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On invariants for measure preserving transformations

by

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Abstract. The classification problem for measure preserving transformations is strictly more complicated than that of graph isomorphism.

1. Preamble. We consider the group M_∞ of all invertible measure preserving transformations either on the unit interval or any other reasonable measure space. It seems natural to say that two of these transformations, σ_1, σ_2, are equivalent or *isomorphic* if there is a third, π, so that

$$\pi \circ \sigma_1 \circ \pi^{-1} = \sigma_2 \quad \text{a.e.}$$

To what extent can this equivalence relation be considered *classifiable*?

In specific cases—for instance σ_1, σ_2 both *Bernoulli* or *discrete spectrum*—there are well accepted systems of complete invariants. However, in the completely general context of arbitrary measure preserving transformations there is no known satisfactory system of complete invariants nor even a clear statement of what this would entail.

For instance, Halmos in [7] despairs of precisely formulating the problem but at page 1029 suggests that its solution should fulfill the "the vague task of finding a complete set of invariants..." At page 75 of [8] he proposes that the central problem is to "find usable necessary and sufficient conditions for the conjugacy of two measure preserving transformations." Some time later Weiss at page 670 of [23] raises the problem of finding "a set of invariants large enough so that if all invariants agree for two m.p.t. one can conclude that the m.p.t. are isomorphic".

This article considers attempts to make this precise and ask abstractly whether the problem of classifiability could *in principle* have a positive solu-

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tion. We present a clearly identifiable lower bound on the *classification difficulty* of the isomorphism relation for measure preserving transformations.

One precise formulation of the problem would be to understand a *classifiable equivalence relation* on a Polish space to be one for which we may find a Borel assignment of reals or points in some other *standard Borel space* as complete invariants. This is the notion of classifiable suggested by the Glimm–Effros dichotomy of [10].

Indeed, Feldman in [4] takes exactly that position. Appealing to [18] he observes that the isomorphism relation for *Bernoulli* shifts allows real numbers to be assigned in a Borel manner as a complete invariant and uses [19] to remark that such an assignment is already impossible for the class of measure preserving transformations having the *property of K*.

A more generous notion of classification, closer to the kinds we consider below, is already implicit in sources [7], [8], [23]. In each case the results of [9] are accepted as providing a complete classification for the *discrete spectrum* measure preserving transformations. Here the invariants are not real numbers or single points in a standard Borel space, but rather countable sets of complex numbers. The significance of this is not in the use of complex numbers as against reals—this is immaterial since all uncountable Polish spaces are Borel isomorphic. The significant feature of the invariants from [9] is that they have the form of a *countable unordered set* of points in a standard Borel space ([1]).

Thus we may in general ask for an equivalence relation E on a standard Borel space X:

Question Q1. Does there exist a countable sequence $(f_i)_{i\in\mathbb{N}}$ of Borel functions, each $f_i : X \to \mathbb{C}$, such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \ \Leftrightarrow\ \{f_i(x_1) : i \in \mathbb{N}\} = \{f_i(x_2) : i \in \mathbb{N}\}?\ (^2)$$

([1]) Superficially it might be thought that the ability to assign countable sets of points in a standard Borel space as a complete invariant is tantamount to classifiability in the sense of [4]. In fact this is untrue—there is in general no *canonical* way to encode or parameterize a countable set of complex numbers by a real, or by a point in any other separable completely metrizable (i.e. *Polish*) space. Indeed, this can already been seen in the present context, as it is known that there is no Borel function that assigns to a discrete spectrum measure preserving transformation a complex number as a complete invariant (compare the start of §5 below). More starkly, and just to allay suspicions that this may be a consequence of restricting to the Borel category, the techniques of [22] are sufficient to demonstrate the consistency of set theory without the axiom of choice along with the non-existence of *any* function assigning complex numbers as complete invariants for discrete spectrum transformations.

It is intrinsic to the methods of [9] that we obtain countable sets of points as complete invariants, and no amount of modification will squeeze it into the form requested by [4].

([2]) Matt Foreman has shown that the procedure of [9] indeed obtains such a classification, that is to say, in the Borel category, for discrete spectrum m.p.t.'s.

As suggested by Matt Foreman, we might hope that "usable and sufficient conditions" should at least make the relation of isomorphy Borel in $M_\infty \times M_\infty$.

QUESTION Q2. Let $\mathrm{Graph}(E) \subseteq X \times X$ be the set of (x_1, x_2) for which $x_1 E x_2$. Is $\mathrm{Graph}(E)$ Borel in the product Borel structure on $X \times X$? (In future I will refer to this conclusion simply as the statement that E is Borel.)

These are in general distinct notions. For an equivalence relation E on a Polish space X, classifiability in the sense of a single Borel function assigning points as complete invariants implies a positive answer to Q1. A positive answer to Q1 implies one for Q2. Neither of these implications reverses.

Below we consider these questions for the specific case of $X = M_\infty$ and E the equivalence relation of conjugacy, given by setting $\sigma_1 E \sigma_2$ if there exists $\pi \in M_\infty$ such that

$$\sigma_1(x) = \pi \circ \sigma_2 \circ \pi^{-1}(x) \qquad \text{a.e.}$$

We place on M_∞ the customary Borel structure, described by [4] and recalled in Section 2 below.

THEOREM 1.1. *The conjugacy equivalence relation on M_∞ is non-Borel.*

In fact, the situation is much worse than this alone would suggest. It turns out that the classification problem for measure preserving transformations encompasses the classification problem for arbitrary countable discrete structures—countable groups, countable linear orderings, graphs, and so on. For $\mathcal{L}$ a countable language, $\mathrm{Mod}(\mathcal{L})$, the space of all $\mathcal{L}$-structures whose underlying set is $\mathbb{N}$, is naturally a standard Borel space; the details of this definition are recalled in Section 2 below, and discussed at length in many places, such as [15], [13], [12]. A specific example of such a collection is the space of (directed) graphs on $\mathbb{N}$, which by appeal to the corresponding characteristic function of the adjacency relation can be identified with $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ and given a natural topology.

THEOREM 1.2. *If $\mathcal{L}$ is a countable language then there is a Borel function $\theta : \mathrm{Mod}(\mathcal{L}) \to M_\infty$ such that for all $M, N \in \mathrm{Mod}(\mathcal{L})$,*

$$M \cong N \iff \exists \pi \in M_\infty \ (\pi \circ \theta(M) \circ \pi^{-1} = \theta(N) \ a.e.).$$

It is not known whether 1.2 or 1.1 can be obtained for the *ergodic* measure preserving transformations—that is to say, whether we may have θ as in 1.2 but with the further requirement that it always assume a value $\theta(M) \in M_\infty$ such that every Borel $\theta(M)$-invariant set is either null or conull.

These further and still open questions are of interest given the ergodic decomposition theorem, stating that every element of M_∞ may be in some sense written as the integral of its ergodic components (see for instance [8], §2.3, [25], [20]). In this context one could compare the conjugacy equivalence relation on the unitary representations of a discrete group: in the case of irreducible representations this is known to be not only Borel but actually F_σ in an appropriate topology. (See [3].)

For the narrow case of ergodic transformations we only know the following:

THEOREM 1.3. *There is a Polish group G and a "turbulent" Polish G-space X and a Borel function $\theta : X \to M_\infty$ such that*

(i) $\theta(x)$ *is ergodic for all* $x \in X$;
(ii) $x_0 E_G^X x_1$ *if and only if*

$$\exists \pi \in M_\infty \ (\pi \circ \theta(x_0) \circ \pi^{-1} = \theta(x_1))$$

for all $x_0, x_1 \in X$.

In other words, we may embed a *turbulent* orbit equivalence relation into the isomorphism relation on ergodic measure preserving transformations. In light of the results from [12], this provides a succinct anti-classifiability result. In particular, the reduction in 1.2 does not reverse: The classification problem for measure preserving transformations is *strictly* more complicated than for discrete countable structures.

THEOREM 1.4. *For $\mathcal{L}$ a countable language, there is no Borel* $\theta_1 : M_\infty \to \mathrm{Mod}(\mathcal{L})$ *such that for all* $\sigma_1, \sigma_2 \in M_\infty$,

$$\exists \pi \in M_\infty \ (\pi \circ \sigma_1 \circ \pi^{-1} = \sigma_2 \ a.e.) \ \Leftrightarrow \ \theta_1(\sigma_1) \cong \theta_1(\sigma_2).$$

Indeed, there is no such θ_1 even with domain just the ergodic transformations. Actually we find in 1.3(i) that each $\theta(x)$ is in the class of *rank* 2 *generalized discrete spectrum* (see [6]). This is an important detail: Since the discrete spectrum measure preserving transformations *do* admit classification by countable sets of complex numbers, and hence by countable models, we might have hoped for instance that the αth level of the generalized discrete spectrum transformations admit complete invariants in something like the αth iteration of the operation of taking all countable subsets applied to $\mathbb{C}$.

It should not be thought that the results above are *fragile* to the choice of the Borel category. We can define more generous classes of reducibility and show that even with broader but still reasonable classes of functions—of the kind that are encountered in Ulm invariants for abelian p-groups and the Scott analysis for countable structures—there is no reduction of conjugacy on M_∞ to isomorphism on countable structures, or equality on countable sets of reals, or indeed to any Borel equivalence relation.

Finally, I suppose it might be felt that the real problem is not that we are demanding Borel functions but more generally that we are requiring *any sort of definability* whatsoever. In this way we might dream of some manner of classification, only without the invariants being produced in an "effective" manner.

But not even that much can be hoped for. If $\approx$ is the conjugacy equivalence relation on M_∞, and $\cong$ isomorphism on countable structures, then by the techniques of [22] it is consistent with ZF and enough of the axiom of choice to develop most classical mathematics that there be *no* injection:

$$M_\infty/\!\approx\ \hookrightarrow \mathrm{Mod}(\mathcal{L})/\!\cong.$$

In particular, if $\mathcal{P}_{\aleph_0}(A)$ denotes the collection of all countable subsets of a set A, then there will be no injection

$$M_\infty/\!\approx\ \hookrightarrow \mathcal{P}_{\aleph_0}(\mathbb{C}),$$

nor

$$M_\infty/\!\approx\ \hookrightarrow \mathcal{P}_{\aleph_0}(\mathcal{P}_{\aleph_0}(\mathbb{C})),$$

and so on. Similarly, it is consistent with ZF and a large fragment (DC) of choice that for any Borel equivalence relation on a Polish space X there is no injection

$$M_\infty/\!\approx\ \hookrightarrow X/E.$$

In Section 2 we give some definitions and present an outline of the proof for 1.2, which is in turn completed in Sections 3 and 4. Section 5 embeds a turbulent orbit equivalence relation into the generalized discrete spectrum transformations. Section 6 gives a proof of a known result to the effect that the natural equivalence relation on cocycles from the measure preserving action of a countable group into a compact group is Borel; this equivalence relation is closely related to the one needed in §5. It is also noted that the collection of measurable transformations conjugating a transformation T to itself is compact if and only if T has discrete spectrum.

In terms of background material needed for reading this paper, formally it does not assume much more than a general knowledge of elementary analysis, of the kind which would be found in a text such as [25]. However, as a practical matter it would be more than helpful to have some acquaintance with ergodic theory. A knowledge of classical descriptive set theory in the sense of [15] may also make the paper easier to read. Many of the results appeal to the modern theory of Borel equivalence relations; for this [5] and [1] are good references. The theory of turbulence is developed in [12]; the notation here largely follows the notation there.

We indulge in all the usual sins. A measurable square summable function is identified with its equivalence class in L^2. We say "everywhere" when we mean "on all but a null set".

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I am also very much indebted to the referee for an exceptionally thorough and penetrating report, and in particular for finding a serious mathematical error in the first draft. This first draft claimed that one can obtain 1.2 by proving that isomorphism on countable torsion-free abelian groups is Borel complete in the sense of [5]; that proof of the Borel completeness of torsion-free abelian groups was erroneous.

2. Outline of proof of 1.2. The concept of "Borel reducibility" is central to the arguments below.

Definition. Let E and F be equivalence relations on Polish spaces X and Y. We say that E is *Borel reducible* to F, written $E \leq_B F$, if there is a Borel function $\theta : X \to Y$ such that for all $x_1, x_2 \in X$ we have

$$x_1 E x_2 \quad \text{if and only if} \quad \theta(x_1) F \theta(x_2).$$

Naturally we write $E <_B F$ if $E \leq_B F$ holds but $F \leq_B E$ fails.

This relation $\leq_B$ is clearly transitive and reflexive.

Examples. More detail, along with proofs of the various folklore assertions, can be found in [12].

(i) For X a Polish space, $\mathrm{id}(X)$ is the identity equivalence relation on X. If $E \leq_B \mathrm{id}(X)$ for any Polish space X then we say that E is *smooth*.

(ii) E_v is the Vitali equivalence relation on $\mathbb{R}$, given by the cosets of $\mathbb{Q}$. Here it is known that E_v is not smooth.

(iii) E_0 the equivalence relation of eventual agreement on infinite binary sequences. It is known that $E_0 \leq_B E_v \leq_B E_0$.

(iv) Let $2^{\mathbb{N}\times\mathbb{N}}$ be the space of functions from $\mathbb{N}\times\mathbb{N}$ to $\{0,1\}$, with the topology of pointwise convergence (³). Following [5] we define F_2 by $x_1 F_2 x_2$ if and only if

$$\forall n \in \mathbb{N}\ \exists m_1, m_2 \in \mathbb{N}\ \forall k \in \mathbb{N}\ (x_1(n,k) = x_2(m_2,k),\ x_1(m_1,k) = x_2(n,k)).$$

Then $\mathrm{id}(\mathbb{R}) <_B E_0 <_B F_2$.

Two important classes of Polish spaces are those consisting of all measure preserving transformations of a Lebesgue space and those consisting of all $\mathcal{L}$-structures on $\mathbb{N}$ for some countable language $\mathcal{L}$.

Definition. Let M_∞ be the group of Borel measure preserving bijections from $[0,1]$ to $[0,1]$ with identification of maps agreeing on a measure one set. For $(O_n)_{n\in\mathbb{N}}$ a basis of $[0,1]$ we obtain a separable metric d_λ on M_∞ by setting

$$d_\lambda(\pi_1,\pi_2) = \sum 2^{-n}[\lambda(\pi_1(O_n) \vartriangle \pi_2(O_n)) + \lambda(\pi_1^{-1}(O_n) \vartriangle \pi_2^{-1}(O_n))],$$

(³) Here and elsewhere we identify 2 with $\{0,1\}$, and thus $2^{\mathbb{N}\times\mathbb{N}}$ is the space of all functions from $\mathbb{N}\times\mathbb{N}$ to $\{0,1\}$.

where λ is Lebesgue measure and $\triangle$ is used to denote symmetric difference: $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

Let $\approx^*$ denote the conjugacy equivalence relation:

$$\pi_1 \approx^* \pi_2 \;\Leftrightarrow\; \exists \sigma \in M_\infty\ (\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2).$$

Definition. For $\mathcal{L}$ a countable language, let $\mathrm{Mod}(\mathcal{L})$ be the collection of all different ways we may place an $\mathcal{L}$-structure on the natural numbers $\mathbb{N}$. We then place a topology on this space by taking as subbasic open sets those of the form

$$\begin{aligned}
&\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : (n_1, \ldots, n_k) \in R^{\mathcal{M}}\},\\
&\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : (n_1, \ldots, n_k) \notin R^{\mathcal{M}}\},\\
&\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : f^{\mathcal{M}}(n_1, \ldots, n_k) = m\},\\
&\{\mathcal{M} \in \mathrm{Mod}(\mathcal{L}) : f^{\mathcal{M}}(n_1, \ldots, n_k) \neq m\},
\end{aligned}$$

where $n_1, \ldots, n_k, m$ range over finite sequences from $\mathbb{N}$, R ranges over relation symbols in $\mathcal{L}$, and f ranges over function symbols in $\mathcal{L}$.

We let $\cong|_{\mathrm{Mod}(\mathcal{L})}$ denote the isomorphism relation on these $\mathcal{L}$-structures. Thus $\mathcal{M}_1 \cong \mathcal{M}_2$ if and only if there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ with

$$(n_1, \ldots, n_k) \in R^{\mathcal{M}_1} \;\Leftrightarrow\; (\sigma(n_1), \ldots, \sigma(n_k)) \in R^{\mathcal{M}_2}$$

and

$$f^{\mathcal{M}_1}(n_1, \ldots, n_k) = m \;\Leftrightarrow\; f^{\mathcal{M}_2}(\sigma(n_1), \ldots, \sigma(n_k)) = \sigma(m)$$

for all relation symbols R, function symbols f, and $n_1, \ldots, n_k, m \in \mathbb{N}$.

Lemma 2.1. *M_∞ is a Polish group*; *d_λ is complete.*

Proof. This was shown in [4]; a proof can also be found in Chapter 2 of [12]. ∎

Lemma 2.2. *$\mathrm{Mod}(\mathcal{L})$ is a Polish space whenever $\mathcal{L}$ is a countable language.*

Proof. This lemma should be obvious, since the space can be naturally identified with a suitable countable product of the Polish spaces $\mathbb{N}^{\{0,1\}}$ and $\mathbb{N}^{\mathbb{N}}$. ∎

We wish to start working towards a proof that for any countable language $\mathcal{L}$ we have

$$\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B \approx^*.$$

An equivalence relation E on X is said to be *Borel* if it is Borel as a subset of $X \times X$. It is then easily seen that the Borel equivalence relations are closed downwards under $\leq_B$. And thus since it is well known (see [5], or 6.16 in [12]) that for many $\mathcal{L}$ one has $\cong|_{\mathrm{Mod}(\mathcal{L})}$ non-Borel, a proof of

$$\approx^* \leq_B \cong|_{\mathrm{Mod}(\mathcal{L})}$$

will in particular imply that $\approx^*$ is non-Borel.

It is rather cumbersome to be continually working with the full range of possible $\cong|_{\mathrm{Mod}(\mathcal{L})}$ as $\mathcal{L}$ ranges over countable languages. Instead, it will be convenient to work with a canonical example, which is already known to have maximal complexity in the $\leq_B$-ordering.

For us a *graph* will be a directed graph where loops are possible but parallel edges are not. Thus we may naturally identify a graph on the underlying set $\mathbb{N}$ with a binary relation on $\mathbb{N}$, and this in turn may by consideration of the characteristic function be identified with $2^{\mathbb{N}\times\mathbb{N}}$.

Definition. Let Mod(Gph) be the Polish space $2^{\mathbb{N}\times\mathbb{N}}$ equipped with the product topology.

We let the infinite symmetric group, S_∞, consisting of all permutations of the natural numbers, act on $2^{\mathbb{N}\times\mathbb{N}}$ in the following manner: Given $\sigma \in S_\infty$ and $x \in 2^{\mathbb{N}\times\mathbb{N}}$, we define $\sigma \cdot x$ by

$$(\sigma \cdot x)(n, m) = x(\sigma^{-1}(n), \sigma^{-1}(m)).$$

We then let $E_{S_\infty}^{\mathrm{Mod(Gph)}}$ denote the resulting orbit equivalence relation:

$$x_1 E_{S_\infty}^{\mathrm{Mod(Gph)}} x_2 \quad \text{if and only if} \quad \exists \sigma \in S_\infty\ (\sigma \cdot x_1 = x_2).$$

Thus $E_{S_\infty}^{\mathrm{Mod(Gph)}}$ is the isomorphism relation for the space of all binary relations on $\mathbb{N}$. Of course, as a space Mod(Gph) is nothing other than $2^{\mathbb{N}\times\mathbb{N}}$; it will be convenient to have this separate notation, to remind ourselves with Mod(Gph) that we are thinking of $2^{\mathbb{N}\times\mathbb{N}}$ as a Polish S_∞-space in a specific way.

Lemma 2.3. *If $\mathcal{L}$ is a countable language, then*

$$\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B E_{S_\infty}^{\mathrm{Mod(Gph)}}.$$

Proof. A proof of this well known folklore fact can be found in many places, including [5]. ∎

Thus the task of showing $\approx^*$ non-Borel has been reduced to showing that for any countable language $\mathcal{L}$ we have

$$\cong|_{\mathrm{Mod}(\mathcal{L})} \leq_B \approx^*,$$

which has in turn been reduced to showing

$$E_{S_\infty}^{\mathrm{Mod(Gph)}} \leq_B \approx^*.$$

In order to do this we will introduce one further equivalence relation, $E_{G_\infty}^{Y_2}$, defined shortly, and show in Section 3 first that

$$E_{G_\infty}^{Y_2} \leq_B \approx^*,$$

and then in Section 4 that

$$E_{S_\infty}^{\mathrm{Mod(Gph)}} \leq_B E_{G_\infty}^{Y_2}.$$

DEFINITION. Let $M(S_\infty) = \{g \in (S_\infty)^{[0,1]} : g \text{ Borel}\}$ be the group of measurable functions from $[0,1]$ to S_∞, where S_∞ is the infinite symmetric group on $\mathbb{N}$; we multiply pointwise,

$$(g_1 g_2)(x) = g_1(x) g_2(x),$$

and identify functions that agree λ-a.e.

Observe then that if g^{-1} is the group theoretic inverse of $g \in M(S_\infty)$ then $g^{-1}(x) = (g(x))^{-1}$ for (λ-a.e.) $x \in [0,1]$.

DEFINITION. Define $\psi : M_\infty \to \mathrm{Aut}(M(S_\infty))$ by the requirement that for $\pi \in M_\infty$, $x \in [0,1]$, $g \in M(S_\infty)$,

$$((\psi(\pi))(g))(x) = g(\pi^{-1}(x)).$$

So $\psi(\pi)$ is the (group theoretic) automorphism of $M(S_\infty)$ obtained by shift.

In fact, these groups all have natural topologies and ψ is a homeomorphism from M_∞ into the group of continuous automorphisms of $M(S_\infty)$.

DEFINITION. Form the semidirect product $M(S_\infty) \rtimes_\psi M_\infty$ in the usual way, so that for $g_0, g_1 \in M(S_\infty)$, $\pi_0, \pi_1 \in M_\infty$,

$$(g_0, \pi_0)(g_1, \pi_1) = (g_0(\psi(\pi_0))(g_1), \pi_0\pi_1).$$

For short write $G_\infty := M(S_\infty) \rtimes_\psi M_\infty$.

DEFINITION. For $y \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$ we define $y(n,\cdot) \in 2^{\mathbb{N}}$ in the obvious way, by $(y(n,\cdot))(m) = y(n,m)$. Let

$$B_2 = \{x \in 2^{\mathbb{N}\times\mathbb{N}} : n \neq m \Rightarrow x(n,\cdot) \neq x(m,\cdot)\};$$

B_2 is a G_δ subset of the Polish space $2^{\mathbb{N}\times\mathbb{N}}$ and hence is Polish itself (see [15], 3C). Let

$$Y_2 = \{y \in (B_2)^{[0,1]} : y \text{ Borel}\}$$

be the space of measurable functions from $[0,1]$ to B_2 where we identify functions agreeing almost everywhere. We give this space the topology of convergence almost everywhere, so that $f_n \to f$ if for almost every $x \in [0,1]$ we have $f_n(x) \to f(x)$.

LEMMA 2.4. *Y_2 is a Polish space.*

Proof. Let d' be a complete compatible metric on B_2. We obtain a complete metric d_2 by setting

$$d_2(y_0, y_1) = \int d'(y_0(x), y_1(x))\, d\lambda(x)$$

for $y_0, y_1 \in Y_2$. ■

Definition. Let G_∞ act on Y_2 as follows: Given $y \in Y_2$,

$$y : [0,1] \to 2^{\mathbb{N}\times\mathbb{N}},$$

and $(g,\pi) \in G_\infty$,

$$g : [0,1] \to S_\infty, \qquad \pi \in M_\infty,$$

we define $(g,\pi)\cdot y : [0,1] \to 2^{\mathbb{N}\times\mathbb{N}}$ by

$$(((g,\pi)(y))(x))(m,n) = (y(\pi^{-1}(x)))(g^{-1}(x)(m),n).$$

(Here g^{-1} is intended to be the group theoretic inverse of $g \in M(S_\infty)$; thus $g^{-1}(x) = (g(x))^{-1}$.) Observe that

$$\begin{aligned}
&([(g_0,\pi_0)((g_1,\pi_1)(y))](x))(m,n)\\
&\qquad = [((g_1,\pi_1)(y))(\pi_0^{-1}(x))]((g_0^{-1}(x))(m),n)\\
&\qquad = (y(\pi_1^{-1}(\pi_0^{-1}(x))))(g_1^{-1}(\pi_0^{-1}(x))(g_0^{-1}(x))(m),n)\\
&\qquad = (y(\pi_1^{-1}(\pi_0^{-1}(x))))(((((\psi(\pi_0))(g_1))^{-1}g_0^{-1})(x))(m),n).
\end{aligned}$$

The last equality uses the fact that the group operations for $M(S_\infty)$ are calculated pointwise, and hence

$$g_1^{-1}(\pi_0^{-1}(x)) = (g_1(\pi_0^{-1}(x)))^{-1}$$

and

$$[(((\psi(\pi_0))(g_1))^{-1}g_0^{-1})](x) = [((\psi(\pi_0))(g_1^{-1}))(x)][g_0^{-1}(x)].$$

Thus,

$$\begin{aligned}
&[(((g_0,\pi_0)((g_1,\pi_1)(y)))(x))](m,n)\\
&\qquad = [y(\pi_1^{-1}(\pi_0^{-1}(x)))](((((\psi(\pi_0))(g_1))^{-1}g_0^{-1})(x))(m),n)\\
&\qquad = [((g_0(\psi(\pi_0)(g_1)),\pi_0\pi_1)(y))(x)](m,n),
\end{aligned}$$

which establishes this to be an action.

We then let $E^{Y_2}_{G_\infty}$ be the orbit equivalence relation on Y_2 resulting from this action.

Lemma 2.5. *For $y_0, y_1 \in Y_2$ we have $y_0 E^{Y_2}_{G_\infty} y_1$ if and only if there is some $\pi \in M_\infty$ such that*

$$\lambda(\{x : \{(y_0(x))(n,\cdot) : n \in \mathbb{N}\} = \{(y_1(\pi^{-1}(x)))(n,\cdot) : n \in \mathbb{N}\}\}) = 1.$$

Proof. The "only if" part of the lemma is trivial. The "if" direction uses the well known fact, whose proof can be found in 18A of [15], that any Borel set in the plane may be uniformized by a Lebesgue measurable function. ∎

Notation. Following the Kuratowski–Mycielski theorem of 19A of [15], choose $C \subset [0,1]$ to be a perfect set such that for any $k \in \mathbb{N}$ and $x_1,\ldots,x_k \in C$, if $x_i \neq x_j$ for all $i < j \le k$ then $x_1,\ldots,x_k$ are rationally independent. Fix a continuous injection $\varphi_C : 2^{\mathbb{N}} \hookrightarrow C$.

With each element y of Y_2 we wish to associate a measure preserving transformation

$$T_y : [0,1] \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to [0,1] \times \mathbb{R}/\mathbb{Z}^{\mathbb{N}}$$

whose ergodic components have the form $\{x\} \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ for $x \in [0,1]$; on each such ergodic component we will have a discrete spectrum measure preserving transformation with eigenvalues $\{e^{2\pi i \varphi_C((y(x))(n,\cdot))} : n \in \mathbb{N}\}$. The proof that $y_1 E_{G_\infty}^{Y_2} y_2$ if and only if T_{y_1} and T_{y_2} are conjugate is then a consequence of the well known fact that two measure preserving transformations are conjugate if and only if there is a measure one set on which their ergodic components are individually conjugate component by component. Partly for the convenience of the reader, and partly because there seems no easy source listing exactly the facts we need in exactly the form we need them, we write out the proof in §3 without assuming any familiarity with the ergodic decomposition of a measure preserving transformation.

3. The relation $E_{G_\infty}^{Y_2} \le_B \approx^*$

Definition. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a Borel measure preserving map, X a Polish space, μ a Borel probability measure on X, $\mathcal{B}$ the collection of Borel subsets of X. A non-zero $f \in L^2(X,\mu)$ (:= the Hilbert space of all square integrable complex-valued functions on (X,μ), subject to the usual identification of functions that agree almost everywhere) is said to be an *eigenfunction* for T if for some $\lambda \in \mathbb{C}$ we have $f \circ T = \lambda f$ a.e.; we then also say that λ is an *eigenvalue*. T is said to be *ergodic* if all T-invariant Borel sets are either null or conull with respect to μ.

Lemma 3.1. *$T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic if and only if the space of eigenfunctions with eigenvalue* 1 *is one-dimensional.*

Proof. If $f : X \to \mathbb{C}$ is a non-constant eigenfunction for the eigenvalue 1, then for some $U \subset \mathbb{C}$ the pullbacks $f^{-1}[U]$ and $f^{-1}[\mathbb{C} \setminus U]$ are disjoint and non-null. ∎

Hence if T is ergodic then no eigenvalue can have corresponding eigenspace with dimension greater than 1—for if $f_1, f_2 : X \to \mathbb{C}$ are linearly independent non-zero functions with $f_1 \circ T = cf_1$ and $f_2 \circ T = cf_2$ then $f_2/f_1 : X \to \mathbb{C}$ would be a non-constant function with eigenvalue 1.

Definition. Let $X = [0,1] \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ with the product of Lebesgue measure on $[0,1]$ and Haar measure ν on each copy of $\mathbb{R}/\mathbb{Z}$ (so that $\nu(\{x\mathbb{Z} : a \le x \le b\}) = b-a$ for any $0 < a < b \le 1$). For $y \in Y_2$ we define $T_y : X \to X$ by

$$T_y(x, z_0, z_1, \ldots) = (x, \varphi_C((y(x))(0,\cdot)) \oplus z_0, \varphi_C((y(x))(1,\cdot)) \oplus z_1, \ldots).$$

where $\oplus$ is addition modulo 1. For $x \in [0,1]$ we let $T_y^x : (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ be the map resulting from restriction to the fiber above x:

$$T_y^x(z_0, z_1, \ldots) = (\varphi_C((y(x))(0,\cdot)) \oplus z_0, \varphi_C((y(x))(1,\cdot)) \oplus z_1, \ldots).$$

LEMMA 3.2. *For all $y \in Y_2$ and $x \in [0,1]$:*

(i) *the set of eigenvalues of T_y^x is the subgroup of the complex unit circle generated by $\{e^{2\pi i \varphi_C((y(x))(n,\cdot))} : n \in \mathbb{N}\}$;*

(ii) *$T_y^x : (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ is ergodic.*

Proof. For x such that $\{(y(x))(n,\cdot)) : n \in \mathbb{N}\}$ is a rationally independent set we can use the Stone–Weierstrass theorem to see that the sums of the *finite* multiples of the projection functions and their inverses

$$\mathrm{Pr}_k : (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to \mathbb{C}, \qquad \vec{z} \mapsto e^{2\pi i z_k},$$

are dense in $L^2((\mathbb{R}/\mathbb{Z})^{\mathbb{N}}, \mu^{\mathbb{N}})$. Hence the functions of the form

$$(\mathrm{Pr}_{k_1})^{n_1} \cdot \ldots \cdot (\mathrm{Pr}_{k_p})^{n_p},$$

as $\langle n_1, \ldots, n_p\rangle$ and $\langle k_1, \ldots, k_p\rangle$ range over finite sequences from $\mathbb{Z}$ and $\mathbb{N}$, form a Hilbert basis for $L^2((\mathbb{R}/\mathbb{Z})^{\mathbb{N}}, \mu^{\mathbb{N}})$. The rational independence property assumed for C implies

$$(\mathrm{Pr}_{k_1})^{n_1} \cdot \ldots \cdot (\mathrm{Pr}_{k_p})^{n_p} \quad \text{and} \quad (\mathrm{Pr}_{l_1})^{m_1} \cdot \ldots \cdot (\mathrm{Pr}_{l_q})^{m_q}$$

have distinct eigenvalues whenever these functions are distinct, so any eigenfunction must be a finite multiple of these coordinate functions $\mathrm{Pr}_k : (\mathbb{R}/\mathbb{Z})^{\mathbb{N}} \to \mathbb{C}$. Thus up to scalar multiplication the only eigenfunctions are 1 and the finite multiples of the coordinate projections $\{\mathrm{Pr}_k : k \in \mathbb{N}\}$. ∎

LEMMA 3.3. *If $y \in Y_2$ and $A \subset X$ is Borel and T_y-invariant then there is a Borel $B \subset X$ so that $A = B \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ modulo some null set.*

Proof. For each $x \in [0,1]$ the set $A^x = \{\vec{z} : (x, \vec{z}) \in A\}$ is T_y^x-invariant. Thus by 3.2, $\nu^{\mathbb{N}}(A^x) \in \{0,1\}$ (where $\nu^{\mathbb{N}}$ is the $\mathbb{N}$-fold product of Haar measure ν on $\mathbb{R}/\mathbb{Z}$). Thus for $B = \{x \in [0,1] : \nu^{\mathbb{N}}(A^x) = 1\}$ we have $A = B \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ off a null set by Fubini. ∎

LEMMA 3.4. *If $y_1, y_2 \in Y_2$ and $x_1, x_2 \in [0,1]$, then the transformations $T_{y_1}^{x_1}$ and $T_{y_2}^{x_2}$ are conjugate if and only if*

$$\{(y_1(x_1))(n,\cdot) : n \in \mathbb{N}\} = \{(y_2(x_2))(n,\cdot) : n \in \mathbb{N}\}.$$

Proof. By 3.2, the transformations $T_{y_1}^{x_1}$ and $T_{y_2}^{x_2}$ are conjugate only if the sets of eigenvalues are equal, which is to say that the multiplicative subgroups of the complex unit circle generated by $\{e^{2\pi i \varphi_C((y_1(x_1))(n,\cdot))} : n \in \mathbb{N}\}$ and by $\{e^{2\pi i \varphi_C((y_2(x_2))(n,\cdot))} : n \in \mathbb{N}\}$ are equal; and this in turn, by the assumptions on C, holds only if

$$\{e^{2\pi i\varphi_C((y_1(x_1))(n,\cdot))} : n \in \mathbb{N}\} = \{e^{2\pi i\varphi_C((y_2(x_2))(n,\cdot))} : n \in \mathbb{N}\}.$$

The converse direction is trivial. ■

LEMMA 3.5. *If $y_1, y_2 \in Y_2$ with T_{y_1}, T_{y_2} conjugate then we may find $g \in G_\infty$ with $g \cdot y_1 = y_2$.*

Proof. If $\pi : X \to X$ is measure preserving and conjugates T_{y_1} and T_{y_2} then we use 3.3 to find some measure one set M such that for all basic open $O \subset [0,1]$ there are $B_O, D_O \subset [0,1]$ so that

$$\pi[(O \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}) \cap M] = (B_O \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}) \cap M,$$
$$\pi^{-1}[(O \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}) \cap M] = (D_O \times (\mathbb{R}/\mathbb{Z})^{\mathbb{N}}) \cap M.$$

We may also assume that M is invariant under π, T_{y_1}, T_{y_2}. Thus, on the measure one set M for all $(x, \vec{z}) \in M$ there is some $\widehat{\pi}(x) \in [0,1]$ so that $\pi|_{(\{x\}\times\mathbb{R}/\mathbb{Z}^{\mathbb{N}})}$ conjugates

$$T_{y_1}|_{\{x\}\times(\mathbb{R}/\mathbb{Z})^{\mathbb{N}}\cap M} \quad \text{and} \quad T_{y_2}|_{\{\widehat{\pi}(x)\}\times(\mathbb{R}/\mathbb{Z})^{\mathbb{N}}\cap M}.$$

This $\widehat{\pi} : [0,1] \to [0,1]$ is measure preserving since for a.e. $(x, \vec{z}) \in [0,1] \times \mathbb{T}^{\mathbb{N}}$ we have $\pi(x, \vec{z}) = (\widehat{\pi}(x), \vec{z}\,')$ some $\vec{z}\,' \in \mathbb{T}^{\mathbb{N}}$. Thus we are finished by 2.5. ■

LEMMA 3.6. *If $g \in G_\infty$ with $g \cdot y_1 = y_2$ then T_{y_1} and T_{y_2} are conjugate.*

Proof. This is simply unpacking the definitions.

First consider the case of $g = (1, \widehat{\pi})$ for some $\widehat{\pi} \in M_\infty$. Then

$$(y_2(x))(n,\cdot) = ((g \cdot y_1)(x))(n,\cdot) := (y_1(\widehat{\pi}^{-1}(x)))(n,\cdot)$$

for a.e. $x \in [0,1]$ and all $n \in \mathbb{N}$. Thus we can define $\pi : X \to X$ by $\pi(x, \vec{z}) = (\widehat{\pi}(x), \vec{z})$ to obtain $T_{y_2} = \pi \circ T_{y_1} \circ \pi^{-1}$.

Similarly, if $\sigma \in M(S_\infty)$ with $(\sigma, 1) \cdot y_1 = y_2$ then we can define $\pi : X \to X$ by

$$(x, z_0, z_1, \ldots) \mapsto (x, z_{(\sigma(x))(0)}, z_{(\sigma(x))(1)}, \ldots).$$

The above terminates the proof, since any $g \in G_\infty$ can be written in the form $g = (\sigma, 1)(1, \widehat{\pi})$. ■

DEFINITION. Following earlier notation, let $M_\infty(X)$ be the group of Borel measure preserving bijections from X to X, again identifying two maps that agree a.e. with respect to the measure $\lambda \times \nu^{\mathbb{N}}$. For $(O_n)_{n\in\mathbb{N}}$ a basis of X we obtain a complete separable metric d_X on $M_\infty(X)$ by setting

$$d_X(\pi_1, \pi_2) = \sum 2^{-n}[\lambda\times\nu^{\mathbb{N}}(\pi_1(O_n) \vartriangle \pi_2(O_n)) + \lambda\times\nu^{\mathbb{N}}(\pi_1^{-1}(O_n) \vartriangle \pi_2^{-1}(O_n))].$$

Note then that $M_\infty(X)$ is a Polish group under composition. It is in fact isomorphic to M_∞, since we can find a measure isomorphism $\Phi : (X, \lambda\times\nu^{\mathbb{N}}) \to ([0,1], \lambda)$ and then take the induced isomorphism

$$\widehat{\Phi} : M_\infty(X) \to M_\infty, \qquad \pi \mapsto \Phi^{-1} \circ \pi \circ \Phi.$$

Let $\approx^{**}$ denote the conjugacy equivalence relation on $M_\infty(X)$.

LEMMA 3.7. *The map*

$$Y_2 \to M_\infty(X), \qquad y \mapsto T_y,$$

is continuous.

Proof. Let $M(\lambda\times\nu^{\mathbb{N}})$ be the algebra of measurable subsets of $(X,\lambda\times\nu^{\mathbb{N}})$ subject to the usual identification of sets which agree off a null set; this is a Polish space in the metric $d(A,B)=\lambda\times\nu^{\mathbb{N}}(A\triangle B)$ (see for instance [12], 2.2). First note almost immediately from the definitions that if $A\subset[0,1]\times(\mathbb{R}/\mathbb{Z})^{\mathbb{N}}$ is a basic open set of the form $\{\vec{x} : x(i)\in J\}$ for some open interval J included in either $[0,1]$ or $\mathbb{R}/\mathbb{Z}$, then the resulting map into the measure algebra

$$Y_2 \to M(\lambda\times\nu^{\mathbb{N}}), \qquad y \mapsto T_y(A),$$

is continuous. Since the subalgebra of the Boolean algebra of measurable sets in $(X,\lambda\times\nu^{\mathbb{N}})$ generated by the cylinders is dense, we conclude that $y\mapsto T_y$ is continuous. ■

PROPOSITION 3.8. $E^{Y_2}_{G_\infty} \le_B \approx^*$.

Proof. X and $[0,1]$ are two non-atomic, standard Borel probability spaces, and hence (4) they are isomorphic as measure spaces. Thus it suffices to show $E^{Y_2}_{G_\infty} \le_B \approx^{**}$, which is exactly the content of the last three lemmas. ■

There are some details here which were not needed in developing a proof of 3.8 but which might have independent interest. Namely, the group G_∞ is a Polish group, and its action on Y_2 is not only continuous but also turbulent in the sense of [12].

4. The relation $E^{\mathrm{Mod(Gph)}}_{S_\infty} \le_B E^{Y_2}_{G_\infty}$

NOTATION. From now until the end of the section fix continuous one-to-one

$$f_0 : 2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}, \qquad f_1 : 2^{\mathbb{N}}\times 2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$$

with

$$f_0[2^{\mathbb{N}}]\cap f_1[2^{\mathbb{N}}\times 2^{\mathbb{N}}]=\emptyset.$$

It is easily seen that such a pair f_0, f_1 exists. For instance

$$\begin{aligned} (f_0(y))(n+1) &= y(n), & (f_0(y))(0) &= 0,\\ (f_1(y_1,y_2))(2n+1) &= y_1(n), & &\\ (f_1(y_1,y_2))(2n+2) &= y_2(n), & (f_1(y_1,y_2))(0) &= 1. \end{aligned}$$

(4) See for instance 17.41 of [15].

NOTATION. For $\vec{z}=(z_i)_{i\in\mathbb{N}}\in(2^{\mathbb{N}})^{\mathbb{N}}$ and $x\in\mathrm{Mod}(\mathrm{Gph})$ we let

$$\begin{aligned}\mathcal{P}(2^{\mathbb{N}},x,\vec{z})=\{y\in 2^{\mathbb{N}}:\exists n\ (y=f_0(z_n))\}\\ \cup\{y\in 2^{\mathbb{N}}:\exists n,m\ (y=f_1(z_n,z_m),x(n,m)=1)\}.\end{aligned}$$

We have previously defined B_2 to be the set of $w\in 2^{\mathbb{N}\times\mathbb{N}}$ such that for all $n_1\neq n_2$ we have $w(n_1,\cdot)\neq w(n_2,\cdot)$.

LEMMA 4.1. *There is a Borel function* $\varphi_{\mathrm{eva}}:\mathrm{Mod}(\mathrm{Gph})\times(2^{\mathbb{N}})^{\mathbb{N}}\to B_2$ *such that for all* $x\in\mathrm{Mod}(\mathrm{Gph})$ *and* $\vec{z}\in(2^{\mathbb{N}})^{\mathbb{N}}$ *with* $z_i\neq z_j$ *all* $i\neq j$ *we have*

$$\{(\varphi_{\mathrm{eva}}(x,\vec{z}))(n,\cdot):n\in\mathbb{N}\}=\mathcal{P}(2^{\mathbb{N}},x,\vec{z}).$$

Proof. We may partition $\mathrm{Mod}(\mathrm{Gph})$ into Borel sets $A_0,A_1,\ldots,A_{\aleph_0}$ such that for each $\kappa\in\{0,1,\ldots,\aleph_0\}$ and each $x\in A_\kappa$ there are exactly κ many pairs (n,m) with $x(n,m)=1$. It suffices then to show $\varphi_{\mathrm{eva}}|_{A_\kappa\times(2^{\mathbb{N}})^{\mathbb{N}}}$ is Borel for each κ.

Fixing κ, we divide $\mathbb{N}$ into sets $\{a_i:i\in\mathbb{N}\}$, $\{b_j:j<\kappa\}$. Then for a given $x\in A_\kappa$ we can let $(m_j,n_j)_{j<\kappa}$ enumerate, in the ordering obtained by comparing maximums and then adjudicating ties lexicographically, the pairs (m,n) with $x(m,n)=1$. We can then let $\varphi_{\mathrm{eva}}(x,\vec{z})\in 2^{\mathbb{N}\times\mathbb{N}}$ be defined by

$$\begin{aligned}(\varphi_{\mathrm{eva}}(x,\vec{z}))(a_i,m)&=(f_0(z_i))(m),\\ (\varphi_{\mathrm{eva}}(x,\vec{z}))(b_j,m)&=(f_1(z_{m_j},z_{n_j}))(m).\end{aligned}$$

The verification that the resulting function is Borel is routine. ■

NOTATION. Let μ be the usual product measure on $2^{\mathbb{N}}$, so that for each n we have

$$\mu(\{x\in 2^{\mathbb{N}}:x(n)=1\})=\tfrac{1}{2}.$$

Let $\mu^{\mathbb{N}}$ be *its* corresponding product measure on $(2^{\mathbb{N}})^{\mathbb{N}}$. Let $N_2=\{\vec{z}\in(2^{\mathbb{N}})^{\mathbb{N}}:\forall i\neq j\ (z_i\neq z_j)\}$, and note that $\mu^{\mathbb{N}}(N_2)=1$.

Since $((2^{\mathbb{N}})^{\mathbb{N}},\mu^{\mathbb{N}})$ and $([0,1]$, Lebesgue measure$)$ are both non-atomic and both have standard Borel structures, we can find a Borel measure preserving bijection $\varphi_{\mathrm{iso}}:N_2\to[0,1]$.

NOTATION. For $x\in\mathrm{Mod}(\mathrm{Gph})$ define $\psi_x:[0,1]\to 2^{\mathbb{N}\times\mathbb{N}}$ by

$$\psi_x(\alpha)=\varphi_{\mathrm{eva}}(x,\varphi_{\mathrm{iso}}^{-1}(\alpha)).$$

LEMMA 4.2. *The function* $x\mapsto\psi_x$ *is a Borel function from* $\mathrm{Mod}(\mathrm{Gph})$ *to* Y_2.

Proof. The key point is that we can apply 4.1 to see that if

$$f(\cdot,\cdot)=\varphi_{\mathrm{eva}}(\cdot,\varphi_{\mathrm{iso}}^{-1}(\cdot))$$

then for each x the set

$$\{\psi_x\}=\{\psi\in Y_2:\psi(\beta)=f(x,\beta)\text{ a.e. }\beta\}$$

is a Borel singleton uniformly in x; thus the assignment of ψ_x to x is Borel by the uniformization theorem for Borel subsets of the plane with countable sections. ∎

LEMMA 4.3. *If* $x_1, x_2 \in \mathrm{Mod}(\mathrm{Gph})$ *with* $x_1 E_{S_\infty}^{\mathrm{Mod}(\mathrm{Gph})} x_2$, *then*

$$\psi_{x_1} E_{G_\infty}^{Y_2} \psi_{x_2}.$$

Proof. Fix $\sigma \in S_\infty$ with $\sigma \cdot x_1 = x_2$. Thus for all $(n, m) \in \mathbb{N} \times \mathbb{N}$ we have

$$x_1(n, m) = x_2(\sigma(n), \sigma(m)).$$

And so if we define $\widehat{\sigma} : (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by

$$(\widehat{\sigma}(\vec{z}))_n = \vec{z}_{\sigma(n)}$$

then $\widehat{\sigma}$ is an invertible measure preserving transformation such that at each $\vec{z}$,

$$\mathcal{P}(2^{\mathbb{N}}, x_1, \vec{z}) = \mathcal{P}(2^{\mathbb{N}}, x_2, \widehat{\sigma}(\vec{z})).$$

From this we obtain $\psi_{x_1} E_{S_\infty}^{Y_2} \psi_{x_2}$ by 2.5. ∎

LEMMA 4.4. *If* $x_1, x_2 \in \mathrm{Mod}(\mathrm{Gph})$ *with* $\psi_{x_1} E_{G_\infty}^{Y_2} \psi_{x_2}$, *then*

$$x_1 E_{G_\infty}^{\mathrm{Mod}(\mathrm{Gph})} x_2.$$

Proof. The assumption that $\psi_{x_1} E_{G_\infty}^{Y_2} \psi_{x_2}$ in particular implies the existence of some $\vec{z}^1$ and $\vec{z}^2$ with

$$\mathcal{P}(2^{\mathbb{N}}, x_1, \vec{z}^1) = \mathcal{P}(2^{\mathbb{N}}, x_2, \vec{z}^2)$$

and for all $n \neq m$,

$$(\vec{z}^1)_n \neq (\vec{z}^1)_m, \quad (\vec{z}^2)_n \neq (\vec{z}^2)_m.$$

This in turn implies

$$\{f_0((\vec{z}^1)_i) : i \in \mathbb{N}\} = \{f_0((\vec{z}^2)_i) : i \in \mathbb{N}\},$$

and so we can find some $\sigma \in S_\infty$ with

$$(\vec{z}^1)_n = (\vec{z}^2)_{\sigma(n)}$$

for all $n \in \mathbb{N}$. At this point the assumptions on f_0 and f_1 entail that for all n, m,

$$x_1(n, m) = x_2(\sigma(n), \sigma(m)),$$

and so $\sigma \cdot x_1 = x_2$. ∎

PROPOSITION 4.5. $E_{S_\infty}^{\mathrm{Mod}(\mathrm{Gph})} \leq_B E_{G_\infty}^{Y_2}$.

Proof. This is exactly what the last three lemmas show. ∎

5. Turbulence for the generalized discrete spectrum transformations. Now we consider the isomorphism relation for generalized discrete spectrum transformations, or what [2] calls the "measure-distal" actions. The results below show non-classifiability by countable structures, but perhaps raise more questions than they answer. For instance, it is not known if there is a way to embed isomorphism of countable models into the generalized discrete spectrum transformations, or more modestly just embed the equivalence relation F_α, arising from the αth iteration of the operation "countable subset of" applied to some standard Borel space ([5]), into some appropriate level of the generalized discrete spectrum hierarchy.

Just by way of comparison, I should mention that for the very simple subclass consisting of the transformations having completely *discrete* spectrum the situation is totally understood. Here [9] shows that we may assign countable subsets of $\mathbb{C}$ as complete invariants. Indeed, Foreman and Louveau have observed that even in the Borel context this precisely encapsulates the classification difficulty of the discrete spectrum maps.

Theorem 5.1 (Foreman, Louveau). *Let $D \subset M_\infty$ be the class of discrete spectrum measure preserving transformations. Then*:

(i) *The set D is a Borel subset of M_∞.*

(ii) *There is a sequence of Borel functions $f_n : D \to \mathbb{R}$ such that for all $\pi_1, \pi_2 \in D$,*

$$\pi_1 \approx^* \pi_2 \quad \textit{if and only if} \quad \{f_n(\pi_1) : n \in \mathbb{N}\} = \{f_n(\pi_2) : n \in \mathbb{N}\};$$

thus there is a Borel $\theta_1 : D \to 2^{\mathbb{N}\times\mathbb{N}}$ such that for all $\pi_1, \pi_2 \in D$,

$$\pi_1 \approx^* \pi_2 \iff \{(\theta_1(\pi_1))(n,\cdot) : n \in \mathbb{N}\} = \{(\theta_1(\pi_2))(n,\cdot) : n \in \mathbb{N}\},$$

so in other words we have

$$\approx^*|_D \le_B F_2.$$

(iii) *Conversely*,

$$F_2 \le_B \approx^*|_D$$

in the sense that there is a Borel function $\theta_2 : 2^{\mathbb{N}\times\mathbb{N}} \to D$ such that for all $x_1, x_2 \in 2^{\mathbb{N}\times\mathbb{N}}$,

$$\{x_1(n,\cdot) : n \in \mathbb{N}\} = \{x_2(n,\cdot) : n \in \mathbb{N}\} \iff \theta_2(x_1) \approx^* \theta_2(x_2).$$

In particular, the isomorphism relation on the discrete spectrum transformations is *non-smooth.*

Notation. Let $\mathbb{T} = \{e^{2\pi i x} : x \in [0,1]\}$, the complex unit circle, be viewed as a group under multiplication. For the remainder of this section let λ be the usual Lebesgue measure on $\mathbb{T}$ normalized so that $\lambda(\mathbb{T}) = 1$.

([5]) See [5].

Let $H_0 = \{f : \mathbb{T} \to \mathbb{T} : f \text{ is Lebesgue measurable}\}$, where we identify $f_0, f_1 \in H_0$ if they agree λ-a.e. For $f_0, f_1 \in H_0$ the product $f_0 f_1 \in H_0$ is defined by pointwise multiplication:

$$(f_0 f_1)(\zeta) = (f_0(\zeta))(f_1(\zeta)).$$

We give this group the topology of a.e. pointwise convergence, which is to say, the topology induced by the metric

$$d_0(f_0, f_1) = \frac{1}{2}\int |f_0(\xi) - f_1(\xi)|\, d\lambda,$$

where $|\cdot|$ is the usual Euclidean distance in $\mathbb{C}$. The technical advantage of just this choice for d_0 is that whenever f_0 and f_1 differ on a set of measure less than ε we must have

$$d_G((1, \overline{0}, f_0), (1, \overline{0}, f_1)) < \varepsilon.$$

In future I will use 1 to denote the function in H_0 which constantly takes the value 1 for all $\zeta \in \mathbb{T}$. This is the group identity, which of course creates some notational conflict with H_0 being commutative.

Lemma 5.2. *H_0 is an abelian Polish group.*

Proof. The metric d_0 is easily checked to be complete, continuous with respect to the group action, and separable since $L^1(\mathbb{T}, \lambda)$ is separable. ∎

Notation. Let $H_1 = \mathbb{T} \times \mathbb{Z}_2$ be the direct product of the groups $\mathbb{T}$ and $\mathbb{Z}_2$. We define $\varphi : H_1 \to \mathrm{Aut}(H_0)$ by the requirement that

$$((\varphi(\zeta, \overline{1}))(f))(\xi) = (f(\xi\zeta))^{-1}, \qquad ((\varphi(\zeta, \overline{0}))(f))(\xi) = (f(\xi\zeta)).$$

Note that H_1 is a compact Polish group and φ is a group homomorphism.

I will write $\varphi_{(\zeta,\overline{k})}$ for the homomorphism $\varphi(\zeta, \overline{k}) : H_0 \to H_0$. The map φ is continuous in the following sense:

Lemma 5.3. *The function*

$$H_1 \times H_0 \to H_0, \qquad ((\zeta, \overline{i}\,), f) \mapsto \varphi_{(\zeta,\overline{i})}(f)$$

is continuous as a map from $H_1 \times H_0$ to H_0.

Proof. Recall that the step functions consisting of finite linear combinations of the characteristic functions of intervals are dense in $L^1(\mathbb{T}, \lambda)$. This rapidly implies that for $f_0 \in H_0$ and $\varepsilon > 0$ there is $g \in H_0$ of the form

$$g = \sum_{j=1}^{k} c_j \chi_{A_j}$$

with

$$d_0(g, f_0) < \varepsilon/3$$

for some $k \in \mathbb{N}$, $c_1, \ldots, c_k \in \mathbb{C}$, of absolute value 1, and measurable subsets $A_1, \ldots, A_k$ of $\mathbb{T}$, each given by

$$A_n = \{e^{2\pi ix} : a_n \leq x < a_{n+1}\}.$$

For continuity it is enough to check that for $(\zeta, \overline{j}\,) = (e^{2\pi iy}, \overline{j}\,)$ sufficiently close to the identity and $f \in H_0$ sufficiently close to f_0 we have

$$d_0(\varphi_{(e^{2\pi iy}, \overline{j})}(f), f_0) < \varepsilon.$$

But if $\overline{j} = \overline{0}$ and y is close enough to 0 that

$$|c_n y| = |y| < \varepsilon/(3k), \qquad n = 1, \ldots, k,$$

then

$$d_0(\varphi_{(e^{2\pi iy}, \overline{0})}(g), g) < \varepsilon/3,$$

since at each $n \leq k$,

$$\lambda(A_n \vartriangle \{e^{2\pi y}\zeta : \zeta \in A_n\}) \leq |y|.$$

The function $\varphi_{(e^{2\pi iy}, \overline{0})}$ is an isometry, so for f close to f_0 we have

$$d_0(\varphi_{(e^{2\pi iy}, \overline{0})}(g), \varphi_{(e^{2\pi iy}, \overline{0})}(f)) < \varepsilon/3,$$

and thus by the triangle inequality

$$\begin{aligned} &d_0(\varphi_{(e^{2\pi iy}, \overline{j})}(f), f_0) \\ &\qquad \leq d_0(\varphi_{(e^{2\pi iy}, \overline{j})}(f), \varphi_{(e^{2\pi iy}, \overline{0})}(g)) + d_0(\varphi_{(e^{2\pi iy}, \overline{0})}(g), g) + d_0(g, f_0) \\ &\qquad < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \ \blacksquare \end{aligned}$$

NOTATION. Let $G = H_1 \rtimes_\varphi H_0$ be the semidirect product of H_1 and H_0 along φ. Thus for $(\zeta_0, \overline{i}_0, f_0), (\zeta_1, \overline{i}_1, f_1) \in G$,

$$(\zeta_0, \overline{i}_0, f_0)(\zeta_1, \overline{i}_1, f_1) = (\zeta_0\zeta_1, \overline{i}_0 + \overline{i}_1, (\varphi(\zeta_1, \overline{i}_1))(f_0)f_1).$$

This order of taking the semidirect product does give us a group since H_1 is abelian.

Here and elsewhere I simply write $(\zeta, \overline{i}, f)$ for an arbitrary element of G, instead of the more cumbersome but perhaps more formally correct $((\zeta, \overline{i}), f)$.

Fix complete metrics d_0 and d_1 on H_0 and H_1.

LEMMA 5.4. *G is a Polish group.*

Proof. We obtain a complete and compatible metric d_G on G by

$$d_G((\zeta_0, \overline{i}_0, f_0), (\zeta_1, \overline{i}_1, f_1)) = d_1((\zeta_0, \overline{i}_0), (\zeta_1, \overline{i}_1)) + d_0(f_0, f_1).$$

It follows from 5.3 and H_0 and H_1 being topological groups that the group operation of multiplication is continuous on G. Since G is Polish as a space it follows from say [21] that $g \mapsto g^{-1}$ is continuous as well. ■

Notation. Let $X = \{h : \mathbb{T} \to \mathbb{T} : h \text{ is Lebesgue measurable}\}$, where we identify functions agreeing a.e. For future reference we let d_X be the metric given on X by $d_X(h_0, h_1) = \frac{1}{2}\int |h_0(\xi) - h_1(\xi)|\, d\lambda$.

We let G act on X as follows:

$$((\zeta, \bar{i}, f) \cdot h)(\xi) = [f(\xi/\zeta)h(\xi/\zeta)(f(e^{2\pi i\sqrt{2}}\xi/\zeta))^{-1}]^{(-1)^i}.$$

(Of course, literally as a space X is the same as H_0. But here we are thinking of X as coming equipped with a G-action, while we think of H_0 as presented with a *group structure*.)

Lemma 5.5. *This* is *an action.*

Proof. It is trivial to confirm that $(1, \bar{0}, 1) \cdot h = h$ for all $h \in X$; the main task is to show the associativity properties of the action.

Let $\widehat{H}_0$ be the subgroup of G consisting of elements of the form $(1, \bar{0}, f)$ and let $\widehat{H}_1$ consist of those of the form $(\zeta, \bar{i}, 1)$. Every element g of G can be written in the form

$$g = h_0 h_1 = h_1^* h_0^*$$

for suitable $h_0, h_0^* \in \widehat{H}_0$ and $h_1, h_1^* \in \widehat{H}_1$. Hence it suffices to show purely for $k_1, k_2 \in \widehat{H}_0 \cup \widehat{H}_1$ that for all $x \in X$ we have

$$k_1 \cdot (k_2 \cdot x) = (k_1 k_2) \cdot x;$$

the point is that given arbitrary $g_1, g_2 \in G$ we can write

$$g_1 = h_{0,1}h_{1,1}, \qquad g_2 = h_{1,2}h_{0,2}$$

(for suitable $h_{0,i} \in \widehat{H}_0$, $h_{1,i} \in \widehat{H}_1$) and then steadily multiply through to get

$$\begin{aligned} g_1 \cdot (g_2 \cdot x) &= h_{0,1} \cdot h_{1,1} \cdot h_{1,2} \cdot h_{0,2} \cdot x \\ &= h_{0,1} \cdot (h_{1,1}h_{1,2}) \cdot h_{0,2} \cdot x \\ &= (h_{0,1}h_{1,1}h_{1,2}) \cdot h_{0,2} \cdot x = h_1^* h_0^* \cdot h_{0,2} \cdot x \end{aligned}$$

(for suitable $h_0^* \in \widehat{H}_0$, $h_1^* \in \widehat{H}_1$)

$$= h_1^* \cdot (h_0^* h_{0,2}) \cdot x = (h_1^* h_0^* h_{0,2}) \cdot x.$$

So we are left only with checking the associativity of the action for $k_1, k_2 \in \widehat{H}_0 \cup \widehat{H}_1$. There are four possibilities, but only the case $k_2 \in \widehat{H}_1$, $k_1 \in \widehat{H}_0$ requires close inspection.

Here, however, we see that for any $h \in X$, $(\zeta, \bar{i}) \in H_1$, $f \in H_0$,

$$\begin{aligned} &(((1, \bar{0}, f)(\zeta, \bar{i}, 1)) \cdot h)(\xi) \\ &\qquad = ((\zeta, \bar{i}, \varphi_{(\zeta,\bar{i})}(f)) \cdot h)(\xi) \\ &\qquad = [((\varphi_{(\zeta,\bar{i})}(f))(\xi/\zeta))h(\xi/\zeta)((\varphi_{(\zeta,\bar{i})}(f))(\xi e^{2\pi i\sqrt{2}}/\zeta))^{-1}]^{(-1)^i} \end{aligned}$$

$$= [(f(\zeta\xi/\zeta))^{(-1)^i} h(\xi/\zeta)(f(\zeta e^{2\pi i\sqrt{2}}\xi/\zeta))^{(-1)^{i+1}}]^{(-1)^i}$$
$$= (f(\xi))^{(-1)^{2i}}(h(\xi/\zeta))^{(-1)^i}(f(\xi e^{2\pi i\sqrt{2}}))^{(-1)^{2i+1}}$$
$$= f(\xi)(h(\xi/\zeta))^{(-1)^i}(f(\xi e^{2\pi i\sqrt{2}}))^{-1}$$
$$= f(\xi)((((\zeta,\bar{i},0)\cdot h)(\xi))(f(e^{2\pi i\sqrt{2}}\xi))^{-1}$$
$$= ((1,\bar{0},f)\cdot(\zeta,\bar{i},1)\cdot h)(\xi)$$

as required. ∎

The action is clearly continuous, and so X is a Polish G-space.

NOTATION. Let E_G^X denote the orbit equivalence relation induced by this action.

NOTATION. From now until the end of the section, M_∞ is used to denote $M_\infty(\mathbb{T}^2,\lambda^2)$, the group of invertible λ^2 measure preserving functions $\pi:\mathbb{T}^2\to\mathbb{T}^2$, subject to the usual identification in the event of agreement almost everywhere.

Thus we are using M_∞ to denote a different Polish group to the one from Section 2, but since these two are naturally isomorphic the identification would seem harmless.

NOTATION. For $h\in X$, let $T_h:\mathbb{T}^2\to\mathbb{T}^2$ be given by

$$(\zeta,\xi)\mapsto(\zeta e^{2\pi i\sqrt{2}},\xi h(\zeta)).$$

LEMMA 5.6. *The function $h\mapsto T_h$ is a continuous function from X to M_∞.*

Proof. In general, "skew products" of this form give rise to measure preserving transformations (compare §2 of [2] or Chapter 1 of [20]). The further facts that T_h is invertible and that the assignment $h\mapsto T_h$ is continuous follow almost immediately from the definitions. ∎

LEMMA 5.7. *Every G-orbit in X is dense*; *in fact*, *for any $h\in X$*, $\{(1,\bar{0},f)\cdot h : f\in H_0\}$ *is dense in X.*

Proof. Fix $h_0,h_1\in X$ and $\varepsilon>0$. Following the Kakutani–Rokhlin lemma (see page 48 of [20]) we may find $A\subseteq\mathbb{T}$ so that for some n

(i) A, $e^{2\pi i\sqrt{2}}A$, $e^{4\pi i\sqrt{2}}A,\dots,e^{2n\pi i\sqrt{2}}A$ are pairwise disjoint ([6]);

(ii) $\lambda(\bigcup_{l<n} e^{2l\pi i\sqrt{2}}A)>1-\varepsilon$.

([6]) Here $e^{2\pi i\sqrt{2}}A=\{e^{2\pi i\sqrt{2}}\zeta:\zeta\in A\}$.

To obtain the existence of this set A we apply Kakutani–Rokhlin to any $n > 2/\varepsilon$ to obtain A so that

$$\lambda\Big(\bigcup_{l\leq n} e^{2l\pi i\sqrt{2}}A\Big) > 1 - \frac{\varepsilon}{2}$$

and the sets

$$A, e^{2\pi i\sqrt{2}}A, e^{4\pi i\sqrt{2}}A, \ldots, e^{2n\pi i\sqrt{2}}A$$

are disjoint as in (i). Then we must have $\lambda(A) < \varepsilon/2$ and (ii) follows as well.

Now we may simply step around this sequence A, $e^{2\pi i\sqrt{2}}A$, $e^{4\pi i\sqrt{2}}A, \ldots,$ $e^{2n\pi i\sqrt{2}}A$, defining $f|_{e^{2l\pi i\sqrt{2}}A}$ by induction on l so that at each $\xi \in e^{2l\pi i\sqrt{2}}A$ we have $f(\xi)h_0(\xi)(f(\xi e^{2\pi i\sqrt{2}}))^{-1} = h_1(\xi)$.

More formally, we let $f|_A$ just be constantly 1. Assuming inductively that $l < n$ and $f|_{e^{2l\pi i\sqrt{2}}A}$ has been defined, we let

$$f(\xi e^{2\pi i\sqrt{2}}) = f(\xi)h_0(\xi)(h_1(\xi))^{-1}$$

for any $\xi \in e^{2l\pi i\sqrt{2}}A$.

Finally, for all $\xi \in \bigcup_{l<n} e^{2l\pi i\sqrt{2}}A$ we have

$$f(\xi)h_0(\xi)(f(\xi e^{2\pi i\sqrt{2}}))^{-1} = h_1(\xi),$$

and thus

$$d_X((1,\overline{0},f)\cdot h_0, h_1) < \varepsilon. \ \blacksquare$$

Notation. Let $\approx$ denote the conjugacy equivalence relation on M_∞, so that for $\sigma_1, \sigma_2 \in M_\infty$ we write $\sigma_1 \approx \sigma_2$ if there is a $\pi \in M_\infty$ such that for a.e. $(\zeta,\xi) \in \mathbb{T}^2$,

$$\pi \circ \sigma_1 \circ \pi^{-1} = \sigma_2.$$

Lemma 5.8. *For all $h_0, h_1 \in X$,*

$$h_0 E_G^X h_1 \ \Rightarrow\ T_{h_0} \approx T_{h_1}.$$

Proof. We may break this down into three cases.

(i) $(\zeta_0, \overline{0}, 1)\cdot h_0 = h_1$, so that $h_1(\xi) = h_0(\xi/\zeta_0)$ for λ-a.e. ξ. Then we may define $\pi : \mathbb{T}^2 \to \mathbb{T}^2$ by $(\zeta,\xi) \mapsto (\zeta\zeta_0, \xi)$. We then have, for a.e. (ζ,ξ),

$$\begin{aligned}(\pi \circ T_{h_0} \circ \pi^{-1})(\zeta,\xi) &= (\pi \circ T_{h_0})(\zeta/\zeta_0, \xi) = (\pi(e^{2\pi i\sqrt{2}}\zeta/\zeta_0), h_0(\zeta/\zeta_0)\xi)\\ &= (e^{2\pi i\sqrt{2}}\zeta, \xi h_1(\zeta)) = T_{h_1}(\zeta,\xi).\end{aligned}$$

(ii) $(1, \overline{1}, 1)\cdot h_0 = h_1$, so that $h_1(\xi) = (h_0(\xi))^{-1}$ a.e. Then define $\pi : \mathbb{T}^2 \to \mathbb{T}^2$ by $(\zeta,\xi) \mapsto (\zeta, \xi^{-1})$ and note that

$$\begin{aligned}\pi T_{h_0}\pi^{-1}(\zeta,\xi) &= \pi(e^{2\pi i\sqrt{2}}\zeta, h_0(\zeta)/\xi) = (e^{2\pi i\sqrt{2}}\zeta, (h_0(\zeta))^{-1}\xi)\\ &= (e^{2\pi i\sqrt{2}}\zeta, h_1(\zeta)\xi).\end{aligned}$$

(iii) $(1,\overline{0},f)\cdot h_0 = h_1$, so that $f(\zeta)h_0(\zeta)(f(\zeta e^{2\pi i\sqrt{2}}))^{-1} = h_1(\zeta)$ a.e. Define $\pi:\mathbb{T}^2\to\mathbb{T}^2$ by $(\zeta,\xi)\mapsto(\zeta,f(\zeta)\xi)$. Note then that

$$\begin{aligned}\pi^{-1}T_{h_0}\pi(\zeta,\xi) &= \pi^{-1}(e^{2\pi i\sqrt{2}}\zeta, h_0(\zeta)f(\zeta)\xi)\\ &= (e^{2\pi i\sqrt{2}}\zeta, (f(\zeta e^{2\pi i\sqrt{2}}))^{-1}h_0(\zeta)f(\zeta)\xi) = T_{h_1}(\zeta,\xi). \blacksquare\end{aligned}$$

LEMMA 5.9. *The set of $h\in X$ for which T_h is ergodic is a dense G_δ.*

Proof. Recall (compare §2 of [2]) that ergodicity is a G_δ condition on an element of M_∞: $\pi\in M_\infty$ is ergodic if and only if for all open $A,B\subseteq\mathbb{T}^2$ arising as finite unions of basic open sets there is m with

$$\lambda(\pi^m(A)\cap B) > \tfrac{1}{4}\lambda(A)\lambda(B).$$

Since $h\mapsto T_h$ is continuous, the set of h for which T_h is ergodic is again G_δ.

However there is *some* h for which T_h is ergodic—for instance $h:\mathbb{T}\to\mathbb{T}$, $\zeta\mapsto\zeta$ (see [2] or [6]). Thus by 5.7, the set of h for which T_h ergodic is a dense G_δ. ∎

NOTATION. Let $X_0 = \{h\in X : T_h \text{ is ergodic}\}$.

By 5.8 we see that X_0 is G-invariant; since it is G_δ it is a Polish G-space in its own right.

For the convenience of the reader, we give the next definition only in the narrow context that is directly relevant. The more general definitions can be found in [24] or [20], §2.4.

DEFINITION. Let $\varrho:\mathbb{T}^2\to\mathbb{T}^2$ be an invertible measure preserving transformation of the form

$$(\zeta,\xi)\mapsto(\zeta e^{2\pi i\sqrt{2}}, \widehat{\varrho}(\zeta,\xi))$$

where $\widehat{\varrho}:\mathbb{T}^2\to\mathbb{T}$. Then a non-zero function $f\in L^2(\mathbb{T}^2)$ is said to be a *generalized eigenfunction* for ϱ if there is some $g\in L^2(\mathbb{T})$, called a *generalized eigenvalue*, with the property that for all $(\zeta,\xi)\in\mathbb{T}^2$ we have

$$f\circ\varrho(\zeta,\xi) = g(\zeta)(f(\zeta,\xi));$$

in other words, $f\circ\varrho = \widehat{g}f$ for $\widehat{g}$ defined by $\widehat{g}(\zeta,\xi)=g(\zeta)$.

The next couple of lemmas are standard; more general results, along with related facts, can be found in [24].

LEMMA 5.10. *Let $h\in X_0$ and let $f_1,f_2\in L^2(\mathbb{T}^2)$ be generalized eigenfunctions for T_h with a common generalized eigenvalue. Then f_1 is a linear multiple of f_2.*

Proof. $f_1^{-1}f_2$ is invariant under T_h, and hence must be a constant function by ergodicity. ∎

LEMMA 5.11. *Let $h\in X_0$. Then the only generalized eigenfunctions are*

$$(\zeta,\xi)\mapsto\xi^n k(\zeta)$$

for some measurable function $k:\mathbb{T}\to\mathbb{C}$.

Proof. Note that by Stone–Weierstrass, every function $f \in L^2(\mathbb{T}^2)$ can be written as

$$f : (\zeta, \xi) \mapsto \sum_{n \in \mathbb{Z}} \xi^n k_n(\zeta)$$

for some $k_n \in L^2(\mathbb{T})$. Moreover, the decomposition is unique, since

$$(\zeta, \xi) \mapsto \xi^n k_n(\zeta) \quad \text{and} \quad (\zeta, \xi) \mapsto \xi^m k_m(\zeta)$$

are orthogonal for $n \neq m$.

Now suppose that $f : (\zeta, \xi) \mapsto \sum_{n \in \mathbb{Z}} \xi^n k_n(\zeta)$ has g as its generalized eigenfunction. Hence

$$f \circ T_h : (\zeta, \xi) \mapsto \sum_{n \in \mathbb{Z}} \xi^n h(\zeta)^n k_n(\zeta e^{2\pi i \sqrt{2}}).$$

Then the uniqueness of the decomposition of $f \circ T_h$ gives, for a.e. $\zeta \in \mathbb{T}$,

$$\xi^n h(\zeta)^n k_n(\zeta e^{2\pi i \sqrt{2}}) = g(\zeta) \xi^n k_n(\zeta).$$

This means that any k_n not identically zero gives rise to

$$(\zeta, \xi) \mapsto \xi^n k_n(\zeta)$$

as a function with generalized eigenfunction g; thus by 5.10 we have $k_n \equiv 0$ for all but a single n. ∎

Lemma 5.12. *For all $h_0, h_1 \in X_0$,*

$$T_{h_0} \approx T_{h_1} \;\Rightarrow\; h_0 E_G^X h_1.$$

Proof. Fix h_0, h_1 with $T_{h_0} \approx T_{h_1}$, and let $\pi \in M_\infty$ witness this—in that $\pi T_{h_0} \pi^{-1} = T_{h_1}$ λ^2-a.e.

By ergodicity

$$(\zeta, \xi) \mapsto \zeta$$

is up to scalar multiples the only eigenfunction with eigenvalue $e^{2\pi i \sqrt{2}}$ for both T_{h_0} and T_{h_1}. Thus

$$\pi(\zeta, \xi) = (\zeta_0 \zeta, \widehat{\varrho}(\zeta, \xi))$$

for some fixed $\zeta_0 \in \mathbb{T}$ and suitable $\widehat{\varrho}$. By replacing h_0 with $((\zeta_0)^{-1}, \overline{0}, 1) \cdot h_0$ we may assume that $\zeta_0 = 1$ and thus

$$\pi(\zeta, \xi) = (\zeta, \widehat{\varrho}(\zeta, \xi)).$$

By 5.11, the only generalized eigenfunctions for T_{h_i} (i equal to either 0 or 1) are of the form

$$(\zeta, \xi) \mapsto \xi^n k(\zeta)$$

for some $n \in \mathbb{Z}$ and measurable $k : \mathbb{T} \to \mathbb{C}$. Thus we see that the generalized eigenfunctions of the form

$$(\zeta, \xi) \mapsto \xi k(\zeta), \qquad (\zeta, \xi) \mapsto \xi^{-1} k(\zeta)$$

have a privileged status of the only generalized eigenfunctions which are able to generate the space $L^2(\mathbb{T}^2, \lambda^2)$ by the operations of multiplication, addition, and multiplication by linear combinations of the eigenfunctions $(\zeta, \xi) \mapsto \zeta^m$ for some $m \in \mathbb{Z}$ (here by "generate" I mean that they are dense in the sense of the Hilbert space norm on L^2). Note then that $(\zeta, \xi) \mapsto \xi$ must be sent to a generalized eigenfunction for T_{h_1} of the form

$$(\zeta, \xi) \mapsto \xi^j k(\zeta)$$

where j is either 1 or -1.

Thus we may assume

$$\pi(\zeta, \xi) = (\zeta, \xi^{(-1)^i} k(\zeta))$$

for some measurable $k : \mathbb{T} \to \mathbb{T}$, and so

$$\pi^{-1}(\zeta, \xi) = (\zeta, \xi^{(-1)^i} (k(\zeta))^{(-1)^{i+1}}).$$

Thus

$$\begin{aligned}(e^{2\pi i\sqrt{2}}\zeta, h_0(\zeta)\xi) &= \pi^{-1} T_{h_1} \pi(\zeta, \xi) = \pi^{-1} T_{h_1}(\zeta, k(\zeta)(\xi)^{(-1)^i}) \\ &= \pi^{-1}(\zeta e^{2\pi i\sqrt{2}}, k(\zeta) h_1(\zeta)(\xi)^{(-1)^i}) \\ &= (\zeta e^{2\pi i\sqrt{2}}, (k(\zeta e^{2\pi i\sqrt{2}}))^{(-1)^{i+1}} (h_1(\zeta))^{(-1)^i} k(\zeta)^{(-1)^i} \xi).\end{aligned}$$

Thus for a.e. (ζ, ξ) we have

$$[(k(\zeta e^{2\pi i\sqrt{2}}))^{-1} h_1(\zeta) k(\zeta)]^{(-1)^i} = h_0(\zeta),$$

and so $h_0 E_G^X h_1$ as required. ∎

Lemma 5.13. *Every orbit in X is meager.*

Proof. Let

$$A_0 = \mathbb{T} \times \left\{e^{2\pi i x} : \tfrac{1}{4} \le x \le \tfrac{3}{4}\right\}.$$

Claim. *For any $O_1, O_2 \subset X$ open, non-empty and A measurable, there exist $h_1 \in O_1$, $h_2 \in O_2$, $k \in \mathbb{N}$, $k > 0$ with*

$$\lambda^2(A \,\triangle\, T_{h_1}^k(A)) < \tfrac{1}{13}, \qquad \lambda^2(A_0 \,\triangle\, T_{h_2}^k(A_0)) > \tfrac{1}{4}.$$

Proof. Let $\widehat{h}_1$ be the function $\zeta \mapsto 1$ on $\mathbb{T}$, and let $\widehat{h}_{\sqrt{3}}$ be the function

$$\widehat{h}_{\sqrt{3}} : \zeta \mapsto e^{2\pi i\sqrt{3}}.$$

Then by 5.7 we may find $(1, \overline{0}, f_1), (1, \overline{0}, f_2) \in G$ with

$$h_1 := (1, \overline{0}, f_1) \cdot \widehat{h}_1 \in O_1, \qquad h_2 := (1, \overline{0}, f_2) \cdot \widehat{h}_{\sqrt{3}} \in O_2.$$

Since the continuous functions are dense in L^1 we may actually assume that f_1 and f_2 are continuous. Then

$$h_1(e^{2\pi ix}) = f_1(e^{2\pi ix})\widehat{h}_1(e^{2\pi ix})(f_1(e^{2\pi i(x+\sqrt{2})}))^{-1}$$
$$= f_1(e^{2\pi ix})(f_1(e^{2\pi i(x+\sqrt{2})}))^{-1}$$

and thus

$$T_{h_1} : (e^{2\pi ix}, \xi) \mapsto (e^{2\pi i(x+\sqrt{2})}, f_1(e^{2\pi ix})(f_1(e^{2\pi i(x+\sqrt{2})}))^{-1}\xi)$$

and

$$(T_{h_1})^k : (e^{2\pi ix}, \xi) \mapsto (e^{2\pi i(x+k\sqrt{2})}, f_1(e^{2\pi ix})(f_1(e^{2\pi i(x+k\sqrt{2})}))^{-1}\xi).$$

An exactly similar calculation gives

$$(T_{h_2})^k : (e^{2\pi ix}, \xi) \mapsto (e^{2\pi i(x+k\sqrt{2})}, f_2(e^{2\pi ix})e^{2k\pi i\sqrt{3}}(f_2(e^{2\pi i(x+k\sqrt{2})}))^{-1}\xi).$$

The continuity of f_1, f_2 guarantees for each δ some corresponding $\widehat{\delta} > 0$ such that whenever

$$|e^{2k\pi i\sqrt{2}} - 1| < \widehat{\delta}$$

then for all $x \in [0,1]$,

$$|f_1(e^{2\pi ix}) - f_1(e^{2\pi i(x+k\sqrt{2})})|, |f_2(e^{2\pi ix}) - f_2(e^{2\pi i(x+k\sqrt{2})})| < \delta.$$

Since $\sqrt{2}$ and $\sqrt{3}$ are rationally independent, we can therefore apply Kronecker's lemma (Theorem 28 of [16]) and find k with

$$|f_1(e^{2\pi ix}) - f_1(e^{2\pi i(x+k\sqrt{2})})|, \quad |f_2(e^{2\pi ix}) - f_2(e^{2\pi i(x+k\sqrt{2})})|$$

both arbitrarily small for all $x \in [0,1]$ and $e^{2k\pi i\sqrt{3}}$ arbitrarily close to $e^{\pi i}$. Such k clearly suffices. ∎

Now let us choose a sequence of measurable sets $(B_i)_i$ such that for all $\pi \in M_\infty$ there is some $i \in \mathbb{N}$ with

$$\lambda^2(\pi(A_0) \triangle B_i) < \tfrac{1}{13}.$$

Let V_i be

$$\left\{\pi \in M_\infty : \lambda^2(\pi(A_0) \triangle B_i) < \tfrac{1}{13}\right\}.$$

By 5.8 it suffices to show that for each i the set

$$\{(h_1, h_2) \in X \times X : \exists \pi \in V_i(\pi^{-1} \circ T_{h_1} \circ \pi = T_{h_2})\}$$

is nowhere dense.

But given any non-empty open $O_1, O_2 \subset X \times X$ we can by the above claim find some non-empty open $U_1 \subset O_1$, $U_2 \subset O_2$ and $k \in \mathbb{N}$ such that for all $h_1 \in U_1$, $h_2 \in U_2$,

$$\lambda^2(B_i \triangle T^k_{h_1}(B_i)) < \tfrac{1}{13}, \qquad \lambda^2(A_0 \triangle T^k_{h_2}(A_0)) > \tfrac{1}{4}.$$

Fixing such h_1, h_2 and $\pi \in V_i$ we need to show

$$\pi^{-1} \circ T_{h_1} \circ \pi \neq T_{h_2}.$$

But since π is measure preserving we have

$$\begin{aligned}\lambda^2(A_0 \triangle (\pi^{-1}T_{h_1}\pi)^k(A_0)) &= \lambda^2(A_0 \triangle \pi^{-1}(T_{h_1}^k(\pi(A_0)))) \\ &= \lambda^2(\pi(A_0) \triangle T_{h_1}^k(\pi(A_0))),\end{aligned}$$

which by the triangle inequality is bounded by

$$\begin{aligned}&\lambda^2(\pi(A_0), B_i) + \lambda^2(B_i, T_{h_1}^k(B_i)) + \lambda^2(T_{h_1}^k(B_i), T_{h_1}^k(\pi(A_0))) \\ &\qquad = \lambda^2(\pi(A_0), B_i) + \lambda^2(B_i, T_{h_1}^k(B_i)) + \lambda^2(B_i, \pi(A_0))\end{aligned}$$

since $T_{h_1}^k$ is measure preserving, which in turn is bounded by

$$\tfrac{1}{13} + \tfrac{1}{13} + \tfrac{1}{13} < \tfrac{1}{4}$$

by the assumption of $h_1 \in U_1$ and $\pi \in V_i$.

This is as required to show

$$\lambda^2(A_0 \triangle \pi^{-1}(T_{h_1}^k(\pi(A_0)))) \neq \lambda^2(A_0 \triangle T_{h_2}^k(A_0)). \ \blacksquare_{5.13}$$

Definition. Let H be a Polish group and Y a Polish H-space. The action of H on Y is said to be *turbulent* if

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in Y$, $U \subseteq Y$, $V \subseteq H$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_H$ (the orbit of y) such that for all open U_0 containing y_0 there are $k \in \mathbb{N}$, $(h_i)_{i<k} \subseteq V$ and $(x_i)_{i\le k} \subseteq U$ with

$$x_0 = x, \quad x_{i+1} = h_i \cdot x_i, \quad x_k \in U_0.$$

The usefulness of this concept is that it gives a sufficient condition for a degree of non-classifiability: As in [12] no turbulent action allows a Borel—or even Baire—measurable function reducing its orbit equivalence relation to isomorphism on countable structures. More generally, any equivalence relation into which we can embed a turbulent orbit equivalence relation will similarly be unclassifiable by countable structures considered up to isomorphism.

Lemma 5.14. *The action of G on X_0 is turbulent.*

Proof. We already established that every orbit is dense and meager, so we are only left to show the "local density" condition at (iii) from the definition of turbulence.

For this purpose, fix $h_0, h_1 \in X$ and $\varepsilon > 0$. It suffices to show there are $n \in \mathbb{N}$ and $g_0, g_1, \ldots, g_{n-1} \in G$ such that

$$h_{0,0} = h_0, \quad h_{0,l+1} = g_l \cdot h_{0,l},$$

and

(i) $d_G(g_l, 1_G) < \varepsilon$ for each $l < n$;
(ii) $d_X(h_{0,l}, h_0) < d_X(h_0, h_1) + \varepsilon$ for each $l \leq n$;
(iii) $d_X(h_{0,n}, h_1) < \varepsilon$.

We will do this in a manner resembling the proof of 5.7.

Choose $n \in \mathbb{N}$ such that $n > 3/\varepsilon$. Appealing to Kakutani–Rokhlin we find $A \subseteq \mathbb{T}$ with

(iv) A, $e^{2\pi i\sqrt{2}}A$, $e^{4\pi i\sqrt{2}}A, \ldots, e^{2n\pi i\sqrt{2}}A$ all disjoint;
(v) (therefore) $\lambda(A) < \varepsilon/3$;
(vi) $\lambda(\bigcup_{l\leq n} e^{2l\pi i\sqrt{2}}A) > 1 - \varepsilon/3$.

We now define $f_l : \mathbb{T} \to \mathbb{T}$ by induction on $l < n$. The function f_0 is constantly 1. Given the definition of f_l we let

$$f_{l+1}(e^{2\pi i\sqrt{2}}\xi) = f_l(\xi)h_0(\xi)(h_1(\xi))^{-1}$$

for $\xi \in e^{2l\pi i\sqrt{2}}A$, and

$$f_{l+1}(e^{2\pi i\sqrt{2}}\xi) = 1$$

otherwise. We then let $g_i = (1, \overline{0}, f_i)$. Next, $h_{0,0} = h_0$ and $h_{0,l+1} = g_l h_{0,l}$ as indicated above.

At once we have (i), since each f_i is not equal to 1 only on a set of measure $< \varepsilon$. For (ii), note that if $k < n$ then for all $\xi \in \bigcup_{l<k} e^{2l\pi i\sqrt{2}}A$,

$$h_{0,k}(\xi) = h_1(\xi),$$

while for $\xi \in \bigcup_{l>k,\, l<n} e^{2l\pi i\sqrt{2}}A$,

$$h_{0,k}(\xi) = h_0(\xi).$$

Thus

$$\begin{aligned} d_X(h_{0,k}, h_0) &= \int \tfrac{1}{2}|h_{0,k}(\xi) - h_0(\xi)|\, d\lambda \\ &< \lambda(e^{2k\pi i\sqrt{2}}A) + \lambda(e^{2n\pi i\sqrt{2}}A) + \varepsilon/3 + \int \tfrac{1}{2}|h_1(\xi) - h_0(\xi)|\, d\lambda \\ &< \varepsilon + d_X(h_0, h_1). \end{aligned}$$

Finally, for (iii) note that $h_{0,n}$ and h_1 agree except on $\mathbb{T} \setminus \bigcup_{l<n} e^{2l\pi i\sqrt{2}}A$, which has measure less than ε. ■

Summarizing what has been proved:

Theorem 5.15. *There is a Polish group G and a Polish G-space X_0 and a Borel function $\theta : X_0 \to M_\infty$ such that*:

(i) *the action of G on X_0 is turbulent* (*Lemma* 5.14);
(ii) *for each $h \in X_0$ the transformation $\theta(h) \in M_\infty$ is ergodic* (*Lemmas* 5.9, 5.12);

(iii) *in fact, each $\theta(h)$ is "measure-distal" (in the sense of* [2]), *and in fact, it is rank* 2 *generalized discrete spectrum* (*granting* (ii), *this is an immediate consequence of the definition of the assignment $h \mapsto T_h$*);

(iv) *for all $h_0, h_1 \in X_0$,*

$$h_0 E_G^X h_1 \quad \textit{if and only if} \quad \theta(h_0) \approx \theta(h_1),$$

where $\approx$ is the equivalence relation of conjugacy (*Lemmas* 5.8, 5.12).

COROLLARY 5.16. (a) *There is no countable language $\mathcal{L}$ and Borel $\varrho : M_\infty \to \mathrm{Mod}(\mathcal{L})$ such that for all $\sigma_1, \sigma_2 \in M_\infty$,*

$$\sigma_1 \approx \sigma_2 \;\Leftrightarrow\; \varrho(\sigma_1) \cong \varrho(\sigma_2).$$

(b) *In fact, if $B \subseteq M_\infty$ is the subclass consisting of rank* 2 *generalized discrete spectrum ergodic transformations then there is no Borel $\varrho : B \to \mathrm{Mod}(\mathcal{L})$ such that for all $\sigma_1, \sigma_2 \in M_\infty$,*

$$\sigma_1 \approx \sigma_2 \;\Leftrightarrow\; \varrho(\sigma_1) \cong \varrho(\sigma_2).$$

Proof. At once by the results of §3.2 of [12]. ∎

In actual fact there is no compulsion to restrict ourselves so carefully to the Borel category. The methods of [12] are sufficient to obtain non-reducibility to isomorphism on countable structures even using very broad classes of functions, such as C-measurable, absolutely $\underset{\sim}{\Delta}^1_2$, and universally Baire measurable. As mentioned in the introduction, we may even obtain the consistency of ZF + DC along with the non-existence of *any* injection

$$i : P/{\approx} \hookrightarrow \mathrm{Mod}(\mathcal{L}).$$

6. Remarks on the equivalence of cocycles. There is obviously a close relation between the arguments of §5 and the isomorphism relation on cocycles. It might be worth pausing before the finish of this paper to consider what can be drawn out in this fashion.

DEFINITION. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and H a countable group acting by measure preserving transformations on Ω. For later purposes assume that Ω is a Lebesgue space (that is to say, measurably isomorphic to $([0,1], \mathrm{Borel}, \lambda)$). Let K be a compact metric group. A measurable map $\alpha : H \times \Omega \to K$ is a *cocycle* if for all $h, h' \in H$ and $s \in \Omega$,

$$\alpha(hh', s) = \alpha(h, h's)\alpha(h', s).$$

(Here the measurability requirement is that for all $h \in H$ we have $s \mapsto \alpha(s, h)$ measurable.)

In the case of $H = \mathbb{Z}$ the cocycle condition becomes especially transparent, since we can exactly specify a cocycle by its value on a generator of $\mathbb{Z}$. Hence we can naturally identify a cocycle for $\mathbb{Z}$ with a measurable function from the space Ω to K.

The perspective of [24] is to only consider the case when K is a compact metric group; the remarks below persist in some form even in the more general context of K being a locally compact Polish group with a bi-invariant metric. If d_K is a compatible complete metric on a compact metric group K then we obtain an invariant metric with $d_{K,\mathrm{inv}}(g_0, g_1)$ set equal to

$$\int_{K\times K} d_K(h_0 g_0 h_1, h_0 g_1 h_1)\, d(\mu_K \times \mu_K).$$

Definition. Let $(\Omega, \mathcal{B}, \mu)$, K, H be as above. Two cocycles $\alpha, \beta : H \times \Omega \to K$ are said to be *equivalent* if there is a measurable function $\varphi : \Omega \to K$ such that for every $h \in H$ and a.e. $s \in \Omega$,

$$\varphi(h \cdot s)^{-1} \alpha(h, s) \varphi(s) = \beta(h, s).$$

Note that—as exploited in §5—this equivalence relation is induced by a Polish group action.

Notation. Let $(\Omega, \mathcal{B}, \mu)$, K, H be as above. Let d_K be an invariant metric on K; by possibly replacing it with $d_K/(1 + d_K)$ we may assume it is bounded by 1. Let $X(\Omega, K, H, \mu)$ be the space of all cocycles from $H \times \Omega$ to K. Let $G(\Omega, K, H, \mu)$ be the group of all measurable $\varphi : \Omega \to K$ under the operation of pointwise multiplication. We let $G(\Omega, K, H, \mu)$ act on $X(\Omega, K, H, \mu)$ by the specification that

$$(\varphi \cdot \alpha)(h, s) = \varphi(h \cdot s)^{-1} \alpha(h, s) \varphi(s).$$

For $\alpha \in X(\Omega, K, H, \mu)$ use $G(\Omega, K, H, \mu)_\alpha$ to denote the *stabilizer* of α, that is to say, the set of φ for which $\varphi \cdot \alpha = \alpha$.

The action of $G(\Omega, K, H, \mu)$ on $X(\Omega, K, H, \mu)$ is exactly chosen so that the resulting orbit equivalence relation is the cocycle equivalence relation. $G(\Omega, K, H, \mu)$ and $X(\Omega, K, H, \mu)$ are Polish spaces, and $X(\Omega, K, H, \mu)$ is a Polish $G(\Omega, K, H, \mu)$-space. For the group we obtain a complete and in fact invariant metric by

$$d_{G(\Omega,K,H,\mu)}(\varphi_0, \varphi_1) = \int_\Omega d_K(\varphi_0(s), \varphi_1(s))\, d\mu.$$

For the cocycles we can choose an enumeration $(h_i)_{i\in\mathbb{N}}$ of H and let

$$d_{X(\Omega,K,H,\mu)}(\alpha, \beta) = \sum_{i\in\mathbb{N}} 2^{-i} \int_\Omega d_K(\alpha(h_i, s), \beta(h_i, s))\, d\mu.$$

A special case of the above is when $H = \mathbb{Z}$, $\Omega = \mathbb{T}$, $K = \mathbb{T}$, and the action of $\mathbb{Z}$ on $\mathbb{T}$ is given by

$$l \cdot \zeta = e^{2l\pi i\sqrt{2}} \zeta.$$

Clearly, the arguments of §5 are sufficient to conclude that in this case—and in many others—the action of $G(\Omega, K, H, \mu)$ on $X(\Omega, K, H, \mu)$ is turbulent.

In showing conditions (i) and (iii) from the definitions of turbulence for G we only used the $G(\Omega, K, H, \mu)$ part in 5.7 and 5.14. Clearly, the property of every orbit being meager goes down to this sub-action by $G(\Omega, K, H, \mu)$. Thus in general the cocycle equivalence relation refuses classification by countable models.

In the presence of ergodicity, the stabilizers are all compact.

LEMMA 6.1. *Let $(\Omega, \mathcal{B}, \mu)$, K, H be as above. Suppose that the action of H on Ω is ergodic. Then $G(\Omega, K, H, \mu)_\alpha$ is compact for every $\alpha \in X(\Omega, K, H, \mu)$.*

Proof. $G(\Omega, K, H, \mu)_\alpha$ is a complete metric space, so we just need to show that it is ε-bounded for each ε.

Let $(\varphi_i)_{i\in\mathbb{N}}$ be a countable dense subset of $G(\Omega, K, H, \mu)_\alpha$. Using the fact that K is a compact metric space we may find a finite sequence of balls $B_0, B_1, \ldots, B_n$ of radius $< \varepsilon/3$ covering K. For each $i_0, i_1 \in \mathbb{N}$, $j \leq n$ let $A_{i_0,i_1,j} = \{s \in \Omega : \varphi_{i_0}(s), \varphi_{i_1}(s) \in B_j\}$. Let $\mathcal{A}$ be the *countable* collection $\{A_{i_0,i_1,j} : i_0, i_1 \in \mathbb{N},\ j \leq n\}$. We may clearly find $s_0 \in \Omega$ such that for all $A \in \mathcal{A}$, if $s_0 \in A$ then A is not null.

Now we may choose a finite collection $\varphi_{l_0}, \varphi_{l_1}, \ldots, \varphi_{l_k}$ from our sequence $(\varphi_i)_{i\in\mathbb{N}}$ such that for any $j \leq n$, if there exists $i \in \mathbb{N}$ with $\varphi_i(s_0) \in B_j$ then there is some $i' \leq k$ with $\varphi_{l_{i'}}(s_0) \in B_j$.

CLAIM. *If $\varphi_i(s_0) \in B_j$ and $\varphi_{l_{i'}}(s_0) \in B_j$ then for a.e. $s \in \Omega$,*

$$d_K(\varphi_i(s), \varphi_{l_{i'}}(s)) \leq 2\varepsilon/3.$$

Proof. Let $A = \{s \in \Omega : d_K(\varphi_i(s), \varphi_{l_{i'}}(s)) \leq 2\varepsilon/3\}$. By the assumption on s_0 this set A has non-zero measure. By ergodicity it suffices to show A is H-invariant.

But for any $h \in H$ and $s \in A$,

$$\alpha(h, s) = (\varphi_i \cdot \alpha)(h, s) = \varphi_i(h \cdot s)^{-1}\alpha(h, s)\varphi_i(s),$$

hence

$$\varphi_i(h \cdot s) = \alpha(h, s)\varphi_i(s)\alpha(h, s)^{-1},$$

and similarly

$$\varphi_{l_{i'}}(h \cdot s) = \alpha(h, s)\varphi_{l_{i'}}(s)\alpha(h, s)^{-1}.$$

Thus by the invariance of the metric $d_K(\varphi_i(h \cdot s), \varphi_{l_{i'}}(h \cdot s)) \leq 2\varepsilon/3$. ∎

Now, it is immediate from the definition of the complete metric on $G(\Omega, K, H, \mu)$ and the density of the set $(\varphi_i)_{i\in\mathbb{N}}$ that every element in $G(\Omega, K, H, \mu)_\alpha$ is within ε of some $\varphi_{l_{i'}}$. ∎6.1

Thus for any two cocycles α and β the set of $\varphi \in G(\Omega, K, H, \mu)$ with $\varphi \cdot \alpha = \beta$ is either empty or compact. Therefore the set

$$\{(\alpha,\beta)\in X(\Omega,K,H,\mu)^2 : \exists\varphi\in G(\Omega,K,H,\mu)\ (\varphi\cdot\alpha=\beta)\}$$

is the projection of a Borel set all of whose sections are compact, and is, by the Arsenin–Kunugui theorem (see §18 of [15] and §4F of [17]), itself Borel.

We consequently have a short proof of the equivalence relation being Borel for ergodic actions. I believe a much deeper proof of this fact has been previously extracted by Foreman and Weiss from the results of [24].

Lemma 6.1 should *not* be thought of as implying that the measure preserving transformations considered in §5 themselves have compact stabilizers in the natural action of M_∞ (= group of invertible measure preserving transformations on the unit interval) on M_∞. In fact, the ergodic transformations having compact stabilizer in M_∞ are exactly the discrete spectrum transformations.

LEMMA 6.2. *Let $T\in M_\infty$ be ergodic. Then the set $\{\pi\in M_\infty : \pi\circ T\circ\pi^{-1}=T\}$ is compact if and only if T is a discrete spectrum transformation.*

Proof. First let us take the case that T has discrete spectrum. Then by [9] we can assume that there is a compact abelian metric group G with a Haar measure μ and corresponding $g\in G$ with $\{n\cdot g : n\in\mathbb{Z}\}$ dense in G and T as a measure preserving transformation on $[0,1]$ isomorphic to translation of (G,μ) by g. For each $k\in G$ let

$$T_k : G\to G,\qquad h\mapsto kh,$$

be the transformation given by k-translation. Let $M_\infty(G)$ be the group of invertible measure preserving transformation on (G,μ). Since the assignment

$$G\to M_\infty(G),\qquad k\mapsto T_k,$$

is continuous we need only show that $\{\pi\in M_\infty(G) : \pi\circ T_g\circ\pi^{-1}=T_g\}$ equals $\{T_k : k\in G\}$.

It is clear that the compact group $\{T_k : k\in G\}$ of measure preserving transformations is included in $\{\pi\in M_\infty(G) : \pi\circ T_g\circ\pi^{-1}=T_g\}$, so fix π with $\pi\circ T_g\circ\pi^{-1}=T_g$ a.e. For each $x\in G$ let $h_x\in G$ be such that

$$h_x x=\pi(x).$$

We then have

$$h_x(T_g x)=h_x(gx)$$

by the definition of T_g, which in turn equals $gh_x x$ by G being abelian, which now equals $g\pi(x)=T_g\pi(x)=(T_g\circ\pi)(x)$, which by assumption on π equals

$$(\pi\circ T_g)(x)=\pi(T_g(x))$$

on a measure one set; and thus the function $x\mapsto h_x$ is a T_g-invariant function on a measure one set, and hence by ergodicity constant almost everywhere. Hence $\pi=T_{h_x}$ for some $x\in G$ on a measure one set, and we are done.

Conversely, if $\{\pi \in M_\infty : \pi \circ T \circ \pi^{-1} = T\}$ is compact, then T is an element of a compact subgroup of M_∞. Thus by the Peter–Weyl theorem, as for instance found in [25], we may find a sequence $H_0, H_1, H_2, \ldots$ of finite-dimensional subspaces of $L^2([0,1],\lambda)$ which are invariant under the unitary operator $f \mapsto f \circ T$ and which jointly sum up to give

$$L^2([0,1],\lambda) = \bigoplus_n H_n.$$

Then diagonalizing this unitary on each of these finite-dimensional subspaces we finish. ■

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