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On the Leibniz–Mycielski axiom in set theory

by

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Abstract. Motivated by Leibniz's thesis on the identity of indiscernibles, Mycielski introduced a set-theoretic axiom, here dubbed the *Leibniz–Mycielski axiom* LM, which asserts that for each pair of distinct sets x and y there exists an ordinal α exceeding the ranks of x and y, and a formula $\varphi(v)$, such that $(V_\alpha, \in)$ satisfies $\varphi(x) \wedge \neg\varphi(y)$.

We examine the relationship between LM and some other axioms of set theory. Our principal results are as follows:

1. *In the presence of* ZF, *the following are equivalent:*
   (a) LM.
   (b) *The existence of a parameter free definable class function* $\mathbf{F}$ *such that for all sets* x *with at least two elements*, $\emptyset \neq \mathbf{F}(x) \subsetneq x$.
   (c) *The existence of a parameter free definable injection of the universe into the class of subsets of ordinals.*
2. Con(ZF) $\Rightarrow$ Con(ZFC + $\neg$LM).
3. [Solovay] Con(ZF) $\Rightarrow$ Con(ZF + LM + $\neg$AC).

1. Introduction. The principle of the *identity of indiscernibles*, formulated by Leibniz [L, p. 308], states that no two distinct substances exactly resemble each other. Leibniz's principle can be construed as prescribing a logical relationship between *objects* and *properties*: any two distinct objects must differ in at least one property. A natural interpretation of this principle is offered by model theory: fix a model $\mathfrak{M} = (M, \ldots)$ in a language $\mathcal{L}$, let the "objects" refer to the elements of M, and the "properties" refer to properties that are $\mathcal{L}$-expressible in $\mathfrak{M}$ via first order formulas with one free variable. Under this interpretation, a model $\mathfrak{M}$ satisfies Leibniz's principle iff $\mathfrak{M}$ has no set of indiscernibles of length 2 or higher in the sense of model theory, in other words, $\mathfrak{M}$ contains no pair of distinct elements a and b such that for every first order formula $\varphi(x)$ of $\mathcal{L}$ with precisely one free variable x,

$$(\mathfrak{M}, a, b) \vDash \varphi(a) \leftrightarrow \varphi(b).$$

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This formulation of Leibniz's principle cannot be expressed in first order logic since the classical theorem of Ehrenfeucht and Mostowski [CK, Theorem 3.3.10] shows that every first order theory with an infinite model has a model with indiscernibles. However, Mycielski [My-1] has introduced the following first order axiom in the language of set theory $\{\in\}$, here referred to as the *Leibniz–Mycielski axiom* LM, which captures the spirit of Leibniz's principle for models of set theory ([1]).

$$\text{LM: } \forall x\ \forall y\ [x \neq y \to \exists \alpha > \max\{\varrho(x), \varrho(y)\}\ \mathrm{Th}(V_\alpha, \in, x) \neq \mathrm{Th}(V_\alpha, \in, y)].$$

Here $\varrho(x)$ is the ordinal rank of x, V_α is the αth level of the von Neumann hierarchy consisting of sets of ordinal rank less than α, and $\mathrm{Th}(V_\alpha, \in, a)$ is the *first order theory* of the structure $(V_\alpha, \in, a)$, where a is viewed as a distinguished constant. The following result directly relates LM to the first order interpretation of Leibniz's principle.

Theorem 1 (Mycielski [My-1]). *A complete extension T of* ZF *proves* LM *iff T has a model with no indiscernibles* ([2]).

In this paper we examine the relationship between LM and other axioms of set theory within the framework of ZF set theory. In Section 2 we show that the "logical" axiom LM is equivalent to *global* forms of the Kinna–Wagner choice principles [KW].

GKW_1: There is a definable (without parameters) map $\mathbf{F}$ such that

$$\forall x\ (|x| > 1 \to (\emptyset \neq \mathbf{F}(x) \subsetneq x)).$$

GKW_2: There is a definable (without parameters) map $\mathbf{G}$ such that

"$\mathbf{G}$ injects $\mathbf{V}$ into the class of subsets of $\mathbf{Ord}$".

This characterization shows that LM is a theorem of $\mathrm{ZF} + \mathbf{V} = \mathbf{OD}$, and imbues LM with the flavor of a choice principle. In Section 3 we examine the relationship between LM and the axiom of choice AC. In Theorem 3.1 we use a standard symmetry argument involving Cohen forcing to show that LM is independent of ZFC. The independence of AC from $\mathrm{ZF} + \mathrm{LM}$ is due to Robert Solovay, and is presented here as Theorem 3.3 with his kind permission. Solovay uses the technology of Jensen's minimal Π^1_2-singletons [Jn] to show that AC fails, but LM remains true in the *symmetric inner model* of the universe obtained by adding countably many mutually generic Jensen reals to the constructible universe. In conclusion, Section 4 focuses

([1]) Mycielski refers to this axiom as A_2' in [My-1], and L in [My-2].
([2]) For more on models of set theory without indiscernibles, see [En].

on variants of LM, and includes a number of open questions. Figure 1.1 summarizes our principal results.

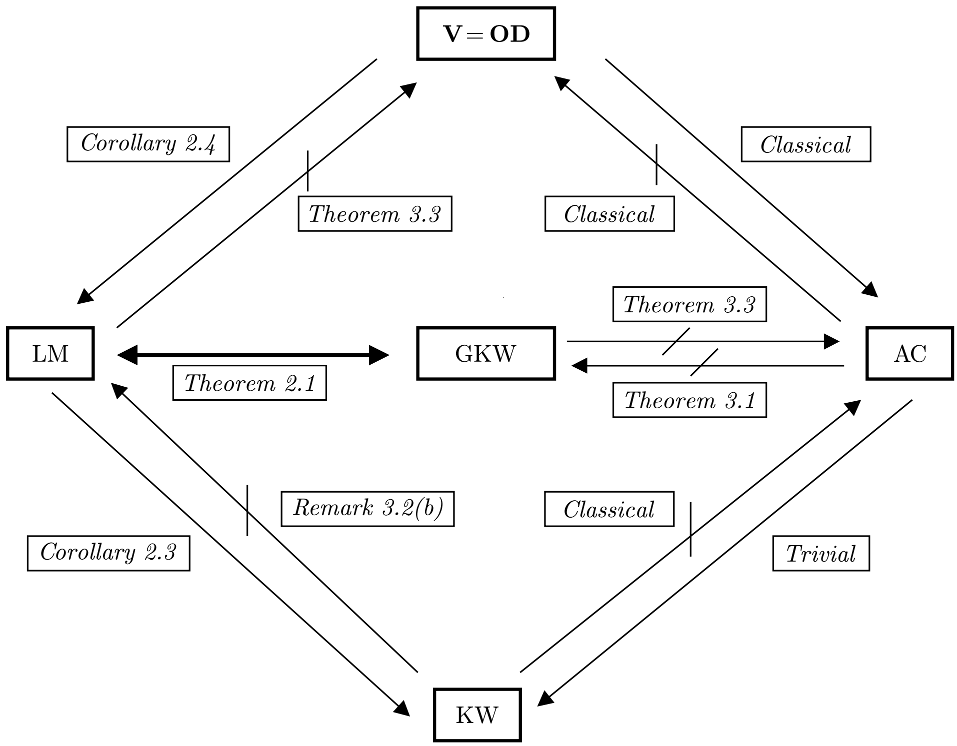

Fig. 1.1. LM among some related axioms of set theory. The arrows indicate provable implications within ZF.

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2. LM and the Global Kinna–Wagner Principles. Myhill and Scott [MS] formulated the axiom $\mathbf{V} = \mathbf{OD}$ in terms of the existence of a parameter free definable global choice function. In the same spirit, Theorem 2.1 shows that LM can be reformulated in terms of the *global* Kinna–Wagner selection principles introduced in Section 1. Recall that the *local* Kinna–Wagner principles are the following weaker forms of the axiom of choice (3), which are known to be equivalent within ZF [KW].

(3) KW_1 is often referred to as the *Kinna–Wagner selection principle* KW.

KW_1: For every family $\mathcal{F}$ of sets there is a function f such that

$$\forall x \in \mathcal{F}\ (|x| \geq 2 \to \emptyset \neq f(x) \subsetneq x).$$

KW_2: Every set can be injected into the power set of some ordinal.

THEOREM 2.1. *Suppose M is a model of* ZF. *The following are equivalent*:

(i) M *satisfies* GKW_1.
(ii) M *satisfies* GKW_2.
(iii) M *satisfies* LM.

Proof. We shall prove (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). The classical argument of Kinna and Wagner can readily be adapted to the global context. Suppose that there is a parameter free definable map $\mathbf{F}$ in M such that

$$M \models \forall x\ (|x| \geq 2 \to \emptyset \neq \mathbf{F}(x) \subsetneq x).$$

Furthermore, we may assume without loss of generality that $\mathbf{F}(x) = x$ whenever x is the empty set or a singleton. We now define an auxiliary class function $\mathbf{C}(x,t)$ within M, where x is any set, and t is a transfinite binary sequence. In what follows $t^\frown s$ denotes the concatenation of t with s, and $\sqsubset$ denotes the "end extension" relation between sequences.

- If $|x| < 2$, then set $\mathbf{C}(x,t) = x$ for all $t \in \bigcup_{\alpha\in\mathbf{Ord}}\{0,1\}^\alpha$.
- For $|x| \geq 2$, define $\mathbf{C}(x,t)$ by recursion on the length of t:

$$\mathbf{C}(x,\emptyset) = x,$$
$$\mathbf{C}(x,t^\frown\langle 0\rangle) = \mathbf{F}(\mathbf{C}(x,t)),$$
$$\mathbf{C}(x,t^\frown\langle 1\rangle) = \mathbf{C}(x,t) \setminus \mathbf{F}(\mathbf{C}(x,t)),$$
$$\mathbf{C}(x,t) = \bigcap_{s\sqsubset t} \mathbf{C}(x,s) \quad \text{for } t \text{ of limit length.}$$

Using Hartogs's theorem [Jc-2, Section 3] we observe that

$$\forall x\ \forall y \in x\ \exists \alpha \in \mathbf{Ord}\ \exists s \in \{0,1\}^\alpha\ \mathbf{C}(x,s) = \{y\}.$$

Given $y \in x$, let $\alpha(x,y)$ be the *least* ordinal α such that

$$\exists s \in \{0,1\}^\alpha\ (\mathrm{length}(s) = \alpha \text{ and } \mathbf{C}(x,s) = \{y\}).$$

It is easy to see that

$$\forall x\ \forall y \in x\ \exists! s \in \{0,1\}^{\alpha(x,y)}\ \mathbf{C}(x,s) = \{y\}.$$

We can therefore describe an injection

$$\mathbf{H} : \mathbf{V} \to \mathbf{Ord} \times \bigcup_{\alpha\in\mathbf{Ord}} \{0,1\}^\alpha$$

as follows: Given an input y of rank θ, let $\beta = \alpha(V_{\theta+1}, y)$, and define $\mathbf{H}(y)$ by

$$\mathbf{H}(y) = \langle \theta, s \rangle \quad \text{if } s \in \{0,1\}^\beta \text{ and } \mathbf{C}(V_{\theta+1}, s) = \{y\}.$$

Finally, follow $\mathbf{H}$ with a definable injection from $\mathbf{Ord} \times \bigcup_{\alpha\in\mathbf{Ord}}\{0,1\}^\alpha$ into the class of subsets of $\mathbf{Ord}$ to obtain the desired $\mathbf{G}$ injecting the universe into the class of subsets of ordinals. This concludes the proof of the implication (i)⇒(ii).

(ii)⇒(iii). Assume that M is a model of ZF, and that $\mathbf{G}$ is a class function definable without parameters, such that

$$M \vDash \text{“}\mathbf{G} \text{ injects } \mathbf{V} \text{ into } \bigcup_{\alpha\in\mathbf{Ord}} \mathcal{P}(\alpha)\text{”}.$$

To see that LM holds in M, suppose a and b are distinct elements of M. Since $\mathbf{G}(a) \neq \mathbf{G}(b)$ holds in M, we may assume without loss of generality that for some $\beta \in \mathbf{Ord}^M$,

$$M \vDash \ \beta \in \mathbf{G}(a) \setminus \mathbf{G}(b).$$

By the Myhill–Scott Extended Reflection Theorem [MS, p. 273] there is an ordinal α such that (1) and (2) below hold in M:

(1) β is first order definable in $(V_\alpha, \in)$.
(2) $(\beta \in \mathbf{G}(a))^{V_\alpha}$ and $(\beta \notin \mathbf{G}(b))^{V_\alpha}$.

This makes it evident that $\mathrm{Th}(V_\alpha, \in, a) \neq \mathrm{Th}(V_\alpha, \in, b)$ holds in M, so (iii) holds.

(iii)⇒(i). Suppose M satisfies LM. Given a set x with at least two elements, we wish to define, in terms of x alone, a proper nonempty subset of x. Let α_x be the least ordinal such that

$$\exists u \in x\ \exists v \in x\ \mathrm{Th}(V_{\alpha_x}, \in, u) \neq \mathrm{Th}(V_{\alpha_x}, \in, v).$$

Choose φ_x to be the first order formula of least Gödel number such that

$$\exists u \in x\ \exists v \in x\ V_{\alpha_x} \vDash \varphi_x(u) \wedge \neg\varphi_x(v).$$

Now let $\mathbf{F}(x) = \{y \in x : V_{\alpha_x} \vDash \varphi_x(y)\}$. Clearly

$$\emptyset \neq \mathbf{F}(x) \subsetneq x. \ \blacksquare$$

REMARK 2.2. (a) Pincus [P] showed that the Kinna–Wagner principles are equivalent to the following statements SDO (*selective dense order principle*) and SUO (*selective unbounded order principle*) over ZF:

SDO: $\forall\mathcal{F}\ \exists f\ \forall x \in \mathcal{F}\ (x \text{ is infinite} \to f(x) \text{ is a dense linear order on } x)$.

SUO: $\forall\mathcal{F}\ \exists f\ \forall x \in \mathcal{F}\ (x \text{ is infinite} \to f(x) \text{ is a linear order on } x$ with no first or last element$)$.

Theorem 2.1 and results of Pincus [P, Theorem 1.5, and the equivalents of KW on p. 446] together imply that LM *is also equivalent to the following global forms* GSDO *and* GSUO of SDO *and* SUO:

GSDO: For some parameter free definable class function $\mathbf{F}$,

$$\forall x\ (x \text{ is infinite} \to \mathbf{F}(x) \text{ is a dense linear order on } x).$$

GSUO: For some parameter free definable class function $\mathbf{F}$,

$$\forall x\ (x \text{ is infinite} \to \mathbf{F}(x) \text{ is a linear order on } x \text{ with no first or last element}).$$

(b) The usual "back-and-forth" proof of the Cantor–Schröder–Bernstein theorem [Jc-2, Theorem 7] can be adapted to show that LM is equivalent to the existence of a definable *bijection* between the universe and the class of subsets of ordinals.

(c) In ZFC the universe can be definably injected into the class of *subsets of subsets of ordinals.* To define the desired injection $\mathbf{H}$, let $\Gamma : \mathbf{Ord}^2 \to \mathbf{Ord}$ be Gödel's canonical pairing function [Jc-2, p. 20] and for every set s let $\kappa_s = |\mathrm{trcl}(\{s\})|$, where $\mathrm{trcl}(x)$ is the transitive closure of x. Then define $\mathbf{H}(s)$ to be

$$\{x \subseteq \kappa_s : s \text{ is the } \in\text{-maximum element of the Mostowski collapse of } \Gamma^{-1}(x)\}.$$

To see that $\mathbf{H}(s) \neq \emptyset$, note that any bijection f between the cardinal κ_s and $\mathrm{trcl}(\{s\})$ induces a binary relation $R \subseteq \kappa_s^2$ such that $(\kappa_s, R) \cong (\mathrm{trcl}(\{s\}), \in)$. Let $r \subseteq \kappa_s$ be the image of R under Γ. Clearly $r \in \mathbf{H}(s)$.

(d) Consider the following sentences in the language of *class theory* asserting the class forms of the Kinna–Wagner principles:

CKW_1: $\exists \mathbf{F}\ \forall x\ (|x| \geq 2 \to \emptyset \neq \mathbf{F}(x) \subsetneq x)$;

CKW_2: $\exists \mathbf{G}$ ($\mathbf{G}$ injects $\mathbf{V}$ into the class of subsets of ordinals).

Similarly, consider the following class form of the Leibniz–Mycielski axiom:

$$\mathrm{CLM}: \ \exists \mathbf{X}\ \forall x\ \forall y\ [x \neq y \to \exists \alpha > \max\{\varrho(x), \varrho(y)\}\ \mathrm{Th}(V_\alpha, \in, \mathbf{X} \cap V_\alpha, x) \neq \mathrm{Th}(V_\alpha, \in, \mathbf{X} \cap V_\alpha, y)].$$

The proof of Theorem 2.1 can be adapted to establish the equivalence of CKW_1, CKW_2, and CLM within GB (the Gödel–Bernays system of class theory without the axiom of choice). Moreover, a forcing argument (similar to the one used in [F]) shows that *every countable model of* $\mathrm{ZF} + \mathrm{KW}$ *can be expanded to a model of* $\mathrm{GB} + \mathrm{CLM}$. In particular, this shows that $\mathrm{GB} + \mathrm{CLM}$ is a *conservative extension* of $\mathrm{ZF} + \mathrm{KW}$ relative to sentences of set theory (i.e., a sentence φ in the language of set theory is provable within $\mathrm{GB} + \mathrm{CLM}$ iff φ is already provable within $\mathrm{ZF} + \mathrm{KW}$).

(e) Let $\mathrm{LM}(c)$ denote the *parametric* version of LM that asserts:

$$\forall x\ \forall y\ [x \neq y \to \exists \alpha > \max\{\varrho(x), \varrho(y), c\}\ \mathrm{Th}(V_\alpha, \in, c, x) \neq \mathrm{Th}(V_\alpha, \in, c, y).$$

Similarly, let $\mathrm{GKW}_1(c)$ and $\mathrm{GKW}_2(c)$ denote the parametric versions of the global Kinna–Wagner principles involving the parameter c. The proof of Theorem 2.1 can be *uniformly* carried out relative to the parameter c to show the equivalence of $\mathrm{LM}(c)$, $\mathrm{GKW}_1(c)$, and $\mathrm{GKW}_2(c)$.

We conclude this section with some immediate consequences of Theorem 2.1.

Corollary 2.3. $\mathrm{ZF} + \mathrm{LM} \vdash \mathrm{KW}$.

Corollary 2.4 ([4]). $\mathrm{ZF} + \mathbf{V} = \mathbf{OD} \vdash \mathrm{LM}$.

Proof. The universe can be globally well-ordered in the presence of $\mathbf{V} = \mathbf{OD}$, so there is a definable injection of the universe into the class of *singletons* of ordinals. This corollary can also be derived from Theorem 1, since every completion of $\mathrm{ZF} + \mathbf{V} = \mathbf{OD}$ has a model all of whose elements are definable (and such a model cannot contain indiscernibles). ■

Corollary 2.5. *In the presence of* $\mathrm{ZF} + \mathrm{LM}$ *there is a parameter free definable global linear ordering of the universe.*

Proof. In light of Theorem 2.1 it suffices to observe that $\bigcup_{\alpha \in \mathbf{Ord}} \{0,1\}^\alpha$ can be linearly ordered, first by length, then lexicographically. ■

Corollary 2.6. $\mathrm{ZF} + \mathrm{LM}$ *proves* $\mathrm{GC}_{<\omega}$ (*global choice for collections of finite sets*).

Proof. This is a consequence of Corollary 2.5, and the fact that within ZF every finite linearly ordered set is well-ordered. ■

Corollary 2.7. $\mathrm{ZF} + \mathrm{LM}$ *proves the existence of a definable set of real numbers that is not Lebesgue measurable and does not have the Baire property.*

Proof ([5]). This is a consequence of putting Corollary 2.6 together with a classical argument of Sierpiński ([Si], [Jc-1, Problem 11]) that establishes, within $\mathrm{ZF} + \mathrm{AC}_2$ (axiom of choice for pairs), the existence of a set of real numbers that is neither Lebesgue measurable, nor has the Baire property. ■

3. Independence results. *In what follows*, *we use the blanket assumption that* ZF *is consistent so as to avoid having to state our theorems in the awkward conditional form "If* ZF *has a model then ...". Moreover*, *since our independence results can all be established via forcing over a model of*

([4]) This result was first established by Mycielski [My-1, Theorem 4].

([5]) This proof is due to Mycielski, and simplifies earlier arguments of Solovay and the author.

ZF + $\mathbf{V} = \mathbf{L}$ *using partial orders of cardinality at most* $\aleph_1$, *we further assume that* $\aleph_2^{\mathbf{L}}$ *is countable.* This will guarantee in all cases considered here that the generic object exists outright within $\mathbf{V}$. It is well known that our assumption comes at no cost at the consistency level.

We wish to examine the relationship between the axiom of choice and LM within ZF. Of course ZFC + LM is consistent because the constructible universe $\mathbf{L}$ already satisfies LM (thanks to Corollary 2.3). Theorem 2.1 leads one to surmise that within ZFC one cannot establish LM. The next result confirms this expectation.

THEOREM 3.1. *There is a model of* ZFC *in which* LM *fails.*

Proof. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2) \in \mathbf{L}$ be Cohen's notion of forcing for adding a generic real. Forcing with $\mathbb{P}^2$ produces a generic filter $G = G_1 \times G_2$, where G_1 and G_2 are *mutually* $\mathbb{P}$*-generic.* Thanks to the equivalence of LM with GKW_2 (established in Theorem 2.1), the failure of LM follows immediately from the existence of indiscernible elements a_1 and a_2 in $\mathbf{L}[G]$ such that:

(1) For all unary formulas $\varphi(x, \theta)$, where θ is an ordinal parameter,
$$\mathbf{L}[G] \vDash \varphi(a_1, \theta) \leftrightarrow \varphi(a_2, \theta).$$

The candidates for the indiscernibles are $a_1 = \mathcal{P}(\omega)^{\mathbf{L}[G_1]}$ and $a_2 = \mathcal{P}(\omega)^{\mathbf{L}[G_2]}$. By mutual genericity of G_1 and G_2, $a_1 \neq a_2$. To prove (1), we shall prove the following stronger assertion:

(1*) For all binary formulas $\varphi(x, y, \theta)$, where θ is an ordinal parameter,
$$\mathbf{L}[G] \vDash \varphi(a_1, a_2, \theta) \leftrightarrow \varphi(a_2, a_1, \theta).$$

For $i \in \{1, 2\}$ let π_i be the canonical $\mathbb{P}^2$-term for a_i. The automorphism $\langle p, q\rangle \mapsto_f \langle q, p\rangle$ of $\mathbb{P}^2$ induces a permutation $\overline{f}$ of order 2 on $\mathbb{P}$-names. Clearly,

$$\overline{f} \text{ transposes } \pi_1 \text{ and } \pi_2. \tag{2}$$

To see that (1*) holds, suppose $\mathbf{L}[G] \vDash \varphi(a_1, a_2, \theta)$. Then for some $p_1 \in G_1$ and $p_2 \in G_2$, $\langle p_1, p_2\rangle \Vdash \varphi(\pi_1, \pi_2)$. Hence by (2) and the symmetry lemma (as in [Jc, Lemma 19.10] or [K, Lemma 7.13(c)]),

$$\langle p_2, p_1\rangle \Vdash \varphi(\pi_2, \pi_1). \tag{3}$$

Without loss of generality, p_1 and p_2 have the same domain. It follows from a standard argument that there is an automorphism h of $\mathbb{P}$ of finite support that transposes p_1 and p_2. Let H_1 and H_2 be the images of G_1 and G_2 respectively under h and let $H = H_1 \times H_2$. Note that $p_1 \in H_2$ and $p_2 \in H_1$. Clearly $\mathbf{L}[H] = \mathbf{L}[G]$ since H_i and G_i differ only by finitely many elements. Also, thanks to the symmetry lemma, H is $\mathbb{P}^2$-generic. Hence $\mathbf{L}[G] \vDash \varphi(a_2, a_1, \theta)$ by (3). ■

Remark 3.2. (a) Since forcing with Cohen's partial order $\mathbb{P}$ is equivalent to forcing with $\mathbb{P}^2$, the above argument shows that LM *fails if a Cohen real is generically added to any universe of set theory.*

(b) A symmetry argument similar to the one used in the proof of Theorem 3.1 shows that LM also fails in Cohen's model N of the negation of AC obtained by adjoining a countably infinite set S of mutually generic Cohen reals to M without adding an enumeration of S itself [Jc-2, Ex. 1, p. 203]. This model was intensively studied by Halpern and Lévy ([HL], [Jc-1, Chapter 5]) who proved that (1) N satisfies the Boolean Prime Ideal theorem, and (2) there is a global injection of the universe into the class of subsets of ordinals that is definable in N from the parameter S. Note that (2) immediately implies that KW holds in N. The failure of LM in N indicates that there is no *parameter free* definable injection of the universe into the class of subsets of ordinals in N. Hence N is a model of $\exists c$ LM(c), but not a model of LM.

(c) Easton [Ea] constructed a model of ZFC in which a definable class of pairs has no definable choice function. In light of Corollary 2.4, this provides an alternative proof of Theorem 3.1. Indeed, using Remark 2.2(e) it is easy to see that $\forall c\ \neg$LM(c) holds in Easton's model.

Theorem 3.3 (Solovay). *There is a model of* ZF *in which* LM *holds but* AC *fails.*

Proof. Roughly speaking, the model is obtained by mixing a construction of Jensen [Jn] with Cohen's model mentioned in Remark 3.2(b). Jensen constructed a c.c.c. poset $\mathbb{P}$ of size $\aleph_1$ in $\mathbf{L}$ that adds a real which is a Π^1_2-singleton of minimal constructibility degree. To first approximation, $\mathbb{P}$ is built by marrying Sacks forcing to Jensen's construction of a Suslin tree in $\mathbf{L}$. Jensen showed ([6]):

(1) There is a $\mathbb{P}$-name for a subset x of ω such that for any generic G, $\mathbf{L}[G] = \mathbf{L}[x_G]$ [and there is a uniform definition of G from x_G]. Because of this, we speak of $\mathbb{P}$-generic reals. They are reals of the form x_G for a generic G.

(2) There is a Π^1_2-formula $A(\cdot)$ such that the following conditions (a) and (b) are equivalent for any real y:

   (a) $A(y)$.

   (b) y is $\mathbb{P}$-generic over $\mathbf{L}$.

(3) For any n distinct $\mathbb{P}$-generic reals $x_1, \ldots, x_n$ the tuple $\langle x_1, \ldots, x_n\rangle$ is $\mathbb{P}^n$-generic.

([6]) N.B. (1) and (2) hold for Cohen forcing; (3) is the key property.

Let $\mathbb{Q}$ be the poset that adds a countable sequence of $\mathbb{P}$-generic reals via the usual finite support product. More precisely, $\mathbb{P}$ has an initial condition $1_{\mathbb{P}}$ which "gives no information" and $\mathbb{Q}$ will consist of functions $h : \omega \to \mathbb{P}$ such that for all but finitely many n, $h(n) = 1_{\mathbb{P}}$. We put the obvious order on $\mathbb{Q}$: $h_1 \le h_2$ iff for all n, $h_1(n) \le h_2(n)$.

Let $N = \mathbf{L}[G]$, where G is $\mathbb{Q}$-generic over $\mathbf{L}$. Also let $a_i = x_{G_i}$, where G_i is the image of G under the projection that maps h to $h(i)$. Finally, let $S = \{a_i : i \in \omega\}$, and $T = S \cup \omega$. *Our desired model* N_1 *is* $\mathbf{L}(T)$. A standard symmetry argument (as in [Jc-2, Lemma 19.12]) shows that the axiom of choice fails in N_1. The verification of LM in N_1 is more difficult and relies on the following preliminary lemmas.

LEMMA 3.3.1. $N_1 \models \{x \subseteq \omega : A(x)\} = S$.

Proof. Suppose towards a contradiction that Lemma 3.3.1 is false. Clearly every element of N_1 is ordinal definable [in N_1] from S and some finite subset of S. Let y be a real such that $y \in N_1$, $A(y)$ holds and $y \notin S$. Pick an OD-definition of y from S and $a_0, \ldots, a_n$. Pick a condition $q \in \mathbb{Q}$ that forces that this definition gives a real that satisfies $A(\cdot)$ but is not in S. By increasing n if necessary, we may assume that q's nontrivial components only refer to a_i's with $i \le n$.

We write the OD definition above as $y = D(S)$. D will also reference names for $a_0, \ldots, a_n$ as well as the name for some ordinal, but we suppress this from our notation. We can write $\mathbb{Q}$ up to isomorphism as the product

$$\mathbb{Q} \cong \mathbb{P}^{n+1} \times \mathbb{Q},$$

where the factor $\mathbb{P}^{n+1}$ adds $\{a_0, \ldots, a_n\}$, and the second $\mathbb{Q}$ adjoins

$$\langle a_m : m \ge n+1 \rangle.$$

We write $M_1 = L[a_0, \ldots, a_n]$. Our first goal is to show that $y \in M_1$. Of course N is a generic extension of M_1 via forcing with $\mathbb{Q}$. Let H_1 be a generic filter on $\mathbb{Q}$ that adjoins the sequence $\langle a_m : m \ge n+1 \rangle$.

We now describe a $\mathbb{Q}$-name $\mathring{y}$ [with respect to the ground model M_1]. $\mathring{y}$ will have the following properties:

(1) $\mathrm{val}_{H_1}(\mathring{y}) = y$.
(2) Let K be any M_1-generic filter over $\mathbb{Q}$. Then $\mathrm{val}_K(\mathring{y})$ is computed as follows:
   (a) Let $\langle b_i : i \in \omega \rangle$ be the sequence of reals adjoined by K.
   (b) Let $S^* = \{a_0, \ldots, a_n\} \cup \{b_i : i \in \omega\}$.
   (c) Let $T^* = S^* \cup \omega$.
   (d) $\mathrm{val}_K(\mathring{y})$ will be $D(S^*)$ as computed in $\mathbf{L}(T^*)$. Note that the various parameters appearing in D that we have suppressed from our notation all lie in M_1, and so are available for this definition.

The following are evident:

- There is a $\mathbb{Q}$-name with property (2).
- Property (2) implies property (1).

We now describe a group of automorphisms of $\mathbb{Q}$. Consider the group $\mathcal{G}$ of permutations of ω that have finite support. In an obvious way this group acts on $\mathbb{Q}$. This action lifts to $\mathbb{Q}$-names, and hence to an action on equivalence classes of $\mathbb{Q}$-names [two $\mathbb{Q}$-names are equivalent if they are forced to be equal by every condition]. It is evident that the equivalence class of $\mathring{y}$ is fixed by this automorphism group. Moreover, the only elements of the Boolean algebra $\mathbb{B} = \text{r.o.}(\mathbb{Q})$ which are fixed by all automorphisms of $\mathbb{B}$ induced by elements of $\mathcal{G}$ are $0_{\mathbb{B}}$ and $1_{\mathbb{B}}$. To see this, let $U \in \mathbb{B}$ be a regular nonempty downward closed proper subset of $\mathbb{Q}$. Choose $q \in U$ and $\overline{q} \notin U$ such that $\{x \in \mathbb{Q} : x \leq \overline{q}\}$ and U are disjoint, and let n be large enough so that for all $i \geq n$, $q(i) = \overline{q}(i) = 1_{\mathbb{P}}$. Let f permute $I = \{0, \ldots, 2n-1\}$ by shifting up the first n elements of I by n, and shifting down the remaining n elements of I by n, and let $\widehat{f}$ be the automorphism of $\mathbb{Q}$ induced by f. Consider the condition $r \in \mathbb{Q}$ defined as follows:

$$r(i) = \begin{cases} q(i) & \text{for } 0 \leq i \leq n-1, \\ \overline{q}(i-n) & \text{for } n \leq i \leq 2n-1, \\ 1_{\mathbb{P}} & \text{for } i \geq 2n. \end{cases}$$

Since $r \in U$ and $\widehat{f}(r) \notin U$, U is not fixed by $\widehat{f}$.

It follows from a standard argument (as in [Jc-2, Theorem 59]) that there is an element of the ground model M_1 such that the value of $\mathring{y}$ is forced by every condition to be this element. Hence $y \in M_1$.

At this point it is easy to derive a contradiction. The elements $a_0, \ldots, a_n, y$ are all distinct and all are $\mathbb{P}$-generic over $\mathbf{L}$. So $\langle a_0, \ldots, a_n, y\rangle$ is $\mathbb{P}^{n+1}$-generic over $\mathbf{L}$. So y is $\mathbb{P}$-generic over M_1. It is evident from Jensen's explicit description of $\mathbb{P}$ that if y is $\mathbb{P}$-generic over M_1 then $y \notin M_1$. ∎ (Lemma 3.3.1)

What have we gained from Lemma 3.3.1? We now know that S is OD in N_1 and that every element of N_1 is OD from some finite subset of the a_i's [this uses the fact that there is a canonical ordering of S induced from the usual canonical ordering of $\mathcal{P}(\omega)$].

LEMMA 3.3.2. *Let $x \in N_1$ and suppose that F_1, F_2 are finite subsets of S such that x is* OD *from F_1 and* OD *from F_2. Then x is* OD *from $F_1 \cap F_2$.*

Before proving Lemma 3.3.2 we note the following corollary:

- *For every $x \in N_1$, there is a unique smallest subset $F \subseteq S$ such that x is* OD *from F [F is called the* ***support*** *of x].*

Proof of Lemma 3.3.2. Suppose not, and fix counterexamples x, F_1, and F_2. Let $F = F_1 \cap F_2$, and let $D_1(F_1)$ and $D_2(F_2)$ be two different definitions of x from ordinal parameters. Let q denote a condition in G such that

$$q \Vdash D_1(F_1) = D_2(F_2).$$

We may suppose that every a_i referenced in q appears in one of F_1 and F_2 by suitably increasing F_1 and F_2. We may do so while maintaining the equation $F = F_1 \cap F_2$. By applying the symmetry group $\mathcal{G}$ we may suppose that $F = \{a_0, \ldots, a_n\}$. But now familiar symmetry arguments (as in [Jc-1, Theorem 5.21]) using $\mathcal{G}$ show that x can be ordinal-defined from F as follows: Find elements $b_0, \ldots, b_s$ among $S \setminus F$ such that q is true of

$$\{a_0, \ldots, a_n, b_0, \ldots, b_s\}.$$

If F_1^* is the analogue of F_1 for this new subset of S, then $x = D_1(F_1^*)$. By existentially quantifying the b_i's, this yields an OD definition of x from merely $a_0, \ldots, a_n$. ∎ (Lemma 3.3.2)

Thanks to Lemma 3.3.2, it is easy to describe a map $\mathbf{G}$ in N_1 that injects the universe into the class ($S^{<\omega} \times$ OD terms) whose definition uses only the parameter S. By Lemma 3.3.1, however, the parameter S can be eliminated from the description of $\mathbf{G}$. Therefore by composing $\mathbf{G}$ with a definable injection from ($S^{<\omega} \times$ OD terms) into the class of sets of ordinals, we obtain the desired parameter free definable injection of the universe into the class of sets of ordinals. Hence by Theorem 2.1, LM holds in N_1. ∎ (Theorem 3.3)

4. Further results and open questions

4.1. *A uniform equivalent form of* LM. In this section we present a result of Solovay on "universal formulas" that answers a question of the author by implying that the following apparently stronger form LM* of LM is not only consistent with ZF, but is indeed *equivalent* to LM itself:

LM*: There is a formula $\varphi(x)$ such that for each pair of distinct sets a and b, there exists an ordinal α exceeding the ranks of a and b such that $(V_\alpha, \in)$ satisfies $\varphi(a) \wedge \neg\varphi(b)$.

THEOREM 4.1.1 (Solovay) *There is a formula $\varphi(x)$ in the language of set theory that has precisely the one indicated free variable such that* ZF *proves*: *if α is an ordinal and $\psi(x)$ is a formula of the language of set theory with one free variable x, then there is an ordinal $\beta > \alpha$ such that for any $x \in V_\beta$,*

$$[V_\beta \models \varphi(x)] \quad \textit{iff} \quad [x \in V_\alpha \textit{ and } V_\alpha \models \psi(x)].$$

It will be convenient to give some preliminary definitions before presenting the proof. Let us say that an ordinal is *special* if it is a limit ordinal but not a limit of limit ordinals. Let α be special and β_α be the largest limit ordinal less than α. Then $\alpha = \beta_\alpha + \omega$. Let $\eta(\alpha)$ be the order type of the set of special ordinals less than α. Note that $\eta(\alpha) < \alpha$ since $\eta(\alpha) \leq \beta_\alpha$. For any ordinal γ, there is a unique special ordinal α such that $\eta(\alpha) = \gamma$. An ordinal α *unpacks* to the pair of ordinals β, γ if α corresponds to the pair $\langle \beta, \gamma \rangle$ via the bijection of **Ord** with $\mathbf{Ord}^2$ explicitly constructed by Gödel (as in [Jc-2, p. 20]). Note that it follows that β and γ are $\leq \alpha$.

Proof of Theorem 4.1.1. The desired formula $\varphi(x)$ asserts (1) through (4) below.

(1) There is no largest ordinal.
(2) There is an ordinal γ such that every limit ordinal is less than γ. Hence we can meaningfully define α to be the order type of the set of special ordinals.
(3) Let α unpack to the pair of ordinals α_0 and α_1. Then α_1 is an integer and is the Gödel number of a formula $\chi(v_0)$ having the one free variable v_0 (v_0 is the first free variable).
(4) $x \in V_{\alpha_0}$ and $V_{\alpha_0} \vDash \psi[v_0/x]$. ∎

4.2. LM *relative to other logics.* Since the formulation of LM involves the *first order theories* of models of the form $(V_\alpha, \in, a)$, it is natural to probe axioms of the form $\mathrm{LM}_{\mathbb{L}}$, where $\mathbb{L}$ is some extension of first order logic $L_{\omega,\omega}$ (such as second order logic, or some flavor of infinitary logics).

$\mathrm{LM}_{\mathbb{L}}$: For each pair of distinct sets x and y, there exists an ordinal α exceeding the ranks of x and y, and a formula $\varphi(v)$ in the logic $\mathbb{L}$ such that $(V_\alpha, \in)$ satisfies $\varphi(x) \wedge \neg\varphi(y)$.

Of course in order to formulate $\mathrm{LM}_{\mathbb{L}}$, we need to assume the following definability condition $(*)$ on satisfiability of $\mathbb{L}$-formulae, which is satisfied by practically all "reasonable" logics (such as all brands of infinitary logic, as well as logics with the quantifier "there exist κ many", where κ is a prescribed definable cardinal):

$$(*) \qquad \{(\alpha, \varphi(x), a) : \varphi(x) \text{ is a unary } \mathbb{L}\text{-formula} \\ \text{and } (V_\alpha, \in, a) \vDash \varphi(a)\} \text{ is definable.}$$

The *proof* of (iii)⇒(i) of Theorem 2.1 shows that if (1) $\mathbb{L}$ is a logic satisfying $(*)$ and (2) the formulas of $\mathbb{L}$ are ordinal definable (and therefore can be definably well-ordered), then

$$\mathrm{ZF} \vdash \mathrm{LM}_{\mathbb{L}} \rightarrow \mathrm{GKW}_1.$$

Combined with Theorem 2.1, this shows:

PROPOSITION 4.2.1. *If* $\mathbb{L}$ *is a logic extending first order logic satisfying* $(*)$ *whose formulas are in* **OD**, *then* $\mathrm{ZF} \vdash \mathrm{LM}_{\mathbb{L}} \leftrightarrow \mathrm{LM}$.

Hence for many logics, such as second order logic, nth order logic, or even $(L_{\infty,\infty})^{\mathbf{HOD}}$ (full infinitary logic in the sense of $\mathbf{HOD}$), $\mathrm{LM}_{\mathbb{L}}$ is equivalent to LM over ZF. However, if some formulas of $\mathbb{L}$ lie outside of $\mathbf{OD}$, then $\mathrm{LM}_{\mathbb{L}}$ might be weaker than LM. For example, if r is a Cohen real over $\mathbf{L}$, then LM fails in $\mathbf{L}(r)$ (see Remark 3.2(a)) but $\mathbf{L}(r)$ satisfies $\mathrm{LM}_{\mathbb{L}}$ with $\mathbb{L} = (L_{\omega_1}, \omega)^{\mathbf{L}}(\mathbf{r})$, because $\mathrm{LM}(r)$ holds in $\mathbf{L}(r)$.

4.3. *From Weak* LM *to* LM. Another axiom related to LM is the *Weak Leibniz–Mycielski axiom* WLM in which the role of unary formulas is replaced by binary formulas:

WLM: For each pair of distinct sets x and y, there exists an ordinal α exceeding the ranks of x and y, and a formula $\varphi(u, v)$, such that $(V_\alpha, \in)$ satisfies $\neg(\varphi(x, y) \leftrightarrow \varphi(y, x))$.

Similar to LM, WLM can also be reformulated as a choice principle, as shown in Theorem 4.3.1 below.

THEOREM 4.3.1. *Suppose M is a model of* ZF. *The following are equivalent.*

(i) *For some parameter free definable function* $\mathbf{F}$, $M \models$ "$\mathbf{F}$ *is a choice function on the class of pairs*".
(ii) $M \models$ WLM.

Proof. We first establish (i)⇒(ii). Suppose M thinks that $\mathbf{F}$ is a choice function on the class of pairs, where $\mathbf{F}$ is defined by some parameter free formula in M. Given a pair $\{x, y\}$ of distinct objects in M, by the reflection theorem there is some ordinal α of M such that x and y are in V_α^M and

$$M \models (\mathbf{F} \text{ is a choice function on the class of pairs})^{V_\alpha}.$$

Therefore, if $\varphi(u, v)$ is the formula $u = \mathbf{F}(\{u, v\})$, then

$$M \models \neg(\varphi(x, y) \leftrightarrow \varphi(y, x))^{V_\alpha}.$$

For (ii)⇒(i), suppose that M satisfies WLM. Given a pair $z = \{x, y\}$ of distinct objects x and y, let α_z be the first ordinal exceeding the ranks of x and y such that

(♣) For some formula $\varphi(u, v)$, V_{α_z} satisfies $\neg(\varphi(x, y) \leftrightarrow \varphi(y, x))$.

Let $\varphi_z(u, v)$ be the formula of least Gödel number witnessing (♣), and define $\mathbf{F}$ by

$$\mathbf{F}(\{x, y\}) = \begin{cases} x & \text{if } V_\alpha \text{ satisfies } \varphi_z(x, y), \\ y & \text{if } V_\alpha \text{ satisfies } \varphi_z(y, x). \end{cases} \blacksquare$$

The following list enumerates some interesting axioms intermediate between WLM and $\mathbf{V} = \mathbf{OD}$. Each axiom on the list except for WLM implies the axiom preceding it within ZF. More specifically, $\mathbf{V} = \mathbf{OD} \rightarrow$ LM was established in Corollary 2.3; LM $\to$ GDO can be proved by combining Theorem 2.1 with a result of Pincus on canonically imposing dense linear orders on infinite families of subsets of ordinals [P, Theorem 1.5]; GDO $\to$ DO is trivial; GO $\to$ $\mathrm{GC}_{<\omega}$ is a consequence of the fact that finite linear orders are well-ordered, and $\mathrm{GC}_{<\omega}$ $\to$ WLM follows from Theorem 4.3.1.

1. WLM.
2. $\mathrm{GC}_{<\omega}$ (*parameter free definable global choice function on the class of finite sets*).
3. GO (*parameter free definable global linear ordering*).
4. GDO (*parameter free definable global dense ordering*).
5. LM.
6. $\mathbf{V} = \mathbf{OD}$.

All of the axioms (1) through (6) are independent of ZFC since the *proof* of Theorem 3.1 and part (a) of Remark 3.2 together show that if r is a Cohen real over $\mathbf{L}$ then WLM fails in $\mathbf{L}(r)$. Indeed, Theorem 4.3.1 shows that the statement $\forall c\ \neg\mathrm{WLM}(c)$ holds in Easton's model of ZFC mentioned in part (c) of Remark 3.2. It is known [7] that the *local* versions of the above axioms form a proper hierarchy within ZF. This motivates the following conjecture.

Conjecture 4.3.2. *None of the following implications reverse in* ZF, *or even in* ZFC (N.B. Theorem 3.3 shows that $(\mathrm{ZF} + \mathrm{LM}) \nrightarrow \mathbf{V} = \mathbf{OD}$).

$$\mathbf{V} = \mathbf{OD} \to \mathrm{LM} \to \mathrm{GDO} \to \mathrm{GO} \to \mathrm{GC}_{<\omega} \to \mathrm{WLM}.$$

Question 4.3.3. *Does* LM *hold in the model N of the proof of Theorem* 3.3?

Since $\mathbf{V} \neq \mathbf{OD}$ holds in N, a positive answer to Question 4.5 would establish $(\mathrm{ZFC} + \mathrm{LM}) \nrightarrow \mathbf{V} = \mathbf{OD}$. The author had hoped that the model $\mathbf{L}(r)$ obtained by adding a *Sacks real* r to $\mathbf{L}$ would demonstrate the independence of $\mathbf{V} = \mathbf{OD}$ from ZFC + LM, but Solovay has recently shown that LM fails in $\mathbf{L}(r)$.

Question 4.3.4. *Is there a model of* ZF + LM *in which* $|\mathcal{P}(\omega) \cap \mathbf{OD}| = \aleph_0$?

4.4. *The Leibniz–Gödel axioms.* In this section we introduce "constructible" variants of LM and WLM in which the role of the von Neumann hierarchy of V_α's is replaced by models of the form $L_\alpha(a)$, where a is a transitive set.

[7] [Je-1] contains the classical proofs of $\mathrm{AC}_2 \nrightarrow \mathrm{AC}_{<\omega}$, $\mathrm{AC}_{<\omega} \nrightarrow \mathrm{O}$, $\mathrm{O} \nrightarrow \mathrm{KW}$, and $\mathrm{KW} \nrightarrow \mathrm{AC}$. More recently, González [G] has established $\mathrm{O} \nrightarrow \mathrm{DO}$ over ZF, and Pincus [P] has established $O \nrightarrow \mathrm{DO}$, and $\mathrm{DO} \nrightarrow \mathrm{KW}$ over ZF + The Boolean Prime Ideal Theorem.

The Leibniz–Gödel axiom (LG): For each pair of distinct sets x and y, there exists an ordinal α exceeding the ranks of x and y, and a formula $\varphi(u, v)$, such that $(L_\alpha(a), \in)$ satisfies $\varphi(x) \wedge \neg\varphi(y)$, where a is the transitive closure of $\{x, y\}$.

The Weak Leibniz–Gödel axiom (WLG): For each pair of distinct sets x and y, there exists an ordinal α exceeding the ranks of x and y, and a formula $\varphi(u, v)$, such that $(L_\alpha(a), \in)$ satisfies $\neg(\varphi(x, y) \leftrightarrow \varphi(y, x))$, where a is the transitive closure of $\{x, y\}$.

It is easy to see that $\mathbf{V} = \mathbf{L}$ implies LG. However, LG is weaker than $\mathbf{V} = \mathbf{L}$ since LG also holds in the model $\mathbf{L}(s)$, where s is a Jensen Π^1_2-singleton, since s is a definable and *minimal* real in $\mathbf{L}(s)$. Moreover, it is easy to see that $\mathrm{ZF} \vdash \mathrm{LG} \rightarrow \mathrm{WLG}$, but it is not clear how to establish that WLG is provably weaker than LG within ZFC. Furthermore, even though a routine argument shows that $\mathrm{ZF} \vdash \mathrm{WLG} \rightarrow \mathrm{WLM}$, it is unclear whether LM and LG are related within ZFC. These considerations lead to the following questions:

QUESTION 4.4.1. *Is there a model of* $\mathrm{ZFC} + \mathrm{LM}$ *in which* LG *fails*?

QUESTION 4.4.2. *Is there a model of* $\mathrm{ZFC} + \mathrm{LG}$ *in which* LM *fails*?

QUESTION 4.4.3. *Is there a model of* $\mathrm{ZFC} + \mathrm{WLG}$ *in which* LG *fails?*

References

[CK] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[Ea] W. Easton, *Powers of regular cardinals*, Ann. Math. Logic 1 (1970), 139–178.

[En] A. Enayat, *Leibnizian models of set theory*, J. Symbolic Logic, to appear.

[F] U. Felgner, *Comparisons of the axioms of local and universal choice*, Fund. Math. 71 (1971), 43–62.

[G] C. G. González, *Dense orderings*, *partitions*, *and weak forms of choice*, Fund. Math. 147 (1995), 11–25.

[HL] J. D. Halpern and A. Lévy, *The prime ideal theorem does not imply the axiom of choice*, in: Axiomatic Set Theory, Part I, Proc. Sympos. Pure Math. 13, Amer. Math. Soc., Providence, RI, 1970, 83–134.

[Jc-1] T. Jech, *The Axiom of Choice*, North-Holland, Amsterdam, 1972.

[Jc-2] —, *Set Theory*, Academic Press, New York, 1978.

[Jn] R. Jensen, *Definable sets of minimal degree*, in: Mathematical Logic and Foundations of Set Theory, Y. Bar-Hillel (ed.), North-Holland, Amsterdam, 1970, 122–127.

[KW] W. Kinna und K. Wagner, *Über eine Abschwächung des Auswahlpostulates*, Fund. Math. 42 (1955), 75–82.

[K] K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1980.

[L] G. W. Leibniz, *Philosophical Papers and Letters*, translated and edited by L. Loemaker, 2nd ed., Reidel, Dordrecht, 1969.

[My-1] J. Mycielski, *New set-theoretic axioms derived from a lean metamathematics*, J. Symbolic Logic 60 (1995), 191–198.
[My-2] —, *Axioms which imply GCH*, Fund. Math. 176 (2003), 193–207.
[MS] J. Myhill and D. Scott, *Ordinal definability*, in: Axiomatic Set Theory, Part I, Proc. Sympos. Pure Math. 13, Amer. Math. Soc., Providence, RI, 1970, 271–278.
[P] D. Pincus, *The dense linear ordering principle*, J. Symbolic Logic 62 (1997), 438–456.
[Si] W. Sierpiński, *Sur un problème conduisant à un ensemble non mesurable*, Fund. Math. 10 (1927), 177–179. Reprinted in W. Sierpiński, Oeuvres Choisies, tome II, PWN, Warszawa, 1975, 561–563.

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