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Uniformization and anti-uniformization properties of ladder systems

by

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Abstract. Natural weakenings of uniformizability of a ladder system on ω_1 are considered. It is shown that even assuming CH all the properties may be distinct in a strong sense. In addition, these properties are studied in conjunction with other properties inconsistent with full uniformizability, which we call anti-uniformization properties. The most important conjunction considered is the uniformization property we call countable metacompactness and the anti-uniformization property we call thinness. The existence of a thin, countably metacompact ladder system is used to construct interesting topological spaces: a countably paracompact and nonnormal subspace of ω_1^2, and a countably paracompact, locally compact screenable space which is not paracompact. Whether the existence of a thin, countably metacompact ladder system is consistent is left open. Finally, the relation between the properties introduced and other well known properties of ladder systems, such as ♣, is considered.

1. Introduction. Let S denote a stationary subset of limit ordinals of ω_1. A *ladder system* on S is a sequence $\{L_\alpha : \alpha \in S\}$ such that each L_α is an unbounded subset of α of order type ω.

A ladder system is *uniformizable* if for each sequence $\langle f_\alpha : \alpha \in S\rangle$ of functions $f_\alpha : L_\alpha \to \omega$ there is an $F : \omega_1 \to \omega$ such that $F\restriction L_\alpha =^* f_\alpha$ for each $\alpha \in S$, i.e., for each $\alpha \in S$,

$$\{\beta \in L_\alpha : F(\beta) \neq f_\alpha(\beta)\} \text{ is finite.}$$

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We now formulate natural weakenings of uniformizable denoted, for each $n \in \omega$, by $\mathcal{P}_n$: A ladder system is said to satisfy $\mathcal{P}_n$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{n+1}$ such that for each $\alpha \in S$,

(a) $F \restriction L_\alpha$ is eventually constant with value s_α,
(b) $f(\alpha) \in s_\alpha$.

Note that $\mathcal{P}_0$ is equivalent to the version of uniformizable obtained by considering only sequences of constant functions f_α.

We will say that a ladder system satisfies $\mathcal{P}_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ satisfying (a) and (b) above.

If we drop the requirement that the restrictions $F \restriction L_\alpha$ are eventually constant we obtain uniformization properties that we denote by $\mathcal{M}_n$ and $\mathcal{M}_{<\omega}$. For example, a ladder system is said to satisfy $\mathcal{M}_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ such that for each $\alpha \in S$, $f(\alpha) \in F(\beta)$ for all but finitely many $\beta \in L_\alpha$.

Most of these uniformization properties can be characterized in terms of properties of a certain topological space naturally associated to any ladder system. If L is a ladder system, let X_L denote the topological space $\omega_1 \times \{0\} \cup S \times \{1\}$ where every point $(\alpha, 0)$ is isolated and for each $\alpha \in S$, a basic neighborhood of $(\alpha, 1)$ consists of $\{(\alpha, 1)\}$ along with a cofinite subset of $L_\alpha \times \{0\}$. Such a space is always first countable and locally compact. The stationarity of S implies that it is not collectionwise Hausdorff.

Spaces X_L have been considered by many to construct examples of normal not collectionwise Hausdorff spaces (see [11] and [2]). It is folklore that a ladder system L satisfies $\mathcal{P}_0$ if and only if X_L is normal. The property $\mathcal{M}_{<\omega}$ is characterized by X_L being countably metacompact. For this reason, we will say that a ladder system L is *countably metacompact* if it satisfies $\mathcal{M}_{<\omega}$.

Claim 1. *Let L be a ladder system on a stationary set S. Then X_L is countably metacompact if and only if L satisfies $\mathcal{M}_{<\omega}$.*

Proof. X_L is countably metacompact if and only if each partition of S into countably many sets has an open expansion which is point-finite on the isolated points. Suppose $f : S \to \omega$. By countable metacompactness, fix a point-finite open expansion $\{U_n : n \in \omega\}$. Let $F(\beta) = \{n : \beta \in U_n\}$. Clearly $F : \omega_1 \to [\omega]^{<\aleph_0}$ is as required. Conversely, given a partition and a corresponding function f fix the F given by $\mathcal{M}_{<\omega}$. Define $U_n = f^{-1}(n) \cup \{\beta \in \omega_1 : n \in F(\beta)\}$. It is straightforward to check that $\{U_n : n \in \omega\}$ is the required point-finite open expansion.

The additional conclusion in $\mathcal{P}_{<\omega}$ that $F \restriction L_\alpha$ is eventually constant corresponds to the open expansion being locally finite, hence $\mathcal{P}_{<\omega}$ implies that X_L is countably paracompact. We do not know whether the two properties are equivalent.

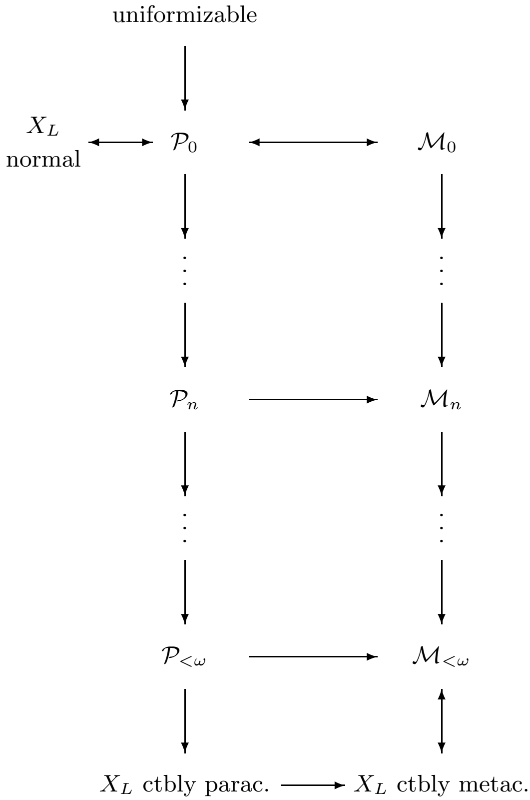

Fig. 1

QUESTION 1. *If X_L is countably paracompact, does L satisfy $\mathcal{P}_{<\omega}$?*

The diagram of known ZFC implications between all the properties is given in Figure 1.

It should be remarked that $\mathrm{MA}(\omega_1)$ implies that every ladder system is uniformizable and that $2^{\aleph_0} < 2^{\aleph_1}$ implies that no ladder system is uniformizable (see [6]). However, there is a ladder system on ω_1 with the property $\mathcal{M}_{<\omega}$. In fact, any ladder system L with the property that the nth element of each ladder L_α is of the form $\beta + n$ with β a limit ordinal has property $\mathcal{M}_{<\omega}$. Indeed, let the function F on ω_1 be defined by $F(\alpha) = \{0, 1, \ldots, n\}$, where α is of the form $\beta + n$ for some limit ordinal β. Then F uniformizes every function f in the sense of $\mathcal{M}_{<\omega}$. However, it is not hard to see that $\mathrm{V} = \mathrm{L}$ implies that no ladder system on a stationary subset of ω_1 can satisfy $\mathcal{P}_{<\omega}$ (see the remark following the proof of Theorem 24).

In Section 2 of this paper we prove that for each $n \in \omega$ it is consistent with CH that for every stationary set S and every ladder L on S, L satisfies $\mathcal{P}_{n+1}$ but does not satisfy $\mathcal{M}_n$. Thus, there are no other ZFC implications between the properties $\mathcal{P}_n$ and $\mathcal{M}_m$ for any $m, n < \omega$. Moreover, by taking

$n = 0$, we find that it is consistent with CH that every ladder system space X_L is countably paracompact (in a strong sense) but not normal.

This leaves a few questions open, including the following:

QUESTION 2. *Is it consistent that all ladder systems satisfy $\mathcal{P}_{<\omega}$ but not $\mathcal{P}_n$ for any n?*

The next set of properties of ladder systems we will consider are in some sense anti-uniformization properties. The following is the strongest of these. A ladder system with this property will also be called *thin*.

(G_1) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |f''L_\alpha| = \aleph_0\}$ is nonstationary.

By strengthening what f is allowed to do on a nonstationary set, we obtain the following weakenings of G_1:

(G_2) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is finite-to-one}\}$ is nonstationary.

(G_3) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is eventually one-to-one}\}$ is nonstationary.

By instead demanding that every f fails to have certain properties on a stationary set, we obtain even weaker properties:

(H_1) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |f''L_\alpha| < \aleph_0\}$ is stationary.

(H_2) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is not finite-to-one}\}$ is stationary.

(H_3) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is not eventually one-to-one}\}$ is stationary.

So, from the definitions we get the diagram of implications in Figure 2.

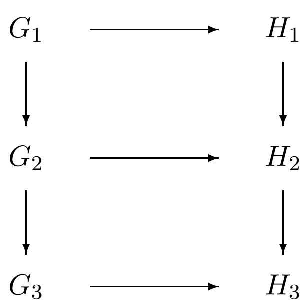

Fig. 2

Note that if a ladder system L is uniformizable, then it fails to satisfy H_3. Indeed, any F uniformizing a sequence of one-to-one functions $\langle f_\alpha \rangle$ will witness the failure of H_3. However, some of the weaker versions of uniformizable can be consistent with some of the anti-uniformization properties. In [11], Shelah proved it consistent that there is a ladder system L on a stationary set such that X_L is normal and the closed discrete set of nonisolated points is not a G_δ-set. Using our terminology, Shelah's ladder system satisfies $\mathcal{P}_0$ and H_2. Burke and Balogh [2] proved it consistent that there is a ladder system defined on a club subset of ω_1 satisfying $\mathcal{M}_{<\omega}$ and H_2.

The most interesting open question is whether it is consistent that there is a ladder system satisfying G_1 and $\mathcal{M}_{<\omega}$. Much of this paper is devoted to these properties and henceforth we adopt the more descriptive names *thin* and *countably metacompact.*

In Section 3 we give two proofs establishing the consistency of the existence of a thin ladder system. We also consider the existence of thin ladders on stationary subsets of cardinals greater than ω_1.

In Section 4 we consider the question whether every countably paracompact subspace of ω_1^2 is normal. With a ladder L on a stationary set S, we associate a nonnormal subspace $Z_L \subseteq \omega_1^2$. The main result of that section is that Z_L is countably paracompact if and only if L is thin and countably metacompact. We also consider another closely related construction of a subspace of ω_1^2.

Section 5 is devoted to the following open problem: Is there (consistently) a countably paracompact, locally compact screenable space which is not paracompact? It is shown that the existence of such a space of cardinality ω_1 is equivalent to the existence of a ladder system on some stationary set that is thin and countably metacompact. In addition, the existence of an example of larger cardinality is characterized by the existence of a thin, countably metacompact ladder system on a stationary subset of the same cardinal with an additional property. Section 6 is a short discussion of the question whether there may exist a thin countably metacompact ladder system on ω_1. In the final section we establish some results connecting our ladder system properties to other known ladder system properties.

2. CH and the uniformization properties. In this section we prove that for each $n \in \omega$ it is consistent with CH that every ladder system satisfies $\mathcal{P}_{n+1}$ but fails to satisfy $\mathcal{M}_n$. Hence, all the uniformization properties are distinct, even assuming CH. In the case $n = 1$ this implies the consistency that every ladder system space X_L is countably paracompact in a strong sense but not normal. Our proof is based on a theorem of Shelah (Theorem 9 below). A proof of Shelah's theorem has only been published for the case $n = 0$. The proof of Shelah's theorem for all other $n \in \omega$ is essentially the same.

Let $\overline{L} = \langle L_\alpha : \alpha \in S\}$ be a ladder system on a stationary set S of countable limit ordinals, and let $f \in {}^{\omega_1}\omega$. We will define a notion of forcing $P = P_{f,\overline{L}}$ that adjoins a function $g : \omega_1 \to [\omega]^{n+2}$ such that for all $\alpha \in S$,

- $g \restriction L_\alpha$ is eventually constant,
- $f(\alpha) \in g(\beta)$ for all but finitely many $\beta \in L_\alpha$.

DEFINITION 2. A condition $p \in P$ satisfies

- $p : \delta + 1 \to [\omega]^{n+2}$ for some $\delta < \omega_1$,
- if $\alpha \leq \delta$ is in S, then $p \restriction L_\alpha$ is eventually constant, say with value F_α, and $p(\alpha) \in F_\alpha$.

LEMMA 3 (Extension Lemma). *Let $p : \delta + 1 \to [\omega]^{n+2}$ be a condition in P. Given $\beta > \delta$ and $x \in [\omega]^{n+2}$, there is an extension q of p with* $\operatorname{dom} q = \beta + 1$ *and $q(\beta) = x$.*

Proof. We prove this by induction on β for all sets $x \in [\omega]^{n+2}$. The cases where β is a successor ordinal or $\beta \notin S$ are trivial. If β is a limit ordinal in S, we let $\{\beta_m : m < \omega\}$ list $L_\beta \setminus \delta + 1$ in increasing order. We apply our induction hypothesis repeatedly to obtain a sequence $\{p_m : m < \omega\}$ such that

- $p_0 \leq p$,
- $p_{m+1} \leq p_m$,
- $\operatorname{dom} p_m = \beta_m + 1$,
- $p_m(\beta_m) = \{f(\beta), \ldots, f(\beta + n + 1)\}$.

Once this is done, we define

$$q = \bigcup_{m \in \omega} p_m \cup \{\langle \beta, x \rangle\}. \qquad (1)$$ ∎

COROLLARY 4. *Given $p : \delta + 1 \to [\omega]^{n+2}$ in P, a finite set A of ordinals in $\omega_1 \setminus (\delta + 1)$, and a set $x \in [\omega]^{n+2}$, there is $q \leq p$ in P such that* $\operatorname{dom} q = \max(A) + 1$ *and $q(\beta) = x$ for all $\beta \in A$.*

PROPOSITION 5. *The set of $N \prec H(\lambda)$ countable with $\{f, \overline{L}, P\} \in N$ satisfying the following forms a club: For any $A \subseteq \delta = N \cap \omega_1$ of order type ω cofinal in δ, and any $x \in [\omega]^{n+2}$, and given $p \in N \cap P$, there is an (N, P) generic sequence $\{p_n : n \in \omega\}$ below p such that the function defined by $\bigcup\{p_n : n \in \omega\}$ is eventually constant with value x on A.*

Proof. We can assume that there is a sequence $\langle N_i : i < \omega \rangle$ of countable elementary submodels of $H(\lambda)$ such that

- $N_i \in N_{i+1}$,
- $N = \bigcup_{n \in \omega} N_i$,
- $\{f, \overline{L}, P, p\} \in N_0$.

Let $\{D_n : n \in \omega\}$ list the dense open subsets of P that are elements of N, and let $\delta_i = N_i \cap \omega_1$ for $i < \omega$. By enlarging A, we may assume that $\{\delta_i : i < \omega\} \subseteq A$—this will not affect the order type of A as the δ_i's are cofinal in δ. We will define $\{p_n : n \in \omega\}$ by induction on n so that

(1) $p_0 = p$,
(2) $p_{n+1} \leq p_n$,

(3) $p_{n+1} \in D_n$,
(4) if $\alpha \in A \cap \operatorname{dom} p_{n+1} \setminus \operatorname{dom} p_n$, then $p_{n+1}(\alpha) = x$.

Given p_n, we first choose i large enough that $\{p_n, D_n\} \in N_i$. Since $A \cap N_i$ is finite, we can apply Corollary 4 inside N_i with p_n and $A \cap N_i$ in place of p and A to obtain a condition which we shall denote by q_n. Now inside N_i, we extend q_n to $p_{n+1} \in D_n$. Clearly p_{n+1} has all the properties required, as does the sequence $\{p_n : n \in \omega\}$. ∎

COROLLARY 6. *P is totally proper. More generally, if $N \prec H(\lambda)$ is countable with $\{\overline{L}, P, f\} \in N$, $p \in N \cap P$, and $x \in [\omega]^{n+2}$, then there is a totally generic $q \leq p$ such that* $\operatorname{dom} q = \delta + 1$ *(where $\delta = N \cap \omega_1$) and* $q(\delta) = x$.

Proof. We apply Proposition 5 with L_δ in place of A, and $\{f(\delta), \ldots, f(\delta) + n + 1\}$ in place of x. The sequence $\{p_n : n < \omega\}$ will have a lower bound q with domain $\delta + 1$, so we can define $q(\delta) = x$ as required. ∎

The following corollary is the place where it is crucial that forcing conditions map into the set $[\omega]^{n+2}$ instead of into $[\omega]^{n+1}$.

COROLLARY 7. *There is a simple $n+2$-completeness system $\mathbb{D}$ such that P is $\mathbb{D}$-complete.*

Proof. Recall that a completeness system is called *simple* if there is a first order formula ψ such that

$$\mathbb{D}(N, P, p) = \{A_x : x \text{ a finitary relation on } N\}, \tag{2}$$

where

$$A_x = \{G \in \operatorname{Gen}(N, P) : \langle N \cup \mathcal{P}(N), \in\rangle \models \psi(G, x, p, P, N)\}. \tag{3}$$

In our case, the formula ψ says that "if x is a pair $\langle y, z\rangle$ such that y is an ω-sequence cofinal in $N \cap \omega_1$ and $z \in \omega$, then $\bigcup G \restriction y$ is eventually constant and z is an element of this limit value."

Note that P is $\mathbb{D}$-complete, because if $x = \langle L_\delta, f(\delta)\rangle$, then any member of A_x has a lower bound. Thus we need only show that $\mathbb{D}$ is an $n+2$-completeness system. To see this, let $\{x_i : i < n+2\}$ be a set of $n+2$ finitary relations on N; we must show that $\bigcap_{i<n+2} A_{x_i}$ is nonempty.

The nontrivial case is where all x_i satisfy the hypothesis of the implication in the formula ψ. Let $x_i = \langle y_i, z_i\rangle$. Let $A = \bigcup_{i<n+2} y_i$. Then A has order type ω. We apply Proposition 5 to this A with $\{z_0, \ldots, z_{n+1}\}$ in place of x. The sequence $\{q_n : n \in \omega\}$ that the conclusion gives us then generates a member of A_{x_i} for every $i < n+2$. ∎

PROPOSITION 8. *P is $<\omega_1$-proper.*

Proof. That P is α-proper for every α follows by induction on α using Corollary 6. ∎

Ensuring the failure of $\mathcal{M}_n$. We need the following result of Shelah (see [10, Chapter VIII, Claim 4.10] for the proof in the case $n = 0$; the proof of the general case is similar).

THEOREM 9. *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \alpha_0 \rangle$ be an iteration with countable support such that each $\dot{Q}_\alpha$ is $<\omega_1$-proper and $\mathbb{D}_\alpha$-complete for some simple $n+2$-completeness system $\mathbb{D}_\alpha$. Suppose that $\langle N_i : i \le \beta \rangle$ is a countable increasing continuous sequence of countable models such that*

- $\langle N_j : j \le i \rangle \in N_{i+1}$,
- $\xi \le \zeta \in N_0 \cap \alpha_0 + 1$,
- $\mathbb{P} \in N_0$.

Suppose further that $\langle N_i \cap \alpha_0 : i \le \beta \rangle$ is long for (ξ, ζ), $(G_i : i < n+2)$ are directed subsets of $P_\xi \cap N_\beta$, $r_i \in P_\xi$ is a lower bound for G_i for each $i < n+2$, $G_i \cap N_0 = G_j \cap N_0$ for all $i, j < n+2$, and for all $\eta \le \beta$, $G_i \cap N_\eta \in \mathrm{Gen}(N_\eta, P_\xi)$. Suppose also that $p \in P_\zeta \cap N_0$ and $p \restriction \xi \in G_0$. Then there is a directed $G^* \subseteq P_\zeta \cap N_0$ such that $G_0 \cap N_0 \subseteq G^*$, $G^* \in \mathrm{Gen}(N_0, P_\zeta, p)$, and

$$r_i \Vdash_{P_\xi} \text{“}\{q \restriction [\xi, \zeta) : q \in G^*\} \textit{ has a lower bound in } P_\zeta / P_\xi\text{”} \tag{4}$$

for all $i < n+2$.

Our iteration $\mathbb{P}$ is a countable support iteration where at even stages we force with $({}^{<\omega_1}\omega, \supseteq)$ and at odd stages we force with $P_{f,\overline{L}}$ for some f and $\overline{L}$. Clearly a book-keeping argument will take care that all ladders satisfy $\mathcal{P}_{n+1}$ in the extension so the crux of the matter is to prove that none of the ladder system spaces satisfy $\mathcal{M}_n$.

Suppose that $p \in P_{\omega_2}$ forces that the ladder system space built from $(\dot{S}, \dot{\overline{L}})$ is $\mathcal{M}_n$. We may assume that $(S, \overline{L})$ are in the ground model as we can find some limit stage $\alpha < \omega_2$ with $(S, \overline{L}) \in V[G_\alpha]$. The first thing we do is force with $({}^{<\omega_1}\omega, \supseteq)$. This adjoins a function $f : \omega_1 \to \omega$. Let $\dot{g}$ be a P_{ω_2}-name such that

$$p \Vdash \dot{g} : \omega_1 \to [\omega]^{n+1} \text{ uniformizes } \check{f} \text{ in the sense of } \mathcal{M}_n. \tag{5}$$

Let $\langle N_i : i \le \beta \rangle$ be a sequence of models as in the assumptions of Theorem 9 with $\xi = 1$ and $\zeta = \omega_2$, and with $\dot{g}$, f all in N_0.

Let $\delta = N_0 \cap \omega_1$, and let G_i be chosen as in the assumptions of Theorem 9 so that $\bigcup G_i(\delta) = i$ for each $i < n+2$. This is easy as $({}^{<\omega_1}\omega, \supseteq)$ is countably complete.

Let G^* be as in the conclusion of the theorem. From G^*, we can decode the values of $\dot{g}(\gamma)$ for all $\gamma < \delta$. Since $p \in G^*$, we know that $\dot{g}$ must uniformize f. Consider the decided sequence of values $(g(\gamma) : \gamma \in L_\delta)$. These sets are of size $n+1$, so there are at most $n+1$ values k such that $k \in g(\gamma)$ for all but finitely many $\gamma \in L_\delta$. This means that from G^* we can decode the value of $f(\delta)$ up to a set of $n+1$ possible values. Take $i < n+2$ such that i is not one of these values. This is a contradiction, as $r_i \Vdash f(\delta) = i$ and r_i can be extended to a lower bound for G^*. ∎

3. Consistency of thin ladder systems. In this section we give two proofs of the consistency of the existence of thin ladder systems. The first proof is from V = L, more specifically from Devlin's $\diamondsuit^{\#}$ (see [5]). For our purposes we will say that a sequence $\{\mathcal{N}_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit^{\#}$-*sequence* if

(1) each $\mathcal{N}_\alpha$ is a countable transitive model of a suitable portion of ZFC,
(2) $\{\mathcal{N}_\alpha \cap \mathcal{P}(\alpha) : \alpha \in \omega_1\}$ forms a $\diamondsuit^{+}$-sequence,
(3) $\{\alpha : \alpha = (\omega_1)^{\mathcal{N}_\alpha}\}$ is stationary.

For a more precise formulation of $\diamondsuit^{\#}$ see [5].

THEOREM 10. *$\diamondsuit^{\#}$ implies the existence of a thin ladder system on a stationary subset of ω_1.*

REMARK. Kunen has shown that V = L implies the existence of a thin ladder system defined on all of ω_1.

Proof. Fix a $\diamondsuit^{\#}$-sequence $\{\mathcal{N}_\alpha : \alpha \in \omega_1\}$ and let $S = \{\alpha : \alpha = (\omega_1)^{\mathcal{N}_\alpha}\}$. So S is stationary. We define the ladder system on S. Fix $\alpha \in S$ and let $\{f_k : k \in \omega\}$ be an enumeration of $\mathcal{N}_\alpha \cap {}^{\alpha}\omega$. Define L_α recursively: Fix n_0 minimal such that

$$\mathcal{N}_\alpha \models \text{“}f_0^{-1}(n_0) \text{ is uncountable.”}$$

We can do this since $\mathcal{N}_\alpha$ "thinks" that α is ω_1 and hence that f_0 is a function from ω_1 to ω. Let $I_0 = f_0^{-1}(n_0)$. Having defined I_k so that $\mathcal{N}_\alpha \models$ "I_k is uncountable", fix n_{k+1} so that

$$\mathcal{N}_\alpha \models \text{“}I_k \cap f_{k+1}^{-1}(n_{k+1}) \text{ is uncountable.”}$$

So $I_0 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and each I_k is unbounded in α. Choose an increasing cofinal ω-sequence L_α in α such that $L_\alpha \subseteq^* I_k$ for each $k \in \omega$. Clearly $f_k \restriction L_\alpha$ is eventually constant for each $k \in \omega$.

To see that $\mathcal{L} = \{L_\alpha : \alpha \in S\}$ is thin, fix $f : \omega_1 \to \omega$ arbitrary. By the property of being a $\diamondsuit^{\#}$-sequence, $C = \{\alpha : f \restriction \alpha \in \mathcal{N}_\alpha\}$ is club. Clearly, by construction, $f \restriction L_\alpha$ is eventually constant for any $\alpha \in S \cap C$. ∎

Our next proof is a forcing construction of a thin ladder system. We obtain the model by collapsing a Mahlo cardinal to ω_1. First we define a single forcing that collapses ω_1 and preserves Mahloness of κ. Let Q be the set of triples (p, F, r) such that

(1) p is a function from $\operatorname{dom} p \in \omega$ to ω_1 such that $n < m < \operatorname{dom} p$ implies $p(n) < p(m)$,
(2) F is a finite subset of ${}^{\omega_1}\omega$,

(3) $r : F \to \omega$,
(4) $\{\alpha \in \omega_1 : \forall f \in F(f(\alpha) = r(f))\}$ is uncountable.

We order Q by $(p, F, r) \leq (q, G, s)$ if

(1) p extends q, $F \supseteq G$, r extends s,
(2) $p(n) \in g^{-1}(s(g))$ for each $n \in \operatorname{dom} p \setminus \operatorname{dom} q$ and each $g \in G$.

Since $|Q| = 2^{\aleph_1}$ we have

Lemma 11. *Q is $(2^{\aleph_1})^+$-cc.*

This cannot be improved: Given $\{f_\alpha : \alpha < 2_1^{\aleph}\} \subseteq {}^{\omega_1}\omega$ such that $\{f_\alpha^{-1}(0) : \alpha < 2^{\aleph_1}\}$ forms an almost disjoint family, the conditions $(\emptyset, \{f_\alpha\}, \langle f_\alpha, 0\rangle)$ are pairwise incompatible (any common extension of two such conditions would violate (4)).

Let g be a Q-name for the generic function $\bigcup\{p : \exists F, r\ (p, F, r) \in \Gamma\}$ and let L be a Q-name for the range of g. Then L is an ω-sequence cofinal in $(\omega_1)^{\mathrm{V}}$, hence ω_1 is collapsed by Q.

It is easy to verify that for each $f \in {}^{\omega_1}\omega$, the set $D_f = \{(p, F, r) \in Q : f \in F\}$ is dense in Q. Thus, the following lemma holds:

Lemma 12. *For any $f \in {}^{\omega_1}\omega$, $\Vdash_Q$ "$\check{f}$ is eventually constant on L."*

Let κ be Mahlo. So, the set of inaccessible ordinals less than κ is stationary in κ. Let $C = \{\kappa_i : i \in \kappa\}$ be an increasing enumeration of all cardinals in κ. So C is club. Let S be the set of inaccessibles. We fix an iteration $\langle P_i, Q_i : i \in \kappa\rangle$ as follows. For each $i \in \kappa$ let Q_i be defined recursively as follows.

(1) If i is a successor and $\Vdash_{P_i}$ "κ_i is uncountable," let Q_i be $\mathrm{Fn}(\omega, \kappa_i)$. So Q_i collapses κ_i to ω.
(2) If κ_i is inaccessible, let Q_i be a P_i-name for Q.
(3) Otherwise let Q_i be the trivial poset.

Let P_κ be the finite support iteration of the Q_i's (countable also works). It easily follows that P_κ has the κ-cc. It also follows that for each inaccessible κ_i, P_i is κ_i-cc and collapses all κ_j for $j < i$ to countable ordinals. Hence $\Vdash_{P_i} \kappa_i = \omega_1$. So Q_i adds an ω-sequence cofinal in κ_i. For each $\delta \in S$ let L_δ be the $P_{\delta+1}$-name for the ladder added by Q_δ. We work with the ladder system $L = \{L_\delta : \delta \in S\}$.

Given any P_κ-name f for a function $f : \kappa \to \omega$, by κ-cc the set of δ for which there is a P_δ-name f_δ such that $\Vdash_{P_\kappa} f_\delta = f{\restriction}\delta$ is club. Thus, by the lemma above, for any P_κ-name f for a function $\kappa \to \omega$, the set of δ for which $\Vdash_{P_\kappa}$ "$f{\restriction}L_\delta$ is eventually constant" is club on S.

Finally, by κ-cc it follows that S remains stationary in V^{P_κ}. Thus the ladder system is thin in V^{P_κ}.

Hence, the existence of thin ladders is consistent. However, such ladder systems are very unstable. If P is a finite support iteration of length ω_1, then P adds a function $g : \omega_1 \to \omega$ that is $\mathrm{Fn}(\omega_1, \omega, {<}\omega_1)$-generic over the universe. For a ladder system L in the ground model, this g has the property that $g \restriction L_\alpha$ has infinite range for every $\alpha \in \omega_1$. Thus, the ladder L fails to have property H_1 in the extension. On the other hand, if P is a countable support iteration of proper posets of length at least ω_1, then P adds a function g that is $\mathrm{Fn}(\omega_1, \omega, {<}\omega_1)$-generic over the universe. This g has the property that $g \restriction L_\alpha$ is eventually one-to-one for stationary many α. This stationary set remains stationary by the tail of the forcing, thus, the ladder L fails to have property G_3 in the extension.

Thus we have the following:

THEOREM 13. *Suppose that L is a ladder system and that $\mathbb{P}$ is a finite or countable support iteration of length at least ω_1 (of proper posets). Then $\Vdash_{\mathbb{P}} L$ is not thin.*

REMARK. F. Hernandez-Hernandez has constructed a proper poset that forces a thin ladder system [8]. The poset is proper but it is an ω_2-length countable support iteration of nonproper posets.

Next we consider ladder systems on stationary subsets of cardinals $\kappa > \omega_1$. For these cardinals, relatively weak assumptions imply that no such ladder is thin.

THEOREM 14. *Let κ be a regular cardinal. Suppose there is a cardinal λ such that $\lambda^\omega < \kappa \leq 2^\lambda$. Then there is no thin ladder system on any stationary subset S of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$.*

Proof. Let $L = \{L_\alpha : \alpha \in S\}$ be a ladder system. We first wish to show that there is a set $\mathcal{F}$ of less than κ functions from κ into ω such that every countable function from κ into ω is extended by some function from $\mathcal{F}$. To this end, identify κ with a subset of the space 2^λ, and let $\mathcal{B}$ be any base for 2^λ of cardinality λ. Given any countable subset $A = \{a_n\}_{n\in\omega}$ of κ, find $B_n \in \mathcal{B}$ with $a_n \in B_n$ and $B_n \cap \{a_i : i < n\} = \emptyset$. Then

$$\mathcal{P} = \Big\{\kappa \cap B_n \setminus \bigcup_{i>n} B_i : n \in \omega\Big\} \cup \Big\{\kappa \setminus \bigcup_{i\in\omega} B_i\Big\}$$

is a partition of κ each element of which contains at most one member of A. Since the hypothesis implies $\kappa > \mathfrak{c}$, we see that there is a set $\mathcal{F}$ of $\lambda \cdot \mathfrak{c}$, in particular less than κ, functions from κ into ω as required.

We may fix, for each $\alpha \in S$, a function $f_\alpha : L_\alpha \to \omega$ such that the range of f_α is unbounded in ω. Since $|\mathcal{F}| < \kappa$ and κ is regular, there is an $f \in \mathcal{F}$ and a stationary $S' \subset S$ such that $f_\alpha = f$ for each $\alpha \in S'$, i.e. the coloring f is unbounded on a stationary set of ladders, so L is not thin. ∎

Corollary 15. (1) *Assume the Continuum Hypothesis. Then there is no thin ladder system on any stationary subset of* $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.

(2) *Assume the Singular Cardinal Hypothesis. If* $\kappa > \mathfrak{c}$ *is regular, and not strongly inaccessible or the successor of a singular strong limit of countable cofinality, then there is no thin ladder system on any stationary subset of* $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$.

Proof. For (1), CH implies that the hypothesis of the theorem is satisfied with $\kappa = \omega_2$ and $\lambda = \omega_1$. For (2), it is not difficult to show that under the Singular Cardinal Hypothesis, the hypothesis on κ in (2) is equivalent to the hypothesis on κ in the theorem. ■

4. Subspaces of ω_1^2. Recently N. Kemoto and others have been systematically studying separation properties of products of ordinals and their subspaces. One of the more interesting questions left open by this investigation is the following:

Question 3. *Is every countably paracompact subspace of* ω_1^2 *normal?*

In [9] the following characterization was proven.

Theorem 16. *For each* $X \subseteq \omega_1^2$, X *is normal if and only if* X *is countably paracompact and strongly collectionwise Hausdorff.*

In addition it was shown in the same paper that ω_1^2 is hereditarily collectionwise Hausdorff. Hence every normal subspace of ω_1^2 is countably paracompact. Also, any countably paracompact nonnormal subspace of ω_1^2 would be an example of a countably paracompact first countable collectionwise Hausdorff not strongly collectionwise Hausdorff space. In any model of ZFC where first countable countably paracompact spaces are strongly collectionwise Hausdorff, we have a positive answer to Question 3. Although it is an open question (due to P. Nyikos) whether V = L implies that countably paracompact first countable spaces are strongly collectionwise Hausdorff, it was shown in [9] that Question 3 still has a positive answer assuming V = L.

In this section we present a general construction of a subspace of ω_1^2 using a ladder system on a stationary subset of ω_1. We consider two special instances of this construction. The first instance we consider is always nonnormal and we prove that it is countably paracompact if and only if the ladder system is thin and countably metacompact. For the second instance we find related conditions on the ladder system that characterize when the space is countably paracompact and nonnormal. Finally, we establish some relationship between the two sets of conditions.

Fix a ladder system L on a stationary $S \subseteq \omega_1$ with the property that each L_α consists of successor ordinals. Let B be a nonstationary subset of

ω_1 such that $B \cap S = \emptyset$. Consider the following subspace of ω_1^2:

$$\{(\xi, \alpha+1) : \alpha \in S,\ \xi \in L_\alpha \cup \{\alpha\}\} \cup \{(\beta, \gamma) : \beta \in B,\ \gamma \in S \setminus \beta + 1\}.$$

EXAMPLE 1. Set $B = \bigcup\{L_\alpha : \alpha \in S\}$ and denote this space by Z_L. Notice that by assumption on the ladder system, for each $(\beta, \gamma) \in Z_L$, if $\beta \notin S$ then β is a successor. Also notice that the closed discrete set $\{(\alpha, \alpha+1) : \alpha \in S\}$ can be separated by a disjoint family of open sets, but, by stationarity of S, it cannot be separated by a discrete family. Hence the space is not normal.

THEOREM 17. *Z_L is countably paracompact if and only if L is thin and countably metacompact.*

Proof. Assume that L is thin and countably metacompact. Fix a decreasing sequence $(D_n)_{n\in\omega}$ of closed subsets of Z_L such that $\bigcap D_n = \emptyset$. Let $E_n = \{\alpha \in S : (\alpha, \alpha+1) \in D_n\}$ and define $f_0 : S \to \omega$ by $f_0(\alpha) = \max\{n : \alpha \in E_n\}$. Fix a corresponding $g_0 : \omega_1 \to [\omega]^{<\aleph_0}$ uniformizing f_0 in the appropriate sense. Let

$$W'_n = \{(\beta, \alpha+1) : \beta \in L_\alpha,\ f_0(\alpha) = n \text{ and } n \in g_0(\beta)\},$$
$$W_n = (\{(\alpha, \alpha+1) : \alpha \in S\} \cap D_n) \cup \bigcup_{i\ge n} W'_i.$$

Notice that W_n is an open neighborhood of the set $D_n \cap \{(\alpha, \alpha+1) : \alpha \in S\}$.

To define neighborhoods at the rest of the points of D_n first note that since ω_1 is countably compact, for each successor β, there is an n_β such that $\{\beta\} \times \omega_1 \cap D_n = \emptyset$ for each $n \ge n_\beta$. Letting $f_1(\beta) = n_\beta$ and applying G_1, we may fix a club C consisting of limit ordinals such that $f_1'' L_\alpha$ is finite for each $\alpha \in C \cap S$.

For each $\gamma \in C$ let γ^+ be the minimal element of C above γ. For each $\gamma \in C$ the space $Z_L \cap (\gamma, \gamma^+]^2$ is a clopen metrizable subspace of Z_L. Thus, there are open sets $O_n^\gamma \supseteq D_n \cap (\gamma, \gamma^+]^2$ such that

$$\bigcap_{n\in\omega} \overline{O}_n^\gamma = \emptyset.$$

For each $(\beta, \eta) \in Z_L \cap D_n$ with $\beta \notin S$ fix an open $U_n(\beta, \eta)$ such that

$$(\beta, \eta) \in U_n(\beta, \eta) \subseteq \{\beta\} \times (0, \eta] \cap Z_L$$

so that

(a) if $(\beta, \eta) \in (\gamma, \gamma^+]^2$ for some $\gamma \in C$ then $U_n(\beta, \eta) \subseteq O_n^\gamma \cap (\gamma, \gamma^+]^2$,
(b) if $\beta \in (\gamma, \gamma^+)$ for some $\gamma \in C$ and $\eta > \gamma^+$ then $U_n(\beta, \eta) \subseteq \{\beta\} \times (\gamma^+, \eta]$.

Now let $O_n = \bigcup\{U_n(\beta, \eta) : (\beta, \eta) \in D_n,\ \beta \notin S\}$ and let

$$G_n = W_n \cup O_n.$$

Claim 18. $\bigcap_{n\in\omega} \overline{G}_n = \emptyset$.

Proof. First fix $\beta \notin S$. Fix n large enough so that for every $m \geq n$ both $\{\beta\}\times\omega_1 \cap D_m = \emptyset$ and $m \notin g_0(\beta)$. Then $\{\beta\}\times\omega_1\cap G_m = \emptyset$ for each $m \geq n$.

Next, fix $\alpha \in S$ and consider the point $(\alpha, \alpha+1)$. For each $m \neq f_0(\alpha)$ it is clear that $\alpha + 1 \times \{\alpha+1\} \cap W'_m = \emptyset$. So

$$(L_\alpha \cup \{\alpha\}) \times \{\alpha+1\} \cap W_m = \emptyset$$

for each $m > f_0(\alpha)$. Thus for $m > f_0(\alpha)$, if $(\alpha, \alpha+1) \in \overline{G}_m$ then $(\alpha, \alpha+1) \in \overline{O}_m$. Now consider two cases:

Case 1: $\alpha \in C$. In this case, $f_1'' L_\alpha$ is finite, so we can fix $n_1 \in \omega$ such that $\{\beta\} \times \omega_1 \cap D_m = \emptyset$ for all $\beta \in L_\alpha$ and all $m \geq n_1$. Thus

$$(L_\alpha \times \omega_1) \cap O_m = \emptyset \quad \text{for each } m \geq n_1.$$

Case 2: $\alpha \notin C$. In this case, fix $\gamma \in C$ such that $\gamma < \alpha < \gamma^+$. Fix n_1 such that $(\alpha, \alpha+1) \notin \overline{O}_m^\gamma$ for each $m \geq n_1$. Thus, by choice of the open sets $U_m(\beta, \eta)$ for $\beta < \gamma$ and by choice of the open sets $U_m(\beta,\eta)$ for $\eta > \gamma^+$ we have $(\alpha, \alpha+1) \notin \overline{O}_m$ for all $m \geq n_1$.

In either case, $(\alpha, \alpha+1) \notin \overline{O}_m$ for all $m \geq n_1$ so Z_L is countably paracompact.

For the converse, first suppose that L is not countably metacompact, and fix a partition $f : S \to \omega$ such that for any $g : \omega_1 \to \mathcal{P}(\omega)$, if for each $\alpha \in S$, $f(\alpha) \in g(\beta)$ for all but finitely many $\beta \in L_\alpha$, then there is a β such that $g(\beta)$ is infinite. Consider the closed sets $E_n = \{(\alpha,\alpha+1) : f(\alpha) = n\}$. Fix open sets $W_n \supseteq E_n$. For each $\beta \notin S$ let $g(\beta) = \{n : (\beta, \alpha+1) \in W_n \text{ for some } \alpha \in f^{-1}(n)\}$. Each W_n is open so $f(\alpha) \in g(\beta)$ for all but finitely many $\beta \in L_\alpha$. Thus we may fix β such that $g(\beta)$ is infinite. Fix $\alpha_n \in f^{-1}(n)$ for each $n \in g(\beta)$. Let γ be a limit of the set $\{\alpha_n : n \in g(\beta)\}$. Then the family $\{W_n : n \in \omega\}$ is not locally finite at (β, γ). So if L is not countably metacompact, then Z_L is not countably paracompact.

On the other hand, assume that L is not thin, and fix a function $f : \omega_1 \to \omega$ such that $T = \{\alpha : |f'' L_\alpha| = \aleph_0\}$ is stationary. Let

$$D_n = \{(\beta, \gamma) \in Z_L : f(\beta) \geq n \text{ and } \gamma \text{ is a limit}\}.$$

Notice that each D_n is closed and $\bigcap_{n\in\omega} D_n = \emptyset$. Suppose that for each n, U_n is an open set containing D_n such that $U_0 \supseteq U_1 \supseteq \cdots$. Fix a countable elementary submodel $\mathcal{M}$ such that everything relevant, e.g., T, $\{D_n : n\in\omega\}$, f, $\{U_n : n \in \omega\}$, lies in $\mathcal{M}$ which can be chosen so that $\mathcal{M}\cap\omega_1 \in T$. Clearly, the following claim will complete the proof.

Claim 19. *Let* $\alpha_0 = \mathcal{M} \cap \omega_1$. *Then* $(\alpha_0, \alpha_0 + 1) \in \bigcap\{\overline{U}_n : n \in \omega\}$.

Proof. For each $\beta \in L_{\alpha_0}$, let $Z_\beta = \{(\beta, \gamma) : \gamma > \beta \text{ is a limit}\} \cap Z_L$. Notice that $Z_\beta \subseteq D_{f(\beta)} \subseteq U_{f(\beta)}$. By the pressing down lemma, there is a γ_β such that $(\{\beta\} \times \omega_1 \setminus \gamma_\beta) \cap Z_L \subseteq U_n$ for each $n \geq f(\beta)$. Since all objects under consideration are in $\mathcal{M}$ we may assume that $\gamma_\beta \in \mathcal{M}$ for each $\beta \in L_{\alpha_0}$. Now fix $n \in \omega$ and $\eta < \alpha_0$. Fix $\beta \in L_{\alpha_0}$ such that $\beta > \eta$ and $f(\beta) \geq n$. This can be done since $\alpha_0 \in T$. Thus, $(\beta, \alpha+1) \in U_{f(\beta)} \cap (\eta, \alpha_0] \times \{\alpha+1\}$. Hence, $(\alpha_0, \alpha_0+1) \in \overline{U}_n$ for every $n \in \omega$. Thus Z_L is not countably paracompact. ∎

We now present a specific case of the general construction.

EXAMPLE 2. Let $B \subseteq \{\alpha+\omega : \alpha \in \mathrm{LIM}\}$. Expand the ladder system on S to a ladder system on $S \cup B$ by letting $L_\beta = \{\alpha+n : n \in \omega\}$, where $\beta = \alpha+\omega \in B$. Denote this space by Y_L. To simplify notation assume that $B \subseteq S$.

Let X_L be the associated ladder system topology on $\omega_1 \times \{0\} \cup S \times \{1\}$ described above. While it is possible to characterize when Y_L is countably paracompact and normal in terms of combinatorial properties of L (see comments after the proof of Theorem 20), it is simpler to consider the corresponding properties of X_L:

THEOREM 20. *Y_L is not normal if and only if X_L satisfies*

(1) *The sets $S \setminus B$ and B have no disjoint open neighborhoods in X_L.*

Y_L is countably paracompact if and only if it satisfies both of the following conditions:

(2) *Every countable partition of B can be extended to a countable family of open sets of X_L which is locally finite at every point of $S \setminus B$.*
(3) *Every countable partition of $S \setminus B$ can be extended to a countable family of open sets of X_L which is locally finite at every point of B.*

Proof. Assume that the ladder system satisfies (1). To see that Y_L is not normal, consider the closed subsets $H_1 = \{(\alpha, \alpha+1) : \alpha \in S \setminus B\}$ and $H_2 = \{(\beta, \gamma) : \beta \in B \text{ and } \gamma \in S \setminus \beta+1\}$. Suppose that U is an open set containing H_1 and V is an open set containing H_2. For each $\alpha \in S \setminus B$, fix $L'_\alpha \subseteq^* L_\alpha$ such that $L'_\alpha \times \{\alpha+1\} \subseteq U$. Let U_X be the corresponding open set in X_L determined by the L'_α's. For each $\beta \in B$ and each $\gamma \in S \setminus \beta+1$, let $\alpha_{\beta,\gamma} < \beta$ and $\delta_{\beta,\gamma} < \gamma$ be such that

$$(\alpha_{\beta,\gamma}, \beta] \times (\delta_{\beta,\gamma}, \gamma] \cap Y_L \subseteq V.$$

For each $\beta \in B$ there is $\alpha_\beta < \beta$ such that $\alpha_\beta = \alpha_{\beta,\gamma}$ for stationarily many $\gamma \in S \setminus \beta+1$. Call this stationary set S_β. For each $\beta \in B$ let $L'_\beta = L_\beta \setminus \alpha_\beta$. Let V_X be the corresponding neighborhood in X_L defined by these L'_β. There is a $\beta \in B$ such that the set of $\alpha \in S \setminus B$ such that $L'_\beta \cap L'_\alpha \neq \emptyset$ is stationary. Let T denote this stationary set of α's. If there were no such β, then B and $S \setminus B$ would be separated in X_L. Choose $\gamma \in S_\beta$ a limit of T. Let $\alpha \in T \cap (\delta_{\beta,\gamma}, \gamma)$. Let $\xi \in L'_\alpha \cap L'_\beta$. Then $(\xi, \alpha+1) \in U \cap V$. So H_1 and H_2 cannot be separated.

Conversely, assume that B and $S \setminus B$ can be separated in X_L. Note that $Y_L = H_1 \cup H_2 \cup I$, where H_1 and H_2 are defined as above and I is the remaining set of isolated points. Since H_1 and H_2 are closed normal subspaces of Y_L, it follows that Y_L is normal if and only if H_1 and H_2 can be separated. Similarly to the argument above, a separation of B and $S \setminus B$ leads to a separation of H_1 and H_2.

Assume (2) and (3). To see that Y_L is countably paracompact, suppose that $(D_n : n \in \omega)$ is a decreasing sequence of closed subsets with empty intersection. For each n let $S_n = \{\alpha \in S \setminus B : (\alpha, \alpha+1) \in D_n\}$. Let $B_n = \{\beta \in B : \{\gamma \in S : (\beta,\gamma) \in D_n\}$ is uncountable$\}$. Then both (S_n) and (B_n) are decreasing sequences of closed subsets of X_L with empty intersection. By (3) we may fix an open expansion (U_n) of (S_n) in X_L such that $B \cap \bigcap \overline{U}_n = \emptyset$. Similarly, by (3) we may fix an open expansion (V_n) of (B_n) with $(S \setminus B) \cap \bigcap \overline{V}_n = \emptyset$. For each $\alpha \in S_n$ let $\alpha_n < \alpha$ be such that $L_\alpha \setminus \alpha_n \subseteq U_n$, and similarly define $\beta_n < \beta$ for each $\beta \in B_n$. Let C be club in ω_1 such that for all $\beta \in B$, $n \in \omega$ and $\delta > \beta$ with $\delta \in C$, if $\beta \notin B_n$, then the countable set $\{\gamma : (\beta,\gamma) \in D_n\}$ is contained in δ. As in the proof that Z_L is countably paracompact, let O_n^γ be an open expansion of $D_n \cap (\gamma, \gamma^+]$ for each $\gamma \in C$ such that the corresponding sequence of open sets $O_n = \bigcup\{O_n^\gamma : \gamma \in C\}$ is locally finite. For $\beta \in B_n$, fix $\gamma \in C$ such that $\gamma < \beta < \gamma^+$ and let $W_n(\beta) = (\beta_n, \beta] \times (\gamma^+, \omega_1) \cap Y_L$. Let S'_n be the set of $\alpha \in S_n$ such that $(\alpha, \alpha+1) \notin O_n$, and let $W_n(\alpha) = (\alpha_n, \alpha] \times \{\alpha+1\} \cap Y_L$. Finally, let

$$W_n = O_n \cup \bigcup\{W_n(\alpha) : \alpha \in S'_n\} \cup \bigcup\{W_n(\beta) : \beta \in B_n\}.$$

By the choice of the sets O_n, U_n and V_n it follows that $\bigcap \overline{W}_n = \emptyset$.

Conversely, suppose that (2) or (3) fails. If (2) fails, then the partition of B leads naturally to a partition of H_2 into a countable discrete family of closed sets that has no open expansion that is locally finite at the points of H_1 (if B_n is one piece of the partition, let $D_n = \{(\beta,\gamma) \in Y_L : \beta \in B_n\}$). Similarly, if (3) fails, then there is a partition of H_1 that witnesses that Y_L is not countably paracompact: If S_n is a piece of the partition of $S \setminus B$, let $E_n = \{(\alpha, \alpha+1) : \alpha \in S_n\}$. ∎

Clearly, items (2) and (3) can be similarly characterized by appropriately uniformizing functions $f : B \to \omega$ and $g : S \setminus B \to \omega$.

We now have two sets of sufficient conditions on a ladder system that provides for the existence of a countably paracompact subspace of ω_1^2 that is not normal. We have the following theorem relating these two sets of conditions:

THEOREM 21. *There is a thin countably metacompact ladder system $\widetilde{L}$ if there is a ladder system $L = \{L_\alpha : \alpha \in S\}$ with both of the following properties.*

(4) X_L *is countably paracompact* (*e.g.*, *this happens if* L *satisfies* $\mathcal{P}_{<\omega}$).
(5) *There is an uncountable set* $B \subseteq S$ *which is discrete in itself if considered with the usual topology of* ω_1, *such that for every open neighborhood* U *of* B *in* X_L, $\overline{\mathrm{Succ} \setminus U}^{X_L}$ *is not stationary in* ω_1.

Moreover, *if* X_L *satisfies* (4) *and* (5), *then there is a ladder system* $\widehat{L}$ *that satisfies* (1)–(3) *of Theorem* 20.

Proof. Let $L = \{L_\alpha : \alpha \in S\}$ be such that X_L satisfies (4) and (5). Without loss of generality, assume that $\bigcup L \subseteq \mathrm{Succ}$. Because B is discrete in itself, we can assume that $\beta < \gamma$ whenever $\alpha_1, \alpha_2 \in B$ with $\alpha_1 < \alpha_2$ and $\beta \in L_{\alpha_1}$ and $\gamma \in L_{\alpha_2}$. Denote the family $\{L_\alpha \cup \{\alpha\} : \alpha \in B\}$ by $\tilde{\mathrm{S}}\mathrm{ucc}$ (for the reason described below). Also, we may assume, without loss of generality, that $\bigcup\{L_\alpha : \alpha \in B\} = \bigcup L$ (since the set $\mathrm{Succ} \setminus \bigcup\{L_\alpha : \alpha \in B\}$ has nonstationary closure we may thin out S and the ladders to obtain this). Let q be the map from $\bigcup L \cup S$ onto $\tilde{\mathrm{S}}\mathrm{ucc} \cup (S \setminus B)$ defined by

(i) for every $\alpha \in B$, q collapses $L_\alpha \cup \{\alpha\}$ into a singleton,
(ii) for every $\beta \in S \setminus B$, $q(\beta) = \beta$.

Then q preserves order in an obvious sense so that $\tilde{\mathrm{S}}\mathrm{ucc} \cup (S \setminus B)$ is order isomorphic to a subset of ω_1 with $\tilde{\mathrm{S}}\mathrm{ucc}$ being a set of successor ordinals and $S \setminus B$ being a stationary set. Define a ladder system $\widetilde{L}$ on the stationary set $S \setminus B$ by the following rule. For every $\alpha \in S \setminus B$ and every $\beta \in \tilde{\mathrm{S}}\mathrm{ucc}$, $\beta \in \widetilde{L}_\alpha$ if and only if $L_\alpha \cap L_\beta \neq \emptyset$.

Now we show that $\widetilde{L}$ is thin. Let $\{\tilde{\mathrm{S}}\mathrm{ucc}^n : n \in \omega\}$ be a countable partition of $\tilde{\mathrm{S}}\mathrm{ucc}$. Then $\mathcal{B} = \{B^n : n \in \omega\}$ is a partition of B if we set $B^n = B \cap q^{-1}(\tilde{\mathrm{S}}\mathrm{ucc}^n)$. Because X_L is countably paracompact, $\mathcal{B}$ can be expanded to a family $\mathcal{U} = \{U^n \subset X_L : n \in \omega\}$ of open subsets of X_L which is locally finite in X_L. Denote the set

$$\overline{\mathrm{Succ} \setminus \bigcup \mathcal{U}}^{X_L} \cap S$$

by F; then F is not stationary by (5). Pick an $\alpha \in S \setminus (B \cup F)$. Consider the ladder $\widetilde{L}_\alpha$ from $\widetilde{L}$.

Because $\alpha \notin F$, $L_\alpha \subseteq^* \bigcup \mathcal{U}$. Because $\mathcal{U}$ is locally finite in X_L, there are only finitely many $k \in \omega$ such that $L_\alpha \cap U^k \neq \emptyset$. Hence if $\alpha \in S \setminus (B \cup F)$, then $\{k \in \omega : L_\alpha \cap \bigcup\{L'_\gamma : \gamma \in B^k\} \neq \emptyset\}$ is a finite set. Because $\{q(L_\gamma) : \gamma \in B^k\} = \tilde{\mathrm{S}}\mathrm{ucc}^k$, the set $\{k \in \omega : \widetilde{L}_\alpha \cap \tilde{\mathrm{S}}\mathrm{ucc}^k \neq \emptyset\}$ is finite as required.

To see that $\widetilde{L}$ is countably metacompact, let $f : S \setminus B \to \omega$. We need to define the appropriate uniformizing function $F : \tilde{\mathrm{S}}\mathrm{ucc} \to [\omega]^{<\omega}$. Since X_L is countably paracompact, we may fix a cofinite subset L'_α of L_α for each $\alpha \in S \setminus B$ such that the family of U_n's defined by

$$U_n = \bigcup\{L'_\alpha : \alpha \in f^{-1}(n)\}$$

is locally finite in X_L. Thus for each $\beta \in B$, $\{n : L_\beta \cap U_n\}$ is finite. Let $F(q(\beta))$ be this finite set. Since $\widetilde{\mathrm{Succ}} = \{q(\beta) : \beta \in B\}$ this defines F. Since $q(L'_\alpha)$ is a cofinite subset of $\widetilde{L}_\alpha$ for each $\alpha \in S \setminus B$, it suffices to check that $f(\alpha) \in F(q(\xi))$ for all $\alpha \in S \setminus B$ and $\xi \in L'_\alpha$. Fix $\alpha \in S \setminus B$ and $\xi \in L'_\alpha$. Let $n = f(\alpha)$. Then by our assumption that $\bigcup\{L_\beta : \beta \in B\} = \bigcup L$, there is a $\beta \in B$ such that $\xi \in L_\beta$. Thus $L_\beta \cap U_n \neq \emptyset$, so $n \in F(q(\beta))$. Since $q(\xi) = q(\beta)$ we are done.

For the rest of the theorem, fix a ladder system L on a stationary set S, satisfying (4) and (5). As above, we may assume that $\bigcup L \subseteq \mathrm{Succ}$. Let C be a club that separates the points of B in the sense that for each $\gamma \in C$, there is a unique element of B lying in the interval (γ, γ^+). Without loss of generality, assume that if $\beta \in B \cap (\gamma, \gamma^+)$ then the ladder L_β also lies in this interval. Without loss of generality we may assume that $S \subseteq C$, and by reordering the points of

$$B \cup \bigcup\{L_\beta : \beta \in B\},$$

we may assume that $B \subseteq \{\gamma + \omega : \gamma \in C\}$ and that $L_{\gamma+\omega} = (\gamma, \gamma + \omega)$ for each such $\gamma + \omega \in B$. Call this resulting ladder system $\widehat{L}$. Now it is easy to verify that (5) implies (1) and that (4) implies (2) and (3). ∎

THEOREM 22. *If there is a model of* ♣ *in which every ladder system space is countably paracompact*, *then there is a ladder system space in this model which satisfies* (1)–(3) *of Theorem* 20.

Proof. Let $B = \{\alpha + \omega : \alpha \in \omega_1\}$ and for each $\alpha \in \mathrm{LIM}$, let $L_{\alpha+\omega} = (\alpha, \alpha+\omega)$. Let $\{L_\alpha : \alpha \in \mathrm{Lim}\}$ be any ♣-sequence extending $\{L_\beta : \beta \in B\}$ in this model. We prove that X_L satisfies (1)–(3). Indeed, (2) and (3) hold since X_L is countably paracompact. To prove (1), assume towards a contradiction that U and V are disjoint open neighborhoods of B and $\mathrm{Lim} \setminus B$ respectively. Since U is an uncountable subset of ω_1, there is $\beta \in \mathrm{Lim}$ such that $L_\beta \subset U$. It follows that $\beta \subset \mathrm{Lim} \setminus B$, hence L_β is contained in V modulo a finite subset, contrary to the assumption that U and V are disjoint. ∎

It is easy to see that ♣ can be replaced in Theorem 22 with a much weaker principle. For example, it is enough to assume the existence of a ladder system $\{L_\alpha : \alpha \in S\}$ such that for every uncountable $A \subset \mathrm{Succ}$ there is $\alpha \in S$ with $A \cap L_\alpha$ infinite. Also, it is enough to assume that the ladder system constructed in the proof is countably paracompact. For a further discussion of similar problems, see Section 7 below.

5. Screenable countably paracompact spaces. A space is *screenable* if every open cover has a σ-disjoint open refinement. Z. Balogh [1] showed that normal locally compact screenable spaces are paracompact (in ZFC). But the question whether or not the same is true with "normal" replaced by "countably paracompact" remains open. P. Daniels showed that it holds under V = L [4] or under MA_{ω_1} + Axiom R [3]. In the result below, we obtain an equivalence of the problem in terms of ladder systems. Note that Daniels's results follow from this equivalence, together with Theorem 24 and Corollary 27 in the next section.

THEOREM 23. *The following are equivalent*:

(1) *There is a countably paracompact*, *locally compact screenable space which is not paracompact.*
(2) *There is an uncountably regular cardinal* κ *and a thin*, *countably metacompact ladder system* L *on a stationary subset* S *of* $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ *such that the ladder space restricted to any* $\alpha < \kappa$ *is CWH.*

Proof. (2)⇒(1). Let $\{L_\alpha : \alpha \in S\}$ be a ladder system satisfying the conditions of (2). The set for our space X is $(\kappa \times 2) \cup E$, where $E = \{\{\alpha, \beta\} : \beta \in L_\alpha \text{ or } \alpha \in L_\beta\}$. The set E is a set of isolated points. Let F be a finite set. A neighborhood of a point of the form $(\beta, 0)$ is

$$N(\beta, 0, F) = \{(\beta, 0)\} \cup \{\{\beta, \alpha\} \in E : \beta \in L_\alpha\} \setminus F,$$

and a neighborhood of $(\alpha, 1)$ is

$$N(\alpha, 1, F) = \{(\alpha, 1)\} \cup \{\{\beta, \alpha\} \in E : \beta \in L_\alpha\} \setminus F.$$

Note the following:

- $\kappa \times 2$ is a closed discrete set in X.
- $N(\alpha, e, F)$ is the one-point compactification of a subset of E.
- For fixed $e < 2$, the sets $N(\alpha, e, \emptyset)$, $\alpha < \kappa$, are pairwise disjoint.
- $N(\beta, 0, \emptyset) \cap N(\alpha, 1, \emptyset)$ equals $\{\alpha, \beta\}$ if $\beta \in L_\alpha$, and is empty otherwise.

It follows that X is locally compact, screenable, and 2-boundedly metacompact. It is not collectionwise Hausdorff, so not paracompact, by the pressing down lemma.

Note that we have not yet used any of the special properties of the ladder system. We will use them in proving that X is countably paracompact. Let $c : \kappa \times 2 \to \omega$ code a countable partition of $\kappa \times 2$. It suffices to show that there is a locally finite expansion.

Let C be a club witnessing thinness of the ladder system with respect to the coloring $c_0(\alpha) = c(\alpha, 0)$. Let $F_\alpha \in [L_\alpha]^{<\omega}$ witness countable metacompactness of the ladder system for the partition $c_1(\alpha) = c(\alpha, 1)$; i.e., if $\beta \in \kappa$, then the set $\{c_1(\alpha) : \beta \in L_\alpha \setminus F_\alpha\}$ is finite. By the CWH property, we may assume that for $\alpha, \alpha' \notin C$, $L_\alpha \setminus F_\alpha \cap L_{\alpha'} \setminus F_{\alpha'} = \emptyset$.

Now let $F_{\beta 0} = \{\{\beta, \alpha\} : \beta \in L_\alpha \setminus F_\alpha,\ \alpha \notin C\}$, and $F_{\alpha 1} = \{\{\beta, \alpha\} : \beta \in F_\alpha\}$. Note that these are finite sets. Let $U_n = \bigcup_{c(\alpha,1)=n} N(\alpha, 1, F_{\alpha 1})$ and $V_n = \bigcup_{c(\beta,0)=n} N(\beta, 0, F_{\beta 0})$.

It remains to prove that $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ are locally finite. Any limit point of $\{V_n : n \in \omega\}$ would have the form $(\alpha, 1)$. Suppose $N(\alpha, 1, F_{\alpha 1}) \cap V_n \neq \emptyset$. Then there is β with $c(\beta, 0) = n$ and $N(\alpha, 1, F_{\alpha 1}) \cap N(\beta, 0, F_{\beta 0}) \neq \emptyset$. Thus $\beta \in L_\alpha \setminus F_\alpha$ and $\alpha \in C$. But for $\alpha \in C$, the set $\{c(\beta, 0) : \beta \in L_\alpha\}$ is finite. It follows that $N(\alpha, 1, F_{\alpha 1})$ meets only finitely many V_n's.

Finally, let us see that $\{U_n : n \in \omega\}$ is locally finite. If not, there would be a limit point of the form $(\beta, 0)$. If $N(\beta, 0, \emptyset) \cap U_n \neq \emptyset$, then there is $\alpha \in S$ with $N(\beta, 0, \emptyset) \cap N(\alpha, 1, F_{\alpha 1}) \neq \emptyset$ and $c(\alpha, 1) = n$. Then $\beta \in L_\alpha \setminus F_\alpha$ and since the F_α's witness countable metacompactness of the ladder space, the set of possible "colors" $n = c(\alpha, 1)$ is finite. So, $N(\beta, 0, \emptyset)$ meets only finitely many U_n's.

(1)⇒(2). Assume that there is a countably paracompact, locally compact, screenable space which is not paracompact. We will be using the following results of P. Daniels [3]. Let $Z(\kappa)$ denote the space $\kappa \cup [\kappa]^2$, where $[\kappa]^2$ is the set of isolated points, and a neighborhood of $\alpha \in \kappa$ has the form

$$N(\alpha, F) = \{\alpha\} \cup \{\{\alpha, \beta\} \in [\kappa]^2 : \beta \in \kappa \setminus F\},$$

where F is finite. Note that κ is a closed discrete set of points in $Z(\kappa)$.

- (i) If there is a space as in (1), then there is a countably paracompact subspace X of $Z(\kappa)$ for some uncountable regular cardinal κ such that X contains κ and κ is unseparated in X, but every initial segment of κ is separated in X.
- (ii) Let $A_\alpha = \{\beta : \{\alpha, \beta\} \in X\}$ and $\Gamma = \{\alpha < \kappa : \exists \beta \geq \alpha\ (|A_\beta \cap \alpha| \geq \omega)\}$. Then Γ is stationary.
- (iii) For each $\alpha \in \Gamma$, choose $\theta(\alpha) \geq \alpha$ such that $A_{\theta(\alpha)} \cap \alpha$ is infinite. Then there is a stationary $\Delta \subset \Gamma$ such that $\theta \restriction \Delta$ is one-to-one. Further, the set $\Omega = \{\alpha \in \Delta : \mathrm{cf}(\alpha) = \omega \text{ and } \sup(A_{\theta(\alpha)} \cap \alpha) = \alpha\}$ is also stationary.

Let Ω be as in (iii) above. By passing to the intersection of Ω with a club if necessary, we may assume that $\alpha < \beta \in \Omega$ implies $\theta(\alpha) < \beta$. For each $\alpha \in \Omega$, let L_α be a subset of $A_{\theta(\alpha)} \cap \alpha$ which is cofinal in α and has order type ω. We claim that $L = \{L_\alpha : \alpha \in \omega\}$ is a ladder system having the desired properties.

That the ladder space of L restricted to any $\alpha < \kappa$ is CWH follows easily from the fact (see (i) above) that X is $<\kappa$-CWH.

CLAIM 1. *L is thin.*

To see this, suppose $g : \kappa \to \omega$. Since X is countably paracompact, for each α we can find $F_\alpha \in [\kappa]^{<\omega}$ such that if $U_n = \bigcup_{g(\alpha)=n} N(\alpha, F_\alpha)$, then $\{U_n\}_{n\in\omega}$ is locally finite, and further, local finiteness at α is witnessed by $N(\alpha, F_\alpha)$. Suppose g has infinite range on L_α for all α in a stationary set T. Find $\alpha \in T$ such that $F_\beta \subset \alpha$ for each $\beta < \alpha$. Then for all sufficiently large $\beta \in L_\alpha$, the point $\{\beta, \theta(\alpha)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha), F_{\theta(\alpha)})$. It follows that $N(\theta(\alpha), F_{\theta(\alpha)})$ meets U_n for infinitely many n, a contradiction.

It remains to prove:

Claim 2. *L is countably metacompact.*

Let $h : \Omega \to \omega$. We need to find finite sets H_α, $\alpha \in \Omega$, such that for each β, the set $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is finite.

To this end, first extend h to $g : \kappa \to \omega$ so that $g(\theta(\alpha)) = h(\alpha)$. Apply countable paracompactness of X to obtain finite sets F_α, $\alpha \in \kappa$, with the same properties as in Claim 1. We claim that $H_\alpha = F_\alpha \cup F_\theta(\alpha)$ works. Suppose on the contrary that β is such that $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is infinite. Choose $k_0 \in \omega$ such that $g(F_\beta) \subset k_0$. Then find $k_0 < k_1 < \cdots$ such that $\beta \in L_{\alpha_n} \setminus H_{\alpha_n}$ for some $\alpha_n \in \Omega$ with $g(\theta(\alpha_n)) = h(\alpha_n) = k_n$. Then the point $\{\beta, \theta(\alpha_n)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha_n), F_{\theta(\alpha_n)})$ for all n, whence $N(\beta, F_\beta)$ meets U_{k_n} for all n, a contradiction. ■

6. Consistency of thin countably metacompact ladders. In the last two sections we have shown two different topological statements to be equivalent to the existence of a ladder system that is both thin and countably metacompact. We conjecture that there may be such a ladder system.

We know various models in which there are no such ladder systems. For example, we have already seen that uncountable finite-support and countable-support iterations, and uniformizability of ladder systems (and hence MA_{ω_1}) kill thin ladder systems. We have also seen that $\diamondsuit^{\#}$ implies the existence of a thin ladder system. However, a weaker diamond principle, $\diamondsuit(S)$, kills the conjunction thin + countably metacompact on S.

Theorem 24. *Let S be a stationary subset of the ordinals of countable cofinality in the regular uncountable cardinal κ. Then $\diamondsuit(S)$ implies there is no thin countably metacompact ladder system on S.*

Proof. $\diamondsuit(S)$ implies that there are $F_\alpha : \alpha \to [\omega]^{<\omega}$ for $\alpha \in S$ such that, given any $F : \kappa \to [\omega]^{<\omega}$, there are stationarily many $\alpha \in S$ with $F\restriction\alpha = F_\alpha$.

Suppose L is a thin, countably metacompact ladder system on S. For each $\alpha \in S$, choose, if possible, $f(\alpha) \in \omega$ such that $f(\alpha) \notin F_\alpha(\beta)$ for cofinally many $\beta \in L_\alpha$; otherwise, let $f(\alpha) = 0$. By countable metacompactness, there is $F : \kappa \to [\omega]^{<\omega}$ such that for each $\alpha \in S$, $f(\alpha) \in F(\beta)$ for all but finitely many $\beta \in L_\alpha$. By thinness, F is eventually constant on a club C of ladders. Considering $\alpha \in C \cap S$ such that $F \restriction \alpha = F_\alpha$ yields a contradiction. ■

REMARK. It is a corollary to the above proof that a thin countably metacompact ladder also satisfies $\mathcal{P}_{<\omega}$ and that $\diamondsuit(S)$ implies that no ladder on S satisfies $\mathcal{P}_{<\omega}$.

As far as higher cardinal versions are concerned, we do not know if it is consistent for there to be any thin ladder system on any stationary subset of a regular cardinal greater than ω_1. Under Fleissner's Axiom R [7], there is no need to consider cardinals higher than ω_1 for the topological application of Section 5 (see Theorem 23).

THEOREM 25 ([7]). (Axiom R) *If X is ω_1-cwH, has local density $\leq \omega_1$, and has countable tightness, then X is CWH.*

COROLLARY 26. (Axiom R) *Let $\kappa > \omega_1$ be regular. Then there is no ladder system on a stationary subset of κ such that the ladder space restricted to any $\alpha < \kappa$ is CWH.*

Proof. A ladder space with the stated properties easily satisfies the hypotheses of Fleissner's theorem but not the conclusion. ■

COROLLARY 27. (MA_{ω_1} + Axiom R) *There is no ladder system satisfying the conditions of Theorem* 23(2).

Finally, we mention that we also do not know if thin + $\mathcal{M}_0$ is consistent, i.e., if there could be a thin, normal ladder system.

7. Relation to other ladder system properties. We now investigate the relationship between ♣-type properties of ladder systems with the properties studied in the previous sections.

It is easy to see that if a ladder system $\{L_\alpha : \alpha \in \omega_1\}$ is a ♣-sequence, then it satisfies H_1. To see this, suppose that $f : \omega_1 \to \omega$ is given. Let n be such that $A = f^{-1}(n)$ is uncountable. Then if the set of α such that $L_\alpha \subseteq^* A$ is stationary, then we see that H_1 is satisfied. Similarly one can see that the weak ♣-principle that requires for each uncountable A that only $L_\alpha \cap A$ is infinite for some α (hence for stationarily many α), gives the property H_2.

On the other hand, if $\{L_\alpha : \alpha \in \omega_1\}$ has any one of our anti-uniformization properties, then so does the sequence $\{L'_\alpha : \alpha \in \omega_1\}$, where

$$L'_\alpha = \{\beta + 1 : \beta \in L_\alpha\}.$$

Each element of this sequence is disjoint from the set of limit ordinals, hence fails even the weak ♣-principle.

It is possible to modify our proofs of the existence of a thin ladder system on ω_1 to make the resulting ladder system a ♣-sequence. On the other hand, it is consistent that there is a ♣-sequence that is not G_3 (both proofs are in [8]).

Question 4. *May there be a ♣-sequence that is not G_3?*

Concerning the uniformization properties, we have the following result:

Proposition 28. *Any ♣-sequence fails property $\mathcal{P}_0$.*

Proof. Let $\{L_\alpha : \alpha \in S\}$ be a ♣-sequence. Fix $f : S \to \omega$ such that $f^{-1}(0)$ is uncountable and nonstationary. Suppose that F uniformizes f as given by $\mathcal{P}_0$. Then $F^{-1}(0)$ is uncountable. Hence the set of α such that $L_\alpha \subseteq^* F^{-1}(0)$ is stationary. Thus there is α such that $f(\alpha) \neq 0$ but F is eventually constant with value 0 on L_α; a contradiction. ∎

Question 5. *May there be a ♣-sequence that satisfies $\mathcal{P}_1$? or even just $\mathcal{M}_{<\omega}$?*

Recall that Theorem 22 of Section 4 raised the following question to which a positive solution would give an example of a countably paracompact subspace of ω_1^2 that is not normal.

Question 6. *Is it consistent that there exists a weak ♣-sequence, and every ladder system satisfies $\mathcal{P}_{<\omega}$?*

In [12] the following ♣-principles are introduced.

Definition 29. $♣_{\mathrm{NS}}$ is the statement: there is a ladder system $L = \{L_\alpha : \alpha \in \omega_1\}$ such that

(1) for each club $C \subseteq \omega_1$, the set of α such that $L_\alpha \subseteq^* C$ contains a club,
(2) for each $A \subseteq \omega_1$ there is a club C such that for each $\alpha \in C$ either $L_\alpha \subseteq^* A$ or $L_\alpha \cap A$ is finite.

Lemma 30. *If L satisfies $♣_{\mathrm{NS}}$, then for every stationary $S \subseteq \omega_1$ and every partition $\{A_n : n \in \omega\}$ of ω_1,*

$$\{\alpha \in S : (\exists n\ (L_\alpha \subseteq^* A_n)) \vee (\forall n\ (|L_\alpha \cap A_n| < \aleph_0)\}$$

is stationary.

Proof. Apply 29(2) to each A_n and let $S' = S \cap \bigcap_n C_n$.

Corollary 31. *If L is a $♣_{\mathrm{NS}}$-sequence, and L is not thin on any stationary set, then for every stationary set there is a partition $\{A_n : n \in \omega\}$ of ω_1 such that*

$$\{\alpha \in S : \forall n\ (|A_n \cap L_\alpha| < \aleph_0)\}$$

is stationary.

For the next definition we need some notation. Let T be stationary and $h : [\omega_1]^{<\omega} \to P(\omega_1)$, and set

$$\mathcal{F}_{T,L} = \{A \subseteq \omega_1 : \{\alpha : L_\alpha \subseteq^* A\} \text{ is club on } T\},$$
$$Z_{h,L} = \{\alpha : \exists \beta \in L_\alpha\ \forall \eta \in L_\alpha \setminus \beta\ (\eta \in h(L_\alpha \cap \eta))\}.$$

Definition 32. $\clubsuit^+_{\rm NS}$ is the statement: there is a ladder system $L = \{L_\alpha : \alpha \in \omega_1\}$ satisfying $\clubsuit_{\rm NS}$ and such that, for all $X \in [P(\omega_1)]^{\omega_1}$ and all stationary S, there is a stationary $T \subseteq S$ and an ultrafilter u on ω_1 such that

(1) $\mathcal{F}_{T,L} \cap X = u \cap X$,
(2) for all $h : [\omega_1]^{<\omega} \to X \cap u$, $T \setminus Z_{h,L}$ is nonstationary.

Theorem 33. *If L satisfies $\clubsuit^+_{\rm NS}$ and is not thin, then $\mathfrak{d} = \omega_1$.*

Proof. Fix a partition $\{A_n : n \in \omega\}$ of ω_1 such that

$$S = \{\alpha : \forall n\ (|L_\alpha \cap A_n| < \aleph_0)\}$$

is stationary.

Let $B_n = \bigcup_{m>n} A_n$ and let $X \subseteq P(\omega_1)$ be any family of sets such that $B_n \in X$ for all n. Let $T \subseteq S$ be stationary and u be an ultrafilter given by the definition of $\clubsuit^+_{\rm NS}$. Note that since $T \subseteq S$, we have $B_n \in \mathcal{F}_{T,L} \cap X$ for each n. So $B_n \in u \cap X$ for each $n \in \omega$.

For each $\alpha \in \omega_1$, let $\{\alpha_n : n \in \omega\}$ be the increasing enumeration of L_α and define $g_\alpha : \omega \to \omega$ by

$$g_\alpha(n) = \min\{m : B_m \cap \{\alpha_i : i \le n\} = \emptyset\}.$$

We claim that the family of g_α is dominating. Fix $f : \omega \to \omega$. Define $h : [\omega]^{<\omega} \to u \cap X$ by $h(a) = B_f(|a|)$. By definition of $\clubsuit^+_{\rm NS}$, we may fix $\alpha \in Z_{h,L} \cap T$. By definition of $Z_{h,L}$ we may fix $\beta \in L_\alpha$ such that $\eta \in h(L_\alpha \cap \eta)$ for all $\eta \in L_\alpha \setminus \beta$. Fix n_0 such that $\alpha(n_0) = \beta$ and fix $n > n_0$. Then

$$\alpha(n) \in h(L_\alpha \cap \alpha(n)) = B_{f(|L_\alpha \cap \alpha(n)|)} = B_{f(n)}.$$

Since $\alpha(n) \notin B_{g_\alpha(n)}$ it follows that $g_\alpha(n) > f(n)$. Thus $f <^* g_\alpha$ and hence $\{g_\alpha : \alpha < \omega_1\}$ is dominating. ■

In [12, Chapter 8], the consistency of $\clubsuit^+_{\rm NS}$ with saturation of the nonstationary ideal in a variant of a $P_{\max}$ extension is obtained. The continuum is $\aleph_2$ in this model. Unfortunately, $\mathfrak{d} = \omega_1$ also holds in this model [13]. Nonetheless, we conjecture that consistency of a thin ladder system satisfying weak uniformization properties should be obtainable using some $P_{\max}$ variation. Indeed, the statement that there is a stationary set carrying a thin ladder system is the negation of a Π_2 sentence, and the sentence asserting that every ladder system is countably metacompact is Π_2.

References

[1] Z. Balogh, *Paracompactness in locally Lindelöf spaces*, Canad. J. Math. 38 (1986), 719–727.

[2] D. K. Burke and Z. Balogh, *A total ladder system space by ccc forcing*, Topology Appl. 44 (1992), 37–44.

[3] P. Daniels, *When countably paracompact, locally compact, screenable spaces are paracompact*, ibid. 26 (1987), 271–279.

[4] —, *On collectionwise Hausdorffness in countably paracompact, locally compact spaces*, ibid. 28 (1988), 113–125.

[5] K. Devlin, *The combinatorial principle* $\diamondsuit^\sharp$, J. Symbolic Logic 47 (1982), 888–899.

[6] K. Devlin and S. Shelah, *A weak version of* $\diamondsuit$ *which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math. 29 (1978), 239–247.

[7] W. G. Fleissner, *Left separated spaces with point-countable bases*, Trans. Amer. Math. Soc. 294 (1986), 665–677.

[8] F. Hernandez-Hernandez, preprint.

[9] N. Kemoto, K. D. Smith and P. J. Szeptycki, *Countable paracompactness versus normality in* ω_1^2, Topology Appl. 104 (2000), 141–154.

[10] S. Shelah, *Proper and Improper Forcing*, Springer, 1998.

[11] —, *A consistent counterexample in the theory of collectionwise Hausdorff spaces*, Israel J. Math. 65 (1989), 219–224.

[12] W. H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, de Gruyter Ser. Logic Appl., de Gruyter, 1999.

[13] —, private communication.

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