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Fixed-point free maps of Euclidean spaces

by

R. Z. Buzyakova and **A. Chigogidze** (Greensboro, NC)

Abstract. Our main result states that every fixed-point free continuous self-map of $\mathbb{R}^n$ is colorable. This result can be reformulated as follows: A continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ is fixed-point free iff $\widetilde{f} : \beta\mathbb{R}^n \to \beta\mathbb{R}^n$ is fixed-point free. We also obtain a generalization of this fact and present some examples.

1. Introduction. It is known that for a continuous map $f\colon X \to X$ the set $\mathrm{Fix}(\widetilde{f})$ of all fixed points of its Stone–Čech extension $\widetilde{f}\colon \beta X \to \beta X$ may differ from $\mathrm{cl}_{\beta X} \mathrm{Fix}(f)$. In particular, $\mathrm{Fix}(\widetilde{f})$ could be non-empty for a fixed-point free f, which is equivalent to non-colorability of f if X is normal. Since this notion of colorability may not be widely known, let us define it.

Definition. $f : X \subset Y \to Y$ is *colorable* if one can cover X by finitely many closed sets each of which misses its image under f.

Metric examples (constructed by Krzysztof Mazur and van Douwen) of fixed-point free non-colorable maps can be found in [5], [3], [6], [8]. It is interesting to note that these examples are either infinite-dimensional or non-locally compact. On the other hand, it is known that: (a) every fixed-point free autohomeomorphism f of a finite-dimensional paracompact space X has the fixed-point free extension $\widetilde{f}$ (see [3]), and (b) every continuous fixed-point free map f of a zero-dimensional sigma-compact space X has the fixed-point free extension $\widetilde{f}$ (see [6]). We also mention here the recent result [1] stating colorability of *every* continuous fixed-point free self-map of the real line $\mathbb{R}$.

Below we show (Theorem 2.6) that every continuous fixed-point free self-map of the Euclidean space $\mathbb{R}^n$ is colorable. In fact, we show more by considering not only self-maps $f\colon X \to X$, but also maps $f\colon X \to Y$, defined on closed subspaces X of Y. This version proves helpful in various situations and in our opinion is of interest. The most general statement we present in

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this paper is Theorem 3.6, which states that every continuous fixed-point free map $f\colon X \to Y$, defined on a closed subspace X of an at most n-dimensional locally compact and paracompact space Y, is colorable in at most $n+3$ colors. We also note that a continuous map $f\colon X \to Y$, defined on a closed subspace X of an at most n-dimensional locally compact and paracompact space Y, is fixed-point free iff $\widetilde{f} : \beta X \to \beta Y$ is fixed-point free. In Section 4, we construct a non-colorable fixed-point free autohomeomorphism of the separable Hilbert space ℓ_2 and a non-colorable continuous fixed-point free self-map of the universal n-dimensional Nöbeling space N_n^{2n+1}. The case $n = 0$ produces (and uses) the same example as that of K. Mazur's since N_0^1 is the space of irrationals. We conclude by proving (Proposition 4.5) that similar examples exist on every zero-dimensional non-σ-compact Polish space.

A good account of the results related to this subject as well as historical comments are given in [8, Section 3.2]. Our notation and terminology related to absolute extensors in dimension n, n-soft maps, universal Nöbeling spaces and inverse spectra follow [2].

When discussing the nth power $\mathbb{R}^n$ of $\mathbb{R}$ we agree that $\mathbb{R}^0$ is the singleton $\{0\}$. We finish the introduction by defining the colorability-related concepts used in this paper. Once again, a continuous map $f : X \to Y$ is *colorable* if there exists a finite cover $\mathcal{F}$ of X by its closed subsets such that $f(F)$ misses F for every $F \in \mathcal{F}$. If $\mathcal{F}$ has n elements we say that f is *colorable in n colors*. Every closed set $F \subset X$ with the property that $f(F) \cap F = \emptyset$ is a *color* (for f). All maps under discussion are continuous.

2. Fixed-point free maps of Euclidean spaces. The following lemma consists of common facts about colorable maps that are proved implicitly for self-maps in other papers on the subject. Since the proof is a straightforward verification we will only sketch it.

Lemma 2.1. *Let X be a closed subspace of Z and $f : X \to Z$ a continuous map.*

1. *If Z is normal, A is a color, and A misses $\overline{f(A)}$, then there is an open neighborhood U of A whose closure is a color.*
2. *If $A \subset Z$ and $f(A) \cap A = \emptyset$ then $f^{-1}(A) \cap A = \emptyset$.*
3. *If $\mathcal{G}$ is a closed cover of Z such that $\overline{f(G) \cap G} = \emptyset$ then $\mathcal{F} = \{f^{-1}(G) : G \in \mathcal{G}\}$ is a cover of X by colors and $\overline{f(F)} \cap F = \emptyset$ for all $F \in \mathcal{F}$.*

Proof. To prove (2), assume the contrary and pick x in $f^{-1}(A) \cap A = \emptyset$. Since $f^{-1}(A)$ is a subset of X, the value $f(x)$ is defined. Hence $f \circ f^{-1}(A)$ meets $f(A)$, contradicting $f(A) \cap A = \emptyset$.

To prove (3), apply (2) to conclude that $f^{-1}(G)$ is a color. Since $\mathcal{G}$ is a cover of Z and f is defined on X only, $\mathcal{F}$ is a cover of X. Finally, let us show that $\overline{f(F)} \cap F = \emptyset$. We have $F = f^{-1}(G)$ for $G \in \mathcal{G}$. Then $\overline{f(F)} \cap F$ equals $\overline{f \circ f^{-1}(G)} \cap f^{-1}(G)$, which is a subset of $\overline{G} \cap f^{-1}(G)$. Since G is closed $\overline{G} \cap f^{-1}(G)$ equals $G \cap f^{-1}(G)$, which is empty by (2). ∎

We will use the above facts quite often without formally referring to the statement.

Lemma 2.2. *Let X be a normal space of dimension at most n. Suppose $\{A_i, B_i : i \leq n+1\}$ is a family of functionally closed sets such that $A_i \cap B_i = \emptyset$. Then there exists a functionally closed cover $\{\tilde{A}_i, \tilde{B}_i : i \leq n+1\}$ of X with the following properties:*

(1) $\tilde{A}_i \cap \tilde{B}_i = \emptyset$.
(2) $\tilde{A}_i \cap Z = A_i$ *and* $\tilde{B}_i \cap Z = B_i$, *where* $Z = \bigcup\{A_i, B_i : i \leq n+1\}$.

Proof. For each $i = 1, \ldots, n+1$, consider a map $f_i \colon X \to I = [0,1]$ such that $f_i^{-1}(0) = A_i$ and $f_i^{-1}(1) = B_i$. The diagonal product $f = \triangle f_i \colon X \to [0,1]^{n+1}$ sends the union $\bigcup(A_i \cup B_i)$ into ∂I^{n+1}. Since $\dim X \leq n$ and since $\partial I^{n+1} \simeq S^n$ is an absolute extensor in dimension n for normal spaces, it follows that the map $f | \bigcup(A_i \cup B_i) \colon \bigcup(A_i \cup B_i) \to \partial I^{n+1}$ has an extension $g \colon X \to \partial I^{n+1}$. The sets $\widetilde{A}_i$, $\widetilde{B}_i$ can now be defined as inverse images under g of the opposite faces of ∂I^{n+1}. The needed properties hold by construction. ∎

To prove our main result for $\mathbb{R}^n$ and a generalization to n-dimensional locally compact paracompact spaces we consider several cases that can be handled by one approach realized in Lemma 2.4 below. To formulate the lemma in the required generality we need the following definition.

Definition. A family $\{X_\alpha : \alpha \leq \tau, \alpha \text{ isolated}\}$ is a *favorable representation* of a space X if the following hold:

(1) $X = \bigcup_{\alpha \leq \tau} X_\alpha$.
(2) If $A \cap X_\alpha$ is functionally closed in X_α for all isolated $\alpha \leq \tau$, then A is functionally closed in X.
(3) If $X_\alpha \cap X_\beta \neq \emptyset$, then $\alpha = \beta + 1$ or $\beta = \alpha + 1$.
(4) The union of any subcollection of $\{X_\alpha : \alpha \leq \tau, \alpha \text{ isolated}\}$ is functionally closed in X.

A much simpler version of Lemma 2.4 for the real line is proved in [1, Lemma 2.1]. To prove our generalized version we introduce three technical definitions that will be used only to prove Lemma 2.4. In what follows when dealing with a favorable representation $\{X_\alpha : \alpha \leq \tau, \alpha \text{ is isolated}\}$ we always assume that X_α is empty if $\alpha \leq \tau$ is limit.

Definition. An ordinal α will be called *odd* if $|\{\beta < \alpha : |\alpha \backslash \beta| \text{ is finite}\}|$ is an odd number.

Definition. Let $\{Y_\alpha : \alpha \leq \tau, \alpha \text{ is isolated}\}$ be a favorable representation of Y; $\dim Y \leq n$; and $f : X \subset Y \to Y$. We say that $\{A_i, B_i : i \leq n+1\}$ is a *β-coloring* of f if the following hold:

C1. $\{A_i, B_i : i \leq n+1\}$ is a functionally closed cover of $\bigcup_{\gamma\leq\beta} Y_\gamma$.
C2. $A_i \cap B_i = \emptyset$.
C3. $f(A_i) \cap A_i = \emptyset$ and $f(B_i) \cap B_i = \emptyset$.

Definition. Let $\{Y_\alpha : \alpha \leq \tau,\ \alpha \text{ isolated}\}$ be a favorable representation of Y, $\dim Y \leq n$, $f : X \subset Y \to Y$, and let $\{A_i^\gamma, B_i^\gamma : i \leq n+1\}$ and $\{A_i^\beta, B_i^\beta : i \leq n+1\}$ be γ- and β-colorings of f with $\gamma + 2 \leq \beta$. We say that the colorings *agree* if the following hold:

A1. If γ and β are odd then $A_i^\gamma \subset A_i^\beta$ and $B_i^\gamma \subset B_i^\beta$.
A2. If β is odd and γ' is the smallest odd ordinal greater than or equal to γ, then $A_i^\beta \cap Y_\gamma = A_i^{\gamma'+2} \cap Y_\gamma$ and $B_i^\beta \cap Y_\gamma = B_i^{\gamma'+2} \cap Y_\gamma$.

Observe that in A2 of the above definition, $\gamma' + 2 \leq \beta$ because β is odd and $\beta \geq \gamma + 2$. What the requirement A2 states is that A_i^β and A_i^α cut out the same piece from Y_γ as long as α and β are odd and at least two ordinals above γ. This property together with (2) of favorable representation will be used below to show that the union of A_i^β's over add β is closed.

Lemma 2.3. *Let $\{Y_\alpha : \alpha \leq \tau,\ \alpha \text{ isolated}\}$ be a favorable representation of a normal space Y of dimension at most n. Let X be a closed subspace of Y and $f : X \to Y$ a continuous map. Let α be limit and $\{\{A_i^\beta, B_i^\beta : i \leq n+1\} : \beta < \alpha\}$ be a collection of β-colorings of f that agree with each other. Then the following sets form an α-coloring of f that agrees with all β-colorings in the given collection:*

$$A_i^\alpha = \bigcup\{B_i^\beta : \beta < \alpha,\ \beta \text{ odd}\}, \quad B_i^\alpha = \bigcup\{A_i^\beta : \beta < \alpha,\ \beta \text{ odd}\}, \quad i \leq n+1.$$

Proof. Neither A1 nor A2 in the definition of agreeing colorings needs verification since being limit, α is not odd. Let us verify C1–C3 in the definition of α-coloring. For C1, recall that $Y_\alpha = \emptyset$. Therefore, if $x \in \bigcup_{\beta\leq\alpha} Y_\beta$ then $x \in Y_\beta$ for $\beta < \alpha$. Let $\gamma < \alpha$ be an odd ordinal greater than β. Then by C1 for γ, we have $x \in \bigcup\{A_i^\gamma, B_i^\gamma : i \leq n+1\}$. Thus, $\{A_i^\alpha, B_i^\alpha : i \leq n+1\}$ is a cover of $\bigcup_{\beta\leq\alpha} Y_\beta$. Let us show that A_i^α is functionally closed in Y. By property (2) of favorable representations we need to fix an isolated γ and show that $Y_\gamma \cap A_i^\alpha$ is functionally closed in Y_γ. The inclusion $A_i^\alpha \subset \bigcup_{\beta\leq\alpha} Y_\beta$, the equality $Y_\alpha = \emptyset$, and (3) of favorable representation imply that A_i^α may meet only Y_γ with $\gamma < \alpha$. We have $A_i^\alpha \cap Y_\gamma = \bigcup\{B_i^\beta \cap Y_\gamma : \beta < \alpha,\ \beta \text{ odd}\}$. This set can be written as the union of two sets, $T_1 = \bigcup\{B_i^\beta \cap Y_\gamma : \gamma' + 2 \leq \beta < \alpha, \beta \text{ odd}\}$ and $T_2 = \bigcup\{B_i^\beta \cap Y_\gamma : \beta \leq \gamma' + 2, \beta \text{ odd}\}$, where γ' is the smallest odd ordinal greater than or equal to γ. The set T_1 is $B_i^{\gamma'+2} \cap Y_\gamma$ by

A2 for ordinals below α. The set $B_i^{\gamma'+2}$ is functionally closed by inductive assumption C1. Therefore, $B_i^{\gamma'+2} \cap Y_\gamma$ is functionally closed in Y_γ. The set T_2 is $B_i^{\gamma'+2} \cap Y_\gamma$ by A1 for ordinals below α and is functionally closed for the same reasons as the first one. Thus, $A_i^\alpha \cap Y_\gamma$ is the union of two functionally closed sets in Y_γ.

For C2, we need to show that $A_i^\beta \cap B_i^\gamma = \emptyset$ for odd $\beta, \gamma < \alpha$. Assume $\gamma \geq \beta$. By A1 for γ, we have $A_i^\beta \subset A_i^\gamma$. By C2 for γ, we have $A_i^\gamma \cap B_i^\gamma = \emptyset$.

For C3, we need to show that $f(B_i^\beta) \cap B_i^\gamma = \emptyset$ for odd $\beta, \gamma < \alpha$. Assume $\gamma \geq \beta$. By A1 for γ, we have $B_i^\beta \subset B_i^\gamma$. By C3 for γ, we have $f(B_i^\gamma) \cap B_i^\gamma = \emptyset$. ■

LEMMA 2.4. *Let $\{Y_\alpha : \alpha \leq \tau, \alpha$ isolated$\}$ be a favorable representation of a normal space Y of dimension at most n. Let X be a closed subspace of Y and $f : X \to Y$ a fixed-point free continuous map with the following properties:*

(1) *X misses Y_0.*
(2) *If $z \in Y_\alpha \cap X$ then $f(z) \in [\bigcup_{\beta<\alpha} Y_\beta] \setminus Y_\alpha$.*

Then there exists a $(2n+2)$-sized coloring $\mathcal{F}$ of f such that $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ for every $F \in \mathcal{F}$.

Proof. Inductively, we will construct building blocks for our colors.

Step 0: Put $A_1^0 = Y_0$, $A_i^0 = \emptyset$ if $1 < i \leq n+1$, and $B_i^0 = \emptyset$ if $i \leq n+1$.

Induction assumption: Assume that for $\beta < \alpha$ we have defined families $\{A_i^\beta, B_i^\beta : i \leq n+1\}$ that are β-colorings of f and agree with each other.

Observe that for $\beta = 0$ the family $\{A_i^0, B_i^0 : i \leq n+1\}$ meets C1–C3, A1, and A2 in the definitions of β-coloring and agreeing colorings. Indeed, C1 holds because $A_1^0 = Y_0$. C2 holds because $B_i^0 = \emptyset$. C3 holds because f is not defined on Y_0. A1 and A2 are not applicable for $\beta = 0$ since 0 is not odd.

Step $[\alpha = \lim\{\beta < \alpha\}]$: We construct an α-coloring as in Lemma 2.3.

Step $[\alpha = \beta + 1]$: Put

$$C_i^\alpha = [f^{-1}(A_i^\beta) \cap Y_\alpha] \cup B_i^\beta \quad \text{and} \quad D_i^\alpha = [f^{-1}(B_i^\beta) \cap Y_\alpha] \cup A_i^\beta.$$

Let us show that these sets have the following properties:

S1. C_i^α and D_i^α are functionally closed.
S2. $C_i^\alpha \cap D_i^\alpha = \emptyset$.
S3. If $x \in \bigcup_{\gamma\leq\alpha} Y_\gamma \cap X$ then both $f(x)$ and x are in $\bigcup\{C_i^\alpha, D_i^\alpha : i \leq n+1\}$.

For S1, observe that $f^{-1}(A_i^\beta)$ is functionally closed as the inverse image of the set functionally closed by C1 for β. By property (4) of favorable representation, $f^{-1}(A_i^\beta)\cap Y_\alpha$ is functionally closed in Y as well. The set B_i^β is functionally closed by C1 for β. Thus, C_i^α is functionally closed. For S2 we need to verify the following four equalities:

(1) $[f^{-1}(A_i^\beta)\cap Y_\alpha]\cap[f^{-1}(B_i^\beta)\cap Y_\alpha]=\emptyset$.
(2) $[f^{-1}(A_i^\beta)\cap Y_\alpha]\cap A_i^\beta=\emptyset$.
(3) $B_i^\beta\cap[f^{-1}(B_i^\beta)\cap Y_\alpha]=\emptyset$.
(4) $B_i^\beta\cap A_i^\beta=\emptyset$.

Equality (1) holds because $A_i^\beta\cap B_i^\beta=\emptyset$ by C2 for β. Equalities (2) and (3) hold by C3 for β. Equality (4) holds by C2 for β. Let us show S3. By (2) of the lemma's hypothesis, $f(x)\in\bigcup_{\gamma\leq\beta}Y_\gamma$ and $x\in[f^{-1}(\bigcup_{\gamma\leq\beta}Y_\gamma)\cap Y_\alpha]\cup\bigcup_{\gamma\leq\beta}Y_\gamma$. By C1 for β, $\bigcup_{\gamma\leq\beta}Y_\gamma=\bigcup\{A_i^\beta,B_i^\beta : i\leq n+1\}$ and $f^{-1}(\bigcup_{\gamma\leq\beta}Y_\gamma)\cap Y_\alpha\subset\bigcup\{f^{-1}(A_i^\beta)\cap Y_\alpha, f^{-1}(B_i^\beta)\cap Y_\alpha : i\leq n+1\}$. The right-hand sets in these formulas are subsets of $\bigcup\{C_i^\alpha,D_i^\alpha : i\leq n+1\}$ by definition.

Since C_i^α and D_i^α are functionally closed and disjoint, by Lemma 2.2, there exists a family $\{A_i^\alpha,B_i^\alpha : i\leq n+1\}$ of subsets of $\bigcup_{\gamma\leq\alpha}Y_\gamma$ with the following properties:

S4. $\{A_i^\alpha,B_i^\alpha : i\leq n+1\}$ is a functionally closed cover of $\bigcup_{\gamma\leq\alpha}Y_\gamma$.
S5. $A_i^\alpha\cap Z=C_i^\alpha$ and $B_i^\alpha\cap Z=D_i^\alpha$, where $Z=\bigcup\{C_j^\alpha,D_j^\alpha : j\leq n+1\}$.
S6. $A_i^\alpha\cap B_i^\alpha=\emptyset$.

In addition, if $x\in A_i^\alpha\cap X$ then, by S3, $x\in Z=\bigcup\{C_i^\alpha,D_i^\alpha : i\leq n+1\}$. Thus, $x\in A_i^\alpha\cap Z$. By S5, $x\in C_i^\alpha$. In short,

S7. If $x\in A_i^\alpha\cap X$ (resp. $x\in B_i^\alpha\cap X$), then $x\in C_i^\alpha$ (resp. $x\in D_i^\alpha$).

Also, if $x\in A_i^\alpha\setminus C_i^\alpha$ then, by S5, $x\notin Z$. The set Z contains $\bigcup_{\gamma\leq\beta}Y_\gamma$ due to second summands in the definitions of C_i^α and D_i^α and assumption C1 for β. Therefore, $x\notin\bigcup_{\gamma\leq\beta}Y_\gamma$. Hence, $x\in Y_\alpha$. In summary,

S8. If $x\in A_i^\alpha\setminus C_i^\alpha$ then $x\in Y_\alpha$.

Let us check C1–C3, A1, and A2. Property C1 is S4 and C2 is S6. Let us demonstrate C3 for A_i^α. By S7, $f(A_i^\alpha)\cap A_i^\alpha=f(C_i^\alpha)\cap A_i^\alpha$. By S3, $f(C_i^\alpha)\subset Z=\bigcup\{C_i^\alpha,D_i^\alpha : i\leq n+1\}$. By S5, $A_i^\alpha\cap Z=C_i^\alpha$. Therefore $f(C_i^\alpha)\cap A_i^\alpha=f(C_i^\alpha)\cap C_i^\alpha$. To show that the last intersection is empty we need to verify the following equalities:

(1) $f(f^{-1}(A_i^\beta)\cap Y_\alpha)\cap[f^{-1}(A_i^\beta)\cap Y_\alpha]=\emptyset$.
(2) $f(f^{-1}(A_i^\beta)\cap Y_\alpha)\cap B_i^\beta=\emptyset$.
(3) $f(B_i^\beta)\cap[f^{-1}(A_i^\beta)\cap Y_\alpha]=\emptyset$.
(4) $f(B_i^\beta)\cap B_i^\beta=\emptyset$.

Equality (1) holds by C3 for β. Equality (2) holds by C2 for β. Let us show (3). If $\beta=0$ then, by (1) of the lemma's hypothesis, $f(B_i^\beta)$ is empty and (3) holds. Now assume $\beta > 0$. From (2) of the lemma's hypothesis and emptiness of Y_λ for limit λ, we conclude that the first set is in $\bigcup_{\gamma<\beta} Y_\gamma$. The second set is in Y_α by the definition. By (3) of favorable representation, $\bigcup_{\gamma<\beta} Y_\gamma$ misses Y_α. Hence the intersection is empty. Equality (4) is C3 for β.

To verify A1 we assume α is odd and pick an odd $\gamma < \alpha$. By construction, A_i^α contains $[f^{-1}(A_i^\beta) \cap Y_\alpha] \cup B_i^\beta$. It suffices to show that B_i^β contains A_i^γ. If β is limit then, by construction, $B_i^\beta = \bigcup\{A_i^\lambda : \lambda < \beta,\ \lambda \text{ odd}\}$ and the right side contains A_i^γ. If $\beta = \lambda + 1$ then, by construction, B_i^β contains $[f^{-1}(B_i^\lambda) \cap Y_\beta] \cup A_i^\lambda$. By A1 for λ, A_i^λ contains A_i^γ.

Finally to show A2 we assume α is odd and pick a γ such that $\gamma'+2 < \alpha$, where γ' is the smallest odd ordinal greater than or equal to γ. Observe that $A_i^\alpha = C_i^\alpha \cup (A_i^\alpha \setminus C_i^\alpha)$, which in its turn equals $[f^{-1}(A_i^\beta) \cap Y_\alpha] \cup B_i^\beta \cup (A_i^\alpha \setminus C_i^\alpha)$. The first set in this union is in Y_α and so is the third by S8. By (3) of favorable representation, Y_α misses Y_γ. Thus, $A_i^\alpha \cap Y_\gamma = B_i^\beta \cap Y_\gamma$.

If β is limit, then $B_i^\beta \cap Y_\gamma = \bigcup\{A_i^\lambda \cap Y_\gamma : \lambda < \beta,\ \lambda \text{ odd}\}$. The right set can be written as the union of $T_1 = \bigcup\{A_i^\lambda \cap Y_\gamma : \lambda \leq \gamma' + 2,\ \lambda \text{ odd}\}$ and $T_2 = \bigcup\{A_i^\lambda \cap Y_\gamma : \gamma' + 2 \leq \lambda < \beta,\ \lambda \text{ odd}\}$. By A1, $T_1 = A_i^{\gamma'+2} \cap Y_\gamma$. By A2, $T_2 = A_i^{\gamma'+2} \cap Y_\gamma$.

Now assume β is isolated. Since there exists an odd γ' below odd α we conclude that $\beta \neq 0$. Therefore, $\beta = \lambda + 1$. We have $B_i^\beta = D_i^\beta \cup (B_i^\beta \setminus D_i^\beta)$, which in its turn equals the union of three sets, $T_1 = f^{-1}(B_i^\lambda) \cap Y_\beta$, $T_2 = A_i^\lambda$, and $T_3 = B_i^\beta \setminus D_i^\beta$. Since $\alpha > \gamma + 2$, we conclude $\beta > \gamma + 1$. By property (3) of favorable representation, Y_β misses Y_γ. The set T_1 is in Y_β and so is T_3 by S8 for isolated β. Therefore, only T_2 can meet Y_γ. Hence, $B_i^\beta \cap Y_\gamma = A_i^\lambda \cap Y_\gamma$. By A2 for λ, the right side of the last equality is $A_i^{\gamma'+2} \cap Y_\gamma$. The inductive construction is complete.

Put $A_i = A_i^\tau$ and $B_i = B_i^\tau$. By C1–C3 , the family $\mathcal{G} = \{A_i, B_i : i \leq n+1\}$ is a closed cover of Y such that $f(G) \cap G = \emptyset$ for every $G \in \mathcal{G}$. By Lemma 2.1(3), $\mathcal{F} = \{f^{-1}(G) : G \in \mathcal{G}\}$ is the desired coloring of f. The lemma is proved. ∎

Lemma 2.5. *Let X be a compact subspace of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous fixed-point free map. Then f is colorable in at most $4n(n+1)$ colors.*

Proof. For each $i \leq n$, put $A_i = \{x \in X : \pi_i \circ f(x) > \pi_i(x)\}$ and $B_i = \{x \in X : \pi_i \circ f(x) < \pi_i(x)\}$. By continuity of f, each of these sets is open. Since f is fixed-point free, $\{A_i, B_i : i \leq n\}$ is a cover of X. By paracompactness of X, there exists a closed shrinking $\{A_i', B_i' : i \leq n\}$. It suffices to show now that $f|_S$ is colorable in $2(n+1)$ colors for each $S \in \{A_i', B_i' : i \leq n\}$. We will demonstrate it for $A = A_1'$.

Fix $a \in A$. Since A is compact and $\pi_1 \circ f(x) > \pi_1(x)$ for every $x \in A$, we can find $\epsilon_a > 0$ such that

(∗) If $x \in A$ and $\pi_1(x) \in [\pi_1(a) - \epsilon_a, \pi_1(a) + \epsilon_a]$ then $\pi_1 \circ f(x) > \pi_1(a) + \epsilon_a$.

By compactness and (∗) we can select a strictly decreasing sequence of reals $r_1 > \cdots > r_m$ such that

P1. $A \subset [r_m, r_1) \times \mathbb{R}^{n-1}$.
P2. If $x \in [r_{k+1}, r_k] \times \mathbb{R}^{n-1} \cap A$ then $\pi_1 \circ f(x) > r_k$.

Put $X_0 = [r_1, \infty) \times \mathbb{R}^{n-1}$; $X_k = [r_{k+1}, r_k] \times \mathbb{R}^{n-1}$ if $1 \leq k \leq m-1$; and $X_m = (-\infty, r_m] \times \mathbb{R}^{n-1}$. Clearly $\{X_k\}_{k \leq m}$ is a favorable representation of $\mathbb{R}^n$. We only need to show that $\mathbb{R}^n$, $\{X_k\}_{k \leq m}$, $f|_A$, and A meet (1) and (2) of Lemma 2.4. Requirement (1) is met because X_0 misses A by P1; and (2) is met by P2. ∎

THEOREM 2.6. *Let X be a closed subspace of $\mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous fixed-point free map. Then f is colorable.*

Proof. Put $X_1 = [-1, 1]^n$ and $X_{k+1} = [-(k+1), (k+1)]^n \setminus (-k, k)^n$. Let $M = 4n(n+1)$.

CLAIM 1. *Let $A = \{x \in X : x \in X_k$ and $f(x) \in X_m$ for some $m \geq k\}$. Then there exists a finite open cover $\mathcal{U}_A$ of A such that the closure of every element of $\mathcal{U}_A$ is a color.*

Proof of Claim 1. Clearly, A is closed. Fix a strictly increasing sequence $\langle b_k \rangle_k$ of positive integers that have the following property:

P1. $f([-b_k, b_k]^n \cap X) \subset (-b_{k+1}, b_{k+1})^n$.

For every k, select a coloring $\mathcal{S}_k = \{S^k_1, \ldots, S^k_M\}$ of f_k, where f_k is the restriction of f to $A \cap ([-b_k, b_k]^n \setminus (-b_{k-1}, b_{k-1})^n)$. This is possible by Lemma 2.5. By the definition of A, $f(S)$ does not meet $(-b_{k-1}, b_{k-1})^n$ for any $S \in \mathcal{S}_k$. Also $f(S)$ does not meet the complement of $(b_{k+1}, b_{k+1})^n$ by P1. In short, we have

P2. $f(S) \subset (-b_{k+1}, b_{k+1})^n \setminus (-b_{k-1}, b_{k-1})^n$ for every $S \in \mathcal{S}_k$.

Put $O_i = \bigcup\{S^k_i : k \text{ odd}\}$ and $E_i = \bigcup\{S^k_i : k \text{ even}\}$. The sets O_i and E_i are closed as unions of discrete families of closed sets and $\{O_i, E_i : i \leq M\}$ is a cover of A. Let us show that $f(O_1) \cap O_1 = \emptyset$. By P2, we have $f(S^k_1) \cap S^m_1 = \emptyset$ if $k \neq m$. Since S^k_1 is a color, $f(S^k_1) \cap S^k_1 = \emptyset$. Thus we have colored $f|_A$ in $2M = 8n(n+1)$ colors. By Lemma 2.1(1), to find a desired open cover it suffices to show that $f(O_i)$ and $f(E_i)$ are closed. We have $f(O_i) = \bigcup\{f(S^k_i) : i \text{ odd}\}$. Each $f(S^k_i)$ is compact since S^k_i is. By P2, $\{f(S^k_i) : i \text{ odd}\}$ is a locally finite family. Therefore, $f(O_i)$ is closed. The claim is proved.

Claim 2. *Let $B = X \setminus \bigcup \mathcal{U}_A$. Then there exists a finite cover $\mathcal{G}_B$ of B by colors.*

Proof of Claim 2. Since B misses A the following holds:

(∗) If $x \in B \cap X_k$ and $f(x) \in X_m$ then $m < k$.

Since $\{X_k\}_k$ is a favorable representation of $\mathbb{R}^n$ we only need to show that (1) and (2) of Lemma 2.4 are met. Since X_1 is a subset of A, it misses B, so (1) is met; and (2) is given by (∗). The claim is proved.

Clearly $\mathcal{G}_B \cup \{\bar{U} : U \in \mathcal{U}_A\}$ is a finite coloring of f. ■

2.1. Chromatic number. Given a fixed-point free map $f : X \to Y$, the *chromatic number* of f is the smallest size of a coloring of f. First of all we record the following statement for future reference. It extends the main result of [1].

Corollary 2.7. *Every continuous fixed-point free self-map of $\mathbb{R}^n$ is colorable in at most $n + 3$ colors.*

Proof. In [8, Theorem 3.12.17] the following theorem (due to R. Pol) is proved: *Let X be a separable metrizable space and let $f : X \to X$ be a fixed-point free continuous map. If f is finitely colorable and* $\dim X \leq n$ *then f can be colored with $n + 3$ colors.* This theorem together with Theorem 2.6 gives the desired conclusion. ■

We remark that the theorem cited in the proof of Corollary 2.7 holds with "separable metrizable" replaced by "normal". This fact is observed in [7] and is attributed to Pol. Fur further reference, Theorem 3.12.17 of [8] will be referred to as the *Pol theorem.*

Next we estimate the chromatic number of continuous fixed-point free maps $f \colon X \to \mathbb{R}^n$, thus extending Corollary 2.7 and making Theorem 2.6 more precise. For this we need the following observation.

Lemma 2.8. *Let $n \geq 0$ and $f \colon X \to Y$ be a continuous fixed-point free map defined on a closed subspace X of a metrizable n-dimensional $AE(n)$-space Y. Then there exist an embedding $X \subset Y \times [0, \infty)$ and a continuous fixed-point free map $g \colon Y \times [0, \infty) \to Y \times [0, \infty)$ such that $g|X = f$.*

Proof. Let $h \colon Y \to Y$ be a continuous extension of f. Such an h exists since $\dim Y = n$ and $Y \in AE(n)$. Let $\mathrm{Fix}(h) = \{y \in Y \colon h(y) = y\}$. Since f is fixed-point free and $h|X = f$ it follows that $X \cap \mathrm{Fix}(h) = \emptyset$. Identify Y with the subspace $Y \times \{0\}$ of the product $Y \times [0, \infty)$, X with the subspace $X \times \{0\}$, and define g as follows:

$$g(y, r) = (h(y), r + \mathrm{dist}(y, X)).$$

For $(x,0)\in X\times\{0\}$ we have $g(x,0)=(h(x),\operatorname{dist}(x,X))=(f(x),0)$, which shows that g is an extension of f. To see that g is fixed-point free, note that

- if $y\in \operatorname{Fix}(h)$, then $y\notin X$, and thus $\operatorname{dist}(y,X)>0$ and $g(y,r)\neq(y,r)$;
- if $y\notin \operatorname{Fix}(h)$, then $h(y)\neq y$ and thus $g(y,r)\neq(y,r)$. ∎

PROPOSITION 2.9. *Every continuous fixed-point free map $f\colon X\to Y$, defined on a closed subset X of an at most n-dimensional locally compact separable metrizable space Y (e.g. $Y=\mathbb{R}^n$) is colorable in at most $n+3$ colors. In addition, there exists an $(n+3)$-sized coloring $\mathcal{F}$ of f such that $F\cap \operatorname{cl}_Y(f(F))=\emptyset$ for every $F\in\mathcal{F}$.*

Proof. Without loss of generality we may assume that Y is a closed subspace of $\mathbb{R}^{2n+1}$. By Lemma 2.8 and Corollary 2.7, there is a colorable (in at most $2n+5$ colors) map $g\colon \mathbb{R}^{2n+2}\to\mathbb{R}^{2n+2}$ such $g|X=f$. Let $p\colon N^{2n+1}_n\to\mathbb{R}^{2n+2}$ be an n-soft map of the n-dimensional universal Nöbeling space (see [2, Theorem 5.6.4]). Since $\dim Y\le n$ and p is n-soft, there exists a map $j\colon Y\to N^{2n+1}_n$ such that $p\circ j=\mathrm{id}_Y$. Clearly $j(X)$ and $j(Y)$ are copies of X and Y in N^{2n+1}_n and the map $f'\colon j(X)\to j(Y)$ defined as $f'=j\circ f\circ p|j(X)$ is a copy of f. Consequently, it suffices to prove the required upper bound for the chromatic number of f'. Using n-softness of p we conclude that there exists a map $h\colon N^{2n+1}_n\to N^{2n+1}_n$ such that $p\circ h=g\circ p$ and $h|j(X)=j\circ g\circ p|j(X)=j\circ f\circ p|j(X)=f'$. Colorability of g and the equality $p\circ h=g\circ p$ imply that h is also colorable. Since the dimension of N^{2n+1}_n is n, by the Pol theorem [8, Theorem 3.12.17], h is colorable in at most $n+3$ colors. Since h is a self-map, by Lemma 2.1(3), we can find $\mathcal{F}$ with the required properties. ∎

3. Fixed-point free maps of locally compact paracompact spaces

PROPOSITION 3.1. *Every continuous fixed-point free map $f\colon X\to Y$ defined on a closed subspace of an at most n-dimensional locally compact and Lindelöf space Y, is colorable in at most $n+3$ colors. In addition, there exists an $(n+3)$-sized coloring $\mathcal{F}$ of f such that $F\cap\operatorname{cl}_Y(f(F))=\emptyset$ for every $F\in\mathcal{F}$.*

Proof. By Proposition 2.9, we may assume that $\omega(Y)>\omega$. First we need the following observation.

CLAIM. *$Y=\lim\mathcal{S}_Y$, where $\mathcal{S}_Y=\{Y_\alpha,q^\beta_\alpha,A\}$ is a factorizing ω-spectrum consisting of locally compact separable metrizable spaces Y_α and proper projections $p^\beta_\alpha\colon Y_\beta\to Y_\alpha$, $\beta\ge\alpha$.*

Proof of Claim. Let $\widetilde{Y}$ be the one-point compactification of Y. Represent $\widetilde{Y}$ as the limit of a factorizing ω-spectrum $\mathcal{S}_{\widetilde{Y}}=\{\widetilde{Y}_\alpha,\widetilde{q}^\beta_\alpha,B\}$, consisting of metrizable compact spaces $\widetilde{Y}_\alpha$ and surjective projections. Since Y is locally compact and Lindelöf, it follows that Y is functionally open in $\widetilde{Y}$. Since $\mathcal{S}_{\widetilde{Y}}$ is a factorizing spectrum, there exists an index $\alpha_0 \in B$ such that $Y = \widetilde{q}_{\alpha_0}^{-1}(\widetilde{q}_{\alpha_0}(Y))$. Let $A = \{\alpha \in B : \alpha \geq \alpha_0\}$, $Y_\alpha = \widetilde{q}_\alpha(Y)$, and $q_\alpha^\beta = \widetilde{q}_\alpha^\beta | Y_\beta$, $\beta \geq \alpha$, $\alpha, \beta \in A$. Then the limit of the spectrum $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ coincides with Y. Moreover, each Y_α, $\alpha \in A$, is locally compact (as an open subspace of $\widetilde{Y}_\alpha$) and each $q_\alpha^\beta \colon Y_\beta \to Y_\alpha$ is proper (as the restriction of $\widetilde{q}_\alpha^\beta$ onto the inverse image $Y_\beta = (\widetilde{q}_\alpha^\beta)^{-1}(Y_\alpha)$). Finally, by [2, Corollary 1.3.2], $\mathcal{S}_Y$ is a factorizing ω-spectrum. This proves our Claim.

Since X is closed in Y it is the limit of the induced spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$, where $X_\alpha = q_\alpha(X)$ and $p_\alpha^\beta = q_\alpha^\beta | X_\beta$, $\beta \geq \alpha$, $\alpha, \beta \in A$. Note that $\mathcal{S}_X$ is also an ω-spectrum. It is factorizing since so is $\mathcal{S}_Y$ and X is C-embedded in Y.

By [2, Theorems 1.3.4 and 1.3.10], we may assume without loss of generality that each Y_α in the spectrum $\mathcal{S}_Y$ is at most n-dimensional. Again by the Spectral Theorem [2, Theorem 1.3.4], we may assume (if necessary passing to a cofinal and ω-complete subset of A) that for each $\alpha \in A$ there is a map $f_\alpha \colon X_\alpha \to Y_\alpha$ such that $q_\alpha \circ f = f_\alpha \circ p_\alpha$. Since f is fixed-point free and X is Lindelöf, we can find a countable functionally open cover $\{G_i : i \in \omega\}$ of X and a countable collection $\{U_i : \in \omega\}$ of functionally open subsets of Y such that $f(G_i) \subset U_i$ and $U_i \cap G_i = \emptyset$ for each $i \in \omega$. Factorizability of the spectra $\mathcal{S}_X$ and $\mathcal{S}_Y$ guarantees (see [2, Proposition 1.3.1]) the existence of an index $\alpha_i \in A$ such that $G_i = p_{\alpha_i}^{-1}(p_{\alpha_i}(G_i))$ and $U_i = (q_{\alpha_i})^{-1}(q_{\alpha_i}(U_i))$, $i \in \omega$. Choose $\beta \in A$ so that $\beta \geq \alpha_i$ for each $i \in \omega$—this is possible because A is an ω-complete set (see [2, Corollary 1.1.28])—and note that $G_i = p_\beta^{-1}(p_\beta(G_i))$ and $U_i = (q_\beta)^{-1}(q_\beta(U_i))$ for each $i \in \omega$. Obviously, $p_\beta(G_i) \cap q_\beta(U_i) = \emptyset$, $i \in \omega$. We claim that $f_\beta \colon X_\beta \to Y_\beta$ is fixed-point free. Assuming the opposite, pick a point $x \in X_\beta$ with $f_\beta(x) = x$, choose $y \in X$ so that $p_\beta(y) = x$ and pick an index $i \in \omega$ such that $y \in G_i = p_\beta^{-1}(p_\beta(G_i))$. Note that $f(y) \in U_i$ and $q_\beta(f(y)) \in q_\beta(U_i)$. But $q_\beta(f(y)) = f_\beta(p_\beta(y)) = f_\beta(x) = x \in p_\beta(G_i)$, which is impossible since $p_\beta(G_i) \cap q_\beta(U_i) = \emptyset$.

By Proposition 2.9, f_β is colorable in $n+3$ colors and there exists a closed cover $\{F_1, \ldots, F_{n+3}\}$ of X_β such that $\mathrm{cl}_{Y_\beta}(f_\beta(F_j)) \cap F_j = \emptyset$ for each $j = 1, \ldots, n+3$. It only remains to note that $\{p_\beta^{-1}(F_1), \ldots, p_\beta^{-1}(F_{n+3})\}$ is a closed cover of X and $f(p_\beta^{-1}(F_j)) \cap p_\beta^{-1}(F_j) = \emptyset$ for each $j = 1, \ldots, n+3$. ■

Lemma 3.2. *Let X be a closed subspace of a locally compact paracompact space Y and $f : X \to Y$ continuous. Then Y can be written as $\bigoplus\{Y_\alpha : \alpha$ isolated, $\alpha \leq \tau\}$, where Y_α is Lindelöf and $f(X \cap Y_\alpha) \subset \bigcup_{\beta \leq \alpha} Y_\beta$.*

Proof. By Theorem 5.1.27 in [4], Y can be written as $\bigoplus_{\alpha < \kappa} Z_\alpha$, where each Z_α is Lindelöf. Suppose for every isolated $\beta < \alpha$, where α is isolated,

Y_β is defined and the following hold:

(1) If Y_β meets Z_γ then Y_β contains Z_γ.
(2) $f(Y_\beta) \subset \bigcup_{\gamma\leq\beta} Y_\gamma$.
(3) Y_β is clopen in Y.
(4) Y_β is Lindelöf.

If $\bigcup_{\beta<\alpha} Y_\beta$ covers all of Y then put $\tau = \alpha$ if α is limit and $\tau = \lambda$ if $\alpha = \lambda + 1$ (notice that α cannot be 0 if Y is not empty). The family $\bigoplus\{Y_\beta : \beta \leq \tau, \beta \text{ isolated}\}$ is a desired sum by properties (1)–(4). Otherwise, we define the next piece as follows.

Construction of Y_α: Let λ_0 be an ordinal such that Z_{λ_0} does not meet $\bigcup_{\beta<\alpha} Y_\beta$. Put $S_0 = \{\lambda_0\}$ and, recursively, $S_{n+1} = \{\lambda : Z_\lambda \text{ meets } f(Z_\gamma), \gamma \in S_n\}$. Since Z_γ is Lindelöf for all γ, $f(Z_\gamma)$ can meet only countably many summands in our original free sum. Therefore, S_n is countable. Put $Y_\alpha = \bigcup\{Z_\gamma : \gamma \in \bigcup_n S_n,\ Z_\gamma \text{ misses } \bigcup_{\beta<\alpha} Y_\beta\}$. Inductive requirements (1) and (2) are explicitly incorporated in the construction. Requirement (3) is met because Y_α is the union of a discrete subfamily of clopen sets. Requirement (4) follows from the Lindelöf property of every Z_γ and countability of S_n. ∎

Lemma 3.3. *Let* $Y = \bigoplus\{Y_\alpha : \alpha \leq \tau,\ \alpha \text{ isolated}\}$, *where each* Y_α *is locally compact and Lindelöf,* $\dim Y \leq n$, *and let* X *be closed in* Y. *Suppose* $f : X \to Y$ *is continuous and the following hold:*

(∗) X *misses* Y_0;
(∗∗) $f(X \cap Y_\alpha) \subset \bigcup_{\beta<\alpha} Y_\beta$ *if* $\alpha \neq 0$.

Then there exists a coloring $\mathcal{F}$ *of* f *such that* $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ *for every* $F \in \mathcal{F}$.

Proof. Clearly, the free sum in the hypothesis is a favorable representation of Y. Let us show that the conditions of Lemma 2.4 are met. Requirement (1) of Lemma 2.4 is met by (∗); and (2) is met by (∗∗) and the fact that Y_α misses $\bigcup_{\beta<\alpha} Y_\beta$, thanks to our free sum representation. ∎

Theorem 3.4. *Any fixed-point free map* $f : X \to Y$ *defined on a closed subspace* X *of a locally compact and paracompact space* Y *with* $\dim Y \leq n$ *is colorable. In addition, there exists a coloring* $\mathcal{F}$ *of* f *such that* $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ *for every* $F \in \mathcal{F}$.

Proof. Represent Y as $\bigoplus\{Y_\alpha : \alpha \leq \tau,\ \alpha \text{ isolated}\}$ as in the conclusion of Lemma 3.2. For each isolated $\alpha \leq \tau$ not equal to 0, put $Z_\alpha = f^{-1}(\bigoplus_{\beta<\alpha} Y_\beta) \cap Y_\alpha$. Observe that each Z_α is clopen in X, being the inverse image of the clopen set $\bigoplus_{\beta<\alpha} Y_\beta$ intersected with the clopen set Y_α. Put $Z = \bigoplus_{0<\alpha\leq\tau} Z_\alpha$. The triple $\{Z, f|_Z, Y\}$ satisfies the hypothesis of Lemma 3.3. Therefore we can find a finite cover $\mathcal{F}_Z$ of Z by colors such that $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ for every $F \in \mathcal{F}_Z$.

For each isolated $\alpha \le \tau$, put $P_\alpha = (X \cap Y_\alpha) \setminus Z_\alpha$. Since Z_α is open, P_α is closed in Y_α. Since $f(Y_\alpha) \subset \bigcup_{\beta\le\alpha} Y_\beta$ and P_α is outside of Z_α, we conclude that $f(P_\alpha) \subset Y_\alpha$. For each isolated $\alpha \le \tau$ the triple $\{Y_\alpha, P_\alpha, f|_{P_\alpha}\}$ meets the requirement of Proposition 3.1. Therefore, we can fix an $(n+3)$-sized family $\mathcal{F}_\alpha = \{F_1^\alpha, \ldots, F_{n+3}^\alpha\}$ which is a coloring for $f|_{P_\alpha}$ and $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ for every $F \in \mathcal{F}_\alpha$. Put $F_i = \bigoplus\{F_i^\alpha : \alpha \le \tau\}$. Each F_i is closed as the union of a discrete family of closed sets. Since $f(F_i^\alpha) \subset Y_\alpha$, we conclude that $\mathrm{cl}_Y(f(F_i^\alpha))$ misses F_i^β if $\alpha \ne \beta$. By the choice of $\mathcal{F}_\alpha$, $\mathrm{cl}_Y(f(F_i^\alpha))$ misses F_i^α. Therefore, $\mathrm{cl}_Y(f(F_i)) \cap F_i = \emptyset$. Thus, $\mathcal{F}_P = \{F_i : i \le n+3\}$ covers $\bigoplus_\alpha P_\alpha$ by colors and $\mathrm{cl}_Y(f(F)) \cap F = \emptyset$ for every $F \in \mathcal{F}_P$. Thus, $\mathcal{F}_Z \cup \mathcal{F}_P$ is a desired coloring of f. ■

The existence of a special coloring in Theorem 3.4 implies that $\widetilde{f} : \beta X \to \beta Y$ is fixed-point free as well. Therefore, we have the following equivalence statement.

THEOREM 3.5. *Let $f\colon X \to Y$ be a continuous map defined on a closed subspace X of a locally compact and paracompact space Y of dimension $\dim Y \le n$. Then f is fixed-point free iff $\widetilde{f} : \beta X \to \beta Y$ is fixed-point free.*

We finish this section by giving an estimate for the chromatic number of maps as in the hypothesis of Theorem 3.4.

THEOREM 3.6. *Any continuous fixed-point free map $f\colon X \to Y$ defined on a closed subspace X of a locally compact and paracompact space Y with $\dim Y \le n$ is colorable in at most $n+3$ colors.*

Proof. By Theorem 3.5, $\widetilde{f}$ has no fixed points. Consequently, by Proposition 3.1, $\widetilde{f}$ is colorable in at most $n+3$ colors, which implies that the chromatic number of f also does not exceed $n+3$. ■

4. Examples and comments. As mentioned in the Introduction, Mazur constructed a continuous fixed-point free non-colorable map of the space of irrationals into itself. The space P of irrational numbers is the first element in the hierarchy $\{N_n^{2n+1} : n \in \omega\}$ of n-dimensional universal Nöbeling spaces (i.e. $P = N_0^1$).

PROPOSITION 4.1. *Let $n \in \omega$. There exists a continuous fixed-point free non-colorable self-map of the n-dimensional universal Nöbeling space N_n^{2n+1}.*

Proof. As noted, for $n = 0$ this is known. Let $n > 0$ and $f\colon N_0^1 \to N_0^1$ be a continuous fixed-point free non-colorable map. Embed N_0^1 into N_n^{2n+1} as a closed subspace. By Lemma 2.8, since $\dim N_n^{2n+1} = n$ and $N_n^{2n+1} \in AE(n)$, we can embed N_0^1 into the product $N_n^{2n+1} \times [0,\infty)$ as a closed subspace in such a way that $f = g|N_0^1$, where $g\colon N_n^{2n+1} \times [0,\infty) \to N_n^{2n+1} \times [0,\infty)$ is a fixed-point free map. Let $p\colon N_n^{2n+1} \to N_n^{2n+1} \times [0,\infty)$ be an n-soft map. Choose a section $i\colon N^1_0 \to N^{2n+1}_n$ of $p|p^{-1}(N^1_0)$, i.e. $pi = \mathrm{id}_{N^1_0}$, and let $A = i(N^1_0)$. Consider the map $j\colon A \to A$ defined by $j = i \circ f \circ p|A$ and note that $p \circ j = p \circ i \circ f \circ p|A = f \circ p|A = g \circ p|A$. Since p is n-soft, there exists a map $r\colon N^{2n+1}_n \to N^{2n+1}_n$ such that $p \circ r = g \circ p$ and $r|A = j$. The former implies that r is fixed-point free. The latter implies that r is not colorable (otherwise j would be colorable; but since j is equivalent to f, this is impossible). ∎

Proposition 4.2. *There exists a continuous fixed-point free non-colorable self-map of the separable Hilbert space $\ell_2 \simeq \mathbb{R}^\omega$.*

Proof. Let

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}\colon \|x\| = 1/(n+1)\} \times \{0_i\}_{i\geq n+2} \subset \mathbb{R}^\omega.$$

The map $f(\mathbf{x}) = -\mathbf{x}$ is a fixed-point free self-map of $\mathbb{P} = \mathbb{R}^\omega \setminus \{\mathbf{0}\}$ and coincides with the antipodal map on each $\mathbb{S}^n$. According to van Douwen's observation, $\widetilde{g}\colon \beta\mathbb{S} \to \beta\mathbb{S}$, where $\mathbb{S} = \bigcup \mathbb{S}^n$ and $g = f|\mathbb{S}$, fixes a point. Since $\mathbb{S}$ is closed in $\mathbb{P}$, it follows that $\widetilde{f}$ fixes a point, which means that f is not colorable. Since ℓ_2 is homeomorphic to $\mathbb{P}$, the conclusion follows. ∎

Proposition 4.3. *Let $f\colon X \to X$ be a continuous self-map of a finite-dimensional locally compact paracompact space and $\widetilde{f}\colon \beta X \to \beta X$ be its extension. Then* $\mathrm{Fix}(\widetilde{f}) = \mathrm{cl}_{\beta X}\,\mathrm{Fix}(f)$.

Proof. Obviously $\mathrm{cl}_{\beta X}\,\mathrm{Fix}(f) \subset \mathrm{Fix}(\widetilde{f})$. Suppose that there is $p \in \mathrm{Fix}(\widetilde{f}) \setminus \mathrm{cl}_{\beta X}\,\mathrm{Fix}(f)$. Choose a functionally closed set $Z \subset X$ such that $p \in \mathrm{cl}_{\beta X} Z$ and $Z \cap \mathrm{Fix}(f) = \emptyset$. Let $\varphi\colon X \to [0,\infty)$ be a function such that $\varphi^{-1}(0) = Z$. Consider the map $g\colon X \times [0,\infty) \to X \times [0,\infty)$ defined by letting $g(x,r) = (f(x), r + \varphi(x))$, $(x,r) \in X \times [0,\infty)$, and note that g has no fixed points. The product $X \times [0,\infty)$ is still finite-dimensional, locally compact and paracompact. Therefore, by Theorem 3.5, $\widetilde{g}\colon \beta(X \times [0,\infty)) \to \beta(X \times [0,\infty))$ has no fixed points. On the other hand, identifying Z with $Z \times \{0\}$, we see that $f|Z = g|Z$ and consequently $\widetilde{g}(p) = \widetilde{f}(p) = p$. This contradiction completes the proof. ∎

The next observation shows that the assumption of closedness of X (in Y) in Theorem 3.4 (and in Propositions 2.9 and 3.1) cannot be dropped.

Proposition 4.4. *There exists a continuous non-colorable fixed-point free map $g\colon X \to Y$ defined on an open subspace X of a zero-dimensional metrizable compactum Y.*

Proof. Let $f\colon P \to P$ be a continuous non-colorable fixed-point free self-map of the space P of irrational numbers. Since the extension of f onto the Stone–Čech compactification βP has fixed points, there exist a zero-dimensional metrizable compactification Y of P and a map $\widetilde{f}\colon Y \to Y$ such that $\widetilde{f}|P = f$ and $\mathrm{Fix}(\widetilde{f}) \neq \emptyset$. Set $X = \widetilde{f}^{-1}(\mathrm{Fix}(\widetilde{f}))$ and $g = \widetilde{f}|X$. ∎

The following statement can be considered as a converse (among Polish spaces) of [6, Theorem 3.3].

PROPOSITION 4.5. *Let X be a zero-dimensional Polish space. Then the following conditions are equivalent:*

(1) *Every continuous fixed-point free self-map of X is colorable.*
(2) *X is σ-compact.*

Proof. (1)⇒(2). Denote by $LC(X)$ the union of all open compact subsets of X. Let $X^0 = LC(X)$, $X^{\alpha+1} = X^\alpha \cup LC(X \setminus X^\alpha)$ and $X^\alpha = \bigcup\{X^\beta\colon \beta < \alpha\}$ for a limit ordinal. Note that $\{X^\alpha\colon \alpha < \omega_1\}$ is an increasing union of open subsets of X and consequently $X^\beta = X^\alpha$ for some $\beta < \omega_1$ and all $\alpha > \beta$. Clearly $Y = X \setminus X^\beta$ has no points of local compactness. Note also that X^β is σ-compact. Assuming that X is not σ-compact, it follows that $Y \neq \emptyset$. Since Y is completely metrizable (as a closed subspace of X) and zero-dimensional, we conclude that Y is homeomorphic to the space of irrational numbers. Let $g\colon Y \to Y$ be a continuous non-colorable fixed-point free map (Mazur's example) and $r\colon X \to Y$ be a retraction. Then the composition $f = g \circ r\colon X \to X$ is a non-colorable fixed-point free map, contradicting (1).

(2)⇒(1). This follows from [6]. ∎

Our main theorem for locally compact Lindelöf spaces of finite dimension and the Krawczyk–Steprāns result for zero-dimensional σ-compact spaces [6, Theorem 3.3] motivate the following question.

QUESTION 4.6. *Let X be a σ-compact space of finite dimension. Is every continuous fixed-point free self-map on X colorable?*

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R. Z. Buzyakova, A. Chigogidze
Department of Mathematics and Statistics
The University of North Carolina at Greensboro
Greensboro, NC 27402, U.S.A.
E-mail: rzbouzia@uncg.edu
chigogidze@uncg.edu

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