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Analytic partial orders and oriented graphs

by

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Abstract. We prove that there is no maximum element, under Borel reducibility, in the class of analytic partial orders and in the class of analytic oriented graphs. We also provide a natural jump operator for these two classes.

This paper is part of the general program of studying $\mathbf{\Sigma}^1_1$ (analytic) binary relations on Polish spaces, under the Borel reducibility ordering, and variants of it.

If R_1 and R_2 are binary relations on Polish spaces X_1 and X_2 respectively, a *reduction* of R_1 to R_2 is a map $f : X_1 \to X_2$ such that for all x, y in X_1, $xR_1y \leftrightarrow f(x)R_2f(y)$.

We say that R_1 is *Borel reducible* to R_2, or R_2 *Borel reduces* R_1, and write $R_1 \leq_{\mathrm{B}} R_2$, if there is a Borel reduction of R_1 to R_2. If there is an injective Borel reduction, we say that R_1 *Borel embeds* into R_2 and write $R_1 \sqsubseteq_{\mathrm{B}} R_2$.

If $\mathcal{C}$ is a class of binary relations on Polish spaces, a relation R is $\mathcal{C}$-*complete* if $R \in \mathcal{C}$ and R Borel reduces all elements of $\mathcal{C}$.

It is known that many natural classes of $\mathbf{\Sigma}^1_1$ binary relations admit complete elements, e.g. the class of $\mathbf{\Sigma}^1_1$ equivalence relations, the class of $\mathbf{\Sigma}^1_1$ quasi-orderings, or the class of $\mathbf{\Sigma}^1_1$ graphs (see [LR]).

There are also known examples of classes with no complete elements. H. Friedman proved that this is the case for Borel equivalence relations (see [FS] and [L1]), and it implies easily that this is also the case for Borel quasi-orderings. Another example is the class of G_δ quasi-orders, as shown in Louveau [L2].

In this paper, we will add two more examples of this phenomenon to the previous list, maybe somewhat more surprising as they are classes of $\mathbf{\Sigma}^1_1$ objects.

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First, we will consider the case of $\boldsymbol{\Sigma}^1_1$ partial orders, i.e. transitive, reflexive relations R such that xRy and yRx imply $x = y$. We will prove:

THEOREM 1. *The class $\mathcal{C}$ of $\boldsymbol{\Sigma}^1_1$ partial orders admits no complete element. In fact, the Borel partial orders are unbounded in $\mathcal{C}$ for the ordering $\leq_{\mathrm{B}}$.*

This result will be obtained by considering more general binary relations.

If R is a partial order, its *strict part* $<_R$ is defined by $x <_R y \leftrightarrow xRy$ and $x \neq y$. It is a *strict order*, i.e. a transitive irreflexive relation. And conversely, for each strict order S, the relation xSy or $x = y$ is a partial order admitting S as its strict part. Moreover, if f is a reduction of the partial order R_1 to the partial order R_2, then f must reduce equality to equality, i.e. be injective, and also reduce $<_{R_1}$ to $<_{R_2}$. And as the converse also holds, we easily see that

$$R_1 \leq_{\mathrm{B}} R_2 \;\leftrightarrow\; <_{R_1} \sqsubseteq_{\mathrm{B}} <_{R_2}.$$

Strict orders are a particular case of *oriented graphs*, those binary relations R which are antisymmetric, i.e. satisfy $xRy \to \neg\, yRx$. And for these relations, there is another interesting natural ordering, weaker than Borel reducibility, given by using homomorphisms instead of reductions.

DEFINITION 2. If R_1, R_2 are binary relations on Polish spaces X_1, X_2 respectively, a map $f : X_1 \to X_2$ is a *homomorphism* from R_1 to R_2 if for all x, y in X_1,

$$xR_1y \;\to\; f(x)R_2f(y).$$

We write $R_1 \preceq_{\mathrm{B}} R_2$ if there is a Borel homomorphism from R_1 to R_2.

From the above discussion, Theorem 1 is an immediate consequence of the following

THEOREM 3. *No $\boldsymbol{\Sigma}^1_1$ oriented graph can bound all Borel strict orders in $\preceq_{\mathrm{B}}$. In particular, the class of $\boldsymbol{\Sigma}^1_1$ oriented graphs and the class of $\boldsymbol{\Sigma}^1_1$ strict orders admit no complete element (for $\preceq_{\mathrm{B}}$, hence also for $\leq_{\mathrm{B}}$).*

In order to prove Theorem 3, we use an "index method". We first define for each $\boldsymbol{\Sigma}^1_1$ oriented graph R a countable ordinal $\mathrm{ind}(R)$ satisfying

$$R_1 \preceq_{\mathrm{B}} R_2 \;\to\; \mathrm{ind}(R_1) \leq \mathrm{ind}(R_2).$$

Then we prove that this index is unbounded on the class of Borel strict orders. These facts together of course prove Theorem 3.

To define the index, recall from [L1] the notion of potential Borel class: A binary relation R on a Polish space X is *potentially* $\boldsymbol{\Delta}^0_\xi$ if there is a finer Polish topology τ on X such that R is $\boldsymbol{\Delta}^0_\xi$ in the square of (X, τ). Note that if $f : X_1 \to X_2$ is a Borel map and R is potentially $\boldsymbol{\Delta}^0_\xi$ on X_2, so is $f^{-1}(R)$ on X_1. For one can first refine the topology of X_2 so that R is $\boldsymbol{\Delta}^0_\xi$, and then the topology of X_1 to make f continuous with respect to the new Polish topology of X_2.

This leads to the following definition, where $\check{R}$ denotes the dual of R, defined by $x\check{R}y$ whenever yRx.

DEFINITION 4. If R is a $\mathbf{\Sigma}^1_1$ oriented graph, let $\mathrm{ind}(R)$ be the least ordinal $\xi < \omega_1$ such that there exists a potentially $\mathbf{\Delta}^0_\xi$ set C separating R from $\check{R}$, i.e. satisfying $R \subseteq C$ and $C \cap \check{R} = \emptyset$.

Note that $\mathrm{ind}(R)$ is well defined for each $\mathbf{\Sigma}^1_1$ oriented graph R, because R and $\check{R}$ are two disjoint $\mathbf{\Sigma}^1_1$ sets, hence Borel separable. And the above remarks imply immediately that the index ind is increasing, i.e. $R_1 \preceq_{\mathrm{B}} R_2$ implies $\mathrm{ind}(R_1) \le \mathrm{ind}(R_2)$, for a Borel homomorphism f from R_1 to R_2 is also a homomorphism from $\check{R}_1$ to $\check{R}_2$ and hence if C is a potentially $\mathbf{\Delta}^0_\xi$ set separating R_2 from $\check{R}_2$, then $f^{-1}(C)$ is a potentially $\mathbf{\Delta}^0_\xi$ set which separates R_1 from $\check{R}_1$.

So to get Theorem 3, it is enough to build a family (R_ξ) of Borel strict orders with $\mathrm{ind}(R_\xi) > \xi$, at least for ξ a successor ordinal ≥ 2, as we now proceed to do. First we define the domain X_ξ of R_ξ as $2^{D_\xi \times \omega}$, where D_ξ is a countable set, defined inductively by

$$D_2 = \{0\},$$

$$D_{\xi+1} = \begin{cases} \omega \times D_\xi & \text{if } \xi \ge 2 \text{ is successor,} \\ \{(n,i) : i \in D_{\xi_n}\} & \text{if } \xi \text{ is limit and } \xi_n \text{ is a sequence of} \\ & \text{successors converging to } \xi. \end{cases}$$

Note that the two cases in this definition are the same if we set $\xi_n = \xi$ for all n, when ξ is successor. And in both cases we can (and will) view each $\alpha \in X_{\xi+1}$ as a sequence $(\alpha_n)_{n\in\omega}$, with $\alpha_n \in X_{\xi_n}$ for all n. With these conventions, we are now in a position to define inductively the strict orders R_ξ, together with Borel sets L_ξ and L^*_ξ, as follows:

CASE $\xi = 2$. Recall that the equivalence relation E_0 is defined on 2^ω by

$$\alpha E_0 \beta \leftrightarrow \forall^* n\ \alpha(n) = \beta(n),$$

where $\forall^*$ is the quantifier "for all but finitely many".

We have $X_2 = 2^\omega$, and set

$$R_2 = \{(\alpha,\beta) : \alpha \ne \beta \text{ and } \alpha E_0 \beta \text{ and if } n \text{ is maximum with } \alpha(n) \ne \beta(n), \text{ then } \alpha(n) < \beta(n)\},$$

$$L_2 = \{\alpha : \alpha E_0 \underline{0}\}, \qquad L^*_2 = \{\alpha : \alpha E_0 \underline{1}\},$$

where $\underline{0}$ and $\underline{1}$ are the reals which are identically 0 and 1 respectively.

The order R_2 is the strict part of the partial order called $\le_0$ in Kanovei [K]. It orders all E_0-classes (except L_2 and L^*_2) in order type $\mathbb{Z}$.

Inductive case. Using the conventions above, for $\xi \geq 2$ we set

$$R_{\xi+1} = \{(\alpha,\beta) \in X_{\xi+1}^2 : \forall^* n\ \alpha_n R_{\xi_n} \beta_n\},$$
$$L_{\xi+1} = \{\alpha \in X_{\xi+1} : \forall n\ (\alpha_n \in L_{\xi_n} \text{ or } \alpha_n \in L^*_{\xi_n}) \text{ and } \forall^* n\ (\alpha_n \in L_{\xi_n})\},$$
$$L^*_{\xi+1} = \{\alpha \in X_{\xi+1} : \forall n\ (\alpha_n \in L_{\xi_n} \text{ or } \alpha_n \in L^*_{\xi_n}) \text{ and } \forall^* n\ (\alpha_n \in L^*_{\xi_n})\}.$$

One easily checks that each R_ξ is a Borel strict order. So to get Theorem 3, it is enough to prove that for all successor $\xi \geq 2$, $\mathrm{ind}(R_\xi) > \xi$, i.e. that no potentially $\mathbf{\Delta}^0_\xi$ set can separate R_ξ from $\check{R}_\xi$. We will prove this in two steps, by first proving that there is no separation by a $\mathbf{\Delta}^0_\xi$ set, and then dealing with the possible change of topologies.

For the first step, we use the following lemma, which also explains the notation for the sets L_ξ and L^*_ξ: the letter L is for Lebesgue (the same idea was already used in [HKL] for other purposes).

Lemma 5. *Let $\xi > 1$ be successor. There is no $\mathbf{\Delta}^0_\xi$ set separating L_ξ from L^*_ξ.*

Proof. This is a direct consequence of Lebesgue's classical result about the generation of Baire class ξ functions by the operation of taking pointwise limits: For $\xi \geq 1$, Baire class ξ functions are the pointwise limits of sequences of functions of Baire class $< \xi$, and even of Baire class $< \lambda$ if $\xi = \lambda + 1$ with λ limit. This result is valid for real-valued functions on arbitrary Polish spaces, but it is also valid for $\{0,1\}$-valued functions on $\dim 0$ Polish spaces, in particular for Borel subsets of 2^ω. And there, tracing back in the inductive definition above, one easily checks that it exactly means that for any successor $\xi \geq 2$ and any $\mathbf{\Delta}^0_\xi$ set $C \subseteq 2^\omega$, there is a sequence $(C_{i,n})_{(i,n)\in D_\xi \times \omega}$ of clopen subsets of 2^ω such that

(i) for each $\alpha \in 2^\omega$, $1_{\{(i,n)\,:\,\alpha \in C_{i,n}\}} \in L_\xi \cup L^*_\xi$,
(ii) $\alpha \in C \leftrightarrow 1_{\{(i,n)\,:\,\alpha \in C_{i,n}\}} \in L^*_\xi$.

But then, as the map $\alpha \mapsto 1_{\{(i,n)\,:\,\alpha \in C_{i,n}\}}$ is continuous, we find that if some $\mathbf{\Delta}^0_\xi$ set C were separating L^*_ξ from L_ξ, every $\mathbf{\Delta}^0_\xi$ subset of 2^ω would be in the Wadge class of C, which, as $\xi \geq 2$, is a contradiction proving the lemma. ■

Using this lemma, we get

Proposition 6. *For each successor ordinal $\xi \geq 2$, there is a continuous map $f_\xi = (f^0_\xi, f^1_\xi) : X_\xi \to X_\xi \times X_\xi$ satisfying*:

(i) *if $\alpha \in L_\xi$, then $f^0_\xi(\alpha) R_\xi f^1_\xi(\alpha)$,*
(ii) *if $\alpha \in L^*_\xi$, then $f^1_\xi(\alpha) R_\xi f^0_\xi(\alpha)$.*

In particular, R_ξ cannot be separated from $\check{R}_\xi$ by a $\mathbf{\Delta}^0_\xi$ set.

Proof. The second assertion follows from the first and Lemma 5: any separating $\mathbf{\Delta}^0_\xi$ set would yield, by taking the inverse image under f_ξ, a $\mathbf{\Delta}^0_\xi$ set separating L_ξ from L^*_ξ, contradicting Lemma 5.

To prove the first assertion, we define f_ξ by induction.

For $\xi = 2$, $\beta^0 = f^0_2(\alpha)$ and $\beta^1 = f^1_2(\alpha)$ are defined by:

(i) for $n = 0$,

$$\beta^0(0) = \alpha(0), \qquad \beta^1(0) = 1 - \alpha(0);$$

(ii) for $n > 0$,

(0) if $\alpha(n) = \alpha(n-1)$, then $\beta^0(n) = \beta^1(n) = 0$,
(1) if $\alpha(n) < \alpha(n-1)$, then $\beta^0(n) = 0$ and $\beta^1(n) = 1$,
(2) if $\alpha(n) > \alpha(n-1)$, then $\beta^0(n) = 1$ and $\beta^1(n) = 0$.

To check it works, suppose first $\alpha \in L_2$, and let n be smallest with $\alpha(k) = 0$ for $k \geq n$. Then, whether $n = 0$ or not, we are in case (1) of the definition at n, and in case (0) at all $k > n$. This implies that n is largest with $\beta^0(n) \neq \beta^1(n)$. As $\beta^0(n) < \beta^1(n)$, we get $\beta^0(n) R_2 \beta^1(n)$ as wanted.

Suppose now $\alpha \in L^*_2$, and let n be smallest with $\alpha(k) = 1$ for $k \geq n$. Then we are in case (2) of the definition at n, and in case (0) at any $k > n$. Again n is largest with $\beta^0(n) \neq \beta^1(n)$, and as $\beta^0(n) > \beta^1(n)$, we get $\beta^1(n) R_2 \beta^0(n)$ as wanted.

This gives the proposition for $\xi = 2$.

The induction step is easy. Using the same conventions as before, for $\alpha = (\alpha_n)_{n\in\omega}$ in X_ξ, set $f^i_\xi(\alpha) = (f^i_{\xi_n}(\alpha_n))_{n\in\omega}$ for $i = 0, 1$.

By induction we find that if $\alpha \in L_\xi$, then for all but finitely many n's, $\alpha_n \in L_{\xi_n}$, hence for the same n's, $f^0_{\xi_n}(\alpha_n) R_{\xi_n} f^1_{\xi_n}(\alpha_n)$, so $f^0_\xi(\alpha) R_\xi f^1_\xi(\alpha)$, and similarly with L^*_ξ and $\check{R}_\xi$. This proves Proposition 6. ∎

We now get rid of the possible change of topologies. Fix the ordinal ξ. A *good pair* (D, γ) *at level* ξ consists of a subset $D \subseteq D_\xi \times \omega$ such that for all $i \in D_\xi$, the set $D_i = \{n : (i, n) \in D\}$ is infinite, together with a map $\gamma : (D_\xi \times \omega) - D \to 2$.

Such a pair (D, γ) defines a compact set

$$K_{D,\gamma} = \{\alpha \in X_\xi : \alpha|_{(D_\xi\times\omega)-D} = \gamma\},$$

and a natural homeomorphism $h_{D,\gamma} = h$ of X_ξ onto $K_{D,\gamma}$: if for $i \in D_\xi$, d_i is the increasing enumeration of D_i, define $\beta = h(\alpha)$ by $\beta(i, n) = \alpha(i, d_i^{-1}(n))$ if $(i, n) \in D$ and $\beta(i, n) = \gamma(i, n)$ otherwise.

Lemma 7. *For each successor ordinal $\xi > 1$ and good pair (D, γ) at level ξ, the map $h_{D,\gamma}$ is a continuous reduction of R_ξ to $R_\xi|_{K_{D,\gamma}}$.*

Proof. We argue by induction. If $\xi = 2$, then D is an infinite subset of ω with enumeration d, and given α and β in $X_2 = 2^\omega$, the set of integers where $h(\alpha)$ and $h(\beta)$ differ is the d-image of the set of integers where α and

β differ. So we get $\alpha \neq \beta$ if and only if $h(\alpha) \neq h(\beta)$, $\alpha E_0 \beta$ if and only if $h(\alpha) E_0 h(\beta)$, and, as d is increasing, the largest n where $h(\alpha)$ and $h(\beta)$ differ, when it exists, is the image under d of the largest n where α and β differ. So finally $\alpha R_2 \beta$ if and only if $h(\alpha) R_2 h(\beta)$.

Again the induction step is easy. Fix (D, γ) good at level ξ. Recall that with our conventions, $D_\xi = \{(n, i) : i \in D_{\xi_n}\}$. So for each n, we get a pair at level ξ_n by setting $D_n = \{(i, k) \in D_{\xi_n} \times \omega : ((n, i), k) \in D\}$ and $\gamma_n((i, k)) = \gamma((n, i), k)$ for $(i, k) \in D_n$. Clearly (D_n, γ_n) is good at level ξ_n, so by induction the corresponding homeomorphism h_n is such that for α, β in X_{ξ_n}, $\alpha R_{\xi_n} \beta$ if and only if $h_n(\alpha) R_{\xi_n} h_n(\beta)$.

But note that for $\alpha = (\alpha_n)_{n\in\omega}$ in X_ξ, one has $h(\alpha) = (h_n(\alpha_n))_{n\in\omega}$, so that the previous fact implies immediately that $\alpha R_\xi \beta$ if and only if $h(\alpha) R_\xi h(\beta)$, as desired. ■

To finish the proof of Theorem 3, we need the following (essentially classical) lemma:

LEMMA 8. *Let C be a countable set, and H a dense G_δ subset of $2^{C\times\omega}$. Then there exists a subset D of $C \times \omega$ with D_i infinite for all $i \in C$, and a map $\gamma : (C \times \omega) - D \to 2$ such that*

$$K_{D,\gamma} = \{\alpha \in 2^{C\times\omega} : \alpha|_{(C\times\omega)-D} = \gamma\}$$

is a subset of H.

Proof. The pair (D, γ) is constructed by induction. Say that (d, g) is a *finite approximation* if d and $\mathrm{dom}(g)$ are finite disjoint subsets of $C \times \omega$. It is clearly enough to check that given a finite approximation (d, g), an $i \in C$ and a dense open set $U \subseteq 2^{C\times\omega}$, one can extend (d, g) to some (d', g') so that $d' - d$ meets $\{i\} \times \omega$, and the clopen set $V_{g'} = \{\alpha \in 2^{C\times\omega} : \alpha|_{\mathrm{dom}(g')} = g'\}$ is a subset of U. But this is easy: Pick first n with (i, n) outside $d \cup \mathrm{dom}(g)$, and set $d' = d \cup \{(i, n)\}$. Enumerate $2^{d'}$ as $f_1, \ldots, f_N$, and define inductively $g_0, g_1, \ldots, g_N$ so that they all have their domains disjoint from d', $g_0 = g$ and they extend each other, and for each $k \le N$, $V_{f_k \cup g_k} \subseteq U$. This is possible by the density of U. But then $g' = g_N$ works. ■

End of proof of Theorem 3. As said before, we just have to check that for any successor ordinal $\xi \ge 2$, $\mathrm{ind}(R_\xi) > \xi$. Argue by contradiction, and suppose C is a Borel set separating R_ξ from $\check{R}_\xi$, and τ a finer Polish topology on X_ξ such that C is $\mathbf{\Delta}^0_\xi$ in $(X_\xi, \tau)^2$. Fix then a dense G_δ subset H of X_ξ on which τ and the usual topology coincide. Applying Lemma 8 (with D_ξ), we get a good pair (D, γ) at level ξ with $K = K_{D,\gamma} \subseteq H$. But then $C \cap K^2$ is $\mathbf{\Delta}^0_\xi$, and by Lemma 7, $h^{-1}_{D,\gamma}(C)$ is a $\mathbf{\Delta}^0_\xi$ set separating R_ξ from $\check{R}_\xi$, contradicting Proposition 6. ■

REMARK. The minimal expected complexity of a strict order R with $\mathrm{ind}(R) > \xi$ is $\mathbf{\Sigma}^0_\xi$ ($\mathbf{\Pi}^0_\xi$ is not possible because of the separation property of that class). Our examples are more complicated, except for $\xi = 2$ and ξ successor of a limit ordinal. We do not know if $\mathbf{\Sigma}^0_\xi$ is always obtainable.

It is often the case with the index method (see [L1]) that the actual proof of the unboundedness of the index provides a jump operator. This is also the case here, at least in spirit, for we will need a slight variant of the previous proof to get the strongest possible jump result.

If $(R_n)_{n\in\omega}$ is a sequence of $\mathbf{\Sigma}^1_1$ oriented graphs on Polish spaces X_n, define a $\mathbf{\Sigma}^1_1$ oriented graph $(R_n)^+$ on $\prod_n X_n$ by

$$x(R_n)^+y \;\leftrightarrow\; \forall^* n\ x_n R_n y_n.$$

The operator $+$ is clearly increasing for the orderings $\preceq_{\mathrm{B}}$ and $\le_{\mathrm{B}}$: if for all n, $R_n \preceq_{\mathrm{B}} S_n$ (resp. $R_n \le_{\mathrm{B}} S_n$), one also has $(R_n)^+ \preceq_{\mathrm{B}} (S_n)^+$ (resp. $(R_n)^+ \le_{\mathrm{B}} (S_n)^+$), by combining the witnessing maps.

If for all n, $R_n = R$, write R^+ instead of $(R_n)^+$. This defines a $\preceq$- and $\le_{\mathrm{B}}$-increasing operator, which clearly satisfies $R \le_{\mathrm{B}} R^+$ for all $\mathbf{\Sigma}^1_1$ oriented graphs R.

The next result is a direct consequence of the proof of the unboundedness of the index. It shows that for complicated enough $\mathbf{\Sigma}^1_1$ oriented graphs, $+$ is in fact a jump operator.

COROLLARY 9. *Let R be a $\mathbf{\Sigma}^1_1$ oriented graph with $R_2 \preceq_{\mathrm{B}} R$. Then $R^+ \not\preceq_{\mathrm{B}} R$.*

Proof. Assume not, and let R be a counterexample. Then from $R^+ \preceq_{\mathrm{B}} R$ and the fact that $+$ is $\preceq_{\mathrm{B}}$-increasing, one sees easily by induction that for all countable successor ordinals $\xi \ge 2$, $R_\xi \preceq_{\mathrm{B}} R$, contradicting (the proof of) Theorem 3. ∎

The condition $R_2 \preceq_{\mathrm{B}} R$ in the previous corollary is not optimal, and may look a bit unnatural. We now show that at least in the context of $\mathbf{\Sigma}^1_1$ strict orders, it provides the optimal result. Then we will see how to change the arguments to get the optimal result for arbitrary $\mathbf{\Sigma}^1_1$ oriented graphs.

First, let us consider the particular case of $\mathbf{\Sigma}^1_1$ strict orders. Recall from Kanovei [K] that a partial order R is Borel linearizable if it admits an extension which is a Borel linear order. We will use the following result from [L3]:

THEOREM 10. *The following are equivalent, for a $\mathbf{\Sigma}^1_1$ partial order R:*

(i) *R is not Borel linearizable,*

(ii) $R_2 \preceq_{\mathrm{B}} <_R$,

(iii) $\mathrm{ind}(<_R) > 2$.

Corollary 11. *Let R be a $\mathbf{\Sigma}^1_1$ strict order which admits finite chains of arbitrary cardinality. Then $R^+ \not\preceq_{\mathrm{B}} R$.*

Proof. By Corollary 9, it is enough to check that any possible counterexample R satisfies $R_2 \preceq_{\mathrm{B}} R$. By the assumption on R, we have for each $n \in \omega$ an increasing R-chain $(x^i_n)_{i \le n}$. We then use the following

Fact. *One can build in ω^ω an ω_1-sequence $(f_\xi)_{\xi<\omega_1}$ such that for all n, $f_\xi(n) \le n$, and $\eta < \xi$ implies $\forall^* n\ f_\eta(n) < f_\xi(n)$.*

Granted this fact, the sequence $((x_n^{f_\xi(n)})_{n\in\omega})_{\xi<\omega_1}$ is then an increasing R^+-chain. But it is a result of Harrington, Marker and Shelah [HMS] that in any Borel linear order there is no uncountable chain. This implies that R^+ is not Borel linearizable, and, by Theorem 10, $R_2 \preceq_{\mathrm{B}} R^+$. So if $R^+ \preceq_{\mathrm{B}} R$, we get $R_2 \preceq_{\mathrm{B}} R$, and Corollary 9 applies, as wanted.

So it remains to construct the sequence $(f_\xi)_{\xi<\omega_1}$ as above. Consider the subset A of ω^ω consisting of those functions f satisfying $f(n) \le n$ for all n, and $n - f(n) \to \infty$ with n. It is clearly enough to prove that for any sequence $(g_k)_{k\in\omega}$ in A, there is f in A with $\forall k\ \forall^* n\ g_k(n) < f(n)$, as one can then by using it build the ω_1 sequence in A by induction on the countable ordinals. As A is closed under finite pointwise suprema, we may assume the sequence g_k is increasing. Let then n_k be least with $n - g_k(n) > k$ for all $n > n_k$, and let $f(n)$ be 0 for $n < n_0$, and $g_k(n) + 1$ for $n_k \le n < n_{k+1}$. One easily checks that f works. ∎

Remark. Corollary 11 is indeed optimal, for if R is such that all R-chains have size $\le k < \omega$, then $R^+ \preceq_{\mathrm{B}} R$. To see this, note first that if R has this property, then so does R^+. Moreover, it is not hard to show by induction on k that if R on X is a $\mathbf{\Sigma}^1_1$ strict order with no $k+1$-chain, then $R \preceq_{\mathrm{B}} (k, <)$. This is clear if $k = 1$. For $k+1$, consider the $\mathbf{\Sigma}^1_1$ set $A \subseteq X$ of all points which are the maximum element of a $k+1$-chain in R. By the hypothesis, A is a subset of the $\mathbf{\Pi}^1_1$ set C of all R-maximal points. By separation, there is a Borel set B with $A \subseteq B \subseteq C$. On the complement of B, there are no $k+1$-chains in R, so there is by induction a Borel homomorphism into $(k, <)$. Sending the points of B to k then gives the desired Borel homomorphism.

Finally, if k is the least upper bound to the cardinality of R-chains, we deduce by the preceding facts that $R^+ \preceq_{\mathrm{B}} (k, <) \preceq_{\mathrm{B}} R$, as desired.

In some cases, one can even have $R^+ \le_{\mathrm{B}} R$, for example if $X = \{0, 1, 2\}$ with $0R1$, as witnessed by sending the sequences which are eventually i, for $i = 0, 1$, to i, and the other sequences to 2.

We now briefly indicate how to adapt the previous arguments to get the following result, which subsumes both Corollaries 9 and 11 and is valid for arbitrary $\mathbf{\Sigma}^1_1$ oriented graphs:

THEOREM 12. *Let R be a $\mathbf{\Sigma}^1_1$ oriented graph on a Polish space X. Assume that $(*)$ for all $k \in \omega$, there is a sequence $(x^k_i)_{i\le k}$ in X with $x^k_i R x^k_{i+1}$ for all $i < k$. Then $R^+ \not\preceq_{\mathrm{B}} R$.*

Proof. We first introduce an oriented graph G_2—which replaces R_2. It is defined on the space X'_2 of all infinite co-infinite subsets of ω by

$$AG_2B \leftrightarrow A \triangle B \text{ is finite \& } \mathrm{card}(B-A) = \mathrm{card}(A-B)+1.$$

We can then define inductively (as we did for the R_ξ's) graphs G_ξ, for successor $\xi \ge 2$, by setting for successor ξ, $G_{\xi+1} = G^+_\xi$, and for limit λ, with (λ_n) an increasing sequence of successor ordinals converging to it, $G_{\lambda+1} = (G_{\lambda_n})^+_n$. We can of course view G_2 as defined on $X_2 = 2^\omega$, and hence G_ξ as defined on X_ξ.

We now argue as in Corollary 11. First we check that if R is a $\mathbf{\Sigma}^1_1$ oriented graph which satisfies condition $(*)$, then $G_2 \preceq_{\mathrm{B}} R^+$. To see this, let $(x^k_i)_{i<k}$ be a witness for $(*)$, and define, for A an infinite co-infinite subset of ω, $i_A(k) = \mathrm{card}(A\cap k)$ and $f(A) = (x^k_{i_A(k)})_{k\in\omega}$. Then if AG_2B and $n = \sup(A\triangle B)+1$, one sees for $k \ge n$ that $i_{\mathrm{B}}(k) = i_A(k)+1$, hence $x^k_{i_A(k)} R x^k_{i_{\mathrm{B}}(k)}$. So $f(A)R^+f(B)$, and f is a Borel homomorphism from G_2 to R^+.

The second step of the proof is then immediate, by induction on ξ: If R satisfies $(*)$ and $R^+ \preceq_{\mathrm{B}} R$, then for all ξ, $G_\xi \preceq_{\mathrm{B}} R$.

So it remains to show that this is impossible, by proving that for all successor $\xi \ge 2$, $\mathrm{ind}(G_\xi) > \xi$. The proof of this last fact is entirely analogous to the proof we gave for R_ξ. We just have to prove the statements analogous to Proposition 6 and Lemma 7. The analog of Lemma 7 is obvious, with the same proof, using the particular form of G_ξ. For the analog of Proposition 6, it comes down to proving that there is a continuous map $f : 2^\omega \to 2^\omega \times 2^\omega$ sending L_0 to G_2 and L_1 to $\check{G}_2$. To do this, one can use the determinacy of the following usual separation game: players I and II play $\alpha \in 2^\omega$ and $(\beta,\gamma) \in 2^\omega \times 2^\omega$ respectively (bit by bit), and player II wins if $\alpha \in L_0$ implies $\beta G_2 \gamma$, and $\alpha \in L_1$ implies $\gamma G_2 \beta$. This game is clearly Borel, hence determined, and a winning strategy for player II provides the wanted map f. So it is enough to check that player I does not have a winning strategy. But otherwise, we get a continuous map $g : 2^\omega \times 2^\omega \to 2^\omega$ which satisfies $g(2^\omega \times 2^\omega) \subseteq L_0 \cup L_1$, $G_2 \subseteq g^{-1}(L_1) = g^{-1}(2^\omega - L_0)$, and $\check{G}_2 \subseteq g^{-1}(L_0) = g^{-1}(2^\omega - L_1)$, by the definition of the game. Now clearly G_2 and $\check{G}_2$ are both dense in $(X'_2)^2$, so $g^{-1}(L_0)$ and $g^{-1}(L_1)$ are disjoint dense G_δ in it, a clear contradiction. Putting everything together, this proves Theorem 12. ∎

To end up this paper, let us come back to our original motivation, i.e. the unboundedness property of $\mathbf{\Sigma}^1_1$ partial orders under $\le_{\mathrm{B}}$ (Theorem 1), and discuss how this result can be generalized to other types of quasi-orders besides partial orders.

If R is a $\mathbf{\Sigma}^1_1$ quasi-order on a Polish space X, denote by $\equiv_R$ the associated (necessarily $\mathbf{\Sigma}^1_1$) equivalence relation, defined by $\equiv_R = R \cap \check{R}$. And if E is a given $\mathbf{\Sigma}^1_1$ equivalence relation on X, let $\mathcal{C}_E$ be the class of $\mathbf{\Sigma}^1_1$ quasi-orders R on X with $\equiv_R = E$. So partial orders correspond to the case of equality, on say $X = 2^\omega$. And Theorem 1 says that when E is equality, $\mathcal{C}_E$ admits no complete element. What is the situation for other E's?

First, it is proved in Louveau–Rosendal [LR] that if E is a complete $\mathbf{\Sigma}^1_1$ equivalence relation, one has $E = \equiv_R$ for some complete $\mathbf{\Sigma}^1_1$ quasi-order R, which is *a fortiori* complete in $\mathcal{C}_E$. So this gives an example of a $\mathbf{\Sigma}^1_1$ equivalence relation E for which there exists a complete element in $\mathcal{C}_E$. Also, the same is true at the other extreme, if E has only countably many classes, for then $\mathcal{C}_E$ corresponds (up to Borel bi-reducibility) to countable partial orders, and it is well known that there exists a complete countable partial order.

Here we have:

Corollary 13. *Let E be a Borel equivalence relation on some Polish space X, with uncountably many classes. Then $\mathcal{C}_E$ has no complete element, and in fact the Borel elements in $\mathcal{C}_E$ are unbounded in $\mathcal{C}_E$.*

Proof. Suppose, towards a contradiction, that some $R \in \mathcal{C}_E$ Borel reduces all Borel elements in $\mathcal{C}_E$. Consider the strict order $<_R = R - E$, which is $\mathbf{\Sigma}^1_1$ as E is Borel. We get the desired contradiction by proving that $<_R$ Borel reduces all Borel strict orders on 2^ω, contradicting Theorem 3. So let S be a Borel strict order on 2^ω. By our assumption and Silver's theorem, there is a one-to-one continuous map $f : 2^\omega \to X$ which reduces equality to E. Define then S' on X by

$$xS'y \;\leftrightarrow\; xEy \text{ or } \exists\alpha \in 2^\omega\ \exists\beta \in 2^\omega\ (f(\alpha)Ex \text{ and } f(\beta)Ey \text{ and } \alpha S\beta).$$

It is easy to check that S' is a quasi-order with $\equiv_{S'} = E$, and that S' is Borel (for the α, β in the definition are unique, when they exist). Moreover, f is a witness that $S \leq_{\mathrm{B}} S'$, and as $S' \in \mathcal{C}_E$, also $S' \leq_{\mathrm{B}} R$, hence we get the desired contradiction. ∎

We do not know whether there is a complete element in $\mathcal{C}_E$ when E is a $\mathbf{\Sigma}^1_1$ equivalence relation which is neither complete nor Borel.

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