frac{\prod_{t=l_k(u)+1}^{s} \|Df^{m_0}|E(q^k_{-m_0t})\|}{\prod_{t=l_k(u)+1}^{r} \|Df^{m_0}|E(q^k_{-m_0t})\|} < \frac{\gamma_0^{s-l_k(u)}}{\gamma_0^{r-l_k(u)}} = \gamma_0^{s-r}. \tag{8.36}$$

Since, for $0 \leq r < l_k(u)$,

$$\begin{aligned} \prod_{t=r+1}^{s} &\|Df^{m_0}|E(q^k_{-m_0t})\| \\ &= \prod_{t=r+1}^{l_k(u)} \|Df^{m_0}|E(q^k_{-m_0t})\| \cdot \prod_{t=l_k(u)+1}^{s} \|Df^{m_0}|E(q^k_{-m_0t})\| \\ &< \gamma_0^{l_k(u)-r} \cdot \gamma_0^{s-l_k(u)} = \gamma_0^{s-r}, \end{aligned} \tag{8.37}$$

from (8.36) and (8.37) we have

$$\prod_{t=r+1}^{s} \|Df^{m_0}|E(q^k_{-m_0t})\| \leq \gamma_0^{s-r}$$

for $0 \leq r < s$, which contradicts the choice of $l_k(u)$. Therefore Claim 5 holds. ∎

By Claim 5 and (8.25) we have

$$\begin{aligned} \|D\widetilde{f}^{-m_0i}|F(\widetilde{f}^{-m_0l_k(u)}(\widetilde{q}^k))\| &\leq \prod_{t=l_k(u)}^{l_k(u)+i-1} \|D\widetilde{f}^{-m_0}|F(\widetilde{f}^{-m_0t}(\widetilde{q}^k))\| \\ &\leq \prod_{t=l_k(u)+1}^{l_k(u)+i} \lambda_0\|Df^{m_0}|E(q^k_{-m_0t})\|^{-1} \\ &\leq (\lambda_0\gamma_0^{-1})^i < 1 \end{aligned}$$

for $k \geq K_1(u)$ and $0 < i \leq [s_k(u)/m_0] - l_k(u)$. Thus the following statement is easily checked from (b) and (c) of Claim 3: for every $\varepsilon > 0$ we can take a small number $0 < \theta < \varepsilon$ such that if $y \in Z^{\mathrm{u}}_{\theta}(\widetilde{f}^{-m_0 l_k(u)}(\widetilde{q}^k), f^{m_0})$, then there is a string $(y_0, \dots, y_{-\overline{s}_k(u)})$ with $y_0 = y$ satisfying

$$d(y_{-j}, q^k_{-m_0 l_k(u)-j}) \le \varepsilon \quad (0 \le j \le \overline{s}_k(u)) \tag{8.38}$$

where

$$\overline{s}_k(u) = s_k(u) + 1 - m_0 l_k(u). \tag{8.39}$$

For fixed $u \ge 1$ let $K_1(u)$ be as in Claim 5. For $k \ge K_1(u)$ define

$$\xi^k_1(u) = (q^k_{-m_0 l_k(u)}, q^k_{-m_0 l_k(u)-1}, \dots, q^k_{-m_k(u)+2}, q^k_{-m_k(u)+1})$$

where $m_k(u)$ is as in Claim 2.

Claim 6. *For every $v \ge 1$ there is $K(u,v) \ge K_1(u)$ such that for $k \ge K(u,v)$,*

$$\xi^k_1(u) \cap V(r_{N(\xi^k)-u-v}, \Lambda_a) = \emptyset,$$

where $N(\xi^k)$ is as in (8.19).

Proof. Suppose that this is false. Then there is $v \ge 1$ such that for infinitely many k with $k \ge K_1(u)$,

$$\xi^k_1(u) \cap V(r_{N(\xi^k)-u-v}, \Lambda_a) \ne \emptyset. \tag{8.40}$$

Without loss of generality we suppose that (8.40) holds for $k \ge K_1(u)$.

It is clear that $\{\xi^k_1(u)\} \subset U_0$. Since $q^k_{-m_0 l_k(u)}, q^k_{-m_k(u)+1} \notin V(r_0, \Lambda_a)$ for $k \ge K_1(u)$, $\xi^k_1(u)$ contains a $(\xi^k_1(u), N(\xi^k) - u - v; a)$-string, and so by (8.21) and Lemma 8.4(1),

$$m_k(u) - 1 - m_0 l_k(u) \ge \sharp\{\xi^k_1(u) \cap V(r_0, \Lambda_a)\} \to \infty$$

as $k \to \infty$. For $\xi^k_1(u)$ we define

$$\mu^1_k = \frac{1}{(m_k(u)-1) - m_0 l_k(u)} \sum_{j=1}^{(m_k(u)-1)-m_0 l_k(u)} \delta_{q^k_{-m_0 l_k(u)-j}}$$

and let μ^1 be an accumulation point of μ^1_k. If we establish that

$$\mu^1(\Lambda_a) > 0, \tag{8.41}$$

then $\{\xi^k_1(u)\}$ satisfies the conditions (1)–(3) of Lemma 8.1. Thus there are sufficiently large integers n, k and a $(\xi^k_1(u), n+1; a)$-string σ_1 such that for every $(\xi^k_1(u), 0; a)$-string $\sigma \ne \sigma_1$,

$$\sigma \cap V(r_n, \Lambda_a) = \emptyset.$$

Since this implies the condition (C) for n large enough, we have a contradiction.

Thus it is enough to show (8.41) to obtain Claim 6. For $k \geq K_1(u)$ define the Borel probability measures ν_k by

$$\nu_k = \frac{1}{[s_k(u)/m_0]} \sum_{j=1}^{[s_k(u)/m_0]} \delta_{q^k_{-m_0 l_k(u) - m_0 j}}.$$

Then ν_k converges to $\nu \in \mathcal{M}(f^{m_0}|\mathrm{cl}(I_{i_0+1}(f)))$ (take a subsequence if necessary). Since

$$\begin{aligned}\int_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\| \, d\nu &= \lim_{k\to\infty} \int_{\mathrm{cl}(I_{i_0+1}(f))} \log \|Df^{m_0}|E\| \, d\nu_k \\ &\geq \log \gamma_0 > \log \lambda_0\end{aligned}$$

by Claim 5, we find that $\nu(\Lambda(i_0)) > 0$ by Lemma 6.1.

For $k \geq K_1(u)$ define a string

$$\zeta_1^k(u) = (q^k_{-m_0 l_k(u)}, q^k_{-m_0 l_k(u)-1}, \ldots, q^k_{-m_k(u)}, \ldots, q^k_{-s_k(u)}, q^k_{-s_k(u)-1}).$$

Then $\xi_1^k(u) \subset \zeta_1^k(u)$ since $m_k(u) < s_k(u)$. For $\zeta_1^k(u)$ we define

$$\nu_k^1 = \frac{1}{s_k(u) + 1 - m_0 l_k(u)} \sum_{j=1}^{s_k(u)+1-m_0 l_k(u)} \delta_{q^k_{-m_0 l_k(u) - j}}.$$

Then ν_k^1 converges to $\nu^1 \in \mathcal{M}(f)$ by (8.35). By the same calculation as in (8.27) we have

$$\nu^1(\Lambda(i_0)) \geq \frac{1}{m_0} \nu(\Lambda(i_0)) > 0,$$

and so

$$\nu^1(\Lambda_b) > 0 \tag{8.42}$$

for some $1 \leq b \leq s$.

Since $\zeta_1^k(u)$ is a substring of ξ^k, we have $N_b(\xi^k) \geq N_b(\zeta_1^k(u))$, and so by (8.19),

$$N(\xi^k) \geq N_b(\zeta_1^k(u)) \quad (k \geq K_1(u)).$$

Thus, by (8.40),

$$N_b(\zeta_1^k(u)) - N_a(\xi_1^k(u)) \leq N(\xi^k) - (N(\xi^k) - u - v) = u + v.$$

Since $\xi_1^k(u)$ is a substring of $\zeta_1^k(u)$, by (8.42) and Lemma 8.2 we have $\mu^1(\Lambda_a) > 0$. Thus (8.41) was proved. ∎

Claim 7. *Let u_1 and u_2 be integers with $1 \leq u_2 < u_1$, and let $K(u_1, 1)$ and $K(u_2, 1)$ be as in Claim* 6. *Then, for $k \geq \max\{K(u_1,1), K(u_2,1)\}$,*

$$m_0 l_k(u_1) < m_0 l_k(u_2).$$

Proof. Since $m_0 l_k(u_1) < s_k(u_1)$ by (8.34), it is enough to show that $s_k(u_1) < m_0 l_k(u_2)$. Otherwise $s_k(u_1) \geq m_0 l_k(u_2)$ for some $k \geq \max\{K(u_1, 1),$

$K(u_2, 1)\}$. By (8.24) we have

$$q^k_{-t} \in V(r_0, \Lambda_a) \quad (m_k(u_1) \le t \le s_k(u_1)).$$

Since $q^k_{-m_0 l_k(u_2)} \notin V(r_0, \Lambda_a)$ by (8.33), we have

$$m_0 l_k(u_2) < m_k(u_1),$$

and so by Claim 2(3),

$$q^k_{-t} \notin V(r_{N(\xi^k)-u_1}, \Lambda_a) \supset V(r_{N(\xi^k)-u_2-1}, \Lambda_a)$$

for $s_k + 1 \le t \le m_0 l_k(u_2)$. Combining this result and Claim 6, we have

$$q^k_{-t} \notin V(r_{N(\xi^k)-u_2-1}, \Lambda_a)$$

for $s_k + 1 \le t \le m_k(u_2) - 1$. Then

$$\sigma \cap V(r_{N(\xi^k)-u_2-1}, \Lambda_a) = \emptyset$$

for every $(\xi^k, 0; a)$-string σ with $\sigma^k < \sigma < \sigma^k(u_2)$. Since σ^k and $\sigma^k(u_2)$ are $(\xi^k, N(\xi^k) - u_2; a)$-strings, the condition (C) holds. Since $N(\xi^k)$ is large enough, we have a contradiction. Thus we have Claim 7. ∎

Let γ be as in Lemma 8.3, and $\overline{n}$ and $\overline{N}$ be as in Lemma 8.5. Let r_0 be a sufficiently small positive number as in (8.1). Choose $\varepsilon > 0$ such that

$$\varepsilon < \min\{(1-\lambda)r_0, \gamma r_0/3\}, \tag{8.43}$$

and take a small number $\theta > 0$ satisfying (8.32) and (8.38). Let $\delta = \delta(\theta) > 0$ be as in Claim 3(d). Since M is compact, there is $v_0 > 0$ such that

$$\max\{\widetilde{d}(\widetilde{x}, \widetilde{y}) : \widetilde{x}, \widetilde{y} \in \mathbb{M}\}/v_0 \le \delta.$$

If a subset G of $\mathbb{M}$ satisfies $\sharp G \ge v_0$, then we can find $\widetilde{x}, \widetilde{y} \in G$ such that $\widetilde{x} \ne \widetilde{y}$ and $\widetilde{d}(\widetilde{x}, \widetilde{y}) \le \delta$. Define

$$K = \max\{K(u, v) : 1 \le u \le v_0(2\overline{n}+1),\ 1 \le v \le v_0(2\overline{n}+1)\} \tag{8.44}$$

where $K(u, v)$ is as in Claim 6. Fix a sufficiently large integer $k \ge K$ satisfying

$$N(\xi^k) - v_0(2\overline{n}+1) \ge \overline{N} \quad \text{and} \quad r_{N(\xi^k)-v_0(2\overline{n}+1)} < \gamma r_0/3. \tag{8.45}$$

For $1 \le u_2 < u_1 \le v_0(2\overline{n}+1)$ by Claim 7 we have

$$\widetilde{f}^{-m_0 l_k(u_1)}(\widetilde{q}^k) \ne \widetilde{f}^{-m_0 l_k(u_2)}(\widetilde{q}^k)$$

where $\widetilde{q}^k$ is a point of $I_{i_0+1}(f)_f$ satisfying (8.17), and so

$$\sharp\{\widetilde{f}^{-m_0 l_k(j(2\overline{n}+1))}(\widetilde{q}^k) : 1 \le j \le v_0\} = v_0.$$

Thus,

$$\widetilde{d}(\widetilde{f}^{-m_0 l_k(j_1(2\overline{n}+1))}(\widetilde{q}^k), \widetilde{f}^{-m_0 l_k(j_2(2\overline{n}+1))}(\widetilde{q}^k)) \le \delta$$

for some $1 \le j_2 < j_1 \le v_0$. Put

$$u_1 = j_1(2\overline{n}+1) \quad \text{and} \quad u_2 = j_2(2\overline{n}+1). \tag{8.46}$$

Then

$$\text{(8.47)}\qquad \begin{aligned} &\text{(i) } 0 \le u_2 < u_1 \le v_0(2\overline{n}+1),\\ &\text{(ii) } 2\overline{n}+1 \le u_1 - u_2 \le v_0(2\overline{n}+1),\\ &\text{(iii) } \widetilde{d}(\widetilde{f}^{-m_0 l_k(u_1)}(\widetilde{q}^k), \widetilde{f}^{-m_0 l_k(u_2)}(\widetilde{q}^k)) \le \delta. \end{aligned}$$

By Claim 3(d),

$$Z^{\mathrm{s}}_\theta(\widetilde{f}^{-m_0 l_k(u_1)}(\widetilde{q}^k), f^{m_0}) \cap Z^{\mathrm{u}}_\theta(\widetilde{f}^{-m_0 l_k(u_2)}(\widetilde{q}^k), f^{m_0})$$

is one point; denote it by z.

Claim 8. *Let $\overline{s}_k(u_2)$ be as in* (8.39) *and $l_k(u)$ be as in* (8.31). *For the above point z there is a string*

$$\eta = (z_1, z_0, z_{-1}, \ldots, z_{-m_0 l_k(u_1)}, \ldots, z_{-m_0 l_k(u_1) - \overline{s}_k(u_2)}, z_{-m_0 l_k(u_1) - \overline{s}_k(u_2) - 1})$$

such that

(i) $z_{-m_0 l_k(u_1)} = z$,
(ii) $d(z_{-j}, q^k_{-j}) \le \varepsilon \ (0 \le j \le m_0 l_k(u_1))$,
(iii) $d(z_{-m_0 l_k(u_1) - j}, q^k_{-m_0 l_k(u_2) - j}) \le \varepsilon \ (0 \le j \le \overline{s}_k(u_2))$,
(iv) *either* $z_1 \notin V(r_0, \Lambda_a)$, *or* $z_0 \notin V(r_0, \Lambda_a)$,
(v) *either* $z_{-m_0 l_k(u_1) - \overline{s}_k(u_2)} \notin V(r_0, \Lambda_a)$, *or* $z_{-m_0 l_k(u_1) - \overline{s}_k(u_2) - 1} \notin V(r_0, \Lambda_a)$.

Proof. For $-1 \le j \le m_0 l_k(u_1)$ put

$$z_{-j} = f^{m_0 l_k(u_1) - j}(z).$$

Since $z \in Z^{\mathrm{u}}_\theta(\widetilde{f}^{-m_0 l_k(u_2)}(\widetilde{q}^k), f^{m_0})$, by (8.38) we can take a string $(z_{-m_0 l_k(u_1)}, \ldots, z_{-m_0 l_k(u_1) - \overline{s}_k(u_2)})$ with $z_{-m_0 l_k(u_1)} = z$ to satisfy (iii). Let $z_{-m_0 l_k(u_1) - \overline{s}_k(u_2) - 1}$ be an arbitrary point belonging to the inverse image of $z_{-m_0 l_k(u_1) - \overline{s}_k(u_2)}$. Then $\eta = (z_1, \ldots, z_{-m_0 l_k(u_1) - \overline{s}_k(u_2) - 1})$ is a string.

Clearly (i) holds. Since $z \in Z^{\mathrm{s}}_\theta(\widetilde{f}^{-m_0 l_k(u_1)}(\widetilde{q}^k), f^{m_0})$, by (8.32) we see that η satisfies (ii).

It remains to show (iv) and (v). Since $q^k_0 \notin V(r_0, \Lambda_a)$, we can check that $d(q^k_0, V^+_a) > r_0$. If $z_0 \in V(r_0, \Lambda_a)$, then by (ii) we have

$$d(z_0, V^+_a) \ge d(q^k_0, V^+_a) - d(z_0, q^k_0) > r_0 - \varepsilon.$$

By Lemma 8.3(2) and (8.43),

$$d(z_1, V^+_a) = d(f(z_0), V^+_a) \ge \frac{1}{\lambda} d(z_0, V^+_a) > \frac{1}{\lambda}(r_0 - \varepsilon) > r_0,$$

and so $z_1 \notin V(r_0, \Lambda_a)$. Thus (iv) is proved. Similarly we can check (v). Therefore Claim 8 holds. ∎

Hereafter let K be as in (8.44) and k be an integer so large that $k \geq K$. Since $q^k_{-m_0 l_k(u_1)} \notin U_{2\varepsilon_0}(\Lambda(i_0))$ by (8.33), it follows from Claim 8(ii) that

$$z_{-m_0 l_k(u_1)} \notin V(r_0, \Lambda_a). \tag{8.48}$$

Let $\sigma^k = (q^k_{-1}, \ldots, q^k_{-t_k}, \ldots, q^k_{-s_k})$ be the $(\xi^k, N(\xi^k); a)$-string of (8.22). Then, by (8.20) and (8.45),

$$q^k_{-t_k} \in \sigma^k \cap V(r_{N(\xi^k)}, \Lambda_a) \quad \text{and} \quad N(\xi^k) \geq \overline{N}.$$

By (8.34) we have $t_k < s_k < m_0 l_k(u_1)$, and so by Claim 8(ii) and (8.43),

$$d(q^k_{-t_k+j}, z_{-t_k+j}) \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t_k - 1).$$

Thus we have the conditions (1) and (2) of Lemma 8.5, and so there is $1 \leq t_1 \leq t_k$ such that

$$z_{-t_1} \in V(r_{N(\xi^k)-\overline{n}}, \Lambda_a) \quad \text{and} \quad z_{-j} \in V(r_0, \Lambda_a) \quad (t_1 \leq j \leq t_k). \tag{8.49}$$

Since $0 \leq t_1 \leq t_k \leq m_0 l_k(u_1)$ and $z_{-m_0 l_k(u_1)} \notin V(r_0, \Lambda_a)$, by (8.43) and Claim 8(iv) there exists an $(\eta, 0; a)$-string σ_1 containing z_{-t_1}.

Let u_2 be as in (8.46). For u_2 let $\sigma^k(u_2) = (q^k_{-m_k(u_2)}, \ldots, q^k_{-t_k(u_2)}, \ldots, q^k_{-s_k(u_2)})$ be the $(\xi^k, N(\xi^k) - u_2; a)$-string defined as in (8.24). By (8.23) and (8.45) we have

$$q^k_{-t_k(u_2)} \in V(r_{N(\xi^k)-u_2}, \Lambda_a), \qquad N(\xi^k) - u_2 \geq N(\xi^k) - v_0(2\overline{n} + 1) \geq \overline{N}.$$

For $0 \leq j \leq t_k(u_2) - m_k(u_2)$, by (8.34) and Claim 2(1) we have

$$0 < -j + t_k(u_2) - m_0 l_k(u_2) < \overline{s}_k(u_2),$$

and so by Claim 8(iii) and (8.43),

$$\begin{aligned} &d(q^k_{-t_k(u_2)+j}, z_{-t_k(u_2)+w(k)+j}) \\ &\qquad = d(q^k_{-m_0 l_k(u_2)-\{-j+t_k(u_2)-m_0 l_k(u_2)\}}, z_{-m_0 l_k(u_1)-\{-j+t_k(u_2)-m_0 l_k(u_2)\}}) \\ &\qquad \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t_k(u_2) - m_k(u_2)) \end{aligned}$$

where

$$w(k) = m_0 l_k(u_2) - m_0 l_k(u_1).$$

Thus we have the conditions (1) and (2) of Lemma 8.5, and so there is t_2 with $m_k(u_2) - w(k) \leq t_2 \leq t_k(u_2) - w(k)$ such that

$$\begin{aligned} &z_{-t_2} \in V(r_{N(\xi^k)-u_2-\overline{n}}, \Lambda_a), \\ &z_{-j} \in V(r_0, \Lambda_a) \quad (t_2 \leq j \leq t_k(u_2) - w(k)). \end{aligned} \tag{8.50}$$

Since $m_0 l_k(u_1) < m_k(u_2) - w(k) \leq t_2$ and $z_{-m_0 l_k(u_1)} \notin V(r_0, \Lambda_a)$, by (8.43) and Claim 8(v) there exists an $(\eta, 0; a)$-string σ_2 containing z_{-t_2}.

Since, by the choice of t_1 and t_2,

$$t_1 \leq t_k < m_0 l_k(u_1) < m_k(u_2) - w(k) \leq t_2,$$

we have $\sigma_1 \neq \sigma_2$. By (8.47)(ii),

$$N(\xi^k) - \overline{n} \geq N(\xi^k) - u_2 - \overline{n} \geq N(\xi^k) - u_1 + \overline{n} + 1,$$

and so by (8.49) and (8.50),

$$z_{-t_1}, z_{-t_2} \in V(r_{N(\xi^k)-u_1+\overline{n}+1}, \Lambda_a).$$

Therefore σ_1 and σ_2 are $(\eta, N(\xi^k) - u_1 + \overline{n} + 1, \Lambda_a)$-strings and $\sigma_1 \neq \sigma_2$.

Claim 9. *Let σ_1 and σ_2 be as above. For every $(\eta, 0; a)$-string σ with $\sigma_1 < \sigma < \sigma_2$ we have*

$$\sigma \cap V(r_{N(\xi^k)-u_1+\overline{n}}, \Lambda_a) = \emptyset$$

for $k \geq K$.

If we establish Claim 9, then the condition (C) holds for $N(\xi^k) - u_1 + \overline{n}$ $(\geq N(\xi^k) - v_0(2\overline{n}+1))$ large enough. This implies the existence of a 1-cycle for Λ_a, which is inconsistent with Lemma 7.1. This contradiction has been derived through the nine claims under the assumption given in $(*)$.

Therefore the assumption $\Lambda_a \cap \mathrm{cl}(I_{i_0+1}(f)) \neq \emptyset$ is invalid, which yields Proposition 4(b). To finish the proof it thus suffices to check that Claim 9 is true.

Proof of Claim 9. If Claim 9 is false, then there is an $(\eta, 0; a)$-string σ such that $\sigma_1 < \sigma < \sigma_2$ and

$$\sigma \cap V(r_{N(\xi^k)-u_1+\overline{n}}, \Lambda_a) \neq \emptyset$$

for some $k \geq K$. Write

$$\sigma = (z_{-l}, \ldots, z_{-s}) \quad (\subset V(r_0, \Lambda_a))$$

for some l, s with $l \leq s$. Choose $l \leq t \leq s$ such that

$$z_{-t} \in \sigma \cap V(r_{N(\xi^k)-u_1+\overline{n}}, \Lambda_a). \tag{8.51}$$

Since $\sigma_1 < \sigma < \sigma_2$, we have $t_1 < l \leq t \leq s < t_2$. Since $z_{-m_0 l_k(u_1)} \notin V(r_0, \Lambda_0)$ by (8.48), we have two cases to consider:

$$\text{(a) } s < m_0 l_k(u_1), \qquad \text{(b) } m_0 l_k(u_1) < l.$$

Case (a): By Claim 8(ii), (8.45) and (8.51) we have

$$z_{-t} \in V(r_{N(\xi^k)-u_1+\overline{n}}, \Lambda_a),$$
$$N(\xi^k) - u_1 + \overline{n} \geq N(\xi^k) - v_0(2\overline{n}+1) \geq \overline{N},$$
$$d(z_{-t+j}, q^k_{-t+j}) \leq \varepsilon < r_0/2 \quad (0 \leq j \leq t - l),$$

and so σ satisfies the conditions (1) and (2) of Lemma 8.5. Replacing ξ by η and η by ξ^k in Lemma 8.5, we can take $l \leq t_3 \leq t$ such that

$$q^k_{-t_3} \in V(r_{N(\xi^k)-u_1}, \Lambda_a), \qquad q^k_{-j} \in V(r_0, \Lambda_a) \quad (t_3 \leq j \leq t). \tag{8.52}$$

By (8.34),

$$t_3 \le t \le s < m_0 l_k(u_1) \le m_k(u_1) - 1,$$

and so by Claim 2(3),

$$q^k_{-t_3} \in V(r_{N(\xi^k)-u_1}, \Lambda_a) \cap \{q^k_{-1}, \ldots, q^k_{-m_k(u_1)+1}\} \subset \sigma^k = (q^k_{-1}, \ldots, q^k_{-s_k}).$$

Since $q^k_{-s_k-1} \notin V(r_0, \Lambda_a)$, by (8.52) we have

$$q^k_{-j} \in \sigma^k \subset V(r_0, \Lambda_a) \quad (1 \le j \le t). \tag{8.53}$$

Since $\sigma = (z_{-l}, \ldots, z_{-s})$ is an $(\eta, 0; a)$-string, we have

$$z_{-l} \in V(r_0, \Lambda_a) \quad \text{and} \quad z_{-l+1} \notin V(r_0, \Lambda_a).$$

Thus it is easily checked by using Lemma 8.3 that $d(z_{-l+1}, V^+_a) > r_0$. Thus,

$$d(z_{-l}, V^+_a) \ge \gamma d(f(z_{-l}), V^+_a) = \gamma d(z_{-l+1}, V^+_a) > \gamma r_0. \tag{8.54}$$

Since $t_1 < l \le t$ and $f^{l-t_1}(q^k_{-l}) = q^k_{-t_1}$, by (8.53) and Lemma 8.3(2) we have

$$d(q^k_{-l}, V^+_a) \le \lambda^{l-t_1} d(f^{l-t_1}(q^k_{-l}), V^+_a) < d(q^k_{-t_1}, V^+_a).$$

Thus, by Claim 8(ii), (8.49) and (8.54),

$$\begin{aligned} 0 &< d(q^k_{-t_1}, V^+_a) - d(q^k_{-l}, V^+_a) \\ &\le (d(q^k_{-t_1}, z_{-t_1}) + d(z_{-t_1}, V^+_a)) - (d(z^k_{-l}, V^+_a) - d(z^k_{-l}, q^k_{-l})) \\ &\le (\varepsilon + r_{N(\xi^k)-\overline{n}}) - (\gamma r_0 - \varepsilon) < r_{N(\xi^k)-\overline{n}} - \gamma r_0/3 < 0. \end{aligned}$$

This is a contradiction.

Case (b): By Claim 8(iii), (8.45) and (8.51) we have

$$\begin{gathered} z_{-t} \in V(r_{N(\xi^k)-u_1+\overline{n}}, \Lambda_a), \\ N(\xi^k) - u_1 + \overline{n} \ge N(\xi^k) - v_0(2\overline{n}+1) \ge \overline{N}, \\ d(z_{-t+j}, q^k_{-t-w(k)+j}) \le \varepsilon < r_0/2 \quad (0 \le j \le t-l), \end{gathered}$$

and so σ satisfies the conditions (1) and (2) of Lemma 8.5. Thus,

$$q^k_{-t_4} \in V(r_{N(\xi^k)-u_1}, \Lambda_a), \quad q^k_{-j} \in V(r_0, \Lambda_a) \quad (t_4 \le j \le t + w(k)) \tag{8.55}$$

for some t_4 with $l + w(k) \le t_4 \le t + w(k)$. Since $u_1 - u_2 \le v_0(2\overline{n}+1)$ by (8.47)(ii), it follows that

$$N(\xi^k) - u_1 \ge N(\xi^k) - u_2 - v_0(2\overline{n}+1),$$

and so by (8.55) and Claim 6,

$$\begin{aligned} q^k_{-t_4} &\in V(r_{N(\xi^k)-u_2-v_0(2\overline{n}+1)}) \cap \{q^k_{-m_o l_k(u_2)}, \ldots, q^k_{-s_k(u_2)-1}\} \\ &\subset \sigma^k(u_2) = (q^k_{-m_k(u_2)}, \ldots, q^k_{-s_k(u_2)}) \end{aligned}$$

because $k \ge K \ge K(u_2, v_0(2\overline{n}+1))$ by (8.44). Since $q^k_{-m_k(u_2)+1} \notin V(r_0, \Lambda_a)$, by (8.55) we have

$$q^k_{-j} \in \sigma^k(u_2) \subset V(r_0, \Lambda_a) \quad (t + w(k) \le j \le s_k(u_2)).$$

Since σ and σ_2 contain z_{-t} and z_{-t_2} respectively and satisfy $\sigma < \sigma_2$, there is $t < t_2' < t_2$ such that

$$z_{-t_2'} \in V(r_0, \Lambda_a) \quad \text{and} \quad z_{-t_2'+1} \notin V(r_0, \Lambda_a),$$

and so by Claim 8(iii) and (8.50),

$$\begin{aligned} 0 &< d(q^k_{-t-w(k)}, V_a^+) - d(q^k_{-t_2'-w(k)}, V_a^+) \\ &\le (d(q^k_{-t-w(k)}, z_{-t}) + d(z_{-t}, V_a^+)) - (d(z^k_{-t_2'}, V_a^+) - d(z^k_{-t_2'}, q^k_{-t_2'-w(k)})) \\ &\le (\varepsilon + r_{N(\xi^k)-u_1+\overline{n}}) - (\gamma r_0 - \varepsilon) < r_{N(\xi^k)-u_1+\overline{n}} - \gamma r_0/3 < 0. \end{aligned}$$

This is a contradiction. Therefore we have Claim 9.

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