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Oligomorphic transformation monoids and homomorphism-homogeneous structures

by

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Abstract. A structure is called homomorphism-homogeneous if every homomorphism between finitely generated substructures of the structure extends to an endomorphism of the structure (P. J. Cameron and J. Nešetřil, 2006). In this paper we introduce oligomorphic transformation monoids in full analogy to oligomorphic permutation groups and use this notion to propose a solution to a problem, posed by Cameron and Nešetřil in 2006, to characterize endomorphism monoids of homomorphism-homogeneous relational structures over finite signatures.

However, the main goal of this paper is to provide more evidence that the concept of homomorphism-homogeneity is analogous to that of ultrahomogeneity. It turns out that many results that hold for ultrahomogeneous or ω-categorical structures have their analogues in the class of countable homomorphism-homogeneous structures, or countable weakly oligomorphic structures (these are structures whose endomorphism monoids are oligomorphic). For example, we characterize countable weakly oligomorphic structures in terms of the Ryll-Nardzewski property with respect to positive formulas; we prove that for countable weakly oligomorphic structures homomorphism-homogeneity is equivalent to quantifier elimination for positive formulas; finally, we prove that an ω-categorical structure is both ultrahomogeneous and homomorphism-homogeneous if and only if it has quantifier elimination where positive formulas reduce to positive quantifier-free formulas.

1. Introduction. Oligomorphic permutation groups have been introduced in the 1970s by Peter Cameron and have played a major role in understanding countable structures ever since (see [3]). A permutation group $G \leq \mathrm{Sym}(A)$ is *oligomorphic* if for every $n \in \mathbb{N}$, the number of orbits in the action of G on A^n is finite. In one of its many incarnations, the well known Ryll-Nardzewski theorem states that the following are equivalent for a countable structure $\mathcal{A}$ (see [3, 5]):

(1) $\mathcal{A}$ is *ω-categorical* (that is, if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ for some countable structure $\mathcal{B}$ over the same signature as $\mathcal{A}$, then $\mathcal{A} \cong \mathcal{B}$);

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(2) $\mathcal{A}$ has the *Ryll-Nardzewski property* (that is, for each $n \in \mathbb{N}$ there exist only finitely many $\mathcal{R}$-formulas in variables $x_1, \dots, x_n$ which are pairwise inequivalent in $\mathcal{A}$);
(3) $\mathrm{Aut}(\mathcal{A})$ is an oligomorphic permutation group.

Oligomorphic permutation groups are closely related to ultrahomogeneous structures and quantifier elimination. Recall that a structure $\mathcal{A}$ is *ultrahomogeneous* if every isomorphism between two finitely generated substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$, and that $\mathcal{A}$ *has quantifier elimination* if every formula is equivalent in $\mathcal{A}$ to a quantifier-free formula. The following facts are well-known (see [3]):

- if $\mathcal{A}$ is an ultrahomogeneous countable structure over a finite relational signature, then $\mathrm{Aut}(\mathcal{A})$ is an oligomorphic permutation group;
- if $\mathcal{A}$ is an ω-categorical structure, then $\mathcal{A}$ is ultrahomogeneous if and only if $\mathcal{A}$ has quantifier elimination.

In their recent paper [4] Cameron and Nešetřil discuss a generalization of homogeneity to various types of morphisms between structures, and in particular introduce the notion of homomorphism-homogeneous structures:

DEFINITION 1.1 (Cameron and Nešetřil [4]). A structure is called *homomorphism-homogeneous* if every homomorphism between finitely generated substructures of the structure extends to an endomorphism of the structure.

It was demonstrated in [4] that the notion of homomorphism-homogeneity parallels that of ultrahomogeneity in many aspects, although there are significant differences. For example, every countable chain $(L, \leq)$ is homomorphism-homogeneous, but not neccesarily ultrahomogeneous.

As for the similarities, assume that A is an infinite set endowed with the discrete topology. Then A^A is endowed with the Tikhonov product topology where the closure operator $X \mapsto \overline{X}$ has a straightforward algebraic description: for $X \subseteq A^A$ we know that $f \in \overline{X}$ if and only if for every nonempty finite subset $B \subseteq A$ there is a $g \in X$ such that $f|_B = g|_B$. As usual, we say that $X \subseteq A^A$ is closed if $X = \overline{X}$. The following representation theorem was proved in [4]:

THEOREM 1.2 ([4]). *Let A be a an infinite set and let $M \subseteq A^A$ be a transformation monoid. The following are equivalent:*

(1) *M is closed in A^A;*
(2) *$M = \mathrm{End}(\mathcal{A})$ for some homomorphism-homogeneous relational structure $\mathcal{A}$ on A.*

The paper [4] ends with the following question:

> "It would be interesting to recognise the monoids which are the endomorphism monoids [...] of [homomorphism-]homogeneous structures with only finitely many relations of each arity (these would be the analogue of the closed oligomorphic permutation groups [...])."

In this paper we introduce oligomorphic transformation monoids in full analogy to oligomorphic permutation groups and, inter alia, propose a solution to the above problem (Theorem 2.7): we show that M is an endomorphism monoid of a homomorphism-homogeneous relational structure over a *residually finite* relational signature (Definition 2.4) if and only if M is closed and oligomorphic.

However, the main goal of this paper is to provide even more evidence that the concept of homomorphism-homogeneity is analogous to that of ultrahomogeneity. It turns out that many results that hold for ultrahomogeneous or ω-categorical structures have their analogues in the class of countable homomorphism-homogeneous structures, or countable structures whose endomorphism monoids are oligomorphic. Certain results such as the characterization of countable structures whose endomorphism monoids are oligomorphic in terms of the Ryll-Nardzewski property (Section 3) can almost be obtained by syntactically replacing certain notions with their analogues, as in Table 1.

Table 1. Analogous notions

ultrahomogeneous structure	$\leftrightarrow$	homomorphism-homogeneous structure
automorphism group	$\leftrightarrow$	endomorphism monoid
oligomorphic permutation group	$\leftrightarrow$	oligomorphic transformation monoid
quantifier elimination	$\leftrightarrow$	quantifier elimination for positive formulas

In Section 2, after an overview of some standard terminology, we introduce oligomorphic transformation monoids, residually finite signatures and characterize endomorphism monoids of homomorphism-homogeneous relational structures over residually finite signatures. Section 3 is devoted to the characterization of countable structures whose endomorphism monoids are oligomorphic in terms of the Ryll-Nardzewski property. Another characterization of structures whose endomorphism monoids are oligomorphic can be found in Section 4 where we prove that such structures are definitionally equivalent to homomorphism-homogeneous structures over residually finite signatures. The paper concludes with Section 5 which describes homomorphism-homogeneous structures in terms of quantifier elimination.

We prove that for countable structures whose endomorphism monoids are oligomorphic, homomorphism-homogeneity is equivalent to quantifier elimination for positive formulas. At the very end of the paper, under the assumption that the structure is ω-categorical, we prove that the structure is both ultrahomogeneous and homomorphism-homogeneous if and only if it has quantifier elimination where positive formulas reduce to positive quantifier-free formulas.

Remark. Note that by positive formulas we understand formulas that some authors refer to as *positive existential* formulas; that is, formulas built using existential quantification along with positive logical connectives $\wedge$ and $\vee$; in particular, $\top$ and $\bot$ are positive formulas.

2. Preliminaries and motivation. Let $\mathcal{R} = (R_i)_{i\in I}$ be a relational signature, and let $r_i \geq 1$ denote the arity of the relational symbol R_i, $i \in I$. By $\mathcal{R}_k$ we denote the relational signature $(R_j)_{j\in J}$ where $J = \{i \in I : r_i = k\}$. That is, $\mathcal{R}_k$ consists of all k-ary relational symbols from $\mathcal{R}$. For an $\mathcal{R}$-formula $\varphi(x_1, \ldots, x_n)$, an $\mathcal{R}$-structure $\mathcal{A} = \langle A, (R_i^{\mathcal{A}})_{i\in I}\rangle$ and some $\boldsymbol{a} = (a_1, \ldots, a_n) \in A^n$, by $\mathcal{A} \models \varphi[\boldsymbol{a}]$ we denote that the formula φ is satisfied in $\mathcal{A}$ under the valuation $\{x_1 := a_1, \ldots, x_n := a_n\}$. Let $\varphi^{\mathcal{A}} = \{\boldsymbol{a} \in A^n : \mathcal{A} \models \varphi[\boldsymbol{a}]\}$. A relation $\rho \subseteq A^h$ is *definable by a formula in an $\mathcal{R}$-structure $\mathcal{A}$* if there is an $\mathcal{R}$-formula $\varphi(x_1, \ldots, x_h)$ such that $\rho = \varphi^{\mathcal{A}}$. We also say that φ defines ρ in $\mathcal{A}$. Formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(y_1, \ldots, y_n)$ are *equivalent in $\mathcal{A}$*, in symbols $\varphi \equiv_{\mathcal{A}} \psi$, if $\mathcal{A} \models \forall \boldsymbol{x}\ (\varphi(\boldsymbol{x}) \Leftrightarrow \psi(\boldsymbol{x}))$, or, equivalently, if $\varphi^{\mathcal{A}} = \psi^{\mathcal{A}}$.

A mapping $f : A \to B$ from an $\mathcal{R}$-structure $\mathcal{A} = \langle A, (R_i^{\mathcal{A}})_{i\in I}\rangle$ into an $\mathcal{R}$-structure $\mathcal{B} = \langle B, (R_i^{\mathcal{B}})_{i\in I}\rangle$ *preserves a formula* $\varphi(x_1, \ldots, x_k)$ if the following holds for every $\boldsymbol{a} \in A^k$: if $\mathcal{A} \models \varphi[\boldsymbol{a}]$ then $\mathcal{B} \models \varphi[f(\boldsymbol{a})]$, where $f(\boldsymbol{a})$ denotes the tuple $(f(a_1), \ldots, f(a_k))$. Clearly, a mapping f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if and only if f preserves all positive $\mathcal{R}$-formulas.

A mapping $f : A \to A$ *preserves* a finitary relation $\rho \subseteq A^h$ if $f(\boldsymbol{a}) \in \rho$ for all $\boldsymbol{a} \in \rho$. We also say that ρ is *invariant* under f. A finitary relation ρ is *invariant* under a transformation monoid M if ρ is invariant under every $f \in M$. If a relation ρ is definable by a formula φ in a structure $\mathcal{A}$, then f preserves ρ if and only if f preserves φ.

The set of all endomorphisms of a relational structure $\mathcal{A}$ will be denoted by $\mathrm{End}(\mathcal{A})$, and the set of all automorphisms of $\mathcal{A}$ by $\mathrm{Aut}(\mathcal{A})$.

A transformation monoid $M \subseteq A^A$ acts on A in a natural way: we let $a^m = m(a)$ for all $m \in M$ and all $a \in A$. Analogously, a transformation monoid $M \subseteq A^A$ acts on A^n if $(a_1, \ldots, a_n)^m = (m(a_1), \ldots, m(a_n))$ for all $m \in M$ and all $(a_1, \ldots, a_n) \in A^n$. The set $\boldsymbol{a}^M = \{\boldsymbol{a}^m : m \in M\}$ is called the *trace* of $\boldsymbol{a} \in A^n$ under the action of M. For every $n \geq 1$ we define an equivalence relation $\sim_M$ on A^n by

$$\boldsymbol{a} \sim_M \boldsymbol{b} \quad \text{if and only if} \quad \boldsymbol{a}^M = \boldsymbol{b}^M.$$

The equivalence classes of $\sim_M$ will be referred to as *orbits* of the action of M on A^n, $n \geq 1$.

DEFINITION 2.1. A transformation monoid $M \subseteq A^A$ is *oligomorphic* if for every $n \in \mathbb{N}$, there are only finitely many orbits of the action of M on A^n.

In the case of group actions the notions of trace and orbit coincide. This is not the case for actions of transformation monoids. However, the following lemma shows that in the above definition we can use traces instead of orbits.

LEMMA 2.2. *A transformation monoid $M \subseteq A^A$ is oligomorphic if and only if for every $n \in \mathbb{N}$, there are only finitely many traces of the action of M on A^n. In particular, if A is finite, every transformation monoid $M \subseteq A^A$ is oligomorphic.*

Proof. ($\Rightarrow$) Every trace is a union of some orbits, so if there are k orbits in the action of M on A^n then there are at most $2^k - 1$ traces in the action of M on A^n (since every trace is a nonempty set of tuples).

($\Leftarrow$) Fix a positive integer n and let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be the set of traces of the action of $M = \mathrm{End}(\mathcal{A})$ on A^n. For each $j \in \{1, \ldots, k\}$, let $O_j = T_j \setminus \bigcup\{T \in \mathcal{T} : T \subsetneq T_j\}$. Since there are finitely many O_j's, it suffices to show that O_j's are the orbits of the action of M on A^n. Clearly, each O_j is nonempty (if $T_j = \boldsymbol{a}_j^M$ then $\boldsymbol{a}_j \in O_j$) and $O_1 \cup \cdots \cup O_k = A^n$. In order to complete the proof, it therefore suffices to show that each O_j is an orbit of the action of M on A^n.

Take any $j \in \{1, \ldots, k\}$ and let $\boldsymbol{a}, \boldsymbol{b} \in O_j$. Since $\boldsymbol{a}^M$ is a trace in the action of M and $\boldsymbol{a} \in T_j$ it follows that $\boldsymbol{a}^M = T_j$ or $\boldsymbol{a}^M \subsetneq T_j$. But $\boldsymbol{a}^M \subsetneq T_j$ implies that $\boldsymbol{a} \notin O_j$, which is not true. Therefore, $\boldsymbol{a}^M = T_j$. By the same argument, $\boldsymbol{b}^M = T_j$, so $\boldsymbol{a}^M = \boldsymbol{b}^M$. This shows that each O_j is contained in some orbit of the action of M. Conversely, consider an orbit $\boldsymbol{a}/{\sim_M}$ and let $\boldsymbol{a} \in O_j$. The same argument yields $\boldsymbol{a}/{\sim_M} \subseteq O_j$. This completes the proof that each O_j is an orbit of the action of M on A^n. ∎

It is a well-known fact that if $\mathcal{A}$ is an ultrahomogeneous structure over a finite relational signature, then $\mathrm{Aut}(\mathcal{A})$ is an oligomorphic permutation group. One can easily imitate the proof of that fact to show the following.

PROPOSITION 2.3. *If $\mathcal{A}$ is a homomorphism-homogeneous structure over a finite relational signature, then* $\mathrm{End}(\mathcal{A})$ *is an oligomorphic transformation monoid.*

The requirement that the signature of $\mathcal{A}$ be finite is needed to ensure that for every $n \in \mathbb{N}$ there are, up to isomorphism, only finitely many n-element substructures of $\mathcal{A}$. The same effect can be achieved by countable signatures provided that every symbol $R \in \mathcal{R}$ of large arity is equivalent in a small substructure of $\mathcal{A}$ to a relation symbol of sufficiently small arity. We shall now present a formal definition and prove the corresponding generalization of Proposition 2.3.

For each pair of integers m, k such that $m \geq k$ and each surjective mapping $f : \{1, \ldots, m\} \to \{1, \ldots, k\}$ fix a right inverse f^* of f, that is, a mapping $f^* : \{1, \ldots, k\} \to \{1, \ldots, m\}$ satisfying $f \circ f^* = \mathrm{id}$. For example, one possibility might be to take $f^*(y) = \min\{x : f(x) = y\}$.

Definition 2.4. We say that a signature $\mathcal{R}$ is *residually finite* in an $\mathcal{R}$-structure $\mathcal{A}$ if

- $\mathcal{R}_k$ is finite for every $k \in \mathbb{N}$,
- for every $n \in \mathbb{N}$ there exists an $l > n$ such that for every $m \geq l$, every $k \leq n$, every relation symbol $R \in \mathcal{R}_m$ and every surjective mapping $f : \{1, \ldots, m\} \to \{1, \ldots, k\}$ there is a relation symbol $R^f \in \mathcal{R}_k$ such that

$$(RF) \qquad \mathcal{A} \models \forall x_1 \ldots \forall x_m \Big(\Big(\bigwedge_{(i,j) \in \ker f} x_i = x_j \Big) \Rightarrow \big(R^f(x_{f^*(1)}, \ldots, x_{f^*(k)}) \Leftrightarrow R(x_1, \ldots, x_m) \big) \Big)$$

Note that every finite relational signature $\mathcal{R}$ is residually finite in every $\mathcal{R}$-structure. On the other hand, if $\mathcal{R}$ is residually finite in some $\mathcal{R}$-structure, then $\mathcal{R}$ is countable.

Lemma 2.5. *Let $\mathcal{R}$ be a countable relational signature and let $\mathcal{A}$ be an $\mathcal{R}$-structure such that the signature $\mathcal{R}$ is residually finite in $\mathcal{A}$. Then for every $n \in \mathbb{N}$ there are, up to isomorphism, only finitely many n-element substructures of $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an $\mathcal{R}$-structure with universe A such that $\mathcal{R}$ is residually finite in $\mathcal{A}$. Fix any $n \in \mathbb{N}$. Then there exists an $l > n$ such that for every $m \geq l$, every $k \leq n$, every relation symbol $R \in \mathcal{R}_m$ and every surjective mapping $f : \{1, \ldots, m\} \to \{1, \ldots, k\}$ there is a relation symbol $R^f \in \mathcal{R}_k$ such that (RF) holds. Clearly, $\mathcal{R}' = \bigcup_{j<l} \mathcal{R}_j$ is finite. Therefore, if we can show that every n-element substructure of $\mathcal{A}$ is uniquely determined by its $\mathcal{R}'$-reduct, then the fact that $\mathcal{R}'$ is finite implies that there are, up to isomorphism, only finitely many n-element substructures of $\mathcal{A}$.

We are going to show that every n-element substructure of $\mathcal{A}$ is uniquely determined by its $\mathcal{R}'$-reduct by showing that for every $R \in \mathcal{R}$ with $\mathrm{ar}(R) \geq l$ there exists an $\mathcal{R}'$-formula φ_R such that R is equivalent to φ_R in every n-element substructure of $\mathcal{A}$. The idea is now straightforward: since $m = \mathrm{ar}(R) > n$, every m-tuple of elements from an n-element set has at most n distinct entries, so all there is to know about R in an n-element structure is already contained in the relation symbols R^f whose arities are at most n. Formally, take an $R \in \mathcal{R}$ such that $m = \mathrm{ar}(R) \geq l$ and let

$$\varphi_R(x_1, \ldots, x_m) = \bigvee_{k=1}^{n} \bigvee_{f \in S_k^m} \Big(R^f(x_{f^*(1)}, \ldots, x_{f^*(k)}) \wedge \bigwedge_{(i,j) \in \ker f} x_i = x_j \Big),$$

where S_k^m is the set of all surjective mappings $\{1, \ldots, m\} \to \{1, \ldots, k\}$. Then it easily follows from (RF) that $\varphi_R \equiv_{\mathcal{B}} R$ for every n-element substructure $\mathcal{B}$ of $\mathcal{A}$. ∎

THEOREM 2.6. *Let $\mathcal{R}$ be a countable relational signature and let $\mathcal{A}$ be a homomorphism-homogeneous $\mathcal{R}$-structure such that the signature $\mathcal{R}$ is residually finite in $\mathcal{A}$. Then* $\mathrm{End}(\mathcal{A})$ *is an oligomorphic transformation monoid.*

Proof. Let $M = \mathrm{End}(\mathcal{A})$. Clearly, it suffices to show that for each n, the number of traces $(a_1, \ldots, a_n)^M$ where all a_i's are distinct is finite (see Lemma 2.2).

Let $(a_1, \ldots, a_n)^M$ and $(b_1, \ldots, b_n)^M$ be two distinct traces of $\mathrm{End}(\mathcal{A})$ such that all a_i's are distinct, and all b_i's are distinct. Without loss of generality we can assume that $(b_1, \ldots, b_n)^M \nsubseteq (a_1, \ldots, a_n)^M$. Let us show that $\langle a_1, \ldots, a_n \rangle_{\mathcal{A}} \ncong \langle b_1, \ldots, b_n \rangle_{\mathcal{A}}$, where $\langle a_1, \ldots, a_n \rangle_{\mathcal{A}}$ denotes the substructure of $\mathcal{A}$ generated by $\{a_1, \ldots, a_n\}$. Suppose to the contrary that $\langle a_1, \ldots, a_n \rangle_{\mathcal{A}} \cong \langle b_1, \ldots, b_n \rangle_{\mathcal{A}}$ and that $f : a_i \mapsto b_i$ is an isomorphism between the two substructures. Then, by the homomorphism-homogeneity of $\mathcal{A}$, f extends to an endomorphism $f^* \in M$. Then $(b_1, \ldots, b_n) \in (a_1, \ldots, a_n)^M$, whence it follows that $(b_1, \ldots, b_n)^M \subseteq (a_1, \ldots, a_n)^M$. Contradiction.

Therefore, the number of traces $(a_1, \ldots, a_n)^M$ of the action of M on A^n where all a_i's are distinct is not greater than $n!$ times the number of n-element nonisomorphic substructures of $\mathcal{A}$, which is finite by Lemma 2.5. ∎

The following theorem characterizes transformation monoids of homomorphism-homogeneous relational structures over residually finite signatures providing, thus, an answer to the problem posed in [4] which was quoted in the Introduction.

THEOREM 2.7. *Let A be a an infinite set and let $M \subseteq A^A$ be a transformation monoid. The following are equivalent:*

(1) *M is closed in A^A and oligomorphic;*
(2) *$M = \mathrm{End}(\mathcal{A})$ for some homomorphism-homogeneous $\mathcal{R}$-structure $\mathcal{A}$ on A, where $\mathcal{R}$ is a countable relational signature which is residually finite in $\mathcal{A}$.*

Proof. (2)⇒(1). Follows from Theorems 1.2 and 2.6.

(1)⇒(2). Since M is oligomorphic, for each $n \in \mathbb{N}$ there are only finitely many distinct traces

$$\boldsymbol{a}^M_{n,1},\ \boldsymbol{a}^M_{n,2},\ \ldots,\ \boldsymbol{a}^M_{n,q_n}$$

of the action of M on A^n. Let $\mathcal{S} = (S_{n,j})_{n\in\mathbb{N},\,1\le j\le q_n}$ be a relational signature such that $\mathrm{ar}(S_{n,j}) = n$ and let $\mathcal{A}$ be an $\mathcal{S}$-structure on A where $S^{\mathcal{A}}_{n,j} = \boldsymbol{a}^M_{n,j}$ for all $n \in \mathbb{N}$ and $1 \le j \le q_n$.

Let us show that $M = \mathrm{End}(\mathcal{A})$. Clearly, $M \subseteq \mathrm{End}(\mathcal{A})$. In order to show that the other inclusion holds, it suffices to show that $\mathrm{End}(\mathcal{A}) \subseteq \overline{M}$ since M is closed. Take any $f \in \mathrm{End}(\mathcal{A})$ and any $B = \{a_1,\ldots,a_n\}$. Since $(a_1,\ldots,a_n)^M$ is one of the q_n traces of the action of M on A^n and since $f \in \mathrm{End}(\mathcal{A})$, we find that f preserves $(a_1,\ldots,a_n)^M$, that is, $(f(a_1),\ldots,f(a_n)) \in (a_1,\ldots,a_n)^M$. Therefore, there is a $g \in M$ such that $(f(a_1),\ldots,f(a_n)) = (g(a_1),\ldots,g(a_n))$. This means that $f|_B = g|_B$, which concludes the proof that $f \in \overline{M}$.

Next, let us show that $\mathcal{A}$ is a homomorphism-homogeneous structure. Let $f : \langle a_1,\ldots,a_n\rangle_{\mathcal{A}} \to \langle b_1,\ldots,b_n\rangle_{\mathcal{A}}$ be a homomorphism between two finitely generated substructures of $\mathcal{A}$. Since $(a_1,\ldots,a_n)^M$ is an interpretation of some relational symbol from $\mathcal{S}$ in $\mathcal{A}$ and since f is a homomorphism from $\langle a_1,\ldots,a_n\rangle_{\mathcal{A}}$ to $\langle b_1,\ldots,b_n\rangle_{\mathcal{A}}$, it follows that $f(\rho|_{\{a_1,\ldots,a_n\}}) \subseteq \rho|_{\{b_1,\ldots,b_n\}}$. In particular, $(f(a_1),\ldots,f(a_n)) \in \rho|_{\{b_1,\ldots,b_n\}} \subseteq \rho = (a_1,\ldots,a_n)^M$. So, there is a $g \in M$ such that $(f(a_1),\ldots,f(a_n)) = (g(a_1),\ldots,g(a_n))$. This g is then an endomorphism of $\mathcal{A}$ which extends f.

Finally, let us show that $\mathcal{S}$ is residually finite in $\mathcal{A}$. Clearly, $\mathcal{S}_n$ is finite for all $n \in \mathbb{N}$. Let us show that for every $n \in \mathbb{N}$ there exists an $l > n$ such that for every $m \ge l$, every $k \le n$, every relation symbol $S \in \mathcal{S}_m$ and every surjective mapping $f : \{1,\ldots,m\} \to \{1,\ldots,k\}$ there is a relation symbol $S^f \in \mathcal{S}_k$ such that (RF) holds. Take any $n \in \mathbb{N}$ and let $l = n+1$. Now, take any $m \ge n+1$, any $k \le n$, a relation symbol $S_{m,i} \in \mathcal{S}_m$ and a surjective mapping $f : \{1,\ldots,m\} \to \{1,\ldots,k\}$. Then $(a_{f(1)},\ldots,a_{f(m)})^M$ is a trace of the action of M on A^k, so $(a_{f(1)},\ldots,a_{f(m)})^M = \boldsymbol{a}^M_{k,j}$ for some j. Therefore, in (RF) we can take $S^f_{m,i}$ to be $S_{k,j}$. ∎

If A is finite, then every transformation monoid $M \subseteq A^A$ is closed and oligomorphic, so Theorem 2.7 implies that every transformation monoid on a finite set is an endomorphism monoid of a homomorphism-homogeneous structure. A careful analysis of the above proof reveals that more is true:

THEOREM 2.8. *Let A be a finite set. For every transformation monoid $M \subseteq A^A$ there is a finite relational signature $\mathcal{S}$ and a homomorphism-homogeneous $\mathcal{S}$-structure $\mathcal{A}$ on A such that $M = \mathrm{End}(\mathcal{A})$.*

Proof. Let A be a finite set and $|A| = m$. For each $n \in \mathbb{N}$ there are only finitely many distinct traces

$$\boldsymbol{a}^M_{n,1},\ \boldsymbol{a}^M_{n,2},\ \ldots,\ \boldsymbol{a}^M_{n,q_n}$$

of the action of M on A^n. Moreover, if $n > m$, the traces of the action of M on A^n are none other than the traces of the action of M on A^m with some entries duplicated. Therefore, it suffices to consider only the traces of the action of M on A^n where $n \leq m$.

Let $\mathcal{S} = (S_{n,j})_{1\leq n\leq m, 1\leq j\leq q_n}$ be a finite relational signature such that $\operatorname{ar}(S_{n,j}) = n$ and let $\mathcal{A}$ be an $\mathcal{S}$-structure on A where $S_{n,j}^{\mathcal{A}} = \boldsymbol{a}_{n,j}^M$ for all $1 \leq n \leq m$ and $1 \leq j \leq q_n$. It is now easy to show that $M = \operatorname{End}(\mathcal{A})$ and that $\mathcal{A}$ is a homomorphism-homogeneous structure. ∎

We introduce the notion of weakly oligomorphic structures in analogy to oligomorphic structures:

Definition 2.9. A relational structure $\mathcal{A}$ is *weakly oligomorphic* if $\operatorname{End}(\mathcal{A})$ is an oligomorphic transformation monoid.

Clearly, every relational structure on a finite universe is weakly oligomorphic. The following lemma justifies the terminology:

Lemma 2.10. *Every relational structure with oligomorphic automorphism group is weakly oligomorphic.*

Proof. Let $\mathcal{A}$ be an $\mathcal{R}$-structure such that $G = \operatorname{Aut}(\mathcal{A})$ is an oligomorphic permutation group. Let us show that $M = \operatorname{End}(\mathcal{A})$ is an oligomorphic transformation monoid by showing that for every $n \in \mathbb{N}$ there are only finitely many traces of the action of M on A^n (Lemma 2.2). Fix an $n \in \mathbb{N}$. Clearly, $G \subseteq M$, so $\boldsymbol{a}^G \subseteq \boldsymbol{a}^M$ for every $\boldsymbol{a} \in A^n$. Therefore, every trace of the action of M on A^n is a union of some orbits of the action of G on A^n. Since there are finitely many orbits in the action of G on A^n, there can be only finitely many traces in the action of M on A^n. ∎

In general, the converse of the above lemma is not true:

Example 2.11. Let $\mathcal{N} = (\mathbb{N}, \leq)$ be the chain of integers. It is easy to see that every chain is homomorphism-homogeneous (the proof can be found in [6]). Therefore, $\mathcal{N}$ is homomorphism-homogeneous, so $\operatorname{End}(\mathcal{N})$ is an oligomorphic transformation monoid by Theorem 2.6. On the other hand, $\operatorname{Aut}(\mathcal{N}) = \{\mathrm{id}\}$, so $\operatorname{Aut}(\mathcal{N})$ is not an oligomorphic permutation group.

3. The Ryll-Nardzewski property. In this section we present a characterization of first order theories of countable weakly oligomorphic structures in terms of an analogue of the Ryll-Nardzewski property. Let $\mathcal{R}$ be a relational signature and T a first order theory in signature $\mathcal{R}$. Formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(y_1, \ldots, y_n)$ are *equivalent in* T, in symbols $\varphi \equiv_T \psi$, if $T \models \forall \boldsymbol{x}\ (\varphi(\boldsymbol{x}) \Leftrightarrow \psi(\boldsymbol{x}))$. We say that T *has the positive Ryll-Nardzewski property* if for each $n \in \mathbb{N}$ there exist only finitely many positive $\mathcal{R}$-formulas in variables $x_1, \ldots, x_n$ which are pairwise inequivalent in T. If $\mathcal{A}$ is an $\mathcal{R}$-

structure, we say that $\mathcal{A}$ *has the positive Ryll-Nardzewski property* if the first-order theory of $\mathcal{A}$, $\mathrm{Th}(\mathcal{A})$, has the positive Ryll-Nardzewski property.

The *complete positive n-type* of an n-tuple $\boldsymbol{a} \in A^n$ in an $\mathcal{R}$-structure $\mathcal{A}$ is the set of all positive formulas that are satisfied by $\boldsymbol{a}$ in $\mathcal{A}$:

$$\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) = \{\varphi(x_1, \ldots, x_n) : \varphi \text{ is a positive } \mathcal{R}\text{-formula and } \mathcal{A} \models \varphi[\boldsymbol{a}]\}.$$

We say that an $\mathcal{R}$-structure $\mathcal{A}$ *realizes* a set $\Phi(\boldsymbol{x})$ of positive formulas if there exists an n-tuple $\boldsymbol{a} \in A^n$ such that $\Phi \subseteq \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a})$. A *complete positive type of a complete theory* T is a complete positive type of a tuple of some model of T. The set of all complete positive n-types of a theory T will be denoted by $S^+_n(T)$.

A set $\Phi(\boldsymbol{x})$ of formulas is *principal* with respect to T if there is a formula $\psi(\boldsymbol{x}) \in \Phi(\boldsymbol{x})$ such that $T \models \forall \boldsymbol{x}\ (\psi(\boldsymbol{x}) \Rightarrow \bigwedge \Phi(\boldsymbol{x}))$. We then say that Φ is *generated* by ψ. A first-order formula $\chi(\boldsymbol{x})$ is a *characteristic formula for* a set $\Phi(\boldsymbol{x})$ of positive formulas with respect to T if $T \models \forall \boldsymbol{x}\ (\chi(\boldsymbol{x}) \Leftrightarrow \bigwedge \Phi(\boldsymbol{x}) \wedge \bigwedge \Phi^{\neg}(\boldsymbol{x}))$, where $\Phi^{\neg}(\boldsymbol{x}) = \{\neg\psi(\boldsymbol{x}) : \psi(\boldsymbol{x}) \text{ is a positive formula and } \psi(\boldsymbol{x}) \notin \Phi(\boldsymbol{x})\}$.

LEMMA 3.1. *Let T be a first-order theory and let $\chi(\boldsymbol{x})$ be a characteristic formula for a set of positive formulas $\Phi(\boldsymbol{x})$ with respect to T. Then for an arbitrary model $\mathcal{A}$ of T and an arbitrary tuple $\boldsymbol{a}$ of elements of A, $\mathcal{A} \models \chi[\boldsymbol{a}]$ if and only if $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) = \Phi$.*

A set Φ of positive formulas is principal (resp. has a characteristic formula) in an $\mathcal{R}$-structure $\mathcal{A}$ if it is principal (resp. has a characteristic formula) with respect to $\mathrm{Th}(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{R}$-structures. Since homomorphisms preserve positive formulas, for every $k \in \mathbb{N}$, every $\boldsymbol{a} \in A^k$ and every homomorphism $f : \mathcal{A} \to \mathcal{B}$ we have $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{B}}(f(\boldsymbol{a}))$. Consequently, if $\boldsymbol{a} \in A^k$ and $\boldsymbol{b} \in B^k$ are k-tuples such that $f(\boldsymbol{a}) = \boldsymbol{b}$ and $g(\boldsymbol{b}) = \boldsymbol{a}$ for some homomorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ then $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) = \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b})$. The converse holds under additional assumptions (note that $(\boldsymbol{a}, c)$ is shorthand for $(a_1, \ldots, a_n, c)$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$):

LEMMA 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable $\mathcal{R}$-structures such that every complete positive type in $\mathcal{A}$ is principal and every complete positive type in $\mathcal{B}$ is principal.*

(1) *If $\boldsymbol{a} \in A^n$, $\boldsymbol{b} \in B^n$ and $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b})$, then for every $c \in A$ there exists a $d \in B$ such that $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b}, d)$.*
(2) *Let $\boldsymbol{a} \in A^n$ and $\boldsymbol{b} \in B^n$ satisfy $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b})$. Then there exists a homomorphism $f : \mathcal{A} \to \mathcal{B}$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$.*

Proof. (1) As $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c)$ is a complete positive type in $\mathcal{A}$, it is principal by assumption so it is generated by a formula $\theta(\boldsymbol{x}, y) \in \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c)$. Clearly,

$\mathcal{A} \models \theta[\boldsymbol{a}, c]$, whence $\mathcal{A} \models (\exists y\ \theta(\boldsymbol{x}, y))[\boldsymbol{a}]$. Therefore,

$$\exists y\ \theta(\boldsymbol{x}, y) \in \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b}),$$

so $\mathcal{B} \models (\exists y\ \theta(\boldsymbol{x}, y))[\boldsymbol{b}]$. Then there is a $d \in B$ such that $\mathcal{B} \models \theta[\boldsymbol{b}, d]$. Let us show that $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b}, d)$. Take any $\varphi(\boldsymbol{x}, y) \in \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c)$. Since $\theta(\boldsymbol{x}, y)$ generates $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, c)$, we have that $\mathcal{A} \models \forall \boldsymbol{x}, y\ (\theta(\boldsymbol{x}, y) \Rightarrow \varphi(\boldsymbol{x}, y))$. Since $\mathcal{B} \models \theta[\boldsymbol{b}, d]$, we have $\mathcal{B} \models \varphi[\boldsymbol{b}, d]$ and thus $\varphi \in \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b}, d)$.

(2) Let $a_{n+1}, a_{n+2}, \ldots$ be the list of all the elements of $\mathcal{A}$ that do not appear in $\boldsymbol{a}$. We know from (1) that there is a sequence $b_{n+1}, b_{n+2}, \ldots \in B$ such that for each $k \in \mathbb{N}$ we have $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}, a_{n+1}, \ldots, a_{n+k}) \subseteq \mathrm{tp}^+_{\mathcal{B}}(\boldsymbol{b}, b_{n+1}, \ldots, b_{n+k})$. Therefore, the mapping $f : A \to B$ defined by $f(\boldsymbol{a}) = \boldsymbol{b}$ and $f(a_i) = b_i$ for all $i > n$ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ which takes $\boldsymbol{a}$ to $\boldsymbol{b}$. ∎

COROLLARY 3.3. *Let $\mathcal{A}$ be a countable $\mathcal{R}$-structure such that every complete positive type in $\mathcal{A}$ is principal. Let $M = \mathrm{End}(\mathcal{A})$ and let $\boldsymbol{a}, \boldsymbol{b} \in A^n$ be n-tuples on A. Then $\boldsymbol{a} \sim_M \boldsymbol{b}$ if and only if $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) = \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$.*

Proof. The conclusion follows directly from Lemma 3.2. It just suffices to note that $\boldsymbol{a}^M = \boldsymbol{b}^M$ if and only if there exist $f, g \in M$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$ and $g(\boldsymbol{b}) = \boldsymbol{a}$. ∎

LEMMA 3.4. *Let $\mathcal{A}$ be a weakly oligomorphic $\mathcal{R}$-structure. Then:*

(1) *$\mathcal{A}$ realizes only finitely many complete positive n-types for each $n \in \mathbb{N}$.*
(2) *$\mathcal{A}$ has the positive Ryll-Nardzewski property.*
(3) *Every complete positive type in $\mathcal{A}$ is principal.*

Proof. (1) Fix a positive integer n and let $\mathcal{O} = \{O_1, \ldots, O_k\}$ be the set of orbits of the action of $M = \mathrm{End}(\mathcal{A})$ on A^n. Clearly, if $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the same orbit O_j then there exist $f, g \in M$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$ and $f(\boldsymbol{b}) = \boldsymbol{a}$, so $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) = \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$. Therefore, for each j all the tuples in O_j realize the same complete positive n-type and hence $\mathcal{A}$ realizes only finitely many complete positive n-types for each $n \in \mathbb{N}$.

(2) Let us show that $\mathcal{A}$ has the positive Ryll-Nardzewski property. Take any $n \in \mathbb{N}$ and assume that $\mathcal{A}$ realizes k complete positive n-types, say, $\Phi_1, \ldots, \Phi_k$. It is easy to see that if $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are positive formulas, then $\varphi \equiv_{\mathcal{A}} \psi$ if and only if for every complete positive n-type $\Phi_i(x_1, \ldots, x_n)$ in $\mathcal{A}$, either $\{\varphi, \psi\} \subseteq \Phi_i$ or $\{\varphi, \psi\} \cap \Phi_i = \emptyset$. Therefore, there are at most 2^k positive formulas in variables $x_1, \ldots, x_n$ which are pairwise inequivalent in $\mathcal{A}$.

(3) Finally, let us show that every complete positive type in $\mathcal{A}$ is principal. Fix an $n \in \mathbb{N}$ and let $\Phi(x_1, \ldots, x_n)$ be a complete positive n-type in $\mathcal{A}$. Since $\mathcal{A}$ has the positive Ryll-Nardzewski property, there are only finitely many positive $\mathcal{R}$-formulas in variables $x_1, \ldots, x_n$ which are pairwise inequiv-

alent in $\mathcal{A}$, say, $\theta_1, \dots, \theta_k$. Let $\{\theta_1, \dots, \theta_k\} \cap \Phi = \{\theta_1, \dots, \theta_s\}$, $s \le k$. Then Φ is generated by $\theta_1 \wedge \dots \wedge \theta_s \in \Phi$. ■

THEOREM 3.5. *Let T be a complete theory over a signature $\mathcal{R}$ and assume that T has infinite models. The following are equivalent:*

(1) *Every countable model of T is weakly oligomorphic.*
(2) *There exists a countable model of T which is weakly oligomorphic.*
(3) *T has the positive Ryll-Nardzewski property.*
(4) *For every $n \ge 1$, every type in $S_n^+(T)$ has a characteristic formula.*
(5) *For every $n \ge 1$, $S_n^+(T)$ is finite.*

Proof. (1)⇒(2). Trivial.

(2)⇒(3). Let $\mathcal{A}$ be a countable model of T which is weakly oligomorphic. Then $\mathcal{A}$ has the positive Ryll-Nardzewski property by Lemma 3.4. On the other hand, $T = \mathrm{Th}(\mathcal{A})$ since T is complete. Therefore, T has the positive Ryll-Nardzewski property.

(3)⇒(4). Fix an $n \in \mathbb{N}$ and let $\Phi(x_1, \dots, x_n)$ be a complete positive n-type of T. Since T has the positive Ryll-Nardzevski property, there are only finitely many positive $\mathcal{R}$-formulas in variables $x_1, \dots, x_n$ which are pairwise inequivalent in T, say, $\theta_1, \dots, \theta_k$. Let $\{\theta_1, \dots, \theta_k\} \cap \Phi = \{\theta_1, \dots, \theta_s\}$, $s \le k$. Then $\theta_1 \wedge \dots \wedge \theta_s \wedge \neg\theta_{s+1} \wedge \dots \wedge \neg\theta_k$ is a characteristic formula of Φ.

(4)⇒(5). Take any $n \in \mathbb{N}$, let $S_n^+(T) = \{\Phi_i(\boldsymbol{x}) : i \in I\}$ and let $\chi_i(\boldsymbol{x})$ be a characteristic formula for $\Phi_i(\boldsymbol{x})$, $i \in I$. Let $\boldsymbol{c}$ be an n-tuple of new constants. Then $T \cup \{\neg\chi_i(\boldsymbol{c}) : i \in I\}$ is not consistent. Due to compactness of first-order logic, there is a finite set $\{i_1, \dots, i_m\} \subseteq I$ such that $T \cup \{\neg\chi_{i_1}(\boldsymbol{c}), \dots, \neg\chi_{i_m}(\boldsymbol{c})\}$ is not consistent. Therefore, $T \models \chi_{i_1}(\boldsymbol{c}) \vee \dots \vee \chi_{i_m}(\boldsymbol{c})$ since T is complete, so $T \models \forall \boldsymbol{x}\ (\chi_{i_1}(\boldsymbol{x}) \vee \dots \vee \chi_{i_m}(\boldsymbol{x}))$. Hence, for every model $\mathcal{A}$ of T and every n-tuple $\boldsymbol{a} \in A^n$, $\mathcal{A} \models (\chi_{i_1} \vee \dots \vee \chi_{i_m})[\boldsymbol{a}]$. Since χ_{i_j} are characteristic formulas of complete positive types, Lemma 3.1 implies that $S_n^+(T) \subseteq \{\Phi_{i_1}, \dots, \Phi_{i_m}\}$.

(5)⇒(1). Let $\mathcal{A}$ be a countable model of T. Since $S_n^+(T)$ is finite for every $n \ge 1$, $\mathcal{A}$ realizes only finitely many complete positive n-types for every $n \ge 1$, so $\mathcal{A}$ has the Ryll-Nardzewski property. It follows, using the same argument as in the proof of Lemma 3.4(3), that every complete positive type in $\mathcal{A}$ is principal. ■

4. Representations by homomorphism-homogeneous structures. We have seen in Theorem 2.6 that homomorphism-homogeneous structures over residually finite signatures have oligomorphic transformation monoids. Although a countable structure with an oligomorphic transformation monoid need not be homomorphism-homogeneous (we treat this problem in more detail in Section 5), we prove that such a structure is always definitionally equivalent to a homomorphism-homogeneous structure over a residually finite signature.

Let $\mathcal{R} = (R_i)_{i\in I}$ and $\mathcal{S} = (S_j)_{j\in J}$ be relational signatures, let $\mathcal{A}$ be an $\mathcal{R}$-structure, and let $\mathcal{B}$ be an $\mathcal{S}$-structure, both over the same universe A. A *positive translation of $\mathcal{A}$ into the signature of $\mathcal{B}$* is a mapping $(-)^\tau$ from $\{R_i : i \in I\}$ to the set of all positive $\mathcal{S}$-formulas such that:

- for all $i \in I$, $R_i^\tau = R_i^\tau(x_1, \dots, x_{r_i})$;
- for every $i \in I$ and every $\boldsymbol{a} \in A^{r_i}$, $\mathcal{A} \models R_i[\boldsymbol{a}]$ if and only if $\mathcal{B} \models R_i^\tau[\boldsymbol{a}]$.

We say that $\mathcal{A}$ and $\mathcal{B}$ over the same universe A are *positively definitionally equivalent* if there exists a positive translation of $\mathcal{A}$ into the signature of $\mathcal{B}$, and a positive translation of $\mathcal{B}$ into the signature of $\mathcal{A}$.

LEMMA 4.1. *Let $\mathcal{A}$ be a countable weakly oligomorphic $\mathcal{R}$-structure on A, and let $\mathcal{F}$ be a set of positive $\mathcal{R}$-formulas such that*

- *$x = y \in \mathcal{F}$ for all variables x and y;*
- *if $\varphi, \psi \in \mathcal{F}$ then $\varphi \wedge \psi \in \mathcal{F}$ and $\varphi \vee \psi \in \mathcal{F}$;*
- *if $g : \{a_1, \dots, a_h\} \to \{b_1, \dots, b_h\}$ defined by $g(a_i) = b_i$, $1 \le i \le h$, cannot be extended to an endomorphism of $\mathcal{A}$, where $a_1, \dots, a_h$ are distinct elements of A and $b_1, \dots, b_h$ are arbitrary elements of A, then there is a formula $\psi(x_1, \dots, x_h) \in \mathcal{F}$ such that $\mathcal{A} \models \psi[a_1, \dots, a_h]$ and $\mathcal{A} \not\models \psi[b_1, \dots, b_h]$.*

Then for every $h \ge 1$ and every relation $\rho \subseteq A^h$, ρ can be defined in $\mathcal{A}$ by a formula from $\mathcal{F}$ if and only if ρ is invariant under $\mathrm{End}(\mathcal{A})$.

Proof. ($\Rightarrow$) Let P be a positive formula from $\mathcal{F}$ which defines ρ in $\mathcal{A}$. Since every endomorphism of $\mathcal{A}$ preserves every positive $\mathcal{R}$-formula, it follows that ρ is invariant under $\mathrm{End}(\mathcal{A})$.

($\Leftarrow$) Let $M = \mathrm{End}(\mathcal{A})$ and let Q be the set of all finitary relations on A that are definable in $\mathcal{A}$ by positive formulas from $\mathcal{F}$. Let us first show that every trace in the action of M on A^h lies in Q.

Let $\boldsymbol{a} = (a_1, \dots, a_h)$ and assume, for a start, that all the entries of $\boldsymbol{a}$ are distinct. Consider

$$\gamma = \bigcap \{\eta \in Q : \mathrm{ar}(\eta) = h \text{ and } \boldsymbol{a} \in \eta\}.$$

According to Lemma 3.4, for each $n \in \mathbb{N}$, there exist only finitely many positive $\mathcal{R}$-formulas in variables $x_1, \dots, x_n$ which are pairwise inequivalent in $\mathcal{A}$. Therefore, there are finitely many h-ary relations $\eta_1, \dots, \eta_s \in Q$ that contain $\boldsymbol{a}$. Each η_i is definable in $\mathcal{A}$ by a positive formula $H_i(x_1, \dots, x_h) \in \mathcal{F}$, $1 \le i \le s$, so

$$\Sigma(x_1, \dots, x_h) = H_1(x_1, \dots, x_h) \wedge \dots \wedge H_s(x_1, \dots, x_h)$$

is a positive formula from $\mathcal{F}$ that defines γ in $\mathcal{A}$. We are going to show that $\boldsymbol{a}^M \in Q$ by provading that $\boldsymbol{a}^M = \gamma$.

Let us first show that $\boldsymbol{a}^M \subseteq \gamma$. Take any $\eta \in Q$ such that $\boldsymbol{a} \in \eta$ and let $H(x_1, \dots, x_h)$ be a positive formula from $\mathcal{F}$ that defines η in $\mathcal{A}$. Since every $f \in M$ preserves H, it follows that $f(\boldsymbol{a}) \in \eta$ for all $f \in M$. Therefore, $\boldsymbol{a}^M \subseteq \eta$ whenever $\boldsymbol{a} \in \eta \in Q$, so $\boldsymbol{a}^M \subseteq \bigcap\{\eta \in Q : \mathrm{ar}(\eta) = h \text{ and } \boldsymbol{a} \in \eta\} = \gamma$.

Assume now that $\boldsymbol{a}^M \subsetneq \gamma$, take any $\boldsymbol{b} = (b_1, \dots, b_h) \in \gamma \setminus \boldsymbol{a}^M$ and define a mapping $g : \{a_1, \dots, a_h\} \to \{b_1, \dots, b_h\}$ by $g(a_i) = b_i$, $1 \le i \le h$. Since $\boldsymbol{b} \notin \boldsymbol{a}^M$ it follows that g cannot be extended to an endomorphism of $\mathcal{A}$. Then, according to the assumptions about $\mathcal{F}$, there is a positive formula $\Psi(x_1, \dots, x_h) \in \mathcal{F}$ such that $\mathcal{A} \models \Psi[\boldsymbol{a}]$ and $\mathcal{A} \not\models \Psi[\boldsymbol{b}]$. Consider now the following positive $\mathcal{R}$-formula:

$$\Delta(x_1, \dots, x_h) = \Sigma(x_1, \dots, x_h) \wedge \Psi(x_1, \dots, x_h).$$

Let $\delta = \Delta^{\mathcal{A}}$. From $\mathcal{A} \models \Psi[\boldsymbol{a}]$ and $\mathcal{A} \models \Sigma[\boldsymbol{a}]$ it follows that $\mathcal{A} \models \Delta[\boldsymbol{a}]$. Therefore, $\boldsymbol{a} \in \delta$ and $\delta \in Q$, so the definition of γ ensures that $\gamma \subseteq \delta$. Since $\boldsymbol{b} \in \gamma \subseteq \delta$, we see that $\mathcal{A} \models \Delta[\boldsymbol{b}]$, so $\mathcal{A} \models \Psi[\boldsymbol{b}]$. Contradiction. Hence, Σ defines $\boldsymbol{a}^M$ in $\mathcal{A}$.

As the next step, let $\boldsymbol{a} = (a_1, \dots, a_h)$ where not all the entries of $\boldsymbol{a}$ are distinct. Let $\ker \boldsymbol{a} = \{(i,j) : a_i = a_j\}$ and let $q = |\{1, \dots, h\}/\ker \boldsymbol{a}|$ denote the number of equivalence classes of $\ker \boldsymbol{a}$. Let $\{1, \dots, h\}/\ker \boldsymbol{a} = \{S_1, \dots, S_q\}$, and let $g : \{1, \dots, q\} \to \{1, \dots, h\}$ be a mapping satisfying $g(i) \in S_i$, $1 \le i \le q$. Then $\boldsymbol{c} = (a_{g(1)}, \dots, a_{g(q)})$ is a tuple with distinct entries, so by the argument above, $\boldsymbol{c}^M$ can be defined in $\mathcal{A}$ by a positive formula from $\mathcal{F}$, say by $\Theta(x_1, \dots, x_q) \in \mathcal{F}$. Then it is easy to see that the following positive formula belongs to $\mathcal{F}$ and defines $\boldsymbol{a}^M$ in $\mathcal{A}$:

$$\Sigma(x_1, \dots, x_h) = \Theta(x_{g(1)}, \dots, x_{g(q)}) \quad \wedge \bigwedge_{(i,j) \in \ker \boldsymbol{a}} x_i = x_j.$$

Finally, let $\rho \subseteq A^h$ be a relation on A with $h \ge 1$, and assume that ρ is invariant under M. For every $\boldsymbol{r} \in \rho$ let $\Sigma_{\boldsymbol{r}}(x_1, \dots, x_h)$ be a positive formula from $\mathcal{F}$ that defines $\boldsymbol{r}^M$ in $\mathcal{A}$. Since ρ is invariant under M, it follows that $\rho = \bigcup\{\boldsymbol{r}^M : \boldsymbol{r} \in \rho\}$. Therefore,

$$P(x_1, \dots, x_h) = \bigvee_{\boldsymbol{r} \in \rho} \Sigma_{\boldsymbol{r}}(x_1, \dots, x_h)$$

is a possibly infinitary positive formula that defines ρ in $\mathcal{A}$. Since for each $n \in \mathbb{N}$ there exist only finitely many positive $\mathcal{R}$-formulas in variables $x_1, \dots, x_n$ which are pairwise inequivalent in $\mathcal{A}$, there exist $\boldsymbol{r}_1, \dots, \boldsymbol{r}_k \in \rho$ such that

$$\bigvee_{\boldsymbol{r} \in \rho} \Sigma_{\boldsymbol{r}}(\boldsymbol{x}) \equiv_{\mathcal{A}} \Sigma_{\boldsymbol{r}_1}(\boldsymbol{x}) \vee \dots \vee \Sigma_{\boldsymbol{r}_k}(\boldsymbol{x}).$$

Therefore, $\Sigma_{r_1}(\boldsymbol{x}) \vee \cdots \vee \Sigma_{r_k}(\boldsymbol{x})$ is a positive formula from $\mathcal{F}$ that defines ρ in $\mathcal{A}$. ∎

Let $\mathcal{L}_{\infty\omega}(\mathcal{R})$ denote the set of all infinitary $\mathcal{R}$-formulas where conjunctions and disjunctions are allowed over arbitrarily large sets of formulas, and let $\mathcal{L}^{+}_{\infty\omega}(\mathcal{R})$ denote the positive fragment of $\mathcal{L}_{\infty\omega}(\mathcal{R})$. Note that formulas from $\mathcal{L}_{\infty\omega}(\mathcal{R})$ may have infinite length, but they are required to have finite depth: they can be constructed from atomic formulas in finitely many steps involving quantification, negation and infinitary conjunction and disjunction.

Theorem 4.2. *Let $\mathcal{A}$ be a countable weakly oligomorphic $\mathcal{R}$-structure on A.*

(1) *For every $h \geq 1$ and every relation $\rho \subseteq A^h$, ρ can be defined in $\mathcal{A}$ by a positive $\mathcal{R}$-formula if and only if ρ is invariant under* $\mathrm{End}(\mathcal{A})$.
(2) *For every formula $\lambda(\boldsymbol{x}) \in \mathcal{L}^{+}_{\infty\omega}(\mathcal{R})$ there exists a positive $\mathcal{R}$-formula $\varphi(\boldsymbol{x})$ such that $\lambda(\boldsymbol{x}) \equiv_{\mathcal{A}} \varphi(\boldsymbol{x})$ (that is, $\mathcal{L}^{+}_{\infty\omega}(\mathcal{R})$ has the same expressive power in $\mathcal{A}$ as the set of all positive $\mathcal{R}$-formulas).*

Proof. (1) Let $\mathcal{F}$ be the set of all positive $\mathcal{R}$-formulas and let us show that it meets the requirements from Lemma 4.1. The first two items are trivially satisfied. As for the third item, let $g : \{a_1, \ldots, a_h\} \to \{b_1, \ldots, b_h\}$ be a mapping defined by $g(a_i) = b_i$, $1 \leq i \leq h$, and assume that g cannot be extended to an endomorphism of $\mathcal{A}$. Since every complete positive type in $\mathcal{A}$ is principal (Lemma 3.4), Lemma 3.2 implies that $\mathrm{tp}^{+}_{\mathcal{A}}(a_1, \ldots, a_h) \nsubseteq \mathrm{tp}^{+}_{\mathcal{A}}(b_1, \ldots, b_h)$. Therefore, there is a positive formula $\psi(x_1, \ldots, x_h)$ such that $\mathcal{A} \models \psi[a_1, \ldots, a_h]$ and $\mathcal{A} \not\models \psi[b_1, \ldots, b_h]$.

(2) Clearly, it suffices to show that for every nonempty set $\Phi(\boldsymbol{x})$ of positive formulas there exist positive formulas $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ such that $\varphi(\boldsymbol{x}) \equiv_{\mathcal{A}} \bigwedge \Phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x}) \equiv_{\mathcal{A}} \bigvee \Phi(\boldsymbol{x})$. Let $\rho = (\bigwedge \Phi(\boldsymbol{x}))^{\mathcal{A}}$ and $\sigma = (\bigvee \Phi(\boldsymbol{x}))^{\mathcal{A}}$. It is easy to see that both ρ and σ are invariant under $\mathrm{End}(\mathcal{A})$, so, by (1) both ρ and σ can be defined in $\mathcal{A}$ by positive formulas, say, $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$, respectively. Then $\varphi(\boldsymbol{x}) \equiv_{\mathcal{A}} \bigwedge \Phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x}) \equiv_{\mathcal{A}} \bigvee \Phi(\boldsymbol{x})$. ∎

Note that the claim (1) in the above theorem is analogous to [2, Theorem 5]. See also [1].

As a direct consequence of Theorem 4.2 we obtain an analogue of the fact that two ω-categorical relational structures over the same universe are definitionally equivalent if and only if their automorphism groups coincide:

Corollary 4.3. *Let Ω_A be a class of all weakly oligomorphic relational structures of all signatures on a countable universe A. Then $\mathcal{A}_1, \mathcal{A}_2 \in \Omega_A$ are positively definitionally equivalent if and only if* $\mathrm{End}(\mathcal{A}_1) = \mathrm{End}(\mathcal{A}_2)$.

The following two corollaries are converses of Theorems 2.6 and 2.8, respectively.

COROLLARY 4.4. *Let $\mathcal{A}$ be a countable weakly oligomorphic relational structure with universe A. Then $\mathcal{A}$ is positively definitionally equivalent to a homomorphism-homogeneous structure $\mathcal{A}^*$ with universe A and over a countable relational signature which is residually finite in $\mathcal{A}^*$.*

Proof. Let $M = \mathrm{End}(\mathcal{A})$. According to Theorem 2.7, there is a homomorphism-homogeneous structure $\mathcal{A}^*$ with universe A and over a countable relational signature which is residually finite in $\mathcal{A}^*$ such that $M = \mathrm{End}(\mathcal{A}^*)$. The structures $\mathcal{A}$ and $\mathcal{A}^*$ are positively definitionally equivalent by Corollary 4.3. ∎

COROLLARY 4.5. *Let $\mathcal{A}$ be a finite relational structure with universe A. Then $\mathcal{A}$ is positively definitionally equivalent to a homomorphism-homogeneous structure $\mathcal{A}^*$ with universe A and over a finite relational signature.*

Proof. Let $M = \mathrm{End}(\mathcal{A})$. According to Theorem 2.8, there is a homomorphism-homogeneous structure $\mathcal{A}^*$ with universe A and over a finite relational signature such that $M = \mathrm{End}(\mathcal{A}^*)$. The structures $\mathcal{A}$ and $\mathcal{A}^*$ are positively definitionally equivalent by Corollary 4.3. ∎

5. Quantifier elimination. We say that an $\mathcal{R}$-structure $\mathcal{A}$ has *quantifier elimination (QE)* if for every formula φ there exists a quantifier-free formula φ^0 such that $\varphi \equiv_{\mathcal{A}} \varphi^0$. An $\mathcal{R}$-structure $\mathcal{A}$ has *quantifier elimination for positive formulas (QEPF)* if for every positive formula φ there exists a quantifier-free positive formula φ^0 such that $\varphi \equiv_{\mathcal{A}} \varphi^0$.

Let $\mathcal{A}$ be an $\mathcal{R}$-structure and let $\boldsymbol{a} \in A^n$. The *complete positive quantifier-free type* of $\boldsymbol{a}$ in $\mathcal{A}$ is the set of positive quantifier-free formulas that are satisfied by $\boldsymbol{a}$ in $\mathcal{A}$:

$$\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) = \{\varphi(x_1, \ldots, x_n) : \varphi \text{ is a positive quantifier-free } \mathcal{R}\text{-formula and } \mathcal{A} \models \varphi[\boldsymbol{a}]\}.$$

LEMMA 5.1. *Let $\mathcal{A}$ be a countable structure on a universe A.*

(1) *If $\mathcal{A}$ is homomorphism-homogeneous, then for every $n \in \mathbb{N}$ and every $\boldsymbol{a}, \boldsymbol{b} \in A^n$, if $\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{b})$ then $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$.*
(2) *Assume that every complete positive type in $\mathcal{A}$ is principal and that for every $n \in \mathbb{N}$ and every $\boldsymbol{a}, \boldsymbol{b} \in A^n$ the inclusion $\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{b})$ implies $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$. Then $\mathcal{A}$ is homomorphism-homogeneous.*
(3) *Assume that $\mathcal{A}$ has QEPF and that every complete positive type in $\mathcal{A}$ is principal. Then $\mathcal{A}$ is homomorphism-homogeneous.*

Proof. Note, first, that if A is finite then it is weakly oligomorphic, so every complete positive type in $\mathcal{A}$ is principal by Lemma 3.4.

(1) Obvious.

(2) Let f be a homomorphism from $\langle a_1, \ldots, a_h \rangle_{\mathcal{A}}$ to $\langle b_1, \ldots, b_h \rangle_{\mathcal{A}}$. Then $\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{b})$. The assumption now yields $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$, whence there is an endomorphism f^* of $\mathcal{A}$ such that $f^*(\boldsymbol{a}) = \boldsymbol{b}$ (Lemma 3.2).

(3) Since $\mathcal{A}$ has QEPF, it is easy to see that for every $n \geq 1$ and every $\boldsymbol{a}, \boldsymbol{b} \in A^n$ the inclusion $\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{b})$ implies $\mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{a}) \subseteq \mathrm{tp}^+_{\mathcal{A}}(\boldsymbol{b})$. Therefore, $\mathcal{A}$ is homomorphism-homogeneous according to (2). ∎

THEOREM 5.2. *Let $\mathcal{A}$ be a countable weakly oligomorphic homomorphism-homogeneous $\mathcal{R}$-structure on A. Then for every $h \geq 1$ and every $\rho \subseteq A^h$, ρ can be defined in $\mathcal{A}$ by a positive quantifier-free $\mathcal{R}$-formula if and only if ρ is invariant under* $\mathrm{End}(\mathcal{A})$.

Proof. Let $\mathcal{F}$ be the set of all positive quantifier-free $\mathcal{R}$-formulas and let us show that it meets the requirements from Lemma 4.1. The first two items are trivially satisfied. As for the third item, let $g : \{a_1, \ldots, a_h\} \to \{b_1, \ldots, b_h\}$ be a mapping defined by $g(a_i) = b_i$, $1 \leq i \leq h$, and assume that g cannot be extended to an endomorphism of $\mathcal{A}$. The structure $\mathcal{A}$ is homomorphism-homogeneous, so g cannot be a local homomorphism from $\langle a_1, \ldots, a_h \rangle_{\mathcal{A}}$ to $\langle b_1, \ldots, b_h \rangle_{\mathcal{A}}$. Therefore, $\mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{a}) \nsubseteq \mathrm{tp}^0_{\mathcal{A}}(\boldsymbol{b})$. In other words, there is a positive quantifier-free formula $\psi(x_1, \ldots, x_h)$ such that $\mathcal{A} \models \psi[\boldsymbol{a}]$ and $\mathcal{A} \not\models \psi[\boldsymbol{b}]$. ∎

THEOREM 5.3. *Let $\mathcal{A}$ be a countable $\mathcal{R}$-structure such that $\mathcal{R}$ is residually finite in $\mathcal{A}$. Then $\mathcal{A}$ is homomorphism-homogeneous if and only if $\mathcal{A}$ is weakly oligomorphic and has QEPF.*

Proof. ($\Rightarrow$) According to Theorem 2.6, $\mathrm{End}(\mathcal{A})$ is an oligomorphic transformation monoid, and Theorems 5.2 and 4.2 ensure that $\mathcal{A}$ has QEPF.

($\Leftarrow$) Since $\mathcal{A}$ is weakly oligomorphic, Lemma 3.4 ensures that every complete positive n-type in $\mathcal{A}$ is principal. Therefore, $\mathcal{A}$ is homomorphism-homogeneous by Lemma 5.1 since $\mathcal{A}$ has QEPF. ∎

THEOREM 5.4. *Let $\mathcal{A}$ be a countable weakly oligomorphic structure. Then $\mathcal{A}$ is homomorphism-homogeneous if and only if $\mathcal{A}$ has QEPF.*

In particular, if $\mathcal{A}$ is a finite structure, then $\mathcal{A}$ is homomorphism-homogeneous if and only if $\mathcal{A}$ has QEPF.

Proof. ($\Rightarrow$) Directly from Theorems 5.2 and 4.2.

($\Leftarrow$) Since $\mathcal{A}$ is weakly oligomorphic, Lemma 3.4 ensures that every complete positive n-type in $\mathcal{A}$ is principal. Therefore, $\mathcal{A}$ is homomorphism-homogeneous by Lemma 5.1 since $\mathcal{A}$ has QEPF.

As for the final part of the theorem, note that if $\mathcal{A}$ is finite then it is trivially weakly oligomorphic. ∎

We say that a structure $\mathcal{A}$ has *quantifier elimination that respects positive formulas (QERPF)* if $\mathcal{A}$ has both QE and QEPF, or equivalently, if $\mathcal{A}$ has quantifier elimination where every positive formula reduces to a positive quantifier-free formula.

COROLLARY 5.5. *Let $\mathcal{A}$ be an ω-categorical structure. Then $\mathcal{A}$ is both ultrahomogeneous and homomorphism-homogeneous if and only if $\mathcal{A}$ has QERPF.*

In particular, if $\mathcal{A}$ is a finite structure, then $\mathcal{A}$ is both ultrahomogeneous and homomorphism-homogeneous if and only if $\mathcal{A}$ has QERPF.

Proof. Note, first, that $\mathcal{A}$ is oligomorphic, so Lemma 2.10 shows that $\mathcal{A}$ is weakly oligomorphic. Since $\mathcal{A}$ is ω-categorical, we know that $\mathcal{A}$ is ultrahomogeneous if and only if $\mathcal{A}$ has quantifier elimination (QE). On the other hand, since $\mathcal{A}$ is weakly oligomorphic we know that $\mathcal{A}$ is homomorphism-homogeneous if and only if $\mathcal{A}$ has QEPF. Therefore, $\mathcal{A}$ is both ultrahomogeneous and homomorphism-homogeneous if and only if $\mathcal{A}$ has both QE and QEPF, which is equivalent to the requirement that $\mathcal{A}$ has QERPF. ∎

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