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Properties of dynamical systems with the asymptotic average shadowing property

by

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Abstract. This article investigates under what conditions nontransitivity can coexist with the asymptotic average shadowing property. We show that there is a large class of maps satisfying both conditions simultaneously and that it is possible to find such examples even among maps on a compact interval. We also study the limit shadowing property and its relation to the asymptotic average shadowing property.

1. Introduction. The theory of shadowing is one of the most important topics in the modern theory of dynamical systems. During the last fifty years many results on shadowing have been published, and there are books devoted entirely to this subject (e.g. [13, 14]).

In recent years the notions of average shadowing property and asymptotic average shadowing property were introduced respectively by Blank [1] and Gu [7]. The idea is to allow in a pseudo-orbit infinitely many errors of high magnitude provided that such errors are sparse enough. These innovations allow one to apply the theory of shadowing in a wider context than was possible before and smoothly integrate it into several existing fields of study (e.g. these definitions are natural in the case of random dynamical systems). What makes these new concepts even more appealing is the fact that they prove to be complex and nontrivial. The main aim of this article is to continue the investigation of the properties of continuous maps with the asymptotic average shadowing property (abbreviated AASP).

The definition of the AASP is stated in terms of limits of averages, which in a natural way brings to mind the Birkhoff ergodic theorem. Therefore, when looking at the setting from the topological point of view, one could expect maps with the AASP to be transitive. Existing results further reinforce this impression—namely, maps with the AASP are chain transitive [7],

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and equicontinuous maps with the AASP are topologically ergodic [8] (which roughly speaking means that the set of hits $N(U,V) = \{n : f^{-n}(V) \cap U \neq \emptyset\}$ has positive density for any two nonempty open sets U, V). We proved in [10] that surjections with the specification property always have the AASP and, in some cases, there is equivalence between these two properties. The only attempt to construct a nontransitive map with the AASP that we are aware of was undertaken in [9, Example B]. This result, however, is problematic, as the constructed map does not have the AASP (it is defined on an interval, 0 and $1-\varepsilon$ are its fixed points and $[1/2, 1]$ is its invariant set, and therefore, by [10, Thm. 3.1], it cannot have this property). The proof in [9] does establish that it has the limit shadowing property, though.

We construct (in Theorem 3.3) a class of nontransitive maps with the AASP, though this result most probably does not cover the full spectrum of possibilities. In particular, if we have a compact set A and a map f on it that has the AASP, then this property will not be affected if we extend the space by adding orbits with ω-limit sets contained in A as long as the new, larger space is still compact. But obviously the dynamics on the extended space will not be transitive. This family of examples does not exhaust the full generality of Theorem 3.3.

The article is organized in the following way: In Section 2 we recall the definitions and basic facts used in the later parts of the article. Section 3 contains the main results. Among other things it provides a tool for recursively constructing complicated examples of nontransitive dynamical systems with the AASP (Theorem 3.11). In the next section we deal with the special case of maps with maximal ω-limit sets. We also provide an example of a nontransitive map on a compact interval which has the AASP (a map with a "wandering interval"). Finally, in Section 5, we study the relations between the limit shadowing property and the AASP.

2. Preliminaries. Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers. A set $J \subset \mathbb{N}$ is said to be *of density zero* provided that

$$\lim_{n\to\infty} \frac{1}{n+1} \#(J \cap [0,n]) = 0.$$

Notation. Let $J \subset \mathbb{N}$ be a set such that $\mathbb{N} \setminus J$ is unbounded. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers. If there is $b \in \mathbb{R}$ such that the sequence obtained from $\{a_i\}_{i=0}^{\infty}$ by deleting the terms with indices from J has limit b, then we write $\lim_{i \notin J} a_i = b$. In an analogous way we define $\limsup_{i \notin J} a_i$.

Throughout the paper we will assume that (X, d) is a compact metric space and $f : X \to X$ is a continuous map. Given a point $x \in X$ we call the set $\{f^n(x) : n \in \mathbb{N}\}$ the *(positive) orbit* of x and denote it by $\mathrm{Orb}^+(x, f)$. The *ω-limit set* (or the *positive limit set*) of a point x is the set

$\omega(x,f)=\bigcap_{i=1}^{\infty}\overline{\mathrm{Orb}^+(f^i(x),f)}$. A point x is *periodic* if $f^n(x)=x$ for some $n>0$, and *eventually periodic* if $f^n(x)$ is periodic for some $n>0$.

A set $M\subset X$ is *minimal* if it is closed, nonempty, invariant (that is, $f(M)\subset M$) and contains no proper subset with these three properties. It is well known that if M is minimal then the orbit of every point of M is dense in M. A point $x\in X$ is called *minimal* if it belongs to a minimal set.

The map f is *transitive* if for any pair of nonempty open sets $U,V\subset X$ there exists $n>0$ such that $f^n(U)\cap V\neq\emptyset$. We say that f is *totally transitive* if f^n is transitive for all $n\geq 1$, and *(topologically) weakly mixing* if $f\times f$ is transitive on $X\times X$.

The map f has the *specification property* if for every $\varepsilon>0$ there exists $M>0$ such that for every $n\in\mathbb{N}$ and any $y_1,\dots,y_n\in X$, given any sequence of natural numbers $a_1\leq b_1<a_2\leq b_2<\cdots\leq b_n$ such that for every $2\leq i\leq n$ we have $a_i-b_{i-1}\geq M$ one can find $z\in X$ such that $d(f^k(y_i),f^k(z))\leq\varepsilon$ for all $1\leq i\leq n$ and $a_i\leq k\leq b_i$.

Recall that a finite sequence $\{x_i\}_{i=0}^n$ is called a *δ-pseudo-orbit* of f (from x_0 to x_n) if $d(f(x_i),x_{i+1})<\delta$ for every $0\leq i<n$. A map f is *chain transitive* if for every $x,y\in X$ and every $\delta>0$ there is a δ-pseudo-orbit from x to y. A map f is said to be *chain mixing* if for any $\delta>0$ and any $x,y\in X$ there is a positive integer N such that for every integer $n>N$ there is a δ-pseudo-orbit from x to y of length n.

The next three definitions were introduced by Gu [7]:

Definition 2.1. The sequence $\{x_i\}_{i=0}^{\infty}\subset X$ is an *asymptotic average pseudo-orbit* of f provided that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}d(f(x_i),x_{i+1})=0.$$

Definition 2.2. The sequence $\{x_i\}_{i=0}^{\infty}\subset X$ is *asymptotically shadowed on average* by a point $z\in X$ provided that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}d(x_i,f^i(z))=0.$$

Definition 2.3. The map f has the *asymptotic average shadowing property* (abbreviated *AASP*) provided that every asymptotic average pseudo-orbit of f is asymptotically shadowed on average by some point in X.

The reader can find several comments on the specification property and its relationship to the AASP in [10].

3. Nontransitive maps with the AASP. Gu has proven in [7] that surjections with the AASP are chain transitive. This result can be strengthened as follows:

Theorem 3.1. *If f is a surjection with the AASP then it is chain mixing.*

Proof. Gu has proven that if f has the AASP then so does f^n for every $n \geq 1$ and that every map with the AASP is chain transitive (see [7, Prop. 2.2 and Thm. 3.1]). But if f^n is chain transitive for every $n \geq 1$ then f is chain mixing [15, Cor. 12]. ∎

Theorem 3.1 was first observed by Gu in the context of maps on connected spaces, but as we can see above, this fact does not actually depend on the structure of the space X (note that the results of Gu [7] were published before [15] appeared).

If f is a surjection with the AASP then X is the only attractor for f. If we skip the assumption that f is onto then the only attractor for f is the maximal set A such that $f(A) = A$. One might therefore suspect that maps with the AASP are always at least transitive. This statement turns out to be false, however, as will be demonstrated in Theorem 3.3 and Example 3.13.

We have found the following lemma to be a very useful tool for studying the AASP. It was extensively used by Gu in his first paper on the subject [7].

Lemma 3.2 ([16, Thm. 1.20]). *Let $\{a_i\}_{i=0}^{\infty}$ be a bounded sequence of non-negative real numbers. The following conditions are equivalent:*

(1) $\lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} a_i = 0$,

(2) *There exists a set $J \subset \mathbb{N}$ of density zero such that* $\lim_{n\notin J} a_n = 0$.

Note that by virtue of this lemma the notion of AASP is purely topological and independent of the metric (as long as it induces the same topology).

The next theorem is the main result of this section.

Theorem 3.3. *Let (X, d) be a compact metric space, let $f : X \to X$ be a continuous map and let $A \subset X$ be a closed set which is invariant under f (that is, $f(A) \subset A$). Assume that the map $f|_A$ has the AASP and that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every $x \in X$ we have*

$$\frac{1}{n}\#\{0 \leq i < n : d(f^i(x), A) < \varepsilon\} > 1 - \varepsilon. \tag{3.1}$$

Then the map f also has the AASP.

We split the proof into a few steps that are stated in the form of several technical lemmas. For brevity we do not repeat the assumptions of Theorem 3.3 when stating the lemmas.

Lemma 3.4. *For every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\delta > 0$ such that for every finite δ-pseudo-orbit $\{x_i\}_{i=0}^{n-1}$ of f we have*

$$\frac{1}{n}\#\{0 \leq i < n : d(x_i, A) < \varepsilon\} > 1 - \varepsilon. \tag{3.2}$$

Proof. Given $\varepsilon > 0$ use (3.1) to choose $n \in \mathbb{N}$ such that for every $x \in X$ we have

$$\frac{1}{n}\#\{0 \le i < n : d(f^i(x), A) < \varepsilon/2\} > 1 - \varepsilon/2.$$

The compactness of X and continuity of f allow one to find in a standard way a constant $\delta > 0$ such that for every finite δ-pseudo-orbit $\{x_i\}_{i=0}^{n-1}$ of f and for every $i \in \{0, 1, \dots, n-1\}$ we have $d(f^i(x_0), x_i) < \varepsilon/2$. Then by the triangle inequality,

$$\frac{1}{n}\#\{0 \le i < n : d(x_i, A) < \varepsilon\} > 1 - \varepsilon/2 > 1 - \varepsilon. \ \blacksquare$$

LEMMA 3.5. *For every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\delta > 0$ such that for every $k \ge n$ and every finite δ-pseudo-orbit $\{x_i\}_{i=0}^{k-1}$ of f we have*

$$\frac{1}{k}\#\{0 \le i < k : d(x_i, A) < \varepsilon\} > 1 - \varepsilon. \tag{3.3}$$

Proof. Given $\varepsilon > 0$ use Lemma 3.4 to obtain $n \in \mathbb{N}$ and $\delta > 0$ such that (3.2) holds with $\varepsilon/2$. Fix $k \ge n$ and a finite δ-pseudo-orbit $\{x_i\}_{i=0}^{k-1}$ of f. Assume that $k = ln + r$, where $l, r \in \mathbb{N}$ and $0 \le r < n$. Define

$$b_j = \#\{j \le i < j + n : d(x_i, A) \ge \varepsilon\}.$$

We can now compute:

$$\begin{aligned}
&\frac{1}{k}\#\{0 \le i < k : d(x_i, A) < \varepsilon\} \\
&\qquad \ge \frac{1}{k}[k - (b_0 + b_n + b_{2n} + \cdots + b_{(l-1)n}) - b_{k-n}] \\
&\qquad \ge 1 - \frac{n}{k}\left[\frac{b_0}{n} + \frac{b_n}{n} + \frac{b_{2n}}{n} + \cdots + \frac{b_{(l-1)n}}{n}\right] - \frac{n}{k}\,\frac{b_{k-n}}{n} \\
&\qquad > 1 - l\frac{n\varepsilon}{2k} - \frac{n\varepsilon}{2k} \ge 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\
&\qquad = 1 - \varepsilon. \ \blacksquare
\end{aligned}$$

LEMMA 3.6. *If $\{x_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo-orbit of f then for every $\varepsilon > 0$ and every $\theta > 0$ there exists a set $J_\varepsilon^\theta \subset \mathbb{N}$ such that*

$$\limsup_{k\to\infty} \frac{1}{k}\,\#(J_\varepsilon^\theta \cap [0, k)) \le \theta \tag{3.4}$$

and $d(x_j, A) < \varepsilon$ for every $j \in \mathbb{N} \setminus J_\varepsilon^\theta$.

Proof. Fix $\{x_i\}_{i=0}^{\infty}$ and $\varepsilon, \theta > 0$. Set $\gamma = \min\{\varepsilon, \theta\}$. Use Lemma 3.5 to obtain $n \in \mathbb{N}$ and $\delta > 0$ such that (3.3) holds with γ in place of ε. Use Lemma 3.2 for the sequence $\{d(f(x_i), x_{i+1})\}_{i=0}^{\infty}$ to obtain a set J of density zero such that $\lim_{i\notin J} d(f(x_i), x_{i+1}) = 0$. We can assume that J is infinite by extending it if necessary to a larger set of density zero. Define J_1 to be

$\mathbb{N}$ minus all sequences of consecutive numbers of length at least n that are disjoint from J. Strictly speaking, we put

$$J_1 = \{i \in \mathbb{N} : j_1 \le i \le j_2 \text{ for some } j_1, j_2 \in J \cup \{0\},\ j_2 - j_1 \le n\}.$$

Note that $J \subset J_1$. Also note that J_1 is of density zero, because

$$\limsup_{k\to\infty} \frac{1}{k+1}\#(J_1 \cap [0,k]) \le \limsup_{k\to\infty} \frac{2n-1}{k+1}\#(J \cap [0,k]) = 0.$$

As $\lim_{i\notin J} d(f(x_i), x_{i+1}) = 0$ we can find $c \in J_1$ such that for every sequence $\{p_1, \ldots, p_j\}$ of consecutive natural numbers that is disjoint from J_1 and such that $p_1 \ge c$ the sequence $\{x_{p_1}, \ldots, x_{p_j}\}$ is a δ-pseudo-orbit of f. Define $J_2 = J_1 \cup \{0, 1, \ldots, c\}$. Note that $J_1 \subset J_2$ and J_2 is of density zero, for

$$\limsup_{k\to\infty} \frac{1}{k+1}\#(J_2 \cap [0,k]) \le \limsup_{k\to\infty} \frac{1}{k+1}\#(J_1 \cap [0,k]) + \limsup_{k\to\infty} \frac{c+1}{k+1} = 0.$$

We can thus select natural numbers $q_0 \le r_0 < q_1 \le r_1 < \cdots$ such that $\mathbb{N} \setminus J_2 = \bigcup_{i=0}^{\infty} \{q_i, \ldots, r_i\}$. Additionally, for every $j \in \mathbb{N}$ we can demand that $n - 1 \le r_j - q_j < 2n - 1$. Note that for every $j \in \mathbb{N}$ the sequence $\{x_{q_j}, \ldots, x_{r_j}\}$ is a δ-pseudo-orbit of f.

Define $J_\varepsilon^\theta = J_2 \cup \{j \in \mathbb{N} \setminus J_2 : d(x_j, A) \ge \varepsilon\}$. It remains to prove that the set J_ε^θ has all the desired properties. For every $j \in \mathbb{N}$ define w_j to be the number of points in $\{x_{q_j}, \ldots, x_{r_j}\}$ that are closer than ε to A. By Lemma 3.5 for every $j \in \mathbb{N}$ we have $w_j/(r_j - q_j + 1) > 1 - \theta$. Given $k \in \mathbb{N}$ let $s(k)$ be the largest natural number such that $q_{s(k)} < k$. To show that (3.4) holds we calculate:

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{k}\#(J_\varepsilon^\theta \cap [0,k)) \\
&\qquad \le \lim_{k\to\infty} \frac{1}{k}\#(J_2 \cap [0,k)) + \limsup_{k\to\infty} \frac{1}{k}\sum_{j=0}^{s(k)} (r_j - q_j + 1 - w_j) \\
&\qquad \le \limsup_{k\to\infty} \frac{1}{k}\sum_{j=0}^{s(k)} (r_j - q_j + 1)\left(1 - \frac{w_j}{r_j - q_j + 1}\right) \\
&\qquad \le \limsup_{k\to\infty} \frac{1}{k}\sum_{j=0}^{s(k)} (r_j - q_j + 1)\theta \\
&\qquad \le \theta \limsup_{k\to\infty} \frac{k+2n}{k} \le \theta.
\end{aligned}$$

Since $d(x_j, A) < \varepsilon$ provided that $j \notin J_\varepsilon^\theta$, the proof is finished. ∎

LEMMA 3.7. *Let $J_i \subset \mathbb{N}$ be a sequence such that*

$$\lim_{i\to\infty} \limsup_{k\to\infty} \frac{1}{k}\#(J_i \cap [0,k)) = 0.$$

Then there is a set J of density zero and increasing sequences $\{m_i\}_{i=0}^{\infty}$, $\{l_i\}_{i=1}^{\infty}$ such that $m_0 = 0$ and for every $i = 1, 2, \dots$ we have

$$J \cap [m_{i-1}, m_i) = J_{l_i} \cap [m_{i-1}, m_i).$$

Additionally, the following conditions hold:

(1) *when each set J_i is of density zero we can take $l_i = i$ for all i,*
(2) *for any given sequence of infinite sets $R_i \subset \mathbb{N}$ we can choose m_i in such a way that $m_i \in R_i$.*

Proof. First fix a sequence $\{l_i\}_{i=1}^{\infty}$ such that for every $i = 1, 2, \dots$ we have $\limsup_{k\to\infty}(1/k)\#(J_{l_i} \cap [0, k)) < 1/2^{i+1}$, and a sequence $\{p_i\}_{i=1}^{\infty}$ such that for every $i = 1, 2, \dots$ and every $k \geq p_i$ we have $(1/k)\#(J_{l_i} \cap [0, k)) < 1/2^i$.

Put $m_0 = 0$ and for $i > 0$ define inductively

$$m_i = \begin{cases} \min\{r \in R_i : r \geq m_{i-1} + 1,\ r \geq p_{i+1}\} & \text{if the sets } R_i \text{ are given,} \\ \max\{m_{i-1} + 1, p_{i+1}\} & \text{if not.} \end{cases}$$

Next define J by the condition $J \cap [m_{i-1}, m_i) = J_{l_i} \cap [m_{i-1}, m_i)$.

It remains to notice that for every $s \in \mathbb{N}_+$ we have

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{k}\#(J \cap [0, k)) \\
&\quad \leq \limsup_{k\to\infty} \frac{1}{k}\Big(\sum_{t=1}^{s} \#(J_{l_t} \cap [m_{t-1}, m_t)) + \sum_{t=s+1}^{\infty} \#(J_{l_t} \cap [m_{t-1}, k))\Big) \\
&\quad \leq \sum_{t=s+1}^{\infty} \frac{1}{2^t} = \frac{1}{2^s}
\end{aligned}$$

where $[a, b)$ is understood to be the empty set for $b \leq a$. ∎

LEMMA 3.8. *If $\{x_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo-orbit of f then for every $\varepsilon > 0$ there exists a set $J_\varepsilon \subset \mathbb{N}$ of density zero such that $d(x_j, A) < \varepsilon$ for every $j \in \mathbb{N} \setminus J_\varepsilon$.*

Proof. Fix $\{x_i\}_{i=0}^{\infty}$ and $\varepsilon > 0$. For every $i \in \mathbb{N}$ use Lemma 3.6 to obtain the set $J_\varepsilon^{1/2^i}$. Next, denote $J_i = J_\varepsilon^{1/2^i}$ and observe that

$$\lim_{i\to\infty} \limsup_{k\to\infty} \frac{1}{k}\,\#(J_i \cap [0, k)) \leq \lim_{i\to\infty} \frac{1}{2^i} = 0,$$

hence we can apply Lemma 3.7 to obtain a set J of density zero and a sequence $\{m_i\}_{i=0}^{\infty}$. In particular, for every $j \in \mathbb{N} \setminus J$ there is i such that $j \in \mathbb{N} \setminus J_i$ and so $d(x_j, A) < \varepsilon$ by Lemma 3.6. The proof is finished by putting $J_\varepsilon = J$. ∎

LEMMA 3.9. *If $\{x_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo-orbit of f then there exists a set $J \subset \mathbb{N}$ of density zero such that $\lim_{i\notin J} d(x_i, A) = 0$.*

Proof. For every $i \in \mathbb{N}$ use Lemma 3.8 to obtain the set $J_{1/2^i}$. Since each $J_{1/2^i}$ is of density zero, the sequence $J_i = J_{1/2^i}$ fulfills the assumptions of Lemma 3.7, and therefore we can produce a set J of density zero and increasing sequences $\{m_i\}_{i=0}^{\infty}, \{l_i\}_{i=1}^{\infty}$ such that $J \cap [m_{i-1}, m_i) = J_{l_i} \cap [m_{i-1}, m_i)$. This guarantees that if $j \in \{m_i + 1, m_i + 2, \ldots\} \setminus J$ then $j \in \mathbb{N} \setminus J_{1/2^n}$ for some $n \geq l_{i+1} \geq i + 1$ and as a consequence $d(x_j, A) < 1/2^{i+1}$. Thus $\lim_{i \notin J} d(x_i, A) = 0$ and the proof is complete. ∎

Proof of Theorem 3.3. Using the compactness of X and continuity of f one can obtain in a standard way for every $i \in \mathbb{N}$ a constant $\delta_i > 0$ such that for every finite δ_i-pseudo-orbit $\{z_0, \ldots, z_{2^{i+1}-1}\}$ of f and for every $u \in X$ such that $d(z_0, u) < \delta_i$ we have $d(f^j(u), z_j) < 1/2^i$ for all $0 \leq j < 2^{i+1}$.

Let $\{x_i\}_{i=0}^{\infty} \subset X$ be an asymptotic average pseudo-orbit of f. Let $Q^{(1)}$ be the set of density zero provided by Lemma 3.9. Use Lemma 3.2 for the sequence $\{d(f(x_i), x_{i+1})\}_{i=0}^{\infty}$ to obtain a set $Q^{(2)}$, also of density zero, such that

$$\lim_{i \notin Q^{(2)}} d(f(x_i), x_{i+1}) = 0.$$

The set $J = Q^{(1)} \cup Q^{(2)}$ is of density zero as the union of two such sets.

For every $n \in \mathbb{N}$ let J_n be the union of all sets of the form $\{l2^n, \ldots, (l+1)2^n - 1\}$ (where $l \in \mathbb{N}$) that contain at least one element from J. In other words

$$J_n = \bigcup_{l \in \Omega_n} \{l2^n, \ldots, (l+1)2^n - 1\}$$

where Ω_n is the collection of all those $l \in \mathbb{N}$ for which

$$\{l2^n, \ldots, (l+1)2^n - 1\} \cap J \neq \emptyset.$$

Note that $\lim_{k \notin J_n} d(x_k, A) = 0$ and $\lim_{k \notin J_n} d(f(x_k), x_{k+1}) = 0$ as these properties carry over from the set J which is a subset of J_n. Additionally

$$\limsup_{k \to \infty} \frac{1}{k+1} \#(J_n \cap [0, k]) \leq \limsup_{k \to \infty} \frac{2^n}{k+1} \#(J \cap [0, k]) = 0.$$

Therefore each J_n is of density zero. Note that because of the properties $\lim_{k \notin J} d(f(x_k), x_{k+1}) = 0$ and $\lim_{k \notin J} d(x_k, A) = 0$ for every $i \in \mathbb{N}$ there is $K_i \in \mathbb{N}$ such that $d(f(x_k), x_{k+1}) < \delta_i$ and $d(x_k, A) < \delta_i$ provided that $k \notin J$ and $k \geq K_i$. Define $R_i = \{l2^{i+1} - 1 : l \in \Omega_{i+1},\, l \geq K_i\}$.

We now apply Lemma 3.7 to obtain a set J'. Notice that by the choice of the sets R_i the following additional properties are satisfied:

- For every $i \in \mathbb{N}$ we have $m_i = l2^{i+1} - 1$ for some $l \in \Omega_{i+1}$.
- $d(f(x_k), x_{k+1}) < \delta_i$ provided that $k \notin J'$ and $k \geq m_i$.
- $d(x_k, A) < \delta_i$ provided that $k \notin J'$ and $k \geq m_i$.

By the above properties the set $\mathbb{N} \setminus J'$ can be written as a union of pairwise disjoint finite sequences of consecutive numbers of nondecreasing length, where the length of each sequence equals 2^i for some $i \in \mathbb{N}$. Denote these sequences by $\{a_i, \dots, b_i\}$ where $a_0 < b_0 < a_1 < b_1 < \cdots$, and put $B = \{b_i : i \in \mathbb{N}\}$. By the definition of the set J_n we may additionally ensure that if $m_i \leq a_k < m_{i+1}$ then $b_k - a_k = 2^{i+1}$. Note that B is of density zero because $b_{k+1} - b_k \geq 2^{i+1}$ for every $b_k > m_{i+1}$.

Pick any $p \in A$. We will now define a sequence $\{y_i\}_{i=0}^{\infty} \subset A$ which will be an asymptotic average pseudo-orbit of f with some additional properties. For all $i \in J'$ put $y_i = p$. For every a_i pick any point y_{a_i} such that $d(x_{a_i}, y_{a_i}) = d(x_{a_i}, A)$ and put $\{y_{a_i+1}, \dots, y_{b_i}\} = \{f(y_{a_i}), \dots, f^{b_i - a_i}(y_{a_i})\}$. Note that the sequence $\{d(f(y_i), y_{i+1})\}_{i=0}^{\infty}$ satisfies the condition (2) of Lemma 3.2 with the set $J' \cup B$ (which is of density zero, being the union of two such sets). It is, therefore, an asymptotic average pseudo-orbit of $f|_A$. Moreover, if $a_i > m_k$ then $\{x_j\}_{j=a_i}^{b_i}$ is a δ_k-pseudo-orbit and so $d(x_{a_i+j}, f^j(y_{a_i})) < 2^{-k}$ for $j = 0, \dots, b_i - a_i$. In particular $\lim_{i \notin J'} d(x_i, y_i) = 0$.

The map $f|_A$ has the AASP, so we can select a point $y \in A$ that asymptotically shadows $\{y_i\}_{i=0}^{\infty}$ on average, that is, $\lim_{i \notin C} d(f^i(y), y_i) = 0$ for some set C of density zero. Observe that

$$\limsup_{i \notin J' \cup C} d(f^i(y), x_i) \leq \limsup_{i \notin J' \cup C} d(f^i(y), y_i) + \limsup_{i \notin J' \cup C} d(y_i, x_i) = 0$$

and the set $J' \cup C$ is of density zero. Therefore, by Lemma 3.2, y asymptotically shadows $\{x_i\}_{i=0}^{\infty}$ on average, and the proof is complete. ∎

In the next lemma we show that if the map has a special structure, then the assumptions of Theorem 3.3 are satisfied.

Lemma 3.10. *Let (X, d) be a compact metric space, let $f : X \to X$ be a continuous map and let*

$$A = \overline{\bigcup_{x \in X} \omega(x, f)}.$$

Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $x \in X$,

$$\frac{1}{N} \#\{0 \leq i < N : d(f^i(x), A) < \varepsilon\} > 1 - \varepsilon. \tag{3.5}$$

Proof. Fix any $\varepsilon > 0$ and denote $U = \{x \in X : d(x, A) < \varepsilon\}$. The set U is open, so $D = X \setminus U$ is compact. Since $\omega(z, f) \subset A$ and f is continuous we know that for every $z \in D$ there is an open neighbourhood V_z of z and $n(z) > 0$ such that for every $y \in V_z$,

$$\frac{1}{n(z) + 1} \#\{0 \leq i \leq n(z) : f^i(y) \notin U\} \leq \frac{\varepsilon}{2}. \tag{3.6}$$

In particular, there are $z_1, \dots, z_k \in D$ such that $D \subset \bigcup_{i=1}^{k} V_{z_i}$. Denote

$m = \max\{n(z_1), \dots, n(z_k)\}$, pick $l > 2/\varepsilon$ and let $N = (l+1)m$. Fix any $x \in X$. We need to verify that (3.5) holds. Set $C = \{0 \le i < N : f^i(x) \notin U\}$. There is a subset $C' \subset C$ and a function $\xi : C' \to \{1, \dots, k\}$ such that

(1) if $i \in C'$ then $f^i(x) \in V_{z_{\xi(i)}}$,
(2) $[i, i + n(z_{\xi(i)})] \cap C' = \{i\}$,
(3) $C \subset \bigcup_{i \in C'} [i, i + n(z_{\xi(i)})]$.

In particular from (2) and (3) it follows that $\sum_{i \in C', i \le lm} (n(z_{\xi(i)}) + 1) \le N$. To finish the proof it is enough to calculate

$$\begin{aligned}\frac{1}{N}\#\{0 \le i < N : f^i(x) \notin U\} \\ &\le \frac{1}{N} \sum_{\substack{i \in C' \\ i \le lm}} \#\{0 \le j \le n(z_{\xi(i)}) : f^{i+j}(x) \notin U\} + \frac{m}{N} \\ &\le \frac{1}{N} \sum_{\substack{i \in C' \\ i \le lm}} \frac{\varepsilon}{2}(n(z_{\xi(i)}) + 1) + \frac{1}{l} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon. \ \blacksquare\end{aligned}$$

THEOREM 3.11. *Let (X, d) be a compact metric space, let $f : X \to X$ be a continuous map and let $A \subset X$ be a closed set that is invariant under f. Assume that the map $f|_A$ has the AASP and $\omega(x, f) \subset A$ for every $x \in X$. Then f has the AASP.*

Proof. This is a direct consequence of Lemma 3.10 and Theorem 3.3. ∎

In particular, as a constant map on a one-point space has the AASP, we have:

COROLLARY 3.12. *Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous map. If there exists a point $y \in X$ such that $\omega(x, f) = \{y\}$ for every $x \in X$ then f has the AASP.*

This allows us to give a simple, yet nontrivial example of a nontransitive map with the AASP—a task that previous research showed to be surprisingly hard.

EXAMPLE 3.13. The map $f : \mathbb{S}^1 \ni e^{2\pi i x} \mapsto e^{2\pi i x^2} \in \mathbb{S}^1$ (where $x \in [0, 1]$ and $\mathbb{S}^1$ denotes the unit circle) has the AASP.

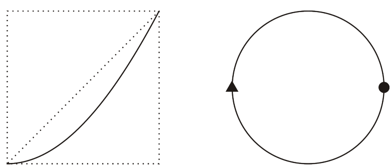

Fig. 1. Graph of the function x^2 over the unit interval and the phase portrait of its lift to $\mathbb{S}^1$

It is easy to see that this map does not have the shadowing property or the specification property, thus proving that the AASP does not imply either of these two properties. In fact, by Theorem 3.1, an onto map with the AASP and shadowing has to be mixing, while the map f is not even transitive.

Remark. We proved in [10] that for surjections the specification property implies the AASP. The above example shows that the converse implication does not hold.

Remark. In view of Example 3.13 one could wonder if the formulation of Lemma 3.10 is not overly complicated. One could suspect that there should exist a global upper bound N on the number of iterations that the orbit of any point $x \in X$ can spend outside any neighbourhood U of A; in other words $\#(\mathrm{Orb}^+(x,f) \setminus U) < N$.

The next example was introduced as [6, Example 17] for a completely different purpose, but it serves nicely to illustrate that this does not necessarily have to be true.

Example 3.14. Let Σ_2 be the set of bi-infinite sequences over the alphabet $\{0,1\}$, i.e. $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$. The set $\{0,1\}$ is given the discrete topology and Σ_2 is endowed with the product topology. We define the shift map $\sigma : \Sigma_2 \to \Sigma_2$ in the standard way, that is, $\sigma(x)_i = x_{i+1}$. It is easy to see that σ is a homeomorphism.

Define a sequence $\{x_n\}_{n=1}^{\infty} \subset \Sigma_2$ by

$$x_n = {}^{\infty}0\dot{0}(10^n)^n 0^{\infty}$$

and put

$$X = \mathrm{cl}\Big(\bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \{\sigma^i(x_n)\}\Big),$$

where cl denotes closure in Σ_2, ${}^{\infty}0$ and 0^{∞} are the sequences of zeros that are infinite respectively to the left and to the right, and the dot marks the symbol at position 0 in a bi-infinite sequence. Taking the closure simply adds the point ${}^{\infty}0\dot{0}0^{\infty}$ to the set. Note that $\sigma(X) = X$ and so $f = \sigma|_X$ is a homeomorphism of X. Let $|y|_1$ denote the number of occurrences of the symbol 1 in $y \in \Sigma_2$. While in general $|y|_1$ can be infinite we know that $|x|_1 < \infty$ for every $x \in X$. Note that $\omega(x,\sigma) = \{{}^{\infty}0\dot{0}0^{\infty}\}$. Put $A = \{{}^{\infty}0\dot{0}0^{\infty}\}$ and observe that the set $U = \{x \in X : x_0 = 0\}$ is an open neighbourhood of A for which

$$\#\{i \in \mathbb{N} : \sigma^i(x_n) \notin U\} = n.$$

4. Maximal ω-limit sets

Definition 4.1. Let $f : X \to X$ be a continuous map. We say that a set $A \subset X$ is a *maximal ω-limit set for f* if there is $x \in X$ such that

$\omega(x,f)=A$ and
$$\omega(y,f)\cap A\neq\emptyset \;\Rightarrow\; \omega(y,f)\subset A$$
for every $y\in X$.

DEFINITION 4.2. Two points x and y in X are said to be *proximal* if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y))=0$.

DEFINITION 4.3. A point $x\in X$ is said to be *distal* if x is not proximal to any point in its orbit closure $\overline{\mathrm{Orb}^+(x,f)}$ other than itself.

LEMMA 4.4. *If there are $x,y\in X$ such that any $z\in X$ is not proximal to at least one of the points x,y then f does not have the AASP.*

Proof. The proof is rather standard (compare e.g. the proof of [10, Theorem 3.1]). We include it for the sake of completeness. Define $\{a_i\}_{i=0}^{\infty}$ in the following way:
$$a_i=\begin{cases} f^i(x) & \text{if } 2^{2k}\le i<2^{2k+1} \text{ for some } k\in\mathbb{N},\\ f^i(y) & \text{otherwise.}\end{cases}$$
Notice that for every $j\in\mathbb{N}$ and for every integer $2^j\le k<2^{j+1}$,
$$\frac{1}{k}\sum_{i=0}^{k-1} d(f(a_i),a_{i+1})\le (j+1)2^{-j}\operatorname{diam} X,$$
which implies that $\{a_i\}_{i=0}^{\infty}$ is an asymptotic average pseudo-orbit of f.

Next, suppose that $\{a_i\}_{i=0}^{\infty}$ is asymptotically shadowed on average by some point $z\in X$. Neither the set $\bigcup_{k=0}^{\infty}\{2^{2k},\dots,2^{2k+1}-1\}$ nor its complement in $\mathbb{N}$ has density zero, so z has to be proximal to both x and y, which is impossible. ∎

THEOREM 4.5. *If f has the AASP then it has at most one maximal ω-limit set.*

Proof. Let ω_1,ω_2 be maximal ω-limit sets. Fix any $x\in\omega_1$ and $y\in\omega_2$. By Lemma 4.4 there is, a point z proximal to both x and y. Hence $\omega(z,f)\cap\omega_1\neq\emptyset$ and $\omega(z,f)\cap\omega_2\neq\emptyset$. But the sets ω_1 and ω_2 are maximal ω-limit sets and therefore $\omega_1=\omega_2$. ∎

The above theorem is useful from at least two points of view. Firstly, for a large class of maps it allows us to quickly verify that they do not have the AASP (for example it works for many maps on topological graphs). Furthermore, a combination of Theorems 4.5 and 3.11 gives an idea where to look for an example of a nontransitive map with the AASP which is more sophisticated than Example 3.13. We now present such a map.

EXAMPLE 4.6. There exists a nontransitive map $f\colon[0,1]\to[0,1]$ with the AASP.

Proof. Start with the standard tent map $T : [0,1] \to [0,1]$, $T(x) = 1 - |1-2x|$. We will perform a construction similar to that of a Denjoy map [4, Example 14.9]. First we choose a point $z \in (0,1)$ with a dense orbit under T. Denote $D_0 = \{z, T(z)\} \cup T^{-1}(\{z\})$ and inductively set $D_{n+1} = T(D_n) \cup T^{-1}(D_n)$. Finally put

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Note that $D \subset (0,1)$ and D is dense in $[0,1]$. Every point $y \in [0,1]$ has at most two preimages under T. Therefore D is countable, say $D = \{x_i : i \in \mathbb{Z}\}$ where $x_i \neq x_j$ for $i \neq j$. Observe that if $T^n(x_i) = x_j$ for some $n > 0$ then $i \neq j$ and $x_i \notin \mathrm{Orb}^+(x_j, T)$, as otherwise z would be an eventually periodic point. Just by the definition, both sets D and $[0,1] \setminus D$ are invariant: $T(D) = D$ and $T([0,1] \setminus D) = [0,1] \setminus D$. There is also a function $\phi \colon \mathbb{Z} \to \mathbb{Z}$ so that $T(x_i) = x_{\phi(i)}$.

As the final step of our construction we remove all points x_i from $[0,1]$ and fill each hole with an interval I_i of length $2^{-|i|}$. This way a new continuous map F is defined on the extended space in such a manner that:

(1) Each interval I_i is mapped homeomorphically onto $I_{\phi(i)}$.
(2) If all intervals I_i are collapsed back into single points then F reverts to the map T.

As the domain of F is isometric to $[0,4]$ we can assume that $F : [0,4] \to [0,4]$. In this way every interval I_i becomes some interval $[a_i, b_i] \subset (0,4)$ and there is a quotient map $\pi : [0,4] \to [0,1]$ that does not increase distances, collapses every interval $[a_i, b_i]$ into a single point (i.e. $\pi([a_i, b_i]) = \{x_i\}$), and has the property that $T \circ \pi = \pi \circ F$. If we fix i, j such that $x_i \notin \mathrm{Orb}^+(x_j)$ then $F^n((a_j, b_j)) \cap (a_i, b_i) = \emptyset$ for all $n > 0$. This shows that F is not transitive. Additionally, if $x \in (a_i, b_i)$ then $\omega(x, F) \cap (a_i, b_i) = \emptyset$, which, since ω-limit sets are closed, implies that $(a_i, b_i) \cap \omega(x, F) = \emptyset$ for every $x \in [0,1]$. Denote $A = [0,4] \setminus \bigcup_{i \in \mathbb{Z}} (a_i, b_i)$ and observe that $\omega(x, F) \subset A$ for every $x \in [0,1]$.

It is known that the tent map T has the specification property (see [2] or [3]). This combined with the fact that T is a surjection implies by [10, Thm. 3.8] that T has the AASP. We will now prove that $F|_A$ also has the AASP.

Let $\{y_i\}_{i \in \mathbb{N}} \subset A$ be an asymptotic average pseudo-orbit of $F|_A$. Since π does not increase distances, the sequence $\{\pi(y_i)\}_{i \in \mathbb{N}}$ is an asymptotic average pseudo-orbit of T and so it is asymptotically shadowed on average by some $x \in [0,1]$. Let $z \in A$ be a point such that $\pi(z) = x$. We claim that $\{y_i\}_{i \in \mathbb{N}}$ is asymptotically shadowed on average by z.

Use Lemma 3.2 to obtain a set $J \subset \mathbb{N}$ for the sequence

$$\{d(\pi(F^i(z)), \pi(y_i))\}_{i=0}^{\infty} = \{d(T^i(x), \pi(y_i))\}_{i=0}^{\infty}$$

so that $\lim_{i \notin J} d(\pi(F^i(z)), \pi(y_i)) = 0$. Notice that $d(F^i(z), y_i)$ is equal to $d(\pi(F^i(z)), \pi(y_i))$ plus the sum of the lengths of all the intervals $[a_j, b_j]$ that are between $F^i(z)$ and y_i.

Let J_i be the set of all those natural numbers j for which the interval $[a_i, b_i]$ is between $F^j(z)$ and y_j. Given any $M \in \mathbb{N}$ one can find a neighbourhood U of x_i so small that if $\pi(F^j(z)) \in U$ then $\pi(F^{j+s}(z)) \notin U$ for $s = 1, \ldots, M$. This combined with the fact that $\lim_{i \notin J} d(\pi(F^i(z)), \pi(y_i)) = 0$ means that for any $M \in \mathbb{N}$ there is $N > 0$ such that for every $k > N$ the following implication holds:

$$J \cap [k, k+M] = \emptyset \;\Rightarrow\; \#\,(J_i \cap [k, k+M]) \leq 1.$$

This implies that J_i is of density zero.

For every $i \in \mathbb{N}$ denote $Q_i = \bigcup_{k=-i}^{i} J_i$ and observe that each Q_i is of density zero, $Q_i \subset Q_{i+1}$ and if $j \notin Q_i$ then $F^j(z)$ and y_j are contained in the same connected component of the set $[0, 4] \setminus \bigcup_{k=-i}^{i} (a_i, b_i)$. In other words

$$d(F^j(z), y_j) \leq d(T^j(x), \pi(y_j)) + \sum_{|k|>i} 2^{-|i|}.$$

By Lemma 3.7 there is an increasing sequence $\{m_i\}_{i=0}^{\infty}$ such that the set

$$Q = J \cup \bigcup_{i \in \mathbb{N}} (Q_i \cap [m_i, m_{i+1}))$$

is also of density zero. Notice that by the definition of the sets Q_i we have $\limsup_{i \notin Q} d(F^i(z), y_i) = 0$ and so $\{y_i\}_{i \in \mathbb{N}}$ is asymptotically shadowed on average by z. This proves that $F|_A$ has the AASP.

Applying Theorem 3.11 to our set A we conclude that the map F also has the AASP. But we have already demonstrated that F is not transitive. The desired map $f\colon [0,1] \to [0,1]$ is obtained by putting $f(x) = (1/4)F(4x)$. ∎

THEOREM 4.7. *Let x be a distal point. Denote $Y = \overline{\mathrm{Orb}^+(x, f)}$. If $f(x) \neq x$ then $f|_Y$ does not have the AASP.*

Proof. Put $y = f(x)$. First observe that there is no point proximal to both x and y, as otherwise, by the definition of a distal point, we have $x = z$, and since $f(x)$ and z are also proximal, we finally get $x = z = f(x)$, which contradicts the assumptions.

Now, it is enough to apply Lemma 4.4. ∎

EXAMPLE 4.8. An irrational rotation of the circle is totally transitive and therefore chain mixing. The only attractor is the whole circle. But such a map does not have the AASP since every point is distal. This shows that even both these properties together are not enough for the AASP.

5. Shadowing, limit shadowing and the AASP. We say that f has the *weak limit shadowing property* if for any sequence $\{x_i\}_{i=0}^{\infty}$ such that

$$\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0$$

there exists a point $z \in X$ such that

$$\lim_{i\to\infty} d(f^i(z), x_i) = 0.$$

This notion was introduced by Eirola et al. [5] under the name of 'limit shadowing'. We reserve the latter name for a stronger property. Namely, following some authors (e.g. [12]), we say that f has the *limit shadowing property* if it has the shadowing property together with the weak limit shadowing property. The reason for this terminology is that in general there is no equivalence between shadowing and weak limit shadowing, but shadowing implies limit shadowing in some important cases (see [12]).

Not much is known about weak limit shadowing. Here we provide two classes of maps with this property which are not expansive. In [10] we have given an example of a class of maps which have limit shadowing but do not have the AASP: transitive c-expansive maps with shadowing which are not mixing. One of the simplest examples of this kind of map is on a finite space with at least two points that are permuted in a single cycle. These examples, however, leave much to be desired—the spaces are not connected and there is an n and a decomposition of the space into pairwise disjoint sets invariant under f^n such that f^n has the AASP on each set from the decomposition. It is therefore natural to ask for a more interesting example:

Is there a map f with limit shadowing such that for any $n \geq 1$ and for any nonempty set A invariant under f^n the map $f^n|_A$ does not have the AASP?

The answer to this question is positive. Odometers serve as an example.

Definition 5.1. Let $\mu = \{m_i\}_{i\in\mathbb{N}}$ be an increasing sequence of positive integers such that m_i divides m_{i+1} for all $i \in \mathbb{N}$. Let $p_i : \mathbb{Z}_{m_i} \to \mathbb{Z}_{m_i}$ be the cyclic permutation given by $p_i(n) = n + 1 \bmod m_i$. Define natural projections $\pi_i : \mathbb{Z}_{m_{i+1}} \to \mathbb{Z}_{m_i}$ by putting $\pi_i(n) = n \bmod m_i$. The inverse limit (with π_i as bonding maps)

$$G_\mu = \varprojlim \mathbb{Z}_{m_i} = \{(n_0, n_1, \ldots) : n_i \in \mathbb{Z}_{m_i},\ n_i = \pi_i(n_{i+1}),\ i \in \mathbb{N}\}$$

is a well defined subset of the countable product

$$\mathbb{Z}_{m_0} \times \mathbb{Z}_{m_1} \times \cdots$$

endowed with the metric $\rho(x, y) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i)$ where d is the discrete metric on $\mathbb{Z}$. The map $p : G_\mu \to G_\mu$ given by $p(x)_i = p_i(x_i)$ is called an *odometer* (on the scale μ). Note that in our setting a single periodic orbit is not an odometer.

THEOREM 5.2. *Let f be an odometer on the scale μ. Then*

(1) *f has limit shadowing.*
(2) *For any $n \geq 1$ and for any nonempty set A that is invariant under f^n the map $f^n|_A$ does not have the AASP.*

Proof. To show (1), first observe that the space is totally disconnected and the map f is equicontinuous, so it has shadowing by [11, Prop. 4.7]. It remains to show that it also has the weak limit shadowing property. Let $\{x^i\}_{i=0}^{\infty} \subset G_\mu$ be such that $\lim_{i\to\infty} d(f(x^i), x^{i+1}) = 0$. By the definition of the metric ρ this condition forces that for every $k \in \mathbb{N}$ there are $z_k \in \mathbb{Z}_{m_k}$ and $N_k \in \mathbb{N}$ such that for every $i > N_k$ we have $x_k^i = z_k + i \bmod m_k$. But then the sequence z satisfies the condition $\lim_{i\to\infty} d(f^i(z), x_i) = 0$.

For the proof of (2) pick any $x \in A$ and put $y = f^n(x)$. As the odometer has no periodic points we get $x \neq y$. But there is no point proximal to both x, y in the odometer, so the proof is finished by Lemma 4.4. ∎

THEOREM 5.3. *Let f be a homeomorphism of the interval I with finitely many fixed points $p_0 < \cdots < p_n$. If $\{x_i\}_{i=0}^{\infty}$ is a sequence such that $\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0$ then there is a $j \in \{0, \ldots, n\}$ such that $\lim_{i\to\infty} d(p_j, x_i) = 0$. In particular f has the weak limit shadowing property.*

Proof. The map f has to be either strictly increasing or strictly decreasing on every interval (p_i, p_{i+1}). Suppose it is strictly increasing on some (p_k, p_{k+1}). It is elementary that if δ is small enough one can find open disjoint neighbourhoods U_k and U_{k+1} of p_k and p_{k+1} respectively such that any δ-pseudo-orbit of f can pass from U_k to U_{k+1} but not from U_{k+1} to U_k. A similar statement is true if f is strictly decreasing on (p_k, p_{k+1}) but the allowed direction of travel is reversed. Applying this to every interval (p_i, p_{i+1}) we obtain a collection of open disjoint neighbourhoods $U_0, \ldots, U_n$ of the points $p_0, \ldots, p_n$ such that any δ-pseudo-orbit of f has to be eventually contained in one of the sets U_i. Additionally, if δ is small enough, one can choose the diameters of the sets U_i to be arbitrarily small. It remains to notice that for any $\delta > 0$ the sequence $\{x_i\}_{i=0}^{\infty}$ is eventually a δ-pseudo-orbit. ∎

Applying the above theorem we can generate many examples of homeomorphisms with limit shadowing but without the AASP. Consider a homeomorphism $f : I \to I$ with finitely many fixed points such that each fixed point is an attractor for either f or f^{-1}. Such an f has both shadowing and weak limit shadowing, but it does not have the AASP, since by Theorem 3.1 it would have to be chain mixing, which is impossible.

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