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Lipschitz and uniform embeddings into ℓ_∞

by

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Abstract. We show that there is no uniformly continuous selection of the quotient map $Q : \ell_\infty \to \ell_\infty/c_0$ relative to the unit ball. We use this to construct an answer to a problem of Benyamini and Lindenstrauss; there is a Banach space X such that there is a no Lipschitz retraction of X^{**} onto X; in fact there is no uniformly continuous retraction from $B_{X^{**}}$ onto B_X.

1. Introduction. It is very well-known that c_0 is not complemented in ℓ_∞; an alternate viewpoint is that there is no continuous linear selection (or right inverse) of the quotient map $Q : \ell_\infty \to \ell_\infty/c_0$. On the other hand, Aharoni and Lindenstrauss [1] showed that there is a large subspace of ℓ_∞/c_0 (isometric to $c_0(I)$ where $|I| = \mathfrak{c}$, the cardinality of the continuum) on which a Lipschitz selection of Q exists. It is also known that c_0 is a Lipschitz retract of ℓ_∞ [15], [4]. This raises the question of whether a global Lipschitz or at least uniformly continuous selection can be found. In fact, there are set-theoretic reasons that one cannot find any selection f so that $f \circ Q$ is weak*-Borel [8], which already means that any formula for such an f must be rather unpleasant. It is also worth noting that every Banach space of density character $\aleph_1$ linearly and isometrically embeds into ℓ_∞/c_0 [16].

In this note we show that there is no selection f which is uniformly continuous on the unit ball of ℓ_∞/c_0. We show in fact that B_{ℓ_∞/c_0} cannot be uniformly embedded into ℓ_∞. We prove this by considering the space $\Omega_n = \omega_1^{[n]}$ of all n-subsets $\{\alpha_1, \dots, \alpha_n\}$ of the countable ordinals as a graph where two subsets are adjacent if they interlace. This is the uncountable analogue of a similar graph considered in [13] in connection with uniform embeddings into reflexive spaces. One then shows that if $f : \Omega_n \to \ell_\infty$ is a map such that $\|f(\alpha) - f(\beta)\| \le 1$ whenever α and β are adjacent then there is an uncountable subset Θ of ω_1 so that $f(\Theta^{[n]})$ has diameter at most one.

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This is very similar to the results of [13] for maps into reflexive spaces on the countable version of this graph.

Our main application of this result is to answer a problem raised in [4, p. 183] (see also Problem 10 of [14]). We give an example of a Banach space X so that there is no Lipschitz retraction of X^{**} onto X; for a discussion of the significance of such an example in the extension theory of Lipschitz maps we refer to [14]. In fact for our example there is no uniformly continuous retraction of the unit ball $B_{X^{**}}$ onto B_X. The space X is a Lindenstrauss space, i.e. its dual is isometric to an L_1-space and its bidual is therefore an injective Banach space (and in particular a 1-absolute Lipschitz retract).

We also discuss non-separable spaces which can be Lipschitz embedded into ℓ_∞. For example, it is shown that every Banach space with an unconditional basis of cardinality at most $\mathfrak{c}$ Lipschitz embeds into ℓ_∞. In the final section, we also show that $\ell_\infty \oplus c_0$ does not have unique Lipschitz structure and show indeed that a very wide class of spaces containing c_0 cannot have unique Lipschitz structure.

2. Notation and preliminaries. All Banach spaces will be real. We denote by B_X the closed unit ball of a Banach space X and by ∂B_X the unit sphere. I_X will denote the identity operator on X.

X is called *weakly compactly generated (WCG)* if there is a weakly compact set W such that $\bigcup_{n\in\mathbb{N}} nW$ is dense in X. A *Markushevich basis* for X is a biorthogonal system $\{x_i, x_i^*\}_{i\in I}$ which is total and fundamental. We shall say that X is *Plichko* if it has a Markushevich basis $\{x_i, x_i^*\}_{i\in I}$ so that the subset E of X^* given by $E = \{x^* : |\{i : x^*(x_i) \neq 0\}| \le \aleph_0\}$ is a 1-norming subspace of X^*. If $E = X^*$ we say that X is *weakly Lindelöf determined (WLD)*. These are not the original definitions of Plichko and WLD spaces but are equivalent (see Theorems 5.37 and 5.63 of [10] or [23] and [22]). Note that WCG spaces are WLD and hence also Plichko. X is said to have the *separable complementation property (SCP)* if every separable subspace of X is contained in a complemented separable subspace.

For any Banach space X we write $\operatorname{dens} X$ for the smallest cardinality of a dense subset (the same definition will be used for metric spaces). We write w^*-dens X for the smallest cardinality of a weak*-dense subset of X^*.

Let M and M' be metric spaces. If $f : M \to M'$ is any mapping we define

$$\psi_f(t) = \sup\{d(f(x), f(x')) : d(x, x') \le t\}.$$

f is said to be *Lipschitz* (with Lipschitz constant L) if $\psi_f(t) \le Lt$ for all t. The function f is *uniformly continuous* if $\lim_{t\to 0} \psi_f(t) = 0$ and *coarsely continuous* if $\psi_f(t) < \infty$ for all t. If f is injective, then f is a Lipschitz (respectively uniform, respectively coarse) *embedding* if f and $f^{-1}|_{f(M)}$ are

Lipschitz (respectively uniformly continuous, respectively coarsely continuous).

If M is a metric space and E is a subset of M then a *retraction* $r : M \to E$ is a map such that $r(e) = e$ for $e \in E$.

If M has a base point (labelled 0), we refer to M as a *pointed metric space* and we define $\mathrm{Lip}(M)$ as the Banach space of all real-valued Lipschitz maps $f : M \to \mathbb{R}$ with the usual norm,

$$\|f\|_{\mathrm{Lip}} = \sup\left\{ \frac{|f(x) - f(x')|}{d(x, x')} : x, x' \in M,\ d(x, x') > 0 \right\}.$$

If $M = X$ is a Banach space, the base point is always the origin. The *Arens–Eells space* Æ(M) is defined as the closed linear span of the point evaluations $\delta_s(f) = f(s)$ in $\mathrm{Lip}(M)^*$. The map $\delta : s \mapsto \delta_s$ is then an isometry of M into Æ(M). We refer to [24] and [9] for further details (in [9] the terminology *Lipschitz-free space* and the notation $\mathcal{F}M$ was used). If X is a Banach space there is a canonical quotient map β : Æ$(X) \to X$ and δ is an isometric selection for β, i.e. $\beta \circ \delta = I_X$.

We will record a result here which is not needed in the sequel but represents an improvement over Proposition 4.1 of [9].

THEOREM 2.1. *Let M be a pointed metric space and suppose $F \subset$ Æ(M) is a bounded non-separable subset of density character $\aleph > \aleph_0$. If $\aleph$ has uncountable cofinality* ([1])*, then there is a subset $(\mu_i)_{i\in I}$ of F with $|I| = \aleph$ so that $(\mu_i)_{i\in I}$ is equivalent to the unit vector basis of $\ell_1(I)$. In particular any weakly compact subset of Æ(M) is separable.*

Proof. If G is any subset of M containing 0, for $0 < \delta < 1$, we denote by $G^{[\delta]}$ the subset of M of all x such that $d(x, G) \le \delta d(x, 0)$. We will first claim that there exists $0 < \delta < 1$ so that, for G as above,

$$\text{if } \mathrm{dens}(G) < \aleph, \text{ then } d(\mu, \text{Æ}(G^{[\delta]})) > \delta \text{ for some } \mu \in F. \tag{2.1}$$

Indeed, if not, for $n \in \mathbb{N}$ we may pick G_n containing 0 so that

$$\mathrm{dens}(G_n) < \aleph \text{ and } d(\mu, \text{Æ}(G_n^{[1/n]}) \le 1/n \quad \text{ for every } \mu \in F.$$

Let $G = \bigcup_n G_n$ and assume $\mu \in F$. We show that $\mu \in$ Æ(G). If not there exists $f \in \mathrm{Lip}(M)$ with $f|_G = 0$, $\|f\|_{\mathrm{Lip}(M)} = 1$ and $\langle \mu, f\rangle = \theta \ne 0$. By picking either $f_+ = \max(f, 0)$ or $f_- = \max(-f, 0)$ we can suppose $f \ge 0$. Now define

$$f_n(x) = \min(f(x), (d(x, G) - d(x, 0)/n)_+).$$

Then $\|f_n\|_{\mathrm{Lip}(M)} \le 1 + 1/n$. We also have

$$f(x) - 1/n \le f_n(x) \le f(x), \quad x \in G,$$

([1]) This assumption has been added by the editors.

and so $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in G$. This implies that $\lim_{n\to\infty} f_n = f$ weak*. Now since $f_n(G^{[1/n]}) = 0$ we have

$$\langle \mu, f_n\rangle \le \frac{1}{n}\left(1+\frac{1}{n}\right)$$

and so $\langle \mu, f\rangle = 0$ contrary to assumption. Thus $F \subset Æ(G)$, contrary to the fact that G, and hence $Æ(G)$, has density character $< \aleph$, while F does not ([2]). This establishes (2.1).

Assuming (2.1) we construct a maximal family $\{G_i, \mu_i, f_i\}_{i\in I}$ where:

(i) $\mu_i \in F$, G_i is a countable subset of M and $\mu_i \in Æ(G_i)$.
(ii) $f_i \in \mathrm{Lip}(M)$ with $f_i \ge 0$ and $\|f_i\|_{\mathrm{Lip}(M)} \le 4/\delta^2$.
(iii) $|\langle \mu_i, f_i\rangle| = 1$ and $\langle \mu_i, f_j\rangle = 0$ for $i \ne j$.
(iv) $\operatorname{supp} f_i = \{f_i > 0\} \subset G_i^{[\delta]}$.
(v) $\operatorname{supp} f_i \cap \operatorname{supp} f_j = \emptyset$ for $i \ne j$.

Let $G = \bigcup_i G_i$; if I is empty we set $G = \{0\}$.

We next argue that $|I| = \aleph$ by contradiction. Suppose $|I| < \aleph$. Then $\operatorname{dens} G < \aleph$. Hence there exists $\mu \in F$ so that $d(\mu, Æ(G^{[\delta]})) > \delta$. Thus there exists a function $g \in \mathrm{Lip}(M)$ with $\|g\|_{\mathrm{Lip}(M)} \le 2/\delta$, $g(G^{[\delta]}) = 0$ and $\langle \mu, g\rangle > 2$. By considering either g_+ or g_- we can therefore find $h \ge 0$ with $\|h\|_{\mathrm{Lip}(M)} \le 2/\delta$, $h(G^{[\delta]}) = 0$ and $|\langle \mu, h\rangle| = 1$.

Take a countable set $G' \subset M$ so that $\mu \in Æ(G')$. Then we define

$$f(x) = \max(\sup\{h(y) - 4\delta^{-2} d(x,y) : y \in G'\}, 0).$$

Then $\|f\|_{\mathrm{Lip}(M)} \le 4\delta^{-2}$. If $x \notin G'^{[\delta]}$, then for any $y \in G'$ we have $d(x,y) > \delta d(x,0)$. Hence

$$d(y,0) \le d(x,0) + d(x,y) \le (\delta^{-1}+1) d(x,y) \le 2\delta^{-1} d(x,y).$$

Then $h(y) \le 2\delta^{-1} d(y,0) \le 4\delta^{-2} d(x,y)$, which implies that $\operatorname{supp} f \subset G'^{[\delta]}$.

Now we can add (G', μ, f) to the collection contradicting maximality. The conclusion is that $|I| = \aleph$.

Finally we argue that (μ_i) is equivalent to the unit vector basis of ℓ_1. Indeed, if A is a finite subset of I and $(a_i)_{i\in A}$ are real numbers we define $h \in \mathrm{Lip}(M)$ by setting $h = \sum_{i\in A} \epsilon_i f_i$ where $\epsilon_i a_i \langle \mu_i, f_i\rangle = |a_i|$. Then $\|h\|_{\mathrm{Lip}(M)} \le 8/\delta^2$. Indeed, suppose $x \in \operatorname{supp} f_i$, $y \in \operatorname{supp} f_j$ where $i \ne j$. Then

$$|h(x) - h(y)| \le |f_i(x) - f_i(y)| + |f_j(x) - f_j(y)| \le 8\delta^{-2} d(x,y).$$

Other cases for x, y give better estimates. Hence

$$\Big\|\sum_{i\in A} a_i\mu_i\Big\| \ge \frac{1}{8}\delta^2 \sum_{i\in A} |a_i|. \ \blacksquare$$

([2]) This sentence has been added by the editors. It seems that in this part of the reasoning the author uses the assumption that $\aleph$ has uncountable cofinality.

3. A metric Ramsey theorem. We denote by ω_1 the first uncountable ordinal. Let $\Omega_n = \omega_1^{[n]}$ where $n \geq 0$ denote the collection of all n-subsets of $\Omega_1 = \omega_1 = [0, \omega_1)$; note that in the case $n = 0$, Ω_0 consists of one point, namely the empty set $\emptyset$. We write a typical element of Ω_n in the form $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 < \cdots < \alpha_n$. If $n \geq 1$ and $A \subset \Omega_n$ we define $\partial A \subset \omega_1^{[n-1]}$ by $\{\alpha_1, \ldots, \alpha_{n-1}\} \in \partial A$ if and only if $\{\beta : \{\alpha_1, \ldots, \alpha_{n-1}, \beta\} \in A\}$ is uncountable. If $n = 1$ this amounts to the fact that $\emptyset \in \partial A$ if and only if A is uncountable. We shall say that $A \subset \Omega_n$ is *large* if $\emptyset \in \partial^n A$; otherwise A is *small*. We say that A is *very large* if its complement $\tilde{A}$ is small.

It follows from the next lemma that if A is very large it is also large.

Lemma 3.1. *Let $(A_k)_{k=1}^{\infty}$ be a sequence of small subsets of Ω_n. Then $\bigcup_{k\geq 1} A_k$ is also small.*

Proof. It is trivial that $\partial \bigcup_k A_k = \bigcup_k \partial A_k$. Iterating gives the result. ∎

Lemma 3.2. *If A is a very large subset of Ω_n there is an uncountable subset $\Theta \subset \Omega_1$ so that $\Theta^{[n]} \subset A$.*

Proof. Let Θ be a maximal subset of Ω_1 so that if $\{\alpha_1, \ldots, \alpha_k\} \in \Theta^{[k]}$ with $0 \leq k \leq n$ then $\{\alpha_1, \ldots, \alpha_k\} \notin \partial^{n-k} \tilde{A}$; here we write $\partial^0 \tilde{A} = \tilde{A}$. Such a maximal subset exists by Zorn's Lemma since $\emptyset$ satisfies the conditions. We show that Θ is uncountable. Assume, on the contrary, that Θ is countable. For each $\{\alpha_1, \ldots, \alpha_k\} \in \Theta^{[k]}$ with $0 \leq k \leq n-1$ the set of β such that $\{\alpha_1, \ldots, \alpha_k, \beta\} \in \partial^{n-k+1} \tilde{A}$ is countable. Let

$$h\{\alpha_1, \ldots, \alpha_k\} = \sup\{\beta : \{\alpha_1, \ldots, \alpha_k, \beta\} \in \partial^{n-k+1} \tilde{A}\}$$

so that $h\{\alpha_1, \ldots, \alpha_k\} < \omega_1$. Let σ be the supremum of all $h\{\alpha_1, \ldots, \alpha_k\}$; since Θ is countable we have $\sigma < \omega_1$. Now $\sigma \geq \sup \Theta$ and $\Theta \cup \{\sigma + 1\}$ gives a contradiction to maximality. ∎

Lemma 3.3. *Let $f : \Omega_n \to \mathbb{R}$ be any mapping. Then there is an open set $U \subset \mathbb{R}$ so that $f^{-1}(U)$ is small and if V is any open set with $f^{-1}(V)$ small then $V \subset U$.*

Proof. Let $\mathcal{U}$ be the set of all open sets V so that $f^{-1}(V)$ is small. Then by the Lindelöf theorem and Lemma 3.1, $U \in \mathcal{U}$ where $U = \bigcup\{V : V \in \mathcal{U}\}$. ∎

We will make Ω_n into a graph by declaring $\alpha \neq \beta$ to be adjacent if either

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_n \leq \beta_n$$

or

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n,$$

i.e. α and β *interlace*. Then we define d to be the least path metric on Ω_n, which then becomes a metric space. We write $\alpha \prec \beta$ if

$$\alpha_1 < \cdots < \alpha_n < \beta_1 < \cdots < \beta_n.$$

If $\alpha \prec \beta$ then $d(\alpha, \beta) = n$ so that Ω_n has diameter n.

LEMMA 3.4. *Suppose A and B are large subsets of Ω_n. Then there exist $\alpha \in A$ and $\beta \in B$ so that α, β interlace.*

Proof. Pick $\alpha_1 \in \partial^{n-1}A$ and then, since $\partial^{n-1}B$ is uncountable, we may pick $\beta_1 \in \partial^{n-1}B$ with $\beta_1 > \alpha_1$. Continuing we may pick $\alpha_2 > \beta_1$ so $\{\alpha_1, \alpha_2\} \in \partial^{n-2}A$ and continue. ∎

LEMMA 3.5. *Let $f : (\Omega_n, d) \to \mathbb{R}$ be any Lipschitz function with Lipschitz constant L. Then there exists $\xi \in \mathbb{R}$ so that $\{\alpha : |f(\alpha) - \xi| > L/2\}$ is small.*

Proof. Let U be the maximal open subset of $\mathbb{R}$ so that $f^{-1}(U)$ is small, given by Lemma 3.3. Let E be its complement, which is necessarily nonempty and closed. It suffices to show that the diameter of E is at most L. Suppose $s, t \in E$ with $s-t > L$. Then we can pick $\epsilon > 0$ so small that $s-\epsilon > t+\epsilon+L$. Now the sets $f^{-1}(s-\epsilon, \infty)$ and $f^{-1}(-\infty, t+\epsilon)$ are both large. By Lemma 3.4 there exist α, β which are interlacing and so that $f(\alpha) > s-\epsilon$ and $f(\beta) < t+\epsilon$, which contradicts the definition of the Lipschitz constant. ∎

THEOREM 3.6. *Let $f : (\Omega_n, d) \to \ell_\infty$ be a Lipschitz map with Lipschitz constant L. Then there exists $\xi \in \ell_\infty$ and an uncountable subset Θ of Ω_n so that*

$$\|f(\alpha) - \xi\| \le L/2, \qquad \alpha \in \Theta^{[n]}.$$

Proof. Let $f(\alpha) = (f_n(\alpha))_{n=1}^{\infty}$. According to Lemma 3.5 there exists $\xi_n \in \mathbb{R}$ so that $f_n^{-1}[\xi_n - L/2, \xi_n + L/2]$ is very large. Hence if $\xi = (\xi_n)_{n=1}^{\infty}$, the set of α with $|f_n(\alpha) - \xi_n| \le L/2$ for all n is also very large (using Lemma 3.1). In particular $\xi \in \ell_\infty$. The proof is completed by Lemma 3.2. ∎

4. Embeddings in ℓ_∞. Let X be a Banach lattice. We will say that X has the *monotone transfinite sequence property (MTSP)* if whenever $(x_\mu)_{\mu<\omega_1}$ is a monotone increasing transfinite sequence, then there exists $x \in X$ such that $x_\mu = x$ eventually.

THEOREM 4.1. *Let X be a Banach lattice with the property that B_X can be uniformly embedded into ℓ_∞. Then X has (MTSP).*

Proof. Let us assume X fails (MTSP). Since any transfinite sequence $(x_\mu)_{\mu<\omega_1}$ is necessarily bounded we may assume it takes values in B_X. Let $\theta(\mu) = \sup_{\sigma>\mu} \|x_\sigma - x_\mu\|$. Then $\theta(\mu)$ is decreasing.

Let $f : B_X \to \ell_\infty$ be a uniform embedding, with inverse $g : f(B_X) \to B_X$.

For each n consider the map $f_n : \Omega_n \to \ell_\infty$ given by $f_n(\alpha) = f\big(\frac{1}{n}\sum_{j=1}^{n} x_{\alpha_j}\big)$. If $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_n < \beta_n$ we have

$$\|f_n(\alpha) - f_n(\beta)\| = \Big\|\frac{1}{n}\sum_{j=1}^{n}(x_{\beta_j} - x_{\alpha_j})\Big\| \le \frac{1}{n}\|x_{\beta_n} - x_{\alpha_1}\| \le 2/n.$$

Hence f_n has Lipschitz constant $\psi_f(2/n)$.

By Theorem 3.6 we may pick an uncountable subset Θ_n of Ω_1 so that

$$\|f_n(\alpha) - f_n(\beta)\| \le \psi_f(2/n), \qquad \alpha, \beta \in \Theta_n.$$

Hence

$$\left\| \frac{1}{n}\sum_{j=1}^n x_{\beta_j} - \frac{1}{n}\sum_{j=1}^n x_{\alpha_j} \right\| \le \psi_g(\psi_f(2/n)), \quad \alpha, \beta \in \Theta_n.$$

Pick $\alpha_1 < \cdots < \alpha_n \in \Theta_n$. If $\nu > \mu > \alpha_n$ then we can find $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \nu$ with $\beta_j \in \Theta_n$ for $1 \le j \le n$ and then

$$\|x_\nu - x_\mu\| \le \left\| \frac{1}{n}\sum_{j=1}^n x_{\beta_j} - \frac{1}{n}\sum_{j=1}^n x_{\alpha_j} \right\| \le \psi_g(\psi_f(2/n)),$$

and hence

$$\theta(\mu) \le \psi_g(\psi_f(2/n)), \quad \mu > \alpha_n.$$

Applying this for every n, since $\lim_{n\to\infty} \psi_g(\psi_f(2/n)) = 0$ we have that $\theta(\mu) = 0$ eventually, which implies that x_μ is eventually constant. ■

THEOREM 4.2. *Let $X = \ell_\infty/c_0$ or $\mathcal{C}[1,\omega_1]$. Then B_X cannot be uniformly embedded into ℓ_∞.*

Proof. Since $\mathcal{C}[1,\omega_1]$ embeds into ℓ_∞/c_0 it is not really necessary to prove these separately. However it is easy to observe that both spaces fail (MTSP). For the case of ℓ_∞/c_0 we may, by induction, define a transfinite sequence of infinite subsets $A_\mu \subset \mathbb{N}$ so that if $\mu > \nu$ we have $A_\nu \setminus A_\mu$ infinite and $A_\mu \subset A_\nu$ modulo finite sets. Let B_μ be the complement of A_μ and then $x_\mu = Q(\chi_{B_\mu})$ where $Q : \ell_\infty \to c_0$ is the quotient map. For the case of $\mathcal{C}[1,\omega_1]$ let $x_\mu = \chi_{[1,\mu]}$. In either case the result follows from the preceding Theorem 4.1. ■

REMARK. In [17] it is shown that $\mathcal{C}[1,\omega_1]$ does not uniformly embed into a space $c_0(I)$. In [1] it is shown that if $|I| = \mathfrak{c}$ then $c_0(I)$ Lipschitz embeds in ℓ_∞. Therefore Theorem 4.2 also implies that $\mathcal{C}[1,\omega_1]$ does not uniformly embed into $c_0(I)$ for any set I. Note however that there is an injective linear map $T : \mathcal{C}[1,\omega_1] \to c_0(I)$ where $|I| = \aleph_1$ defined by $Tf(\alpha) = f(\alpha+1) - f(\alpha)$ for $\alpha < \omega_1$ and $Tf(\omega_1) = f(1)$. Thus there is a countable family of Lipschitz functions on $\mathcal{C}[1,\omega_1]$ which separate the points of $\mathcal{C}[1,\omega_1]$.

We now give a slight variation of Theorem 4.2. Let us say that a Banach lattice X has the *countable interpolation property* if whenever $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are two sequences with $x_m \ge y_n$ whenever $m, n \in \mathbb{N}$ then there exists $z \in X$ with $x_m \ge z \ge y_n$ for all $m, n \in \mathbb{N}$. If K is a compact Hausdorff space then $\mathcal{C}(K)$ has the countable interpolation property if and only if K is an F-space [21]; here K is an F-space if the closures of two disjoint open F_σ-sets remain disjoint (see [7]). The space $\beta\mathbb{N} \setminus \mathbb{N}$ is an F-space so the following result implies Theorem 4.2 for the case ℓ_∞/c_0.

Proposition 4.3. *Let X be a Banach lattice with (MTSP) and the countable interpolation property. Then X is order-complete. In particular if K is an F-space such that $\mathcal{C}(K)$ has (MTSP), then K is Stonian.*

Proof. Let A be a subset of X which is bounded above. Let us say that A has *property* (P) if for every countable set B with $x \le y$ for $x \in A$ and $y \in B$ there exists $z \in X$ with $x \le z \le y$ for $x \in A$, $y \in B$. We claim that property (P) implies A has a least upper bound (or supremum). If not let y_1 be any upper bound. We construct a strictly decreasing transfinite sequence $(y_\mu)_{\mu<\omega_1}$ by transfinite induction. If $(y_\nu)_{\nu<\mu}$ has been chosen to be strictly decreasing we pick for each ν an upper bound $y'_\nu \le y_\nu$ with $y'_\nu \ne y_\nu$ and then use (P) to find an upper bound $y_\mu \le y'_\nu$ for all $\nu < \mu$. This contradicts (MTSP) and shows that A has a least upper bound.

Now by the countable interpolation property, countable sets have property (P) and hence have a least upper bound. But this implies that every set A which is bounded above has property (P) and the proof is complete. ∎

We will now turn to an application of the above results. We will need the following elementary lemma, which appears first in [12]; we include the proof for the convenience of the reader.

Lemma 4.4. *Let Y be a Banach space and let $Q : Y \to X$ be a quotient mapping. In order that there exists a uniformly continuous section $f : B_X \to Y$ it is necessary and sufficient that for some $0 < \lambda < 1$ there is a uniformly continuous map $\phi : \partial B_X \to Y$ with $\|Q(\phi(x)) - x\| \le \lambda$ for $x \in \partial B_X$.*

Proof. We may extend ϕ to B_X to be positively homogeneous and ϕ remains uniformly continuous. Define $g(x) = x - Q(\phi(x))$, so that g is also positively homogeneous. Then $\|g(x)\| \le \lambda\|x\|$ and so $\|g^n(x)\| \le \lambda^n\|x\|$ for $x \in B_X$. Let $g^0(x) = x$. Let

$$f(x) = \sum_{n=0}^{\infty} \phi(g^n(x)).$$

The series converges uniformly in $x \in B_X$ and so f is uniformly continuous. Furthermore

$$Qf(x) = \sum_{n=0}^{\infty} (g^n(x) - g^{n+1}(x)) = x. \quad ∎$$

The following theorem answers a question raised in [4, p. 183] (see also the discussion in [14]).

Theorem 4.5. *There exists a (non-separable) Lindenstrauss space Z such that there is no uniformly continuous retraction of $B_{Z^{**}}$ onto B_Z. In particular there is no uniformly continuous retraction of Z^{**} onto Z.*

Proof. The space Z is the same example that was given by Benyamini [3] as an example of a non-separable M-space which is not a $\mathcal{C}(K)$-space.

We start by considering the quotient map $Q : \ell_\infty \to \ell_\infty/c_0$. For each n we pick a maximal set D_n in the interior of B_{ℓ_∞/c_0} so that $\|x-x'\| \geq 1/n$ for $x, x' \in D_n$ and $x \neq x'$. Then for each n we can define a map $h_n : D_n \to B_{\ell_\infty}$ with $Qh_n(x) = x$ for $x \in D_n$.

Now let Y_n denote ℓ_∞ with the equivalent norm

$$\|y\|_{Y_n} = \max\left(\frac{1}{n}\|y\|_{\ell_\infty}, \|Qy\|_{\ell_\infty/c_0}\right).$$

Then Y_n is a Lindenstrauss space; note also that $Q : Y_n \to \ell_\infty/c_0$ remains a quotient map for the usual norm on ℓ_∞/c_0. Let $Z = c_0(Y_n)$, which is also a Lindenstrauss space. Let us assume that there is a uniformly continuous retraction of $B_{Z^{**}}$ onto B_Z. Then it follows that there is a sequence of uniformly continuous retractions $g_n : Y_n^{**} \to Y_n$ which is equi-uniformly continuous, i.e. so that

$$\psi_{g_n}(t) \leq \psi(t), \qquad 0 < t \leq 2,$$

where $\lim_{t\to 0} \psi(t) = 0$.

Consider the map $h_n : D_n \to Y_n$. If $x \neq x' \in D_n$ then

$$\|h_n(x) - h_n(x')\| \leq \max\left(\frac{2}{n}, \|x - x'\|\right) \leq 2\|x - x'\|.$$

Hence since $B_{Y_n^{**}}$ is a 1-absolute Lipschitz retract, there is an extension $f_n : B_{\ell_\infty/c_0} \to B_{Y_n^{**}}$ with $\mathrm{Lip}(f_n) \leq 2$. Now if $x \in B_{\ell_\infty/c_0}$ there exists $x' \in D_n$ with $\|x - x'\| < 2/n$ (actually $1/n$ if x is in the open unit ball). Thus

$$\|g_n(f_n(x)) - g_n(f_n(x'))\| \leq \psi(4/n)$$

and hence

$$\|Q(g_n \circ f_n(x)) - x\| \leq \psi(4/n) + 2/n.$$

For some choice of n we have $\psi(4/n)+2/n < 1$. By Lemma 4.4 this means there is a uniformly continuous section of the quotient map $Q : B_{\ell_\infty/c_0} \to Y_n$. Thus B_{ℓ_∞/c_0} uniformly embeds into Y_n which is isomorphic to ℓ_∞ and thus we have contradicted Theorem 4.2. ∎

Remark. Note that $\mathcal{C}[1,\omega_1]$ isometrically embeds into ℓ_∞ [16]; let F be a subspace of ℓ_∞/c_0 isometric to $\mathcal{C}[1,\omega_1]$. Let $E = Q^{-1}(F)$ and let Z_0 be the subspace of Z of all sequences $(y_n)_{n=1}^\infty$ such that $y_n \in E$. Then arguing in exactly the same way shows that there is no uniformly continuous retraction of $B_{Z_0^{**}}$ onto B_{Z_0}. Thus Z_0 is an example of a Lindenstrauss space with the properties of Theorem 4.5 with the additional property that Z_0^* is isometric to $\ell_1(I)$ where $|I| = \aleph_1$.

5. Banach spaces which Lipschitz embed into ℓ_∞

Proposition 5.1. *The Arens–Eells space $Æ(\ell_\infty)$ of ℓ_∞ is linearly isometric to a closed linear subspace of ℓ_∞.*

Proof. For each $m \in \mathbb{N}$ we may pick a countable collection of functions $\{f_{mn}\}_{n=1}^\infty$ in $\mathrm{Lip}(\ell_\infty^m)$ with $\|f_{mn}\|_{\mathrm{Lip}(\ell_\infty^m)} \le 1$ and such that

$$\|\mu\|_{Æ(\ell_\infty^m)} = \sup_n \langle \mu, f_{mn}\rangle, \qquad \mu \in Æ(\ell_\infty^m).$$

Now for each finite subset $J = \{k_1, \dots, k_m\} \subset \mathbb{N}$ we define the natural quotient $Q_J : \ell_\infty \to \ell_\infty^m$ by $Q\xi = (\xi_{k_j})_{j=1}^m$. We consider the countable family Φ of Lipschitz functions $\varphi = \varphi_{J,n}$ of the form

$$\varphi_{J,n}(\xi) = f_{mn}(Q_J\xi).$$

Clearly $\|\varphi_{J,n}\|_{\mathrm{Lip}(\ell_\infty)} \le 1$ for all J and n.

Now suppose $\mu \in Æ(\ell_\infty)$, with $\mu \neq 0$, has finite support $\xi_1, \dots, \xi_n \in \ell_\infty$ and let F be the linear span of $\xi_1, \dots, \xi_n$. Then $\|\mu\|_{Æ(\ell_\infty)} = \|\mu\|_{Æ(F)}$.

Given $\epsilon > 0$ we can find a finite subset J of $\mathbb{N}$, with $|J| = m$, say, so that

$$\|Q_J\xi\| > (1-\epsilon)\|\xi\|, \qquad \xi \in F.$$

Hence if $\xi = \sum_{j=1}^n a_j\delta_{\xi_j}$,

$$\Big\|\sum_{j=1}^n a_j\delta_{Q_J\xi_j}\Big\|_{Æ(\ell_\infty^m)} > (1-\epsilon)\|\mu\|_{Æ(\ell_\infty)}.$$

Thus we can find n so that

$$\langle \mu, \varphi_{J,n}\rangle > (1-\epsilon)\|\mu\|_{Æ(\ell_\infty)}.$$

Thus the map

$$\mu \mapsto (\langle \mu, \varphi_{J,n}\rangle)_{J,n}$$

defines an isometry of $Æ(\ell_\infty)$ into $\ell_\infty(S)$ where S is a countable set. ∎

Lemma 5.2. *In order that a Banach space X Lipschitz embeds into ℓ_∞ it is necessary and sufficient that there is a Lipschitz function $g : X \to \ell_\infty$ with the property that*

$$\|g(u) - g(v)\| \ge 1, \qquad \|u - v\| \ge 1.$$

Proof. Let $\mathbb{Q}_+$ denote the set of positive rationals and define $G : X \to \ell_\infty(\mathbb{Q}_+ \times \mathbb{N})$ by $G(x)(q,n) = q^{-1}g(qx)_n$. Clearly G has Lipschitz constant at most $\mathrm{Lip}(g)$. If $u, v \in X$ we have

$$\|G(u) - G(v)\| = \sup_{t>0} t^{-1}\|g(tu) - g(tv)\|.$$

If $u \neq v$ let $t = \|u-v\|^{-1}$ and we have

$$\|G(u) - G(v)\| \ge \|u - v\|.$$

Thus G is a Lipschitz embedding. ∎

THEOREM 5.3. *Let X be a Banach space. Then the following conditions on X are equivalent:*

(i) *X uniformly embeds into ℓ_∞.*
(ii) *X coarsely embeds into ℓ_∞.*
(iii) *X Lipschitz embeds into ℓ_∞.*
(iv) *$\text{Æ}(X)$ linearly embeds into ℓ_∞.*
(v) *There is a subspace Z of ℓ_∞ and a linear surjection $Q : Z \to X$ which admits a Lipschitz selection $\varphi : X \to Z$.*
(vi) *There is a subspace Z of ℓ_∞ and a linear surjection $Q : Z \to X$ which admits a uniformly continuous selection $\varphi : X \to Z$.*
(vii) *There is a subspace Z of ℓ_∞ and a linear surjection $Q : Z \to X$ which admits a coarsely continuous selection $\varphi : X \to Z$.*

Proof. (i) or (ii) $\Rightarrow$ (iii). Let $f : X \to \ell_\infty$ be either a coarse embedding or a uniform embedding. Then, in either case, we may find $0 < a, b, c < \infty$ such that

$$\|x - y\| \le a \;\Rightarrow\; \|f(x) - f(y)\| \le b$$

and

$$\|f(x) - f(y)\| \le 5b \;\Rightarrow\; \|x - y\| \le c.$$

Pick a maximal subset S of X such that $\|x-y\| \ge a$ if $x, y \in S$ and $x \ne y$. Then $f|_S$ has Lipschitz constant bounded by $\sup_{\delta \ge a} \omega(\delta)/\delta \le 2b/a$. Since ℓ_∞ is a 1-absolute Lipschitz retract we may find a function $h : X \to \ell_\infty$ with Lipschitz constant at most $2b/a$ and $h|_S = f|_S$. Now assume $\|u-v\| \ge c+2a$, Then there exist $x, y \in S$ with $\|x - u\|, \|y - v\| < a$ and hence $\|x - y\| > c$. Thus $\|f(x) - f(y)\| > 5b$. Also $\|h(x) - f(u)\| < 2b$ and $\|h(y) - f(v)\| < 2b$. Hence $\|h(u) - h(v)\| > b$. If we let $g(x) = b^{-1}h((a + 2c)x)$ then g satisfies the hypotheses of Lemma 5.2 and so (iii) is proved.

(iii) $\Rightarrow$ (iv). $\text{Æ}(X)$ linearly embeds into $\text{Æ}(\ell_\infty)$; then use Proposition 5.1.

(iv) $\Rightarrow$ (v). Take $Z = \text{Æ}(X)$ (cf. [9]).

(v) $\Rightarrow$ (vi) and (vii) is trivial. Clearly (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (ii). ∎

Let $|I| = \mathfrak{c}$. Then $\ell_1(I)$ linearly isometrically embeds into ℓ_∞. On the other hand $c_0(I)$ 2-Lipschitz embeds into ℓ_∞ by the result of [1]. The following theorem shows that we also have a Lipschitz embedding for spaces such as $\ell_p(I)$ where $1 < p < \infty$.

THEOREM 5.4. *Let X be a Banach space with a 1-unconditional basis $(e_i)_{i\in I}$ where $|I| \le \mathfrak{c}$. Then X 2-Lipschitz embeds in ℓ_∞.*

Proof. We will consider the case when $I = \mathbb{R}$, i.e. the basis is indexed by $\mathbb{R}$. Let $(e_t^*)_{t\in\mathbb{R}}$ denote the biorthogonal functionals. If $x \in X$ we write $x(t) = e_t^*(x)$. Suppose $a, b, c \in \mathbb{Q}^n$ for some $n \in \mathbb{N}$. We write $a = (a_1, \ldots, a_n)$ and so on; we denote by $-a$ the sequence $(-a_1, \ldots, -a_n)$. We then define a subset $U(a,b,c) \subset \mathbb{R}^n$ by $(t_1,\dots,t_n) \in U(a,b,c)$ if $b_j < t_j < c_j$ for $j=1,\dots,n$, $t_1 < \cdots < t_n$ and

$$\Big\| \sum_{j=1}^n a_j e_{t_j}^* \Big\|_{X^*} \le 1.$$

Note that the set $U(a,b,c)$ can be empty.

For $t \in \mathbb{R}$ we write $t_+ = \max(t,0)$ and $t_- = t_+ - t$. We next define $f_{a,b,c} : X \to \mathbb{R}$ by $f_{a,b,c} \equiv 0$ if $U(a,b,c)$ is empty and otherwise

$$f_{a,b,c}(x) = \sup\Big\{ \sum_{j=1}^n (a_j x(t_j))_+ : (t_1,\dots,t_n) \in U(a,b,c) \Big\}.$$

It follows from the definition that each $f_{a,b,c}$ is Lipschitz with constant at most one and $f_{a,b,c}(0) = 0$.

Suppose $x, y \in X$ and $\epsilon > 0$. Then we may find a finite subset $\{t_1,\dots,t_n\}$ of $\mathbb{R}$ with $t_1 < \cdots < t_n$ so that

$$\Big\| x - \sum_{j=1}^n x(t_j) e_{t_j} \Big\| < \epsilon/6, \qquad \Big\| y - \sum_{j=1}^n y(t_j) e_{t_j} \Big\| < \epsilon/6.$$

Then pick $a_1,\dots,a_n \in \mathbb{Q}$ so that $\| \sum_{j=1}^n a_j e_{t_j}^* \| \le 1$ and

$$\sum_{j=1}^n a_j (x(t_j) - y(t_j)) > \|x-y\| - \epsilon/3.$$

Finally pick b_j, c_j rationals such that $b_1 < t_1 < c_1 < b_2 < t_2 < \cdots < t_n < c_n$.

It is clear that

$$\Big| f_{a,b,c}(x) - \sum_{j=1}^n (a_j x(t_j))_+ \Big| < \epsilon/6$$

and

$$\Big| f_{-a,b,c}(x) - \sum_{j=1}^n (a_j x(t_j))_- \Big| < \epsilon/6.$$

Thus

$$\Big| f_{a,b,c}(x) - f_{-a,b,c}(x) - \sum_{j=1}^n a_j x(t_j) \Big| < \epsilon/3.$$

Similarly

$$\Big| f_{a,b,c}(y) - f_{-a,b,c}(y) - \sum_{j=1}^n a_j y(t_j) \Big| < \epsilon/3.$$

Hence

$$f_{a,b,c}(x) - f_{a,b,c}(y) - f_{-a,b,c}(x) + f_{-a,b,c}(y) > \|x-y\| - \epsilon$$

and either

$$|f_{a,b,c}(x) - f_{a,b,c}(y)| > \frac{1}{2}(\|x-y\| - \epsilon)$$

or

$$|f_{-a,b,c}(x) - f_{-a,b,c}(y)| > \frac{1}{2}(\|x-y\| - \epsilon).$$

This shows that the map $F(x) = (f_{a,b,c}(x))_{(a,b,c)\in\bigcup_n(\mathbb{Q}^n)^3}$ defines a 2-Lipschitz embedding of X into ℓ_∞. ∎

Remark. The constant 2 is optimal for $c_0(\mathbb{R})$. Indeed suppose $F : c_0(\mathbb{R}) \to \ell_\infty$ satisfies $F(0) = 0$ and

$$\|x-y\| \le \|F(x) - F(y)\| \le \lambda\|x-y\|, \qquad x, y \in c_0(\mathbb{R}),$$

where $\lambda < 2$. We may define the sets $A_t = \{n : |F(e_t)_n - F(-e_t)_n| > \lambda\}$. It is then clear that the sets $\{A_t : t \in \mathbb{R}\}$ are disjoint and non-empty, producing a contradiction. It would, however, be interesting to know the best constant for an embedding of $\ell_p(I)$ where $1 < p < \infty$.

We also note that we do not know whether every reflexive space (or even WCG space) of density character $\mathfrak{c}$ (or even $\aleph_1$) can be Lipschitz embedded into ℓ_∞.

6. Applications of pull-back constructions. Our final section is inspired by a result of Cabello Sánchez and Castillo [5]. We will need a preliminary lemma:

Lemma 6.1. *Let $0 \to X \to Y_1 \to Z_1 \to 0$ be a short exact sequence of Banach spaces and suppose there is a Lipschitz selection $\varphi : Z_1 \to Y_1$ of the quotient map $Y_1 \to Z_1$. Let $T : Z \to Z_1$ be a bounded linear operator and let $0 \to X \to Y \to Z \to 0$ be the short exact sequence obtained by the pull-back construction. Then there is a Lipschitz selection $\psi : Z \to Y$ of the quotient map $Y \to Z$. Thus Y is Lipschitz isomorphic to $X \oplus Z$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle I_X} & & \uparrow{\scriptstyle \tilde{T}} & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

We may define a map $\theta : Z \to Y$, which is a selection for the quotient $Y \to Z$ and such that $\tilde{T} \circ \theta = \varphi \circ T$. We must show that θ is Lipschitz. Assume φ is Lipschitz with constant L. If $z_1, z_2 \in Z$ we may determine $y_1, y_2 \in Y$ with $\|y_1 - y_2\| \le 2\|x_1 - x_2\|$ and $Q_Z y_1 = z_1,\ Q_Z y_2 = x_2$ where Q_Z is the quotient map. Then $\theta(z_j) - y_j \in X$ and $\theta(z_j) - y_j = \varphi(Tz_j) - \tilde{T}y_j$

for $j = 1, 2$. Thus

$$\begin{aligned}\|\theta(z_1) - \theta(z_2)\| &\le 2\|z_1 - z_2\| + \|\theta(z_1) - y_1 - \theta(z_2) + y_2\| \\ &\le 2\|z_1 - z_2\| + \|\varphi(Tz_1) - \varphi(Tz_2)\| + \|\tilde{T}(y_1 - y_2)\| \\ &\le (2 + \|\tilde{T}\| + L\|T\|)\|z_1 - z_2\|. \ \blacksquare\end{aligned}$$

Remark. The same proof works for uniformly continuous selections. A simpler proof works for push-outs.

We now follow the argument of Cabello Sánchez and Castillo [5]. We start with the short exact sequence constructed by Johnson and Lindenstrauss [11] and later used by Aharoni and Lindenstrauss [1] as a counterexample to the Lipschitz isomorphism problem for non-separable Banach spaces:

$$0 \to c_0 \to JL_\infty \to c_0(I) \to 0$$

where $|I| = \mathfrak{c}$, Y embeds in ℓ_∞ and there is a 2-Lipschitz selection of the quotient map of Y onto $c_0(I)$. If we consider the inclusion map $T : \ell_2(I) \to c_0(I)$ and perform the pull-back construction we obtain a short exact sequence

$$0 \to c_0 \to JL_2 \to \ell_2(I) \to 0.$$

Here JL_2 denotes another Johnson–Lindenstrauss space first constructed in [11], as an example in the theory of WCG spaces. JL_2 is a not a WCG space and no non-separable subspace of JL_2 linearly embeds ℓ_∞. An immediate conclusion, using Theorem 5.4 and Lemma 6.1, is:

Proposition 6.2. *JL_2 is Lipschitz isomorphic to $c_0 \oplus \ell_2(I)$ and Lipschitz embeds into ℓ_∞.*

Continuing we note (following [5]) that there is a quotient map $S : \ell_\infty \to \ell_2(I)$ (first proved by Rosenthal [19], see also [10, p. 141]). Performing the pull-back operation again we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & JL_\infty & \longrightarrow & c_0(I) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle I_{c_0}} & & \uparrow{\scriptstyle \tilde{T}} & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & c_0 & \longrightarrow & JL_2 & \longrightarrow & \ell_2(I) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle I_{c_0}} & & \uparrow{\scriptstyle \tilde{S}} & & \uparrow{\scriptstyle S} & & \\
0 & \longrightarrow & c_0 & \longrightarrow & CC & \longrightarrow & \ell_\infty & \longrightarrow & 0
\end{array}$$

Proposition 6.3. *The Banach space CC is linearly isomorphic to a closed linear subspace of ℓ_∞ and Lipschitz isomorphic to $\ell_\infty \oplus c_0$. However it is not linearly isomorphic to $\ell_\infty \oplus c_0$.*

Proof. CC is isomorphic to a subspace of ℓ_∞ because JL_∞ embeds into ℓ_∞ and CC is a subspace of $JL_\infty \oplus \ell_\infty$. The fact that it is not isomorphic to $\ell_\infty \oplus c_0$ is shown in [5], but we include the details. Thus JL_2 is a quotient of CC but every operator from ℓ_∞ to JL_2 is weakly compact ([20]); but since JL_2 is not WCG it cannot be a quotient of $\ell_\infty \oplus c_0$. However CC is Lipschitz isomorphic to $\ell_\infty \oplus c_0$ by an application of Lemma 6.1. ∎

REMARK. This proposition shows that $\ell_\infty \oplus c_0$ does not have unique Lipschitz structure. Other examples of $\mathcal{C}(K)$-spaces with non-unique Lipschitz structure have been given in [1], [6] and [2]. We do not know if CC (which is an $\mathcal{L}_\infty$-space) is linearly isomorphic to any $\mathcal{C}(K)$-space.

We do not know, however, if ℓ_∞ has unique Lipschitz structure.

We conclude by showing that Proposition 6.2 can be generalized considerably:

THEOREM 6.4. *Let X be a Plichko space containing a subspace isomorphic to c_0 and a non-separable subspace X_0 with $\aleph_0 < w^*\text{-}\operatorname{dens} X_0 \le \operatorname{dens} X_0 \le \mathfrak{c}$. Then X fails to have unique Lipschitz structure.*

In particular if X is a non-separable WLD space containing a subspace isomorphic to c_0 then X fails to have unique Lipschitz structure.

Proof. If X is Plichko, then X has the separable complementation property [18] and so every copy of c_0 in X is complemented. In particular X is linearly isomorphic to $X \oplus c_0$.

Let $(x_j, x_j^*)_{j\in J}$ be a Markushevich basis for X. Then there is a subset $I_0 \subset J$ with $\aleph_0 < |I| \le \mathfrak{c}$ such that $X_0 \subset [x_i]_{i\in I_0}$. Let $I \supset I_0$ be a set with $|I| = \mathfrak{c}$. Define the map $T : X \to c_0(I)$ by $Tx(i) = x_i^*(x)$ for $i \in I_0$ and $Tx(i) = 0$ for $i \in I \setminus I_0$.

We now construct the pull-back from the Johnson–Lindenstrauss sequence $0 \to c_0 \to JL_\infty \to c_0(I) \to 0$ as before:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & JL_\infty & \longrightarrow & c_0(I) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle I_{c_0}} & & \uparrow{\scriptstyle \tilde{T}} & & \uparrow{\scriptstyle T} & & \\
0 & \longrightarrow & c_0 & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

We claim that the pull-back sequence cannot split. Indeed, if c_0 is complemented in Z it follows there exists a bounded operator $S : X \to JL_\infty$ so that $QS = T$ where $Q : JL_\infty \to c_0(I)$ is the quotient map. The linear functionals $(x_i^*)_{i\in I_0}$ separate the points of X_0 and so $T|_{X_0}$ is injective: hence S is injective and thus w^*-dens $X_0 = \aleph_0$, which gives a contradiction. Hence Z fails (SCP) and is not linearly isomorphic to X. However Z is Lipschitz isomorphic to $X \oplus c_0$ and hence to X using Lemma 6.1.

If X is WLD we may take X_0 to be any subspace with $\operatorname{dens} X_0 = \aleph_1$. Then X_0 is also WLD and hence (see [10, p. 181]) $\operatorname{dens} X_0 = w^*\text{-}\operatorname{dens} X_0 = \aleph_1$. ∎

REMARK. Thus, for example $X = c_0 \oplus Y$, where Y is any non-separable reflexive space, fails to have unique Lipschitz structure. The theorem also applies to $\mathcal{C}[1,\omega_1]$ which is a Plichko space since $\operatorname{dens}\mathcal{C}[1,\omega_1] = w^*\text{-}\operatorname{dens}\mathcal{C}[1,\omega_1] = \aleph_1$. Thus $\mathcal{C}[1,\omega_1]$ fails to have unique Lipschitz structure.

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