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Holonomy groups of flat manifolds with the R_∞ property

by

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Abstract. Let M be a flat manifold. We say that M has the R_∞ property if the Reidemeister number $R(f)$ is infinite for every homeomorphism $f\colon M \to M$. We investigate relations between the holonomy representation ρ of M and the R_∞ property. When the holonomy group of M is solvable we show that if ρ has a unique $\mathbb{R}$-irreducible subrepresentation of odd degree then M has the R_∞ property. This result is related to Conjecture 4.8 in [K. Dekimpe et al., Topol. Methods Nonlinear Anal. 34 (2009)].

1. Introduction. Let $f\colon M^n \to M^n$ be a continuous map on a closed n-dimensional manifold M^n. From the point of view of fixed point theory the following three numbers are of particular importance: the Lefschetz number $L(f)$, the Nielsen number $N(f)$ and the Reidemeister number $R(f)$. If $n \geq 3$, the Nielsen number $N(f)$ is a sharp lower bound on the number of fixed points of any element in the homotopy class of f. However in general $N(f)$ is difficult to calculate. In 1963, B. Jiang identified a large class of spaces for which

$$N(f) = \begin{cases} 0 & \text{if } L(f) = 0, \\ R(f) & \text{if } L(f) \neq 0, \end{cases}$$

for all continuous maps $f\colon M^n \to M^n$.

In the light of the above relation, since the Nielsen number is always finite, the finiteness of the Reidemeister number is important. This was one of the motivations for introducing

Definition 1.1. A manifold M^n has the R_∞ *property* if $R(f) = \infty$ for every homeomorphism $f\colon M^n \to M^n$.

The Reidemeister number can be defined at the level of the fundamental group $\Gamma = \pi_1(M^n)$. Recall that any continuous map $f\colon M^n \to M^n$ induces

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a morphism $f_\sharp\colon \Gamma \to \Gamma$. We say that two elements $\alpha, \beta \in \Gamma$ are *$f_\sharp$-conjugate* if there exists $\gamma \in \Gamma$ such that $\beta = \gamma\alpha f_\sharp(\gamma)^{-1}$. The $f_\sharp$-conjugacy class $\{\gamma\alpha f_\sharp(\gamma)^{-1} \mid \gamma \in \Gamma\}$ of α is called a *Reidemeister class* of f. The number of Reidemeister classes is called the *Reidemeister number* $R(f)$ of f. It is evident that we can also define $R(\Phi)$ for a countable discrete group E and its automorphism Φ. We say that a group E has the *R_∞ property* if $R(\Phi) = \infty$ for any automorphism Φ. The class of groups with the R_∞ property includes: all non-elementary Gromov-hyperbolic groups, the Baumslag–Solitar groups $BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n\rangle$ except for $BS(1,1)$, the lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ if and only if $2 \mid n$ or $3 \mid n$, the Thompson group F and the symplectic groups $\mathrm{Sp}(2n, \mathbb{Z}), n \in \mathbb{Z}_+$. See [4] and [9] for a more comprehensive list, the history of R_∞-groups and a complete bibliography.

Let M^n be a closed Riemannian manifold of dimension n. We shall call M^n *flat* if, at any point, the sectional curvature is equal to zero. Equivalently, M^n is isometric to the orbit space $\mathbb{R}^n/\Gamma$, where Γ is a discrete, torsion-free and cocompact subgroup of $O(n) \ltimes \mathbb{R}^n = \mathrm{Isom}(\mathbb{R}^n)$. From the Bieberbach theorem (see [1], [10]), Γ defines a short exact sequence of groups

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0, \tag{1.1}$$

where G is a finite group. Here Γ is called a *Bieberbach group* and G its *holonomy group*. We can define the *holonomy representation* $\rho\colon G \to \mathrm{GL}(n, \mathbb{Z})$ by the formula:

$$\forall_{g\in G} \quad \rho(g)(e_i) = \tilde{g}e_i(\tilde{g})^{-1}, \tag{1.2}$$

where $e_i \in \Gamma$, $i = 1, \ldots, n$, are generators of the free abelian group $\mathbb{Z}^n$, and $\tilde{g} \in \Gamma$ is such that $p(\tilde{g}) = g$.

In this article we describe relations between the R_∞ property of the flat manifold M^n (or of the Bieberbach group Γ) and the structure of the holonomy representation. Connections between geometric properties of M^n and algebraic properties of ρ were already considered in different cases. For example, $\mathrm{Out}(\Gamma)$ is finite if and only if the holonomy representation is $\mathbb{Q}$-multiplicity free and any $\mathbb{Q}$-irreducible component of the holonomy representation is $\mathbb{R}$-irreducible (see [8]). A similar equivalence says that an Anosov diffeomorphism $f\colon M^n \to M^n$ exists if and only if any $\mathbb{Q}$-irreducible component of the holonomy representation that occurs with multiplicity one is reducible over $\mathbb{R}$ (see [5]).

We want to define conditions of this kind for the holonomy representation of a flat manifold with the R_∞ property. We already know that, in this way, a complete characterization is not possible. There are examples [3, Th. 5.9] of flat manifolds M_1, M_2 with the same holonomy representation such that M_1 has the R_∞ property and M_2 does not. In [3, Corollary 4.4] it is proved that if there exists an Anosov diffeomorphism $f\colon M^n \to M^n$ then $R(f)$

is finite and M^n does not have the R_∞ property. Moreover there exists M whose holonomy representation has a $\mathbb{Q}$-irreducible component which is irreducible over $\mathbb{R}$ and occurs with multiplicity one and M does not have the R_∞ property, [3, Example 4.6]. Nevertheless in [3, Th. 4.7] the following is proved:

Theorem 1.2. *Let M be a flat manifold with holonomy representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$. Assume that there exists a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation $\rho'\colon G \to \mathrm{GL}(n',\mathbb{Z})$ of ρ such that $\rho'(G)$ is not $\mathbb{Q}$-conjugate to $\tilde\rho(G)$ for any other $\mathbb{Q}$-subrepresentation $\tilde\rho$ of ρ. Suppose moreover that for every $D' \in N_{\mathrm{GL}(n',\mathbb{Z})}(\rho'(G))$, there exists $A \in G$ such that $\rho'(A)D'$ has eigenvalue 1. Then M has the R_∞ property.*

Remark 1.3. If we assume that

$$N_{\mathrm{GL}(n',\mathbb{Q})}(\rho'(G))/C_{\mathrm{GL}(n',\mathbb{Q})}(\rho'(G)) \cong \mathrm{Aut}(G), \tag{1.3}$$

then the above requirement that $\rho'(G)$ is not $\mathbb{Q}$-conjugate to $\tilde\rho(G)$ is equivalent to the condition that $\rho' \subset \rho$ has multiplicity one. For example, if we take the diagonal representation $\rho : (\mathbb{Z}_2)^{2n} \to \mathrm{SL}(2n+1,\mathbb{Z})$ of the elementary abelian 2-group, then (1.3) is not satisfied for any $\mathbb{Q}$-irreducible subrepresentation of ρ.

We shall prove:

Theorem 1.4. *Let M be a flat manifold with holonomy representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ where G is a solvable group. Assume that there exists a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation $\rho'\colon G \to \mathrm{GL}(n',\mathbb{Z})$ of ρ of odd dimension such that $\rho'(G)$ is not $\mathbb{Q}$-conjugate to $\tilde\rho(G)$ for any other $\mathbb{Q}$-subrepresentation $\tilde\rho$ of ρ. Then M has the R_∞ property.*

If we restrict our consideration to the class of finite groups which satisfy the condition (1.3) we have

Theorem 1.5. *Let M be a flat manifold with holonomy representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ where G is a solvable group. Assume that there exists a $\mathbb{Q}$-irreducible $\mathbb{Q}$-subrepresentation $\rho'\colon G \to \mathrm{GL}(n',\mathbb{Z})$ of ρ of multiplicity one and odd dimension which satisfies the condition* (1.3)*. Then M has the R_∞ property.*

The above result is a corollary of [7, Th. 5.4.4], Theorem 1.2 and the following theorem:

Theorem A. *Let G be a finite group with a non-trivial normal abelian subgroup A and let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose n is odd. Then for every $D \in N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue* 1.

The main idea used in the proof of the main result is to apply Clifford's theorem [2, Theorem 49.2], which gives a relation between irreducible kG-modules and kH-modules, where H is a normal subgroup of a finite group G, and k is an arbitrary field.

REMARK 1.6. Conjecture 4.8 in [3] says that the above Theorem A is true for any finite group. We do not know whether it holds in general.

2. Proof of Theorem A

THEOREM 2.1. *Let G be a finite group and n be an odd integer. Let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a faithful representation of G which is irreducible over $\mathbb{R}$. Then ρ is irreducible over $\mathbb{C}$.*

Proof. Assume that ρ is reducible over $\mathbb{C}$ and let τ be any $\mathbb{C}$-irreducible subrepresentation of ρ. By [6, Theorem 2], the representation ρ is uniquely determined by τ and, if χ is the character of τ, then the character of ρ is $\chi + \overline{\chi}$. Hence ρ is of even degree. This proves the theorem. ∎

For the rest of this section we assume that $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ is an absolutely irreducible representation of G, where n is an odd integer.

PROPOSITION 2.2. *If A is a normal abelian subgroup of G, then A is an elementary abelian 2-group.*

Proof. Let τ be an $\mathbb{R}$-irreducible subrepresentation of $\rho_{|A}$. By Clifford's theorem [2, Theorem 49.2], all $\mathbb{R}$-subrepresentations of $\rho_{|A}$ are conjugates of an $\mathbb{R}$-irreducible subrepresentation τ, i.e. there exist $g_1 = 1, g_2, \ldots, g_l \in G$ such that

$$\rho_{|A} = \tau^{(g_1)} \oplus \cdots \oplus \tau^{(g_l)}, \tag{2.1}$$

where

$$\forall_{1\le i\le l}\ \forall_{g\in G} \quad \tau^{(g_i)}(g) = \tau(g_i^{-1} g g_i).$$

Let $a \in A$ be an element of order greater than 2. Since ρ is faithful, there exists $1 \le i \le l$ such that $\tau^{(g_i)}(a)$ is a real matrix of order at least 3. Hence $\deg(\tau^{(g_i)}) = \deg(\tau) = 2$ and $n = \deg(\rho) = \deg(\rho_{|A}) = l\deg(\tau) = 2l$ is an even integer. This contradiction finishes the proof. ∎

Since A is an elementary abelian 2-group, the decomposition (2.1) may be realized over the rationals. By [2, Theorem 49.7] we may assume that

$$\rho_{|A} = e\tau^{(g_1)} \oplus \cdots \oplus e\tau^{(g_k)}, \tag{2.2}$$

i.e. the one-dimensional representations $\tau^{(g_1)}, \ldots, \tau^{(g_k)}$ occur with the same multiplicity $e = n/k$. Let $\rho_i := e\tau^{(g_i)}$ for $i = 1, \ldots, k$. By a suitable choice of basis of $\mathbb{Q}^n$ we may assume that for every $a \in A$, $\rho(a)$ is a diagonal matrix such that

$$\forall_{1\le i\le k} \quad \mathrm{Img}(\rho_k) = \langle -\mathrm{I}\rangle, \tag{2.3}$$

where I is the $e \times e$ identity matrix.

Since $A \lhd G$ and ρ is faithful, we have

$$\rho(A) \lhd \rho(G) \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{m \in \mathrm{GL}(n,\mathbb{Q}) \mid m^{-1}\rho(A)m = \rho(A)\}.$$

In the next two subsections we will focus on the above normalizer.

2.1. Centralizer. First we describe the centralizer

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{m \in \mathrm{GL}(n,\mathbb{Q}) \mid \forall_{a\in A}\ m\rho(a) = \rho(a)m\}.$$

Let $m = (m_{ij}) \in \mathrm{GL}(n,\mathbb{Q})$ be a block matrix such that $m\rho_{|A} = \rho_{|A}m$. We get

$$\begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk}\end{pmatrix}\begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k\end{pmatrix} = \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k\end{pmatrix}\begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk}\end{pmatrix},$$

and thus

$$\forall_{1\le i,j\le k} \quad m_{ij}\rho_j = \rho_i m_{ij}.$$

Since for $i \neq j$, ρ_i and ρ_j have no common subrepresentation, by Schur's Lemma (see [2, (27.3)]) $m_{ij} = 0$ for $i \neq j$ and $m_{ii} \in \mathrm{GL}(n/k,\mathbb{Q})$ for $i = 1,\dots,k$. We have just proved

LEMMA 2.3. *Let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Q})$ be a faithful, absolutely irreducible representation of a finite group G of odd degree n. Let A be a normal abelian subgroup of G such that conditions* (2.2) *and* (2.3) *hold. Then*

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \{\mathrm{diag}(c_1,\dots,c_k) \mid c_i \in \mathrm{GL}(n/k,\mathbb{Q}),\ i = 1,\dots,k\},$$

where k is the number of pairwise non-isomorphic irreducible subrepresentations of $\rho_{|A}$.

2.2. Normalizer. Since the group A is finite, $\mathrm{Aut}(A)$ is a finite group. Moreover, we have a monomorphism

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))/C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \hookrightarrow \mathrm{Aut}(A).$$

Hence any coset $mC_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ with $m \in N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ corresponds to some automorphism of A.

Let $\varphi \in \mathrm{Aut}(A)$ and $m = (m_{ij}) \in \mathrm{GL}(n,\mathbb{Q})$ be a block matrix which represents this automorphism, with blocks of degree n/k, i.e.

$$\forall_{c\in C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))}\ \forall_{a\in A} \quad (mc)\rho(a)(mc)^{-1} = m\rho(a)m^{-1} = \rho(\varphi(a)).$$

We have

$$\begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix} \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k \end{pmatrix} = \begin{pmatrix} \rho_1\varphi & & 0 \\ & \ddots & \\ 0 & & \rho_k\varphi \end{pmatrix} \begin{pmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \dots & m_{kk} \end{pmatrix}.$$

Note that

$$\forall_{1\le i\le k} \quad \mathrm{Img}(\rho_i) = \mathrm{Img}(\rho_i\varphi) = \langle -\mathrm{I}\rangle. \tag{2.4}$$

Since, for $i \neq j$, ρ_i and ρ_j do not have common subrepresentations, the same applies to $\rho_i\varphi$ and $\rho_j\varphi$. Hence, using Schur's lemma again, for every $1 \le i \le k$ there exists exactly one $1 \le j \le k$ such that

$$m_{ji}\rho_i = \rho_j\varphi m_{ji}$$

and $m_{ji} \neq 0$. Moreover, $\det(m) \neq 0$ and also $\det(m_{ij}) \neq 0$. By (2.4), $\rho_i = \rho_j\varphi$ and there exists a permutation $\sigma \in S_k$, where S_k is the symmetric group on k letters, such that

$$m\,\mathrm{diag}(\rho_1,\dots,\rho_k)m^{-1} = \mathrm{diag}(\rho_{\sigma(1)},\dots,\rho_{\sigma(k)}). \tag{2.5}$$

Let $\tau \in S_k$ be any permutation and let $P_\tau \in \mathrm{GL}(n,\mathbb{Q})$ be the block matrix, with all blocks of degree n/k, such that

$$(P_\tau)_{i,j} = \begin{cases} \mathrm{I} & \text{if } \tau(i) = j, \\ 0 & \text{otherwise}, \end{cases} \tag{2.6}$$

where $1 \le i, j \le k$. By (2.5) we may take

$$m = P_\sigma$$

as a representative of a coset in $N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))/C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ which realizes the automorphism φ.

Let

$$S := \{\tau \in S_k \mid P_\tau \in N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))\}.$$

Then S is a subgroup of S_k and

$$P := \{P_\tau \mid \tau \in S\}$$

is a subgroup of the normalizer. By the above and Lemma 2.3, we get

PROPOSITION 2.4. *The normalizer $N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ is a semidirect product of $C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))$ and P. Moreover*

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))\cdot P \cong C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \rtimes S \cong \mathrm{GL}(n/k,\mathbb{Q})\wr S,$$

where $\wr$ denotes wreath product.

2.3. Properties of the group G. Let $C := C_G(A)$ be the centralizer of A in G. Since ρ is faithful, we have

$$C = \rho^{-1}(C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))).$$

By Proposition 2.4, the kernel of the composition

$$G \xrightarrow{\rho} N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \xrightarrow{\nu} N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A))/C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \cong S,$$

where ν is the quotient homomorphism, equals C and hence we have an isomorphism of groups

$$S \cong G/C.$$

The representations ρ_i, $i = 1, \ldots, k$, are defined on the group A. Lemma 2.3 enables us to extend their domains to C. Let V_i be the subspaces of $\mathbb{Q}^n$ corresponding to the representations ρ_i, $i = 1, \ldots, k$. In fact, since $\rho_{|C}$ is in block diagonal form, we have

$$\forall_{1\le i\le k} \quad V_i = \underbrace{\Theta \oplus \cdots \oplus \Theta}_{i-1} \oplus \mathbb{Q}^{n/k} \oplus \Theta \oplus \cdots \oplus \Theta \subset \mathbb{Q}^n,$$

where Θ is considered as a zero-dimensional subspace (zero vector) of $\mathbb{Q}^{n/k}$. Moreover, every element of the group S permutes the set $\{V_1, \ldots, V_k\}$. We want to prove that this action is transitive.

Lemma 2.5. *$S \subset S_k$ is a transitive permutation group.*

Proof. Assume that S is not transitive, so

$$\exists_{1\le j\le k}\ \forall_{i\ne j}\ \forall_{\tau\in S} \quad \tau(i) \ne j.$$

Let

$$\hat{V}_j = \bigoplus_{\substack{i=1\\ i\ne j}}^{k} V_i$$

and $g \in G$. Then $\rho(g) = P_\tau m$ for some $\tau \in S$ and $m \in \bigoplus_{i=1}^k \mathrm{GL}(n/k, \mathbb{Q})$. We get

$$\rho(g)(\hat{V}_j) = P_\tau m \cdot \hat{V}_j = P_\tau \cdot \hat{V}_j = \bigoplus_{\substack{i=1\\ i\ne j}}^{k} V_{\tau(i)} = \hat{V}_j.$$

Thus $\hat{V}_j \subsetneq \mathbb{Q}^n$ is an invariant subspace of ρ and hence ρ is reducible (over $\mathbb{Q}$). This contradiction proves the lemma. ■

The following lemma helps us to understand the structure of the representation ρ.

Lemma 2.6. *The representations $\rho_1, \ldots, \rho_k \colon C \to \mathrm{GL}(n/k, \mathbb{Q})$ are absolutely irreducible.*

Proof. Let $\phi\colon C \to \mathrm{GL}(d, \mathbb{C})$ be a $\mathbb{C}$-irreducible subrepresentation of $\rho_{|C}$. By Clifford's theorem, for the group $C \lhd G$ the representation $\rho_{|C}$ is a sum of conjugates of ϕ, i.e.

$$\rho_{|C} = \bigoplus_{s=1}^{m} \phi^{(g_s)},$$

where $g_s \in G$, $s = 1, \ldots, m$, and $g_1 = 1$. For every $1 \leq s \leq m$, $\phi^{(g_s)}$ is a complex subrepresentation of some ρ_i, $i = 1, \ldots, k$. Counting dimensions, we can see that for every $1 \leq i \leq k$,

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j},$$

where

$$\forall_{1 \leq j \leq m/k} \quad \rho_{i,j} \in \{\phi^{(g_s)} \mid 1 \leq s \leq m\}.$$

Let $V_{i,j} \subset V_i$ be an invariant space under the action of $\rho_{i,j}$ for $1 \leq i \leq k$, $1 \leq j \leq m/k$. Taking a suitable basis for V_i, $1 \leq i \leq k$, we can assume that the decomposition

$$\rho_i = \bigoplus_{j=1}^{m/k} \rho_{i,j}$$

is given in block diagonal form:

$$\forall_{1 \leq j \leq m/k} \quad V_{i,j} = \underbrace{\Theta \oplus \cdots \oplus \Theta}_{j-1} \oplus \mathbb{C}^{n/m} \oplus \Theta \oplus \cdots \oplus \Theta \subset V_i,$$

where Θ is a zero-dimensional subspace (zero vector) of $\mathbb{C}^{n/m}$. Note that the images of $\rho_{i|A}$, $i = 1 \ldots, k$, remain the same in this new basis. Hence the description of the representatives of the normalizer given in Subsection 2.2 remains the same for the group $\mathrm{GL}(n, \mathbb{C})$.

If the representations ρ_i, $i = 1, \ldots, k$, are $\mathbb{C}$-reducible then $m > k$. Let

$$W = \bigoplus_{i=1}^{k} V_{i,1}$$

and $g \in G$. Then $\rho(g) = P_\tau m$ (as in the proof of Lemma 2.5) and we get

$$\rho(g)(W) = P_\tau m \cdot W = P_\tau \cdot W = \bigoplus_{i=1}^{k} V_{\tau(i),1} = W.$$

Hence $W \subsetneq \mathbb{C}^n$ is an invariant subspace of ρ and thus ρ cannot be absolutely irreducible. This contradiction finishes the proof. ∎

2.4. Abelian normal subgroups. Without loss of generality we can assume that A is a maximal abelian normal subgroup of G, i.e. if $A' \lhd G$ is abelian and $A \subset A'$ then $A = A'$. We will show that A is unique in G and hence characteristic.

LEMMA 2.7. *A is unique in C.*

Proof. Let $A' \lhd G$ be an abelian group such that $A' \subset C$. Since all elements of A commute with all elements of C, they commute with all elements of A'. Hence AA' is a normal abelian subgroup of G. Since A is maximal, we have

$$AA' = A \Rightarrow A' \subset A. \ \blacksquare$$

If we can prove that $A \subset C$, then A is going to be unique in G. Recall that we have a short exact sequence

$$1 \to C \to G \xrightarrow{p} S \to 1.$$

Assuming $A \not\subset C$, we get $1 \neq p(A) \lhd S$. We will prove that this is impossible.

LEMMA 2.8. *Let $S \subset S_k$ be a transitive permutation group and k be an odd natural number. Then S contains no non-trivial normal elementary abelian 2-groups.*

Proof. Let $x \in X = \{1, \ldots, k\}$. Let S_x be the stabilizer of x in S, and Sx be the orbit of x. By the transitivity of the action of S on X, we have $Sx = X$ and since we have a bijection

$$Sx \leftrightarrow \{\tau S_x \mid \tau \in S\},$$

the index $[S : S_x]$ is an odd number. Now let B be any normal 2-subgroup of S. Then $B \subset S_x$ and we get

$$B = \bigcap_{\tau \in S} \tau B \tau^{-1} \subset \bigcap_{\tau \in S} \tau S_x \tau^{-1} = \bigcap_{\tau \in S} S_{\tau(x)} = 1,$$

since S acts faithfully on X. $\blacksquare$

We have just proved

PROPOSITION 2.9. *The maximal, normal elementary abelian subgroup $A \lhd G$ is unique maximal in G and hence it is a characteristic subgroup.*

COROLLARY 2.10.

$$N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(G)) \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)).$$

2.5. The proof of Theorem A. Let us recall the statement of the theorem.

THEOREM A. *Let G be a finite group with a non-trivial normal abelian subgroup A and let $\rho\colon G \to \mathrm{GL}(n,\mathbb{Z})$ be a faithful $\mathbb{R}$-irreducible representation. Suppose n is odd. Then for every $D \in N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$, there exists $g \in G$ such that $\rho(g)D$ has eigenvalue* 1.

Proof. Note that, by $\mathbb{R}$-irreducibility of ρ, $N = N_{\mathrm{GL}(n,\mathbb{Z})}(\rho(G))$ is a finite group (see [8, pp. 587–588]).

Since eigenvalues of matrices do not depend on their conjugacy class, we can assume that $\rho(A)$ is a group of diagonal matrices. Using Corollary 2.10, Proposition 2.4 and the fact that

$$N \subset N_{\mathrm{GL}(n,\mathbb{Q})}(\rho(G)),$$

we get

$$N \subset C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) \cdot P.$$

Recall that

$$C_{\mathrm{GL}(n,\mathbb{Q})}(\rho(A)) = \bigoplus_{i=1}^{k} \mathrm{GL}(n/k,\mathbb{Q})$$

and elements of P are "block permutation matrices" (see Lemma 2.3 and (2.6) respectively).

Let $D \in N$. Then D has the form

$$D = P_\sigma \operatorname{diag}(c_1,\dots,c_k),$$

where $\sigma \in S_k$ and $c_i \in \mathrm{GL}(n/k,\mathbb{Q})$ for $i = 1,\dots,k$. Recall that $G/C \cong S$, where $S \subset S_k$ is a transitive permutation group (see Lemma 2.5). Hence there exists $\tau \in S$ such that

$$\tau(1) = \sigma^{-1}(1)$$

and for some $g' \in G$,

$$\rho(g') = \operatorname{diag}(c'_1,\dots,c'_k)P_\tau.$$

We get

$$\begin{aligned}\rho(g')D &= \operatorname{diag}(c'_1,\dots,c'_k)P_\tau P_\sigma \operatorname{diag}(c_1,\dots,c_k)\\ &= \operatorname{diag}(c'_1,\dots,c'_k)P_{\sigma\tau} \operatorname{diag}(c_1,\dots,c_k) = \operatorname{diag}(d,T),\end{aligned}$$

where T is the matrix of rows of $\operatorname{diag}(c_2,\dots,c_k)$ permuted by $\sigma\tau$ and multiplied on the left by $\operatorname{diag}(c'_2,\dots,c'_k)$. Since $d = c'_1c_1 \in \mathrm{GL}(n/k,\mathbb{Q})$ has an odd degree, it must have a real eigenvalue, and since N is of finite order, this eigenvalue is ±1. If the eigenvalue is 1, then we take $g = g'$ and the theorem is proved. Otherwise, by Clifford's theorem and the faithfulness of ρ, we can take $a \in A$ such that $\rho_1(a) = -\mathrm{I}$. Then $\rho_1(a)d$ has an eigenvalue 1 and hence, taking $g = ag'$, the element

$$\begin{aligned}\rho(g)D = \rho(ag')D = \rho(a)\rho(g')D &= \rho(a)\operatorname{diag}(d,T)\\ &= (\rho_1 \oplus \dots \oplus \rho_k)(a) \cdot \operatorname{diag}(d,T)\\ &= \operatorname{diag}(\rho_1(a)d, (\rho_2 \oplus \dots \oplus \rho_k)(a)T)\end{aligned}$$

also has an eigenvalue equal to 1. This finishes the proof. ∎

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