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Dispersing cocycles and mixing flows under functions

by

Klaus Schmidt (Wien)

Abstract. Let T be a measure-preserving and mixing action of a countable abelian group G on a probability space $(X,\mathcal{S},\mu)$ and A a locally compact second countable abelian group. A cocycle $c\colon G\times X\to A$ for T *disperses* if $\lim_{g\to\infty} c(g,\cdot)-\alpha(g)=\infty$ in measure for every map $\alpha\colon G\to A$. We prove that such a cocycle c does not disperse if and only if there exists a compact subgroup $A_0\subset A$ such that the composition $\theta\circ c\colon G\times X\to A/A_0$ of c with the quotient map $\theta\colon A\to A/A_0$ is *trivial* (i.e. cohomologous to a homomorphism $\eta\colon G\to A/A_0$).

This result extends a number of earlier characterizations of coboundaries and trivial cocycles by tightness conditions on the distributions of the maps $\{c(g,\cdot) : g\in G\}$ and has implications for flows under functions: let T be a measure-preserving ergodic automorphism of a probability space $(X,\mathcal{S},\mu)$, $f\colon X\to\mathbb{R}$ be a nonnegative Borel map with $\int f\,d\mu=1$, and T^f be the flow under the function f with base T. Our main result implies that, if T is mixing and T^f is weakly mixing, or if T is ergodic and T^f is mixing, then the cocycle $\mathbf{f}\colon\mathbb{Z}\times X\to\mathbb{R}$ defined by f disperses. The latter statement answers a question raised by Mariusz Lemańczyk in [7].

1. Dispersion of cocycles

Definition 1.1. Let $T\colon g\mapsto T_g$ be a measure-preserving action of a countable additive abelian group G on a standard probability space $(X,\mathcal{S},\mu)$, and let A be a locally compact second countable additive abelian group with identity element 0. A Borel map $c\colon G\times X\to A$ is a *cocycle* for T if

$$c(g,T_hx)+c(h,x)=c(g+h,x)$$

for every $g,h\in G$ and $x\in X$. Two cocycles $c,c'\colon G\times X\to A$ are *cohomologous* if there exists a Borel map $b\colon X\to A$ such that

$$c(g,x)=c'(g,x)+b(T_gx)-b(x) \tag{1.1}$$

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for every $g \in G$ and μ-a.e. $x \in X$. The map b in (1.1) is called a *transfer function*. If c is cohomologous to the zero cocycle $c' \equiv 0$ then c is a *coboundary* with *transfer* (or *cobounding*) function b.

Let $c: G \times X \to A$ be a cocycle. The cocycle c is a *homomorphism* if the map $c(g, \cdot): X \to A$ is constant for every $g \in G$, and c is *trivial* if it is cohomologous to a homomorphism.

The cocycle c is *bounded* (*in measure*) *on a subset* $H \subset G$ if there exists, for every $\varepsilon > 0$, a compact subset $C \subset A$ with

$$\mu(\{x : c(g,x) \in C\}) > 1 - \varepsilon \tag{1.2}$$

for every $g \in H$.

The cocycle c is *translation-bounded on a subset* $H \subset G$ if there exist, for every $\varepsilon > 0$, a map $\alpha: H \to A$ and a compact subset $C \subset A$ with

$$\mu(\{x : c(g,x) - \alpha(g) \in C\}) > 1 - \varepsilon \tag{1.3}$$

for every $g \in H$.

If $H = G$ in (1.2) or (1.3) then c is said to be *bounded* or *translation-bounded*, respectively.

Finally, the cocycle c *disperses* if $\lim_{g\to\infty} c(g, \cdot) - \alpha(g) = \infty$ in measure for every map $\alpha: G \to A$ or, equivalently, if

$$\lim_{g\to\infty} \sup_{a\in A} \mu(\{x \in X : c(g,x) - a \in C\}) = 0 \tag{1.4}$$

for every compact set $C \subset A$.

It has long been known that a cocycle $c: G \times X \to \mathbb{R}$ is a coboundary if and only if it is bounded in the sense of Definition 1.1 (cf. [9, Theorem 11.8]). More generally, if A is a locally compact second countable abelian group and $c: G \times X \to A$ a bounded cocycle, then c is cohomologous to a cocycle taking values in a compact subgroup $A_0 \subset A$ (for extensions of this result with varying degrees of generality see [8, Theorem 5.2], [10, Theorem 4.7] and [1]). Furthermore, if $A = \mathbb{R}$, then c is trivial if and only if it is translation-bounded (cf. [8, Theorem 6.2]).

More recently it was shown that, if T is mixing, then boundedness (or translation-boundedness) of a cocycle $c: G \times X \to \mathbb{R}$ on an infinite subset $H \subset G$ also implies triviality. The first published result in this direction is [1, Theorem 2], where it is proved that, for a mixing action of $G = \mathbb{Z}$, translation-boundedness of a cocycle $c: \mathbb{Z} \times X \to \mathbb{R}$ on some infinite subset $H \subset G$ implies triviality of c, and boundedness of c on H implies that c is a coboundary. These results can break down for $\mathbb{Z}$-actions which are only mildly mixing (cf. [1]).

In this note we prove the following extension of [1, Theorem 2].

THEOREM 1.2. *Let T be a measure-preserving and mixing action of a countable abelian group G on a standard probability space $(X, \mathcal{S}, \mu)$, A a* *locally compact second countable abelian group and* $c: G \times X \to A$ *a cocycle for* T. *The following conditions are equivalent*:

(1) *The cocycle* c *does not disperse* (*cf.* (1.4)).

(2) *There exists a compact subgroup* $A_0 \subset A$ *such that the composition* $\theta \circ c: G \times X \to A/A_0$ *of* c *with the quotient map* $\theta: A \to A/A_0$ *is a trivial cocycle.*

For the proof of Theorem 1.2 we need a little bit of notation. Let T be a continuous action of a countable abelian group G on a compact metrizable space X and μ be a T-invariant Borel probability measure on X. We denote by $\Delta = \{(x, x) : x \in X\}$ the diagonal in $X \times X$ and define the "diagonal" probability measure μ_Δ on $X \times X$ by setting

$$\mu_\Delta(\{(x, x) : x \in B\}) = \mu(B)$$

for every Borel set $B \subset X$. For every $g \in G$, the "off-diagonal" probability measure

$$\nu_g = (T_g \times \mathrm{Id}_X)_*(\mu_\Delta) \tag{1.5}$$

is the self-joining of μ supported on the graph $\{(T_g x, x) : x \in X\}$ of T_g.

Theorem 1.2 is an easy consequence of the following proposition.

Proposition 1.3. *Let* T *be a continuous action of a countable abelian group* G *on a compact metrizable space* X, μ *a* T-*invariant and weakly mixing Borel probability measure on* X, A *a locally compact second countable abelian group and* $c: G \times X \to A$ *a cocycle for* T. *Suppose that there exists a sequence* $(h_n : n \geq 1)$ *in* G *with the following properties*:

(1) $\lim_{n\to\infty} \nu_{h_n} = \mu \times \mu$ *in the topology of weak convergence*;

(2) *There exist an* $\varepsilon > 0$, *a compact set* $C \subset A$ *and elements* $\alpha_n \in A$, $n \geq 1$, *with*

$$\mu(\{x \in X : c(h_n, x) - \alpha_n \in C\}) \geq \varepsilon \quad \textit{for every } n \geq 1. \tag{1.6}$$

Then there exists a compact subgroup $A_0 \subset A$ *such that the composition* $\theta \circ c: G \times X \to A/A_0$ *of* c *with the quotient map* $\theta: A \to A/A_0$ *is trivial.*

For the proof of Proposition 1.3 we require an elementary lemma closely related to [10, Lemma 4.4].

Lemma 1.4. *Let* T *be a measure-preserving and ergodic action of a countable abelian group* G *on a standard probability space* $(X, \mathcal{S}, \mu)$, A *a locally compact second countable abelian group and* $c: G \times X \to A$ *a cocycle for* T. *We define the skew-product action* $T^{(c)}$ *of* G *on* $Y = X \times A$ *by setting*

$$T^{(c)}_g(x, a) = (T_g x, c(g, x) + a)$$

for every $g \in G$ *and* $(x, a) \in Y$.

If there exists a $T^{(c)}$-invariant probability measure ϱ on Y with $\pi_*(\varrho) = \mu$ (where $\pi: Y \to X$ is the first coordinate projection), then c is cohomologous to a cocycle c' taking values in some compact subgroup $A_0 \subset A$.

Conversely, if c is cohomologous, with transfer function b, to a cocycle $c': G \times X \to A_0$, where $A_0 \subset A$ is a compact subgroup with normalized Haar measure λ_{A_0}, then the probability measure ϱ on Y, defined by

$$\int f\, d\varrho = \iint f(x, a + b(x))\, d\lambda_{A_0}(a)\, d\mu(x)$$

for every bounded Borel map $f: Y \to \mathbb{R}$, is $T^{(c)}$-invariant and $\pi_*(\varrho) = \mu$.

Proof. Choose a Borel measurable family of probability measures $\{\varrho_x : x \in X\}$ on A such that

$$\int f\, d\varrho = \iint f(x,t)\, d\varrho_x(t)\, d\varrho(x)$$

for every bounded Borel map $f: Y \to A$. Since ϱ is $T^{(c)}$-invariant,

$$\varrho_{T_g x}(B + c(g,x)) = \varrho_x(B) \qquad \text{for } \mu\text{-a.e. } x \in X,$$

for every Borel set $B \subset A$ and every $g \in G$. We fix a nonnegative continuous map $\phi: A \to \mathbb{R}$ with compact support such that $\phi(0) > 0$. For every $x \in X$, the map $a \mapsto \int \phi(a+s)\, d\varrho_x(s) = \psi(x,a)$ from A to $\mathbb{R}$ is continuous, not everywhere equal to zero, and vanishes at infinity. Furthermore, the resulting Borel map $\psi: X \times A \to \mathbb{R}$ is $T^{(c)}$-invariant, and for some $\varepsilon > 0$ the Borel set

$$K = \{(x,a) \in Y : \psi(x,a) \geq \varepsilon\}$$

is nonempty and again $T^{(c)}$-invariant. For every $x \in X$, the set

$$K_x = \{a \in A : (x,a) \in K\}$$

and the subgroup

$$A_x = \{a \in A : a + K_x = K_x\}$$

are both compact, and the ergodicity of T and the $T^{(c)}$-invariance of K imply that there exists a compact subgroup $A_0 \subset A$ with $A_x = A_0$ for μ-a.e. $x \in X$. By using one of the standard selection theorems (cf. e.g. Kunugui's theorem in [5]–[6]) we can choose a Borel map $b: X \to A$ with $b(x) \in K_x$ for μ-a.e. $x \in X$ and conclude that $c(g,x) + b(x) - b(T_g x) \in A_0$ for every $g \in G$ and μ-a.e. $x \in X$.

The final statement of the lemma is obvious. ∎

Proof of Proposition 1.3. Let $\overline{T}: g \mapsto T_g \times T_g$ denote the diagonal action of G on $\overline{X} = X \times X$, and let $\overline{c}: G \times \overline{X} \to A$ be the cocycle

$$\overline{c}(g,(x_1,x_2)) = c(g,x_1) - c(g,x_2)$$

for $\overline{T}$. The cocycle equation (2.1) yields

$$c(g, T_{h_n} x) - c(g,x) = c(h_n, T_g x) - c(h_n, x) \tag{1.7}$$

for every $g \in G$ and $n \geq 1$. For every $n \geq 1$ we define $b_n: \overline{X} \to A$ by $b_n(x_1, x_2) = c(h_n, x_2) - \alpha_n$ and conclude from (1.7) that

$$\overline{c}(g, (x_1, x_2)) = b_n \circ \overline{T}_g(x_1, x_2) - b_n(x_1, x_2)$$

for every $g \in G$ and $(x_1, x_2) \in (T_{h_n} \times \mathrm{Id}_X)(\Delta)$, i.e. $\overline{c}$ is a coboundary with cobounding function b_n with respect to the $\overline{T}$-invariant measure $\nu_n = \nu_{h_n}$. We denote by ϱ_n the probability measure on $Y = \overline{X} \times A$ with

$$\varrho_n(\{(T_{h_n}x, x, b_n(T_{h_n}x, x)) : x \in B\}) = \mu(B)$$

for every Borel set $B \subset X$ and observe as in Lemma 1.4 that ϱ_n is the unique $T^{(c)}$-invariant probability measure supported on the graph of b_n with $\pi_*(\varrho_n) = \nu_n$, where $\pi: \overline{X} \times A \to \overline{X}$ is the projection map. If

$$A_n = \{x \in X : b_n(x) \in C\},$$

where $C \subset A$ is the compact set appearing in (1.6), then

$$\varrho_n(\overline{X} \times C) = \nu_n(T_h^{-1} A_n \times A_n) = \mu(A_n) \geq \varepsilon \tag{1.8}$$

for every $n \geq 1$.

By going over to a subsequence of (h_n), if necessary, we may take it that the sequence of probability measures (ϱ_n) converges vaguely to a finite measure ϱ on Y, i.e. $\lim_{n\to\infty} \int f\, d\varrho_n = \int f\, d\varrho$ for every continuous function $f: Y \to A$ with compact support. According to (1.8),

$$\varrho(\overline{X} \times C) \geq \limsup_{n\to\infty} \varrho_n(\overline{X} \times C) \geq \varepsilon,$$

which implies that ϱ is nonzero. We set $\nu = \pi_*(\varrho)$ and claim that

$$\nu(B) \leq (\mu \times \mu)(B) \tag{1.9}$$

for every Borel set $B \subset \overline{X}$.

Indeed, let $f: \overline{X} \to A$ be a continuous function, $\mathcal{U} \subset A$ an open neighbourhood of the identity with compact closure, $(D_m : m \geq 1)$ a sequence of compact subsets of A with $D_{m+1} \supset D_m + \mathcal{U}$ for every $m \geq 1$ and $\bigcup_{m\geq 1} D_m = A$, and let, for every $m \geq 1$, $\phi_m: A \to A$ be a continuous map with $\phi_m(a) = 1$ for $a \in D_m$ and $\phi_m(a) = 0$ for $a \notin D_m + \mathcal{U}$. We set $f_m(x_1, x_2, a) = f(x_1, x_2)\phi_m(a)$ and observe that

$$\int f_m\, d\varrho = \lim_{n\to\infty} \int f_m\, d\varrho_n \leq \lim_{n\to\infty} \int f\, d\nu_n = \int f\, d(\mu \times \mu).$$

By letting $m \to \infty$ we obtain

$$\int f\, d\nu = \sup_{m\geq 1} \int f_m\, d\varrho \leq \int f\, d(\mu \times \mu).$$

As f was arbitrary, this proves (1.9).

Since each of the probability measures ϱ_n is invariant under the skew-product action

$$\overline{T}_g^{(\overline{c})}(x_1, x_2, a) = (T_g x_1, T_g x_2, \overline{c}(g, (x_1, x_2)) + a)$$

of G on Y, the same ([1]) is true for ϱ, and hence the measure ν on $\overline{X}$ is invariant under $\overline{T}$. From (1.9) and the ergodicity of $\mu \times \mu$ it is clear that $(1/\nu(\overline{X}))\nu = \nu'$ is a $\overline{T}$-invariant probability measure on $\overline{X}$ which is absolutely continuous with respect to, and hence equal to, $\mu \times \mu$, and that the probability measure $\varrho' = (1/\varrho(Y))\varrho = (1/\nu(\overline{X}))\varrho$ on Y is invariant under $\overline{T}^{(\overline{c})}$ and satisfies $\pi_*(\varrho') = \mu \times \mu$.

By Lemma 1.4 there exists a compact subgroup $A_0 \subset A$ such that $\theta \circ \overline{c}$ is a coboundary, where $\theta: A \to A' = A/A_0$ is the quotient map.

In order to simplify notation a little we set $\overline{c}' = \theta \circ \overline{c}: G \times \overline{X} \to A'$ and $c' = \theta \circ c: G \times X \to A'$. In the notation of [8, (6.1) and Theorem 6.2(4)] we have proved that $j_*(c') = 0$, i.e. $c' \in \mathit{\Gamma}_2(A')$ in the notation of [8, (7.5)]. As μ is weakly mixing, the triple diagonal action $T \times T \times T$ of G on $(X \times X \times X, \mu \times \mu \times \mu)$ is ergodic, and [8, Corollary 7.2] shows that $\mathit{\Gamma}_2(A') = \mathit{\Gamma}_0(A')$ in the notation of [8, (7.3)–(7.5)]. Hence c' is a homomorphism in the terminology of [8] or is trivial in our terminology. ∎

Proof of Theorem 1.2. We assume without loss in generality that X is a compact metric space and that the G-action T on X is continuous (cf. [11]).

If (1.4) is violated, then there exist an $\varepsilon > 0$, a compact set $C \subset A$, an infinite subset $H \subset G$, and elements $\alpha_h \in A$, $h \in H$, with

$$\mu(\{x \in X : c(h,x) - \alpha_h \in C\}) \geq \varepsilon$$

for every $h \in H$. We can thus choose a sequence (h_n) in H with $\lim_{n\to\infty} h_n = \infty$ which satisfies the conditions of Proposition 1.3. Hence there exists a compact subgroup $A_0 \subset A$ such that the composition $\theta \circ c: G \times X \to A/A_0$ of c with the quotient map $\theta: A \to A/A_0$ is trivial. This proves that (1)⇒(2), and the reverse implication (2)⇒(1) is obvious. ∎

2. Mixing flows under functions. In order to apply Theorem 1.2 (or, more precisely, Proposition 1.3) to mixing properties of flows under functions we let T be a measure-preserving automorphism of a standard probability space $(X, \mathcal{S}, \mu)$ and $f: X \to \mathbb{R}$ a Borel map with $\int f\, d\mu = 1$ and $f(x) > 0$ for every $x \in X$. For every $n \in \mathbb{Z}$ and $x \in X$ we set

$$\mathbf{f}(n,x) = \begin{cases} \sum_{k=0}^{n-1} f(T^k x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\mathbf{f}(-n, T^n x) & \text{if } n < 0. \end{cases} \tag{2.1}$$

([1]) In order to see this, consider the ring $\mathcal{R}$ of all Borel sets $B \subset Y$ with the property that $\overline{T}_g^{(\overline{c})}(B)$ has compact closure and boundary measure $\varrho(\partial(\overline{T}_g^{(\overline{c})}B)) = 0$ for every $g \in G$. By assumption, $\varrho(B) = \lim_{n\to\infty} \varrho_n(B) = \lim_{n\to\infty} \varrho_n(\overline{T}_g^{(\overline{c})}B) = \varrho(\overline{T}_g^{(\overline{c})}B)$ for every $g \in G$ and $B \in \mathcal{R}$. Since $\mathcal{R}$ generates the Borel field of Y this proves that ϱ is $\overline{T}^{(\overline{c})}$-invariant.

The resulting map $\mathbf{f}: \mathbb{Z} \times X \to \mathbb{R}$ is a cocycle for T (or for the $\mathbb{Z}$-action $n \mapsto T^n$ on $(X, \mathcal{S}, \mu)$). We define an equivalence relation R^f on $X \times \mathbb{R}$ by saying that

$$(x,t) \sim (T^n x, t - \mathbf{f}(n,x))$$

for every $(x,t) \in X \times \mathbb{T}$ and $n \in \mathbb{Z}$. The "vertical" flow $S_t: (x,t') \mapsto (x, t+t')$, $(x,t') \in X \times \mathbb{R}$, $t \in \mathbb{R}$, preserves this equivalence relation and thus induces a flow $t \mapsto S_t^f$ on the space $(X \times \mathbb{R})_{R^f}$ of equivalence classes of R^f. The set

$$X^f = \{(x,t) : x \in X,\ 0 \le t < f(x)\}$$

intersects each equivalence class of R^f in exactly one point and may thus be identified with $(X \times \mathbb{R})_{R^f}$. We denote by $\mathcal{S}^f$ the Borel field of $X^f \subset X \times \mathbb{R}$, write λ for the Lebesgue measure on $\mathbb{R}$, μ^f for the restriction to $X^f \subset X \times \mathbb{R}$ of the product measure $\mu \times \lambda$, and T^f for the measure-preserving flow induced by S^f on the probability space $(X^f, \mathcal{S}^f, \mu^f)$. This flow is usually called the *flow under the function f with base T*.

Several authors have studied conditions on f for a given ergodic base T which determine whether the flow T^f is mixing (cf. e.g. [3], [4], [2], [7] and the references listed there). In [7] the author proves the following result under the additional hypothesis that T is an irrational rotation on $X = \mathbb{R}/\mathbb{Z}$, and asks whether Corollary 2.1 (under hypothesis (2)) holds for more general classes of ergodic automorphisms ([7, Remarque 2]).

COROLLARY 2.1. *Let T be a measure-preserving automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, $f: X \to \mathbb{R}$ be a Borel map with $\int f\, d\mu = 1$ and $f(x) > 0$ for every $x \in X$, and T^f be the flow under f with base T on the probability space $(X^f, \mathcal{S}^f, \mu^f)$. Suppose that either of the following conditions is satisfied*:

(1) *T is mixing and T^f is weakly mixing*;

(2) *T^f is mixing.*

Then the cocycle $\mathbf{f}: \mathbb{Z} \times X \to \mathbb{R}$ in (2.1) *disperses in the sense of* (1.4).

Proof. If T is mixing, then Theorem 1.2 with $G = \mathbb{Z}$ and $A = \mathbb{R}$ shows that the cocycle $\mathbf{f}$ either disperses or is trivial, in which case the flow T^f is not weakly mixing.

In order to prove dispersion of $\mathbf{f}$ under hypothesis (2) we may replace f by a cohomologous function $f' = f + b \circ T - b$ such that $b: X \to \mathbb{R}$ is measurable and f' is bounded above and below by positive constants; this will affect neither the hypotheses nor the conclusions of the corollary. We assume therefore without loss of generality that there exist positive constants $c_1 < c_2$ such that $c_1 \le f(x) \le c_2$ for every $x \in X$.

Define a map $F\colon X^f \to \mathbb{R}$ by

$$F(x,s) = 1/f(x)$$

for every $x \in X$ and $s \in [0, f(x))$ and consider the cocycle $c\colon \mathbb{R} \times X^f \to \mathbb{R}$ given by

$$c(t,z) = \begin{cases} \int_0^t F(T_s^f z)\,ds & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -c(-t, T_t^f z) & \text{if } t < 0. \end{cases} \tag{2.2}$$

For every positive t we denote by $d(t,z)$ the number of intersections of the set $\{(x,0) : x \in X\} \subset X^f$ with the trajectory $\{T_s^f z : 0 \le s < t\}$. Then

$$|c(t,z) - d(t,z)| \le 2 \quad \text{and} \quad |t - \mathbf{f}(d(t,(x,s)),x)| \le 2c_2 \tag{2.3}$$

for every $t \ge 0$ and $z = (x,s) \in X^f$, and

$$\int c(t,z)\,d\mu^f(z) = t \tag{2.4}$$

for every $t \in \mathbb{R}$.

Suppose that (1.4) is not satisfied, i.e. that there exist an $\varepsilon > 0$, a constant $L > 0$, an increasing sequence (m_n) of positive integers, and a sequence (t_n) in $\mathbb{R}$ with

$$\mu(\{x \in X : |\mathbf{f}(m_n, x) - t_n| \le L\}) \ge \varepsilon \tag{2.5}$$

for every $n \ge 1$. According to (2.3) and (2.5),

$$\mu^f(\{(x,s) \in X^f : |\mathbf{f}(d(t_n,(x,s)),x) - \mathbf{f}(m_n,x)| \le L + 2c_2\}) \ge c_1\varepsilon,$$

hence

$$\mu^f\left(\left\{z \in X^f : |d(t_n,z) - m_n| \le \frac{L+2c_2}{c_1}\right\}\right) \ge c_1\varepsilon,$$

and the first inequality in (2.3) implies that

$$\mu^f\left(\left\{z \in X^f : |c(t_n,z) - m_n| \le \frac{L+2c_2}{c_1} + 2\right\}\right) \ge c_1\varepsilon \tag{2.6}$$

for every $n \ge 1$. Since μ^f is mixing and $\lim_{n\to\infty} t_n = \infty$ by (2.5), $\lim_{n\to\infty} \nu_{t_n} = \mu^f \times \mu^f$, where ν_{t_n} is the off-diagonal measure (1.5) with μ and g replaced by μ^f and t_n.

We choose a countable dense subgroup $G \subset \mathbb{R}$ which contains the sequence (t_n). Since G is dense in $\mathbb{R}$ and $\mu^f \times \mu^f$ is ergodic under the diagonal $\mathbb{R}$-action $T^f \times T^f$ on $X^f \times X^f$, the measure $\mu^f \times \mu^f$ is also ergodic under the restriction of $T^f \times T^f$ to G, which implies that μ^f is weakly mixing under the restriction of T^f to G. According to (2.6) and Proposition 1.3 there exist a homomorphism $\eta\colon G \to \mathbb{R}$ and a Borel map $b\colon X^f \to \mathbb{R}$ such that

$$c(t,z) = \eta(t) + b(T_t^f z) - b(z) \tag{2.7}$$

for every $t \in G$ and μ^f-a.e. $z \in X^f$. From (2.4) we know that $\eta(t) = t$ for every $t \in G$, and the continuity of the map $t \mapsto c(t, \cdot)$ from $\mathbb{R}$ into $L^1(X^f, \mathcal{S}^f, \mu^f)$ in (2.2) guarantees that (2.7) holds for every $t \in \mathbb{R}$.

We re-trace our steps and conclude from (2.7) and (2.3) that the cocycle $\mathbf{f}': \mathbb{Z} \times X \to \mathbb{R}$, defined by

$$\mathbf{f}'(n, \cdot) = \mathbf{f}(n, \cdot) - n, \quad n \in \mathbb{Z},$$

is bounded in μ-measure. By [9, Theorem 11.8], [8, Theorem 5.2] or [10, Theorem 4.7] there exists a Borel map $b: X \to \mathbb{R}$ with $f(x) = 1 + b(Tx) - b(x)$ for μ-a.e. $x \in X$, which implies that T^f is not even weakly mixing. This contradiction resulting from (2.5) proves the corollary. ■

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Mathematics Institute
University of Vienna
Strudlhofgasse 4
A-1090 Wien, Austria
E-mail: klaus.schmidt@univie.ac.at

Erwin Schrödinger Institute
for Mathematical Physics
Boltzmanngasse 9
A-1090 Vienna, Austria

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