

Uncountable cardinals have the same monadic \forall_1^1 positive theory over large sets

by

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Abstract. We show that uncountable cardinals are indistinguishable by sentences of the monadic second-order language of order of the form $(\forall X)\phi(X)$ and $(\exists X)\phi(X)$, for ϕ positive in X and containing no set-quantifiers, when the set variables range over large (= cofinal) subsets of the cardinals. This strengthens the result of Doner–Mostowski–Tarski [3] that (κ, \in) , (λ, \in) are elementarily equivalent when κ, λ are uncountable. It follows that we can consistently postulate that the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$, $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ are indistinguishable with respect to \forall_1^1 positive sentences. A consequence of this postulate is that $2^\kappa = \kappa^+$ iff $2^\lambda = \lambda^+$ for all infinite κ, λ . Moreover, if measurable cardinals do not exist, GCH is true.

1. Preliminaries. Let $\mathcal{L} = \{<\}$ be the first-order language of order with equality. The following is a special case of a much more general result proved in [3]:

THEOREM 1.1. *For any uncountable cardinals κ, λ $(\kappa, <) \equiv_{\mathcal{L}} (\lambda, <)$, where $<$ is the natural ordering of ordinals (i.e., $< \in \in$).*

Proof. The reader is warned that there is a mistake in the definition of congruence modulo ω^ω given in [3, p. 51], as Professor Doner kindly informed me. The correct definition, which can be found e.g. in [2], is as follows: The ordinals α, β are congruent modulo ω^ω if there are ξ, η and $\delta < \omega^\omega$ such that $\alpha = \omega^\omega \cdot \xi + \delta$, $\beta = \omega^\omega \cdot \eta + \delta$ and either $\xi = \eta = 0$ or both $\xi \neq 0$ and $\eta \neq 0$. Since for all cardinals $\kappa, \lambda > \omega$, $\kappa = \omega^\omega \cdot \kappa$ and $\lambda = \omega^\omega \cdot \lambda$, it follows that κ, λ are congruent modulo ω^ω . So the claim follows from Corollary 44 of [3]. ■

Henceforth we write $(\kappa, <) \equiv (\lambda, <)$ instead of $(\kappa, <) \equiv_{\mathcal{L}} (\lambda, <)$. Let x, y, z, \dots range over the individual variables of \mathcal{L} . The monadic (second-order) extension of \mathcal{L} , denoted by \mathcal{L}_{mon} , is \mathcal{L} augmented with \in and set variables X, Y, Z, \dots . Hence \mathcal{L}_{mon} has $x \in X$ as additional atoms. The stan-

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standard interpretations of \mathcal{L}_{mon} are the structures $(A, \mathcal{P}(A), <)$, where $(A, <)$ is an ordered set and $\mathcal{P}(A)$ is the set of its subsets. The expressive power of \mathcal{L}_{mon} is huge compared to that of \mathcal{L} . For instance it is shown in [1] that each of the structures $(\omega_n, <)$, $n \geq 0$, is finitely axiomatizable and categorical with respect to \mathcal{L}_{mon} (under the standard interpretation). That is, for every cardinal ω_n , the ordered set $(\omega_n, <)$ is characterized, up to isomorphism, by a sentence of \mathcal{L}_{mon} .

However there are more general interpretations of \mathcal{L}_{mon} besides the standard one. These are of the form $(A, \mathcal{A}, <)$, where $\mathcal{A} \subset \mathcal{P}(A)$. \mathcal{A} may be some class of “small” sets, i.e., an ideal of $\mathcal{P}(A)$, or, in the opposite direction, a class of “large” sets. Given cardinals $\kappa < \lambda$, let us fix the following notation:

$$[\lambda]^{<\kappa} = \{X \subseteq \lambda : |X| < \kappa\}, \quad [\lambda]^\kappa = \{X \subseteq \lambda : |X| = \kappa\},$$

$$[\lambda]^{>\kappa} = \{X \subseteq \lambda : |X| > \kappa\}, \quad \text{Cof}(\lambda) = \{X \subseteq \lambda : X \text{ is cofinal in } \lambda\}.$$

Clearly if κ is regular then $[\kappa]^\kappa = \text{Cof}(\kappa)$. We often call the elements of $[\kappa]^{<\kappa}$ *small* subsets of κ . Accordingly we call the sets of $[\kappa]^\kappa$ and $\text{Cof}(\kappa)$ *large* and *cofinal*, respectively. Then X is *co-small*, i.e., $-X \in [\kappa]^{<\kappa}$, iff $X \cap Y \neq \emptyset$ for every $Y \in [\kappa]^\kappa$.

The aim of this paper is to strengthen Theorem 1.1 above as follows: For any uncountable κ, λ , the second-order structures $(\kappa, \text{Cof}(\kappa), <)$ and $(\lambda, \text{Cof}(\lambda), <)$ are indistinguishable with respect to \forall_1^1 and \exists_1^1 positive formulas (defined below).

Both positive and \forall_1^1 are natural classes of formulas. As an application, observe that the instance $2^\kappa = \kappa^+$ of the GCH can be expressed by the truth of the \forall_1^1 positive formula $\forall X(X \text{ is cofinal})$ in the structure $(2^\kappa, [2^\kappa]^{>\kappa}, <)$. By the main result of the paper, we can consistently postulate (see Section 4) that for all infinite κ, λ , the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$, $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ are indistinguishable with respect to \forall_1^1 positive formulas.

Concerning the optimality of the result: We do not know whether $(\kappa, \text{Cof}(\kappa), <)$ and $(\lambda, \text{Cof}(\lambda), <)$ are indistinguishable with respect to all \forall_1^1 and \exists_1^1 formulas. Also we do not know if the structures $(\kappa, \mathcal{P}(\kappa), <)$ and $(\lambda, \mathcal{P}(\lambda), <)$ are indistinguishable either with respect to all \forall_1^1 and \exists_1^1 formulas, or with respect to positive \forall_1^1 and \exists_1^1 formulas. What we do know is that the structures $(\omega_1, \text{Cof}(\omega_1), <)$ and $(\omega_2, \text{Cof}(\omega_2), <)$ are distinguishable by a formula of the form $(\forall x)(\exists X)\psi$, where ψ has no second-order quantifiers but is not positive. And analogously for the structures concerning ω_m, ω_n (see the last section).

We come to the definition of positive formulas.

DEFINITION 1.2. Let $\phi(X)$ be a formula of \mathcal{L}_{mon} with at most one set variable X . Then $\phi(X)$ is *normal* if it has no set quantifiers. A normal formula ϕ is *positive in the set variable X* (or just *positive*) if it belongs to the

smallest class of formulas which (i) contains all formulas of \mathcal{L} , (ii) contains the atomic formulas $x \in X$ and (iii) is closed under the logical operations \wedge , \vee , and the first-order quantifiers \exists and \forall .

The following result is due to Moschovakis [6]:

LEMMA 1.3 (Moschovakis [6, p. 59]). *Let $\phi(X)$ be a formula of \mathcal{L}_{mon} positive in X . Then there is a quantifier-free and X -free formula $\theta(\bar{w}, u)$, where $\bar{w} = (w_1, \dots, w_n)$, and a string of quantifiers $\bar{Q} = (Q_1, \dots, Q_n)$ such that, for every structure $(A, \mathcal{P}(A), <)$ and every $Z \neq A$,*

$$\phi(Z) \Leftrightarrow (\bar{Q}\bar{w})(\forall u)(\theta(\bar{w}, u) \Rightarrow u \in Z)$$

holds in $(A, \mathcal{P}(A), <)$.

In view of the above we may assume that every formula positive in X has the form

$$(1) \quad \phi(X) \equiv (\bar{Q}\bar{w})(\forall u)(\theta(\bar{w}, u) \Rightarrow u \in X).$$

We only have to make sure that $X \neq A$ when using the above form in a structure $(A, <)$. We shall refer to (1) as the *canonical form* of ϕ .

We shall further restrict ourselves mainly to \forall_1^1 positive formulas of \mathcal{L}_{mon} , i.e., those (equivalent to one) with prenex form $(\forall X)\phi$, where ϕ is positive in X .

2. The main theorem. From now on we shall be dealing with second-order structures of the form $(\kappa, \text{Cof}(\kappa), <)$, where κ is an uncountable cardinal. If κ is regular this structure is identical to $(\kappa, [\kappa]^\kappa, <)$. Otherwise $[\kappa]^\kappa \subseteq \text{Cof}(\kappa)$. So for brevity let us set

$$[[\kappa]] := (\kappa, <), \quad [[\kappa^2]] := (\kappa, \text{Cof}(\kappa), <).$$

This section contains the proof of the following:

THEOREM 2.1 (Main Theorem). *For all uncountable cardinals κ, λ ,*

$$[[\kappa^2]] \equiv_{\text{pos}}^{\forall_1^1} [[\lambda^2]],$$

i.e.,

$$[[\kappa^2]] \models (\forall X)\phi(X) \Leftrightarrow [[\lambda^2]] \models (\forall X)\phi(X),$$

for every positive $\phi(X)$.

Plan of the proof. To help the reader let us describe briefly the general plan of the proof. Let

$$\phi(X) \equiv (\bar{Q}\bar{w})(\forall u)(\theta(\bar{w}, u) \Rightarrow u \in X)$$

be a positive formula in its canonical form. Our aim is to show that $(\forall X)\phi(X)$ is absolute between $[[\kappa^2]]$ and $[[\lambda^2]]$. The proof of this fact will be split into several cases corresponding to various syntactic forms of $\phi(X)$, namely forms of $\bar{Q}\bar{w}$ and θ . In each case we reduce the truth of $(\forall X)\phi(X)$

in $[[\kappa^2]]$ to that of a first-order formula in $[[\kappa]]$, which, in view of Theorem 1.1, means that $(\forall X)\phi(X)$ is absolute between $[[\kappa^2]]$ and $[[\lambda^2]]$. Some of the cases in question are treated as Lemmas 2.2, 2.8 and 2.9 below (the rest of the lemmas provide technical tools for the treatment of θ). After Lemma 2.9, we concentrate on three remaining cases which are treated explicitly as Cases 1, 2 and 3 with several subcases. After exhausting them all, we give the final proof in which we simply summarize our line of thought.

To start with, given κ , it is convenient to rewrite $[[\kappa^2]] \models \phi(X)$ as follows: We set, for every $\bar{w} \in \kappa$,

$$R_\theta^\kappa(\bar{w}) = \{u \in \kappa : [[\kappa]] \models \theta(\bar{w}, u)\}.$$

Then

$$(\forall u)(\theta(\bar{w}, u) \Rightarrow u \in X) \Leftrightarrow R_\theta^\kappa(\bar{w}) \subseteq X,$$

so, over κ , the following identification will often be used:

$$(2) \quad \phi(X) \equiv (\overline{Q}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X].$$

We consider the following easy case first:

LEMMA 2.2. *Let \overline{Q} contain no existential quantifier, i.e., $\overline{Q} = \overline{\forall}$. Then for all uncountable κ, λ ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \Leftrightarrow [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Suppose $\overline{Q} = \overline{\forall}$. Then, by (2) above,

$$[[\kappa^2]] \models (\forall X)\phi(X) \Leftrightarrow [[\kappa^2]] \models (\forall X)(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X].$$

Now it is easy to see that

$$(3) \quad [[\kappa^2]] \models (\forall X)(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X] \Leftrightarrow [[\kappa]] \models (\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) = \emptyset].$$

Direction “ \Leftarrow ” of (3) is immediate: If $(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) = \emptyset]$ holds in $[[\kappa]]$, then for every cofinal X , $(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X]$ holds in $[[\kappa^2]]$, hence so does $(\forall X)(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X]$. Conversely, suppose $[[\kappa^2]] \models (\forall X)(\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) \subseteq X]$. Pick any two disjoint cofinal Z_1, Z_2 . Then for any tuple $\bar{a} \in \kappa$, we have $R_\theta^\kappa(\bar{a}) \subseteq Z_1$ and $R_\theta^\kappa(\bar{a}) \subseteq Z_2$. Since $Z_1 \cap Z_2 = \emptyset$, it follows that $R_\theta^\kappa(\bar{a}) = \emptyset$. Since this holds for every $\bar{a} \in \kappa$, $[[\kappa]] \models (\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) = \emptyset]$ and (3) is proved. Now by (3) and Theorem 1.1, for uncountable κ, λ we have

$$\begin{aligned} [[\kappa^2]] \models (\forall X)\phi(X) &\Leftrightarrow [[\kappa]] \models (\overline{\forall}\bar{w})[R_\theta^\kappa(\bar{w}) = \emptyset] \\ &\Leftrightarrow [[\lambda]] \models (\overline{\forall}\bar{w})[R_\theta^\lambda(\bar{w}) = \emptyset] \Leftrightarrow [[\lambda^2]] \models (\forall X)\phi(X). \blacksquare \end{aligned}$$

If \overline{Q} in the above canonical form of $\phi(X)$ contains existential quantifiers, then a more thorough analysis of the structure of θ is needed. $\theta(\bar{w}, u)$ is a Boolean combination of atomic formulas of $\mathcal{L} = \{<, =\}$. Therefore θ is a Boolean combination of atoms $t < s$ and $t = s$, where $t, s \in \{u, w_1, \dots, w_n\}$. But over a linearly ordered set the formulas $t \neq s$ and $t \neq s$ are equivalent to $s < t \vee s = t$ and $t < s \vee s < t$, respectively, therefore θ is a

$\{\vee, \wedge\}$ -combination of atoms $t \prec s$ and $t = s$. So $\theta(\bar{w}, u)$ has the following disjunctive normal form:

$$(4) \quad \theta(\bar{w}, u) = \bigvee_{1 \leq p \leq k} \sigma_p(\bar{w}, u),$$

where each σ_p is a conjunction of atoms $t \prec t, t = s$. We call σ_p 's *clauses*. A clause is *consistent* if it does not contain contradictory atoms, e.g. of the form $t \prec s \wedge s \preceq t$, or $t \prec s \wedge t = s$. We may assume that each σ_p is consistent (otherwise we drop it).

DEFINITION 2.3. A clause σ_p is said to be *full* if for any two variables t, s occurring in it, σ_p proves one of the formulas $t \prec s, s \prec t, t = s$. A *filling* of σ_p is a formula σ'_p which is consistent, full, contains the same variables as σ_p , and $\sigma'_p \vdash \sigma_p$. (That is, σ'_p contains some extra atoms that guarantee completeness.)

If σ_p is a full clause containing the variables $t_1, \dots, t_m \in \{u, w_1, \dots, w_n\}$, then it is clear that (by renaming t_i if necessary) σ_p entails (and is equivalent to) a total ordering of t_i of the form $t_1 \preceq t_2 \preceq \dots \preceq t_m$ (where of course \preceq means \prec or $=$). More precisely, the preceding relation is written $t_1 R t_2 R \dots R t_m$, where R is either \prec or $=$. Thus we have shown clause (i) of the following (clause (ii) is straightforward).

LEMMA 2.4. (i) *If σ_p is complete, then it is equivalent to a conjunction of the form*

$$t_1 R t_2 R \dots R t_m,$$

where R is either \prec or $=$ and all t_i are distinct variables from among $\{u, w_1, \dots, w_n\}$.

(ii) *Every (consistent) σ_p has a filling, and if $\sigma'_{pi}, i \leq l$, are all the distinct completions of σ_p , then $\sigma_p = \bigvee_{i \leq l} \sigma'_{pi}$.*

In view of 2.4(ii), we may assume that every clause σ_p in the analysis (4) of θ is full, hence, by 2.4(i), it is of the form

$$t_1 R t_2 R \dots R t_m.$$

σ_p is said to be a *u-clause* if it contains the variable u . Otherwise it is called *u-free*. If δ denotes the disjunction of all *u-free* clauses, then θ is written

$$(5) \quad \theta(\bar{w}, u) \equiv \bigvee_{1 \leq p \leq k} \sigma_p(\bar{w}, u) \vee \delta(\bar{w}),$$

where each σ_p is a *u-clause*. Further, each *u-clause* σ_p consists of *u-atoms* and *u-free atoms*. Writing ζ_p and η_p for the conjunction of *u-atoms* and *u-free atoms* of σ_p respectively, we have

$$\sigma_p(\bar{w}, u) \equiv \zeta_p(\bar{w}, u) \wedge \eta_p(\bar{w}).$$

A u -clause $\sigma_p(\bar{w}, u)$ is said to be *equational in u* (or just *equational*) if it contains an identity $u = t$.

Now if σ_p is equational we may assume that ζ_p consists of a single equation $u = t$, i.e., $\zeta_p \equiv (u = t_{k_p})$. Indeed, first we may assume that σ_p contains only one u -equation. Otherwise we shall have the sequence

$$t_1 R t_2 R \cdots t_i = u = t_j R \cdots R t_m,$$

which can be simply rewritten as

$$t_1 R t_2 R \cdots R t_i = t_j = u R \cdots R t_m,$$

or

$$t_1 R t_2 R \cdots R u = t_i = t_j R \cdots R t_m.$$

If σ_p contains moreover a u -literal which is not an equation, say $u = t_1 \wedge u < t_2$, then $(u = t_1 \wedge u < t_2) \equiv (u = t_1 \wedge t_1 < t_2)$. So ζ_p can be taken to be a single equation and thus an equational σ_p has the form

$$(6) \quad \sigma_p \equiv (u = t_{k_p}) \wedge \eta_p.$$

On the other hand, if a u -clause σ_p is non-equational, then u occurs only in inequalities, so σ_p will be of the form either

$$t_1 R t_2 R \cdots R t_i < u < t_j R \cdots R t_m,$$

or

$$u < t_1 R t_2 R \cdots R t_m,$$

or

$$t_1 R t_2 R \cdots R t_m < u.$$

Therefore every non-equational σ_p has the form

$$(7) \quad \sigma_p \equiv (t_{i_p} < u < t_{j_p}) \wedge \eta_p,$$

where one of the t_{i_p}, t_{j_p} may be missing. For every equational $\sigma_p(\bar{w}, u)$ and for a given cardinal κ , let

$$E_p^\kappa(\bar{w}) = \{u : [[\kappa]] \models \sigma_p(\bar{w}, u)\}$$

(E for “equation”). Similarly for non-equational $\sigma_p(\bar{w}, u)$ and κ , let

$$I_p^\kappa(\bar{w}) = \{u : [[\kappa]] \models \sigma_p(\bar{w}, u)\}$$

(I for “interval”). Finally, let

$$D^\kappa(\bar{w}) = \{u : [[\kappa]] \models \delta(\bar{w})\}.$$

By the above analysis we get the following immediately:

LEMMA 2.5. (i) *If σ_p is equational, then $E_p^\kappa(\bar{w})$ is a singleton or \emptyset , namely,*

$$E_p^\kappa(\bar{w}) = \{u : u = t_{k_p} \wedge \eta_p(\bar{w})\},$$

where $t_{k_p} \in \{w_1, \dots, w_n\}$.

- (ii) If σ_p is non-equational, then $I_p^\kappa(\bar{w})$ is an interval (maybe empty), namely

$$I_p^\kappa(\bar{w}) = \{u : t_{i_p} \prec u \prec t_{j_p} \wedge \eta_p(\bar{w})\},$$

where $t_{i_p}, t_{j_p} \in \{w_1, \dots, w_n\}$ and one of the t_{i_p}, t_{j_p} may be missing.

- (iii) $D^\kappa(\bar{w})$ is either κ or \emptyset .

- (iv) $R_\theta^\kappa(\bar{w}) = \bigcup_{1 \leq p \leq r} E_p^\kappa(\bar{w}) \cup \bigcup_{r < p \leq k} I_p^\kappa(\bar{w}) \cup D^\kappa(\bar{w})$ for some $r \leq k$.

LEMMA 2.6. Let $[[\kappa^2]] \models (\forall X)\phi(X)$. Then

$$[[\kappa^2]] \models (\forall X)(\overline{Q\bar{w}})[R_\theta^\kappa(\bar{w}) \text{ contains no non-trivial interval} \wedge R_\theta^\kappa(\bar{w}) \subseteq X].$$

Proof. Note first that “ $R_\theta^\kappa(\bar{w})$ contains no non-trivial interval” is the informal formulation of a formula of \mathcal{L} . Assume the contrary, i.e., that there is a cofinal set Z such that

$$[[\kappa^2]] \models (\overline{Q'\bar{w}})[R_\theta^\kappa(\bar{w}) \text{ contains a non-trivial interval} \vee R_\theta^\kappa(\bar{w}) \not\subseteq Z],$$

where $\overline{Q'}$ is the dual string of \overline{Q} . But then we can get a cofinal $Y \subseteq Z$ that contains no non-trivial interval (e.g. the limit points of the ordering $(Z, <)$). So, clearly, the sentence “ $R_\theta^\kappa(\bar{w})$ contains a non-trivial interval” implies $R_\theta^\kappa(\bar{w}) \not\subseteq Y$ and also $R_\theta^\kappa(\bar{w}) \not\subseteq Z$ implies $R_\theta^\kappa(\bar{w}) \not\subseteq Y$. Therefore the above disjunction implies

$$[[\kappa^2]] \models (\overline{Q'\bar{w}})[R_\theta^\kappa(\bar{w}) \not\subseteq Y].$$

It follows that $[[\kappa^2]] \models (\exists X)(\overline{Q'\bar{w}})[R_\theta^\kappa(\bar{w}) \not\subseteq X]$. Since

$$\phi(X) \equiv (\overline{Q\bar{w}})[R_\theta^\kappa(\bar{w}) \subseteq X],$$

we get $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, which contradicts our assumption. ■

LEMMA 2.7. Let $\phi(X) \equiv (\overline{Q\bar{w}})(\forall u)(\theta(\bar{w}, u) \Rightarrow u \in X)$, and let $[[\kappa^2]] \models (\forall X)\phi(X)$. Then there is a quantifier free $\theta^*(\bar{w}, \bar{z}, u)$ and a string of quantifiers $\overline{Q^*}$ such that if $\phi^*(X) \equiv (\overline{Q^*\bar{w}\bar{z}})(\forall u)(\theta^*(\bar{w}, u) \Rightarrow u \in X)$, then $\phi(X)$ and $\phi^*(X)$ are equivalent over $[[\kappa^2]]$ and

$$[[\kappa^2]] \models (\forall X)(\overline{Q^*\bar{w}\bar{z}})[R_{\theta^*}^\kappa(\bar{w}, \bar{z}) \text{ contains no non-trivial interval} \\ \wedge R_{\theta^*}^\kappa(\bar{w}, \bar{z}) \subseteq X],$$

and $\theta^*(\bar{w}, \bar{z}, u) = \bigvee_q \tau_q(\bar{w}, \bar{z}, u)$, where all τ_q are equational.

Proof. By the assumption and Lemma 2.6,

$$(8) \quad [[\kappa^2]] \models (\forall X)(\overline{Q\bar{w}})[R_\theta^\kappa(\bar{w}) \text{ contains no non-trivial interval} \\ \wedge R_\theta^\kappa(\bar{w}) \subseteq X].$$

Recall from Lemma 2.5(iv) that $R_\theta^\kappa(\bar{w})$ can be written in the form

$$R_\theta^\kappa(\bar{w}) = \bigcup_{1 \leq p \leq r} E_p^\kappa(\bar{w}) \cup \bigcup_{r < p \leq k} I_p^\kappa(\bar{w}) \cup D^\kappa(\bar{w}).$$

Since the set $D^\kappa(\bar{w})$ is either \emptyset or κ , it follows from (8) that $D^\kappa(\bar{w}) = \emptyset$ (otherwise $D^\kappa(\bar{w}) = R_\theta^\kappa(\bar{w}) = \kappa$, which contains non-trivial intervals).

Therefore

$$R_{\theta}^{\kappa}(\bar{w}) = \bigcup_{1 \leq p \leq r} E_p^{\kappa}(\bar{w}) \cup \bigcup_{r < p \leq k} I_p^{\kappa}(\bar{w}).$$

Let I_p^{κ} be some of the intervals contained above, defined by a non-equational clause (7). By (8),

$$[[\kappa]] \models (\overline{Q}\bar{w})(I_p^{\kappa} = \emptyset \text{ or a singleton}).$$

But the fact that $t_{i_p} \prec u \prec t_{j_p} \wedge \eta_p$ defines at most a singleton implies that the latter formula can be equivalently replaced by

$$(t'_{i_p} = u \wedge u' = t'_{j_p}) \wedge \eta_p,$$

where $t' = s$ abbreviates “ s is the immediate successor of t ”. The above formula is analytically written as

$$(9) \quad (\exists z_1)[t_{i_p} \prec z_1 \prec t_{j_p} \wedge (\forall z_2)(z_2 \preceq t_{i_p} \vee z_2 = u \vee t_{j_p} \preceq z_2)] \wedge \eta_p.$$

But (9) is equational in u , at the cost that it contains some extra bound variables.

Concerning intervals of the form (t_{i_p}, ∞) , or $[0, t_{j_p})$, in view of (8), if σ_p is of the form $(t_{i_p} \prec u) \wedge \eta_p$, then the defined interval must always be \emptyset , therefore we can replace $(t_{i_p} \prec u) \wedge \eta_p$ by \perp . If σ_p is of the form $(u \prec t_{j_p}) \wedge \eta_p$, then, in order for $[0, t_{j_p})$ to be trivial, we must have either $\neg\eta$, or $t_{j_p} = 0$, or $t_{j_p} = 0'$. Therefore, $(u \prec t_{j_p}) \wedge \eta_p$ can be replaced by the formula $(t_{j_p} = 0 \vee t_{j_p} = 0') \wedge \eta$.

Now if we replace each non-equational clause σ_p by a formula of this kind and put the resulting formula in prenex normal form, we shall get a formula $\theta^*(\bar{w}, \bar{z}, u)$ whose disjunctive normal form contains only u -equational clauses. Moreover, if $\phi^*(X)$ is the positive formula resulting from θ^* in the obvious way, then $\phi(X)$ and $\phi^*(X)$ are equivalent over $[[\kappa^2]]$ and

$$[[\kappa^2]] \models (\forall X)(\overline{Q^*}\bar{w}\bar{z})[R_{\theta^*}^{\kappa}(\bar{w}, \bar{z}) \text{ contains no non-trivial interval} \\ \wedge R_{\theta^*}^{\kappa}(\bar{w}, \bar{z}) \subseteq X]. \blacksquare$$

Let \overline{Q} contain existential quantifiers and let $[[\kappa^2]] \models (\forall X)\phi(X)$. By Lemma 2.7, we may assume that if $\theta = \bigvee_p \sigma_p$, then all σ_p are *equational* clauses. This simplifies things considerably and we now turn to examining the form of the string \overline{Q} . Without essential loss of generality, and for the sake of illustrating the argument, assume that \overline{Q} consists of simple alternations of \forall and \exists , starting with \forall . Let henceforth $\bar{w} = (w_1, \dots, w_n)$ denote the string of variables bound by \forall and let $\bar{v} = (v_1, \dots, v_n)$ denote the string of variables bound by \exists . So instead of writing $\overline{Q}\bar{w}$ we henceforth write $\overline{Q}\bar{w}\bar{v}$, where

$$\overline{Q}\bar{w}\bar{v} = (\forall w_1)(\exists v_1)(\forall w_2) \cdots (\forall w_n)(\exists v_n).$$

Then $[[\kappa^2]] \models (\forall X)\phi(X)$ is written as

$$(10) \quad [[\kappa^2]] \models (\forall X)(\forall w_1)(\exists v_1) \cdots (\forall w_n)(\exists v_n)[R_\theta^\kappa(\bar{w}, \bar{v}) \subseteq X],$$

where $R_\theta^\kappa(\bar{w}, \bar{v})$ is a finite union of singletons $\{w_i\}$ or $\{v_j\}$. We divide the equational clauses of θ into $(u = w)$ -clauses, if the u -equation contained is of the form $u = w_i$ (independent variable), and $(u = v)$ -clauses, if the u -equation contained is of the form $u = v_j$ (dependent variable). Also we refer to the independent and dependent variables as w -variables and v -variables respectively.

LEMMA 2.8. *If θ consists of $(u = v)$ -clauses only, then $[[\kappa^2]] \models (\forall X)\phi(X)$, hence for all κ, λ ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \Leftrightarrow [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Let $\theta \equiv \bigvee_p \sigma_p$, where all σ_p are $(u = v)$ -clauses. We have to show that

$$[[\kappa^2]] \models (\forall X)(\overline{Qw\bar{v}})(\forall u) \left(\bigvee_p \sigma_p \Rightarrow u \in X \right).$$

Now given any cofinal X , it suffices to pick an arbitrary $\bar{v} \in X$ no matter what the choice of \bar{w} is. Then each σ_p will define either a singleton $\{v_j\} \subseteq X$, if η_p is satisfied, or \emptyset otherwise. Thus in any case $R_\theta^\kappa(\bar{w}, \bar{v}) \subseteq X$, and we are done. ■

LEMMA 2.9. *If some $(u = w)$ -clause contains only w -variables, then for any κ , $[[\kappa^2]] \not\models (\forall X)\phi(X)$, hence for all κ, λ ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \Leftrightarrow [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Let σ_p be a clause containing only variables of type w_i . Then it has the form

$$w_{i_1} R \cdots R u = w_{i_p} R \cdots R w_{i_l},$$

where R is either \prec or $=$. Let $\overline{Q'}$ be the dual string of the string \overline{Q} of ϕ . Then $\overline{Q'}$ contains $(\exists w_{i_1}) \cdots (\exists w_{i_p}) \cdots (\exists w_{i_l})$ as a substring. Moreover, since the other variables of (\bar{w}, \bar{v}) do not affect σ_p ,

$$[[\kappa]] \models (\overline{Q'w\bar{v}})(\exists u)\sigma_p$$

is equivalent to

$$[[\kappa]] \models (\exists w_{i_1}) \cdots (\exists w_{i_p}) \cdots (\exists w_{i_l})(\exists u)[w_{i_1} R \cdots R u = w_{i_p} R \cdots R w_{i_l}].$$

Now it is clear that, whatever the particular form of the clause

$$w_{i_1} R \cdots R u = w_{i_p} R \cdots R w_{i_l}$$

would be, there is a cofinal set Z such that

$$[[\kappa]] \models (\exists w_{i_1}) \cdots (\exists w_{i_p}) \cdots (\exists w_{i_l})(\exists u) \\ [(w_{i_1} R \cdots R u = w_{i_p} R \cdots R w_{i_l}) \wedge u \notin Z].$$

Equivalently,

$$[[\kappa]] \models (\overline{Q'}\overline{wv})(\exists u)(\sigma_p \wedge u \notin Z).$$

Therefore

$$[[\kappa]] \models (\overline{Q'}\overline{wv})(R_\theta^k(\overline{w}, \overline{v}) \not\subseteq Z),$$

whence $[[\kappa^2]] \not\models (\forall X)\phi(X)$. ■

In view of Lemmas 2.8 and 2.9, where the truth of $(\forall X)\phi(X)$ in every $[[\kappa^2]]$ is settled under the conditions stated, we now assume that these conditions fail, i.e., that θ contains $(u = w)$ -clauses and every $(u = w)$ -clause contains v -variables. Then we have to examine three main cases, where Cases 1 and 2 contain several subcases, sub-subcases etc. We call them all “Cases” and enumerate them by sequences of numbers. For instance, 1.2.1 and 1.2.2 are subcases of 1.2. We mark a case by \rightarrow if it is a subcase of the previous one, by \downarrow if it is a case of equal depth as the previous one, and by \leftarrow if we return to a case of smaller depth than the previous one. The cases concern either syntactic properties of the clauses of θ or the truth in $[[\kappa]]$ of some (first-order) subformula of θ . It turns out that these characteristics fully determine the truth or falsehood of $(\forall X)\phi(X)$ in $[[\kappa^2]]$. Therefore, in view of Theorem 1.1, the same syntactic and semantic conditions determine the truth or falsehood, respectively, of $(\forall X)\phi(X)$ in $[[\lambda^2]]$, for any uncountable cardinal λ . So, roughly, in the cases below we reduce the truth of the second-order formula $(\forall X)\phi(X)$ to the truth of some first-order formula plus some syntactic conditions.

CASE 1. There are w -clauses in which all v -variables precede all w -variables (in the enumeration of variables given in (10)), i.e., for all variables w_i and v_j occurring in these clauses, we have $j < i$. Since every v -variable is bound by \exists and every w -variable is bound by \forall , it follows that every value of the v -variables of the clause is independent of any value of a w -variable contained there.

\rightarrow CASE 1.1. Suppose there is a w -clause $\sigma_p \equiv (u = w_i) \wedge \eta_p$ containing a unique v -variable, say v_j .

\rightarrow CASE 1.1.1. σ_p contains formulas of the form $w_k \prec v_j$, and specifically let σ_p define m elements below v_j , say by the inequalities $w_{i_1} \prec w_{i_2} \prec \dots \prec w_{i_m} \prec v_j$.

\rightarrow CASE 1.1.1.1. There is no clause containing $u = v_j$. Then, choosing $v_j = 0$, we falsify σ_p for any choice of the independent variables, so $[[\kappa^2]] \models (\forall X)\phi(X)$.

\downarrow CASE 1.1.1.2. There are clauses containing $u = v_j$; let us enumerate them as $\sigma_{p_k} \equiv (u = v_j) \wedge \eta_{p_k}$ for $k \leq s$.

→ CASE 1.1.1.2.1. $[[\kappa]] \models (\overline{Q'}\overline{wv})(\bigvee_{k \leq s} \eta_{p_k})$, where in the string $\overline{Q'}\overline{wv}$, we have $(\forall v_j \in \{0, \dots, m\})$ instead of $(\forall v_j)$. Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. Indeed, recall from the above that $w_{i_1} \prec w_{i_2} \prec \dots \prec w_{i_m} \prec v_j$, and $j < i_1, \dots, i_m$, so in order to show that there is Z such that

$$[[\kappa^2]] \models (\overline{Q'})[R_\theta^\kappa(\overline{w}, \overline{v}) \not\subseteq Z],$$

it suffices to show that there is Z such that

$$[[\kappa^2]] \models (\forall v_j)(\exists w_{i_1}) \dots (\exists w_{i_m})(\exists u) \left[\left(w_{i_1} \prec \dots \prec w_{i_m} \prec v_j \dots u = w_i \dots \vee \bigvee_{k \leq s} \eta_{p_k}(\overline{w}, \overline{v}) \right) \wedge u \notin Z \right].$$

Take Z to be a cofinal set such that $\{0, \dots, m\} \cap Z = \emptyset$ and $-Z$ is cofinal. Then it is easy to check that the preceding formula holds in $[[\kappa^2]]$.

↓ CASE 1.1.1.2.2. $[[\kappa]] \not\models (\overline{Q'}\overline{wv}) \bigvee_{k \leq s} \eta_{p_k}$, where in the string $\overline{Q'}\overline{wv}$, we have $(\forall v_j \in \{0, \dots, m\})$ instead of $(\forall v_j)$. That is, $[[\kappa]] \models (\overline{Q}\overline{wv})(\bigwedge_{k \leq s} \neg\eta_{p_k})$, where in the string $\overline{Q}\overline{wv}$, we have $(\exists v_j \in \{0, \dots, m\})$ instead of $(\exists v_j)$. Then, choosing $v_i \in \{0, \dots, m\}$, we simultaneously falsify $w_{i_1} \prec w_{i_2} \prec \dots \prec w_{i_m} \prec v_j \dots u = w_i \dots$ and all σ_{p_k} . Hence $(\forall X)\phi(X)$ holds in $[[\kappa^2]]$.

← CASE 1.1.2. σ_p contains no formula of the form $w_k \prec v_j$, i.e., σ_p is of the form

$$v_j R w_{k_1} R \dots u = w_i R \dots R w_{k_l}.$$

Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. Taking a cofinal set Z whose complement is also cofinal, we have

$$[[\kappa^2]] \models (\forall v_j)(\exists w_{k_1}) \dots (\exists w_{k_l})(\exists u) \left[(v_j R w_{k_1} R \dots R u = w_i R \dots R w_{k_l}) \wedge u \notin Z \right].$$

Consequently,

$$[[\kappa^2]] \models (\overline{Q'})[R_\theta^\kappa(\overline{w}, \overline{v}) \not\subseteq Z],$$

i.e., $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

← CASE 1.2. There is a $(u = w)$ -clause $\sigma_p \equiv (u = w_i \wedge \eta_p)$ containing at least two v -variables. Let v_{j_1}, \dots, v_{j_t} be the list of all v -variables of σ_p . We divide the $(u = w)$ -clauses into two groups: (a) those defining a two-end interval with respect to u , i.e. σ_p contains a subformula of the form $v_{j_a} \preceq u = w_i \preceq v_{j_b}$, and (b) those defining a one-end interval, i.e. in which all v_{j_a} are strictly before or after the equation $u = w_i$. We denote the clauses of the first kind by σ_p^2 and of the second kind by σ_p^1 , and similarly their corresponding subformulas by η_p^2 and η_p^1 respectively. Further, let $\sigma_q \equiv (u = v_{j_q} \wedge \eta_q)$ be the clauses of θ (if any) containing an $(u = v)$ -equation for the above mentioned variables v_{j_1}, \dots, v_{j_t} .

So the relevant subformula of $\phi(X)$ is written as

$$(11) \quad (\exists v_{j_1}) \cdots (\exists v_{j_t})(\forall w_{i_1}) \cdots (\forall w_{i_s})(\forall u) \left[\left(\bigvee_p \sigma_p^2 \vee \bigvee_r \sigma_r^1 \vee \bigvee_q \sigma_q \right) \Rightarrow u \in X \right].$$

→ CASE 1.2.1. In (11) above there are no formulas of type σ_r^1 . Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. We have to show that

$$[[\kappa^2]] \models (\forall X)(\exists v_{j_1}) \cdots (\exists v_{j_t})(\forall w_{i_1}) \cdots (\forall w_{i_s})(\forall u) \left[\left(\bigvee_p \sigma_p^2 \vee \bigvee_q \sigma_q \right) \Rightarrow u \in X \right].$$

Each of the formulas σ_p^2 contains a pair of variables v_{j_a}, v_{j_b} that bound the equation $u = w_i$ from above and below. So given X it suffices to choose $v_{j_a} = v_{j_b} \in X$. Then the intervals they define are null, independently of the choice of w -variables. Moreover if this choice satisfies η_q , then σ_q defines a singleton whose element belongs to X . Otherwise σ_q defines \emptyset and we are done. So the above formula holds. Observe also that the v_{j_a}, v_{j_b} can be chosen arbitrarily high.

↓ CASE 1.2.2. In (11) above there are no formulas of type σ_r^2 . Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. We have to show that

$$[[\kappa^2]] \models (\exists X)(\forall v_{j_1}) \cdots (\forall v_{j_t})(\exists w_{i_1}) \cdots (\exists w_{i_s})(\exists u) \left[\left(\bigvee_r \sigma_r^1 \vee \bigvee_q \sigma_q \right) \wedge u \notin X \right].$$

Now in every formula σ_r , the equation $u = w_i$ is to the left or right of all v -variables, so the situation is quite similar to that of Case 1.1.

↓ CASE 1.2.3. In (11) we have formulas of both types σ_r^1 and σ_r^2 . Then we distinguish two subcases.

→ CASE 1.2.3.1.

$$[[\kappa]] \models (\exists v_{j_1}) \cdots (\exists v_{j_t})(\forall w_{i_1}) \cdots (\forall w_{i_s}) \left[\bigvee_p \eta_p^2 \vee \left(\bigwedge_r \neg\eta_r^1 \wedge \bigwedge_q \neg\eta_q \right) \right].$$

Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. By assumption there is a valuation of the v -variables that makes all η_r^1 formulas defining one-end intervals false. So $[[\kappa^2]] \models (\forall X)\phi(X)$ is shown by the argument of Case 1.2.1 above.

↓ CASE 1.2.3.2.

$$[[\kappa]] \models (\forall v_{j_1}) \cdots (\forall v_{j_t})(\exists w_{i_1}) \cdots (\exists w_{i_s}) \left[\bigwedge_p \neg\eta_p^2 \wedge \left(\bigvee_r \eta_r^1 \vee \bigvee_q \eta_q \right) \right].$$

Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. By assumption for every valuation of the v -variables there is a valuation of the w -variables that makes all η_r^2 formulas defining two-end intervals false. So $[[\kappa^2]] \models (\exists X)\neg\phi(X)$ is shown by the argument of Case 1.2.2 above.

This exhausts the examination of Case 1. It is important to note that whenever in the above subcases $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, then for every X , we can choose each v_j arbitrarily high in X , unless the equation $u = v_i$ does not occur in θ , or the choice of v_j falsifies simultaneously all clauses containing $u = v_j$.

← CASE 2. There is a $(u = w_i)$ -clause σ_p containing a formula $w_k \prec v_j$ with $k \leq j$.

→ CASE 2.1. θ does not contain any $(u = v_j)$ -clause. Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. The relevant subformula of ϕ is

$$(\forall w_k)(\exists v_j)(\forall u)[(u = w_i \wedge w_k \prec v_j) \Rightarrow u \in X].$$

Now since $(u = v_j)$ does not occur in θ , given any X and w_k , we can pick, say, $v_j = w_k$, falsifying thus the hypothesis of the above implication, i.e., σ_p , and so $[[\kappa^2]] \models (\forall X)\phi(X)$.

↓ CASE 2.2. θ contains $(u = v_j)$ -clauses and let $(\sigma_q)_q$ be an enumeration of them. Let η_p, η_q be the corresponding subformulas of them.

→ CASE 2.2.1.

$$[[\kappa]] \models (\forall w_k)(\exists v_j) \left[\neg\eta_p \wedge \bigwedge_q \neg\eta_q \right].$$

Then clearly

$$[[\kappa^2]] \models (\forall X)(\forall w_k)(\exists v_j)(\forall u) \left[\left(\neg\eta_p \wedge \bigwedge_q \neg\eta_q \right) \Rightarrow u \in X \right],$$

that is, $[[\kappa^2]] \models (\forall X)\phi(X)$.

↓ CASE 2.2.2.

$$[[\kappa]] \models (\exists w_k)(\forall v_j) \left[\eta_p \vee \bigvee_q \eta_q \right].$$

We claim that $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. We examine the relative positions of w_i, w_k, v_j in σ_p . They are:

(a) $u = w_i \preceq w_k \prec v_j$, (b) $w_k \prec u = w_i \preceq v_j$, or (c) $w_k \prec v_j \preceq u = w_i$.

(a) Let $u = w_i \preceq w_k \prec v_j$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j) \left[w_i \preceq w_k \prec v_j \vee \bigvee_q \eta_q \right].$$

Let $c \in \kappa$ be a value for w_k satisfying the last formula. Take a cofinal Z such that $Z \cap [0, c] = \emptyset$. Then it is easy to verify that

$$[[\kappa^2]] \models (\forall v_j)(\exists u) \left[\left(u = w_i \preceq c \prec v_j \vee \bigvee_q \eta_q \right) \wedge u \notin Z \right].$$

Indeed, for v_j such that $c \prec v_j$, take $u = c$ for which the first clause holds and $c \notin X$. If $v_j \leq c$, then v_j satisfies $\bigvee_q \eta_q$, where each η_q contains the equation $u = v_j$. So if $u = v_j$, then $u \notin X$ and we are done. It follows that $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

(b) Let $w_k \prec u = w_i \preceq v_j$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j) \left[w_k \prec u = w_i \preceq v_j \vee \bigvee_q \eta_q \right].$$

Let $c \in \kappa$ be a value for w_k satisfying the last formula. Take a cofinal Z such that $Z \cap [0, c+1] = \emptyset$ and Z contains no two consecutive elements. Then it is easy to verify again that

$$[[\kappa^2]] \models (\forall v_j)(\exists u) \left[\left(c \prec u = w_i \preceq v_j \vee \bigvee_q \eta_q \right) \wedge u \notin Z \right].$$

Indeed, for $v_j \leq c+1$, we pick $u = v_j$ which satisfies $\bigvee_q \eta_q$ and $v_j \notin Z$. For $v_j > c+1$, we pick any u such $c < u < v_j$ and $u \notin Z$. So in any case the above holds. Thus again $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

(c) Let $w_k \prec v_j \preceq u = w_i$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j) \left[w_k \prec v_j \preceq u = w_i \vee \bigvee_q \eta_q \right].$$

Take c for w_k satisfying this formula and a cofinal Z as in (b) above. Then we easily see again that

$$[[\kappa^2]] \models (\forall v_j)(\exists u) \left[\left(c \prec v_j \preceq u = w_i \vee \bigvee_q \eta_q \right) \wedge u \notin Z \right].$$

Hence $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

This exhausts the examination of Case 2. Note again that whenever in the above subcases $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, then for every X , we can choose each v_j arbitrarily high in X , unless the equation $u = v_j$ does not occur in θ , or the choice of v_j falsifies simultaneously all clauses containing $u = v_j$.

← CASE 3. Suppose Cases 1 and 2 are false. That means that there is no $(u = w)$ -clause of θ in which for every pair of variables w_i, v_j , either $j < i$ (Case 1) or $i \leq j$ and $w_i \prec v_j$ is in the clause (Case 2). It follows that every $(u = w)$ -clause of θ contains a formula of the form $v_j \prec w_i$ or $v_j = w_i$ such that $i \leq j$. Then we claim that $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. In order for $[[\kappa^2]] \models (\forall X)\phi(X)$ to hold, it suffices, for every X , to pick all v_j so that (a) $v_j \in X$ and (b) these v_j falsify all $(u = w)$ -clauses. We do not need to bother about the truth or falsity of $(u = v)$ -clauses, because for every such choice, they will define either \emptyset or singletons contained in X , hence $R_\theta^\kappa \subseteq X$ is guaranteed. Now since each $(u = w)$ -clause contains a formula of the form $v_j \prec w_i$ or $v_j = w_i$ with $i \leq j$ (i.e., v_j depending on w_i) it suffices, given X and w_i , to pick $v_j \in X$ such that $w_i < v_j$. The same dependent variable v_j may dominate several w_{i_1}, \dots, w_{i_m} . For every such finite sequence w_{i_1}, \dots, w_{i_m} , we can pick $v_j > w_{i_1}, \dots, w_{i_m}$ and $v_j \in X$. This is always possible because every X is cofinal. (This is the first and last place where the cofinality of the sets X is essential.) ■

Proof of the Main Theorem. Let

$$\phi(X) \equiv (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in X)$$

be the canonical form of ϕ , and let $[[\kappa^2]] \models (\forall X)\phi(X)$. Let also λ be another uncountable cardinal. It suffices to show that $[[\lambda^2]] \models (\forall X)\phi(X)$.

If $\overline{Q} = \overline{\forall}$, the claim follows from Lemma 2.2. So let \overline{Q} contain existential quantifiers. In view of Lemmas 2.6 and 2.7, we may assume that $\theta = \bigvee_p \sigma_p$, where all σ_p are equational. If θ contains only $(u = v)$ -clauses, the theorem follows from Lemma 2.8. Hence let $\neg\theta$ contain $(u = w)$ -clauses. If some $(u = w)$ -clause has no v -variables, then by Lemma 2.9, $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, which contradicts our assumption. So suppose every $(u = w)$ -clause of θ contains v -variables.

Then we come to Cases 1, 2, 3 treated above. We see that $[[\kappa^2]] \models (\forall X)\phi(X)$ is compatible only with Cases 1.1.1.1, 1.1.1.2.2, 1.2.1, 1.2.3.1, 2.1, 2.2.1, and 3. That is, every $(u = w)$ -clause of θ satisfies the conditions of some of these cases. Moreover, as explained above, the choice of the value for the dependent variables v_j can be done in such a way that all conditions are met simultaneously. Since these conditions are first-order, they guarantee the truth of $(\forall X)\phi(X)$ in any other structure $[[\lambda^2]]$ for uncountable λ . Therefore $[[\lambda^2]] \models (\forall X)\phi(X)$. This completes the proof of the Main Theorem. ■

3. \exists_1^1 positive formulas. Theorem 2.1 also holds for \exists_1^1 positive formulas, but the proof is much simpler. This is already suggested by the canonical form $(\overline{Q}\overline{w})(R_\theta^\kappa(\overline{w}) \subseteq X)$ of $\phi(X)$. Since the latter obviously holds in κ for $X = \kappa$, it follows that $(\exists X)\phi(X)$ holds in κ and similarly in every uncountable cardinal. However, this argument is fallacious, because as remarked after Lemma 1.3 of the first section, the canonical form holds only for X different from the universal set.

PROPOSITION 3.1. *For any uncountable cardinals κ, λ and every normal positive $\phi(X)$,*

$$[[\kappa^2]] \models (\exists X)\phi(X) \Leftrightarrow [[\lambda^2]] \models (\exists X)\phi(X).$$

Proof. Let $\kappa, \phi(X)$ be given. Observe that

$$(12) \quad [[\kappa^2]] \models (\exists X)\phi(X) \Leftrightarrow [[\kappa^2]] \models \phi(\kappa).$$

\Leftarrow of (12) is straightforward, while \Rightarrow follows from monotonicity of positive formulas: If $\phi(X)$ is true and $X \subseteq Y$, then $\phi(Y)$ is true. Now it is easy to check, by induction on the construction steps, that for every positive $\phi(X)$,

$$(13) \quad \phi(\kappa) \equiv \chi \quad \text{or} \quad \phi(\kappa) \equiv \top,$$

where χ is a first-order formula. Combining (12) and (13), and the fact that $[[\kappa]] \equiv [[\lambda]]$, we have

$$\begin{aligned} [[\kappa^2]] \models (\exists X)\phi(X) &\Leftrightarrow [[\kappa^2]] \models \phi(\kappa) \\ &\Leftrightarrow [[\kappa]] \models \chi \text{ (or } \top) \Leftrightarrow [[\lambda]] \models \chi \text{ (or } \top) \\ &\Leftrightarrow [[\lambda^2]] \models \phi(\lambda) \Leftrightarrow [[\lambda^2]] \models (\exists X)\phi(X). \blacksquare \end{aligned}$$

4. Some consequences of the Main Theorem. Given an infinite cardinal κ , observe that the principle $2^\kappa = \kappa^+$ can be formulated as a \forall_1^1 positive monadic sentence holding in a certain subclass of $\mathcal{P}(2^\kappa)$. Namely,

$$(14) \quad 2^\kappa = \kappa^+ \Leftrightarrow (2^\kappa, [2^\kappa]^{>\kappa}, <) \models (\forall X)(X \text{ is cofinal}).$$

Indeed, if $2^\kappa = \kappa^+$ then every $X \in [2^\kappa]^{>\kappa}$ has cardinality κ^+ and therefore is cofinal in κ^+ . If, conversely, $2^\kappa > \kappa^+$, then there is a set $X \in [2^\kappa]^{>\kappa}$, e.g. κ^+ , which is not cofinal in 2^κ .

“ X is cofinal” is the formula $(\forall x)(\exists y)(x \prec y \wedge y \in X)$, which is positive in X . Hence $(\forall X)(X \text{ is cofinal})$ is \forall_1^1 positive monadic.

As $2^\kappa = \kappa^+$ expresses a certain relationship between 2^κ and κ , one might reasonably argue that a wider class of such simple relationships could be common to all pairs of $\kappa, 2^\kappa$. A specific implementation of this general idea would be the claim that for all infinite κ, λ , the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$ and $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ satisfy the same \forall_1^1 positive monadic sentences. Call this principle the \forall_1^1 *Positive Equity Principle* (\forall_1^1 -PEP). That is,

$$(15) \quad (\forall_1^1\text{-PEP}) \quad (\forall \kappa, \lambda \geq \omega)[(2^\kappa, [2^\kappa]^{>\kappa}, <) \equiv_{\text{pos}}^{\forall_1^1} (2^\lambda, [2^\lambda]^{>\lambda}, <)].$$

The restriction to cofinal sets and to positive \forall_1^1 formulas makes the above principle minimalistic. Of course it would be desirable to have stronger versions for PEP, i.e., Γ -PEP for Γ a class of formulas larger than \forall_1^1 , provided its consistency with ZFC can be established. For the time being, the main theorem of the previous section implies the following.

THEOREM 4.1. *ZFC + \forall_1^1 -PEP is consistent.*

Proof. It suffices to show that $\text{ZFC} + \text{GCH} \vdash \forall_1^1\text{-PEP}$. In the presence of GCH, $\forall_1^1\text{-PEP}$ becomes

$$(\kappa^+, [\kappa^+]^{>\kappa}, <) \equiv_{\text{pos}}^{\forall_1^1} (\lambda^+, [\lambda^+]^{>\lambda}, <).$$

But

$$[\kappa^+]^{>\kappa} = [\kappa^+]^{\kappa^+} = \text{Cof}(\kappa^+)$$

and similarly $[\lambda^+]^{>\lambda} = \text{Cof}(\lambda^+)$. So the above reduces to

$$[[(\kappa^+)^2]] \equiv_{\text{pos}}^{\forall_1^1} [[(\lambda^+)^2]],$$

which follows from the Main Theorem. ■

COROLLARY 4.2. $\text{ZFC} + \forall_1^1\text{-PEP} \vdash (\forall \kappa, \lambda \geq \omega)(2^\kappa = \kappa^+ \Leftrightarrow 2^\lambda = \lambda^+)$.

Proof. Immediate from (14). ■

Does $\forall_1^1\text{-PEP}$ settle GCH? The answer is yes, unless a large cardinal assumption is made. Namely, suppose $\text{ZFC} + \forall_1^1\text{-PEP} \not\vdash \text{GCH}$. Then the consistency of $\forall_1^1\text{-PEP} + \neg\text{GCH}$ implies the consistency of $(\forall \kappa)(2^\kappa > \kappa^+)$, and hence the consistency of $2^\kappa > \kappa^+$ for a strong limit singular cardinal κ . But, by [4] (see also [5, Theorem 36.1]), this is equivalent to the consistency of the existence of a measurable cardinal with Mitchell order κ^{++} .

5. Separating ω_m from ω_n . In this section we use (essentially) the formulas employed in [1, p. 19] to distinguish between the structures $(\omega_1, \text{Cof}(\omega_1), <)$ and $(\omega_2, \text{Cof}(\omega_2), <)$. The formula that makes this distinction is of the form $(\forall x)(\exists X)\psi$, where ψ is normal but not positive.

Consider the formulas:

$$\begin{aligned} \text{succ}(x) &: (\exists y \prec x)(\forall z)(z \preceq y \vee x \preceq z), \\ \text{type}_\omega(X, x) &: (\forall y)(y \in X \ \& \ y \prec x \Rightarrow y = 0 \vee \text{succ}(y)), \\ \text{cof}(X, x) &: x = 0 \vee \text{succ}(x) \vee (\forall y \prec x)(\exists z \in X)(y \prec z \prec x), \\ \text{acc}_\omega(x) &: (\exists X)[\text{cof}(X, x) \wedge \text{type}_\omega(X, x)], \\ \phi_{\omega_1} &: (\forall x)\text{acc}_\omega(x). \end{aligned}$$

It is easy to check that ϕ_{ω_1} says that every element has cofinality ω , hence $(\omega_1, \mathcal{P}(\omega_1), <) \models \phi_{\omega_1}$, while $(\omega_2, \mathcal{P}(\omega_2), <) \models \neg\phi_{\omega_1}$. Now

$$\phi_{\omega_1} \equiv (\forall x)\text{acc}_\omega(x) \equiv (\forall x)(\exists X)\psi,$$

where $\psi \equiv \text{cof}(X, x) \ \& \ \text{type}_\omega(X, x)$. Since $\text{type}_\omega(X, x)$ is not positive, ψ is also not positive.

Now, by the same token, the formula $(\forall x)(\exists X)\psi$ also distinguishes the structures $(\omega_1, \text{Cof}(\omega_1), <)$ and $(\omega_2, \text{Cof}(\omega_2), <)$, since the quantifier $\exists X$ can range only over cofinal sets. Analogous separation formulas can be found for the other cardinals ω_m, ω_n , $m, n \in \omega$.

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