Function spaces and local properties

by

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Abstract. Necessary conditions and sufficient conditions are given for $C_p(X)$ to be a (σ-) m_1- or m_3-space. (A space is an m_1-space if each of its points has a closure-preserving local base.) A compact uncountable space K is given with $C_p(K)$ an m_1-space, which answers questions raised by Dow, Ramírez Martínez and Tkachuk (2010) and Tkachuk (2011).

1. Introduction. The purpose of this note is to investigate local properties of the function space $C_p(X)$ of all continuous real-valued functions on a Tychonoff space X with the topology of pointwise convergence. Every $C_p(X)$ is a dense locally convex topological vector subspace of the Tychonoff power $\mathbb{R}^X$, and hence if $C_p(X)$ is first countable then it is metrizable, and so X is countable. A space is an m_1-*space* if every point of the space has a closure-preserving local base. Clearly, first countable spaces are m_1-spaces, and so Dow et al. [3] and Tkachuk [11] were led to ask:

(1) if $C_p(X)$ is an m_1-space then must X be countable? and
(2) what about the special case when X is compact?

The above questions have positive answers in various restricted cases. These hold for some properties a little weaker than the m_1-property. A collection $\mathcal{P}$ of pairs of subsets of a space is said to be *cushioned* if for every subcollection $\mathcal{P}'$ of $\mathcal{P}$ we have

$$\overline{\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}} \subseteq \bigcup\{\overline{P_2} : (P_1, P_2) \in \mathcal{P}'\}.$$

Observe that a collection of subsets $\mathcal{C}$ of a space is closure-preserving if and only if the collection of pairs $\{(C, C) : C \in \mathcal{C}\}$ is cushioned. Then a space X is an m_3-*space* if every point x of X has a cushioned local pairbase, $\mathcal{P}_x$ (so $\mathcal{P}_x$ is a cushioned family of pairs and for every open U containing x there is a (P_1, P_2) in $\mathcal{P}_x$ such that x is in the interior of P_1 and $P_1 \subseteq P_2 \subseteq U$).

Further, let us call a space a σ-m_1-*space* if each point of the space has a σ-closure-preserving local base, and a σ-m_3-*space* if each of its points has a σ-cushioned local pairbase. See [2] for basic information on the (σ-) m_i-properties. Evidently m_1-spaces are m_3-spaces, σ-m_1-spaces are σ-m_3-spaces, and m_i-spaces are σ-m_i-spaces for $i = 1, 3$.

An important source of m_3-spaces are those that are monotonically normal. A space X is *monotonically normal* if for every point z of X and every open set U containing z, we can find an open set $V(z, U)$ containing z and contained in U, such that if $V(z, U) \cap V(z', U') \neq \emptyset$ then either $z \in U'$ or $z' \in U$. Given a space X, an operator $V(\cdot, \cdot)$ as in the definition of monotone normality, and a point x in X, it is easy to see that

$$\mathcal{P}_x = \{(V(x, U), U) : x \in U \text{ and } U \text{ is open in } X\}$$

is a cushioned pairbase at x. Thus monotonically normal spaces are m_3-spaces. It is unknown if every monotonically normal space is an m_1-space. Gartside [5] showed that if $C_p(X)$ is monotonically normal then X is countable (see also [12] and [9]), and his proof used the fact that monotonically normal spaces are m_3-spaces. However his proof also uses the observation that $C_p(X)$ always has *countable cellularity* (disjoint families of opens sets are countable) and the fact that in monotonically normal spaces countable cellularity implies *calibre* $(\omega_1, \omega, \omega)$ (point-finite families of non-empty open sets are countable) (see [4]), and in general $C_p(X)$ need not have calibre $(\omega_1, \omega, \omega)$.

To summarize the above: if $C_p(X)$ is an m_1-space or an m_3-space for standard reasons (it is first countable or monotonically normal) then X is countable. Further, X is countable if $C_p(X)$ is *separable* and a σ-m_3-space. The latter is due to the fact [6] that separable subspaces of topological groups which are σ-m_3-spaces are stratifiable, and hence monotonically normal.

In Sections 2 and 3 below we give strong restrictions on those spaces, X, whose function space, $C_p(X)$, is a (σ-) m_3-space. In particular, for compact spaces K, if $C_p(K)$ is an m_3-space then K must be separable and scattered.

However, in Sections 4 and 5 we show that there are compact spaces K with $C_p(K)$ a σ-m_1-space but not an m_3-space and other uncountable compact spaces K with $C_p(K)$ an m_1-space. This answers both questions (1) and (2) above in the negative.

The example of a compact K with $C_p(K)$ an m_1-space of Section 4 is such that the locally convex topological vector space $C_p(K)$ is not paracompact (or even normal). For contrast recall that first countable topological groups are metrizable, and so hereditarily paracompact. In Section 5 we show that monotonically normal topological groups are also always hereditarily paracompact, again distinguishing the cases when the m_3-property

arises in the 'standard' ways (first countability and monotone normality) from the general case.

We conclude this introduction by recording two useful lemmas and some convenient notation. The first lemma points to a key difference between the (σ-) m_1-property and the (σ-) m_3-property: for an m_3-space the local pairbases can be 'tidied' to consist of pairs of open sets coming from a specified basis, but it is not true in general that a closure-preserving base for a point in an m_1-space can be 'tidied' to only contain basic open sets. This difference means that the m_3-property is considerably easier to reason about than the m_1-property.

LEMMA 1. *If $\mathcal{P}$ is a (σ-) cushioned local pairbase at a point y in a space Y, and $\mathcal{B}$ is a local base at y, then there is an operator $H : \mathcal{B} \to \mathcal{B}$ such that the collection $\widehat{\mathcal{P}} = \{(H(B), B) : B \in \mathcal{B}\}$ is a (σ-) cushioned local pairbase at y.*

Proof. Fix the space Y and point y of Y. Suppose $\mathcal{P}$ is a cushioned pairbase at y (the σ-cushioned case is similar). For each B in $\mathcal{B}$ pick $(P_1^B, P_2^B) \in \mathcal{P}$ such that $P_1^B \subseteq P_2^B \subseteq B$ and pick $H(B)$ in $\mathcal{B}$ such that $H(B) \subseteq P_1^B$.

Let $\widehat{\mathcal{P}} = \{(H(B), B) : B \in \mathcal{B}\}$. Then $\widehat{\mathcal{P}}$ is clearly a local pairbase at y, and it is cushioned: if $\{(H(B^\lambda), B^\lambda) : \lambda \in \Lambda\} \subseteq \widehat{\mathcal{P}}$ then

$$\overline{\bigcup\{H(B^\lambda) : \lambda \in \Lambda\}} \subseteq \overline{\bigcup\{P_1^{B_\lambda} : \lambda \in \Lambda\}} \subseteq \bigcup\{\overline{P_2^{B_\lambda}} : \lambda \in \Lambda\}$$
$$\subseteq \bigcup\{\overline{B^\lambda} : \lambda \in \Lambda\}. \quad \blacksquare$$

The second lemma simplifies the task of showing that a function space is an m_i-space or a σ-m_i-space, for $i = 1$ or 3.

LEMMA 2. *Let $*$ be a point in a space X. Let $C_p(X; \{*\}) = \{f \in C(X) : f(*) = 0\}$. Then, for $i = 1$ or 3, the function space $C_p(X)$ is an m_i-space if and only if $C_p(X; \{*\})$ is an m_i-space, and is a σ-m_i-space if and only if $C_p(X; \{*\})$ is a σ-m_i-space.*

To see this recall that $C_p(X)$ is naturally homeomorphic to $C_p(X; \{*\}) \times \mathbb{R}$, and observe that each of the (σ-) m_i-properties, for $i = 1$ or 3, is finitely productive.

Some additional notation will be helpful below. For any set Y, let $[Y]^{<\omega}$ be the set of all non-empty finite subsets of Y; and for any family $\mathcal{F}$ of finite subsets of Y, let us call an operator $G : \mathcal{F} \to [Y]^{<\omega}$ an *expander* on $\mathcal{F}$ provided $G(F) \supseteq F$ for all F in $\mathcal{F}$. For any function space $C_p(X)$, let $\mathbf{0}$ be the function which maps X constantly to 0, and $B(f, F, \epsilon) = \{g : g \in C(X)$ and $|f(x) - g(x)| < \epsilon$ for all $x \in F\}$ for any $f \in C(X)$, any finite set $F \subset X$ and $\epsilon > 0$.

2. Restrictions on X when $C_p(X)$ is a σ-m_3-space. The following concepts are related to the σ-m_3 property by Gartside's theorem from [5] (Theorem 3 below). As mentioned in the introduction, a space has calibre $(\omega_1, \omega, \omega)$ if every point-finite family of open sets is countable. A subspace A of a space X is said to be K_1-*embedded* if there is a map k from the subspace topology τA on A to the topology τX on X such that $k(U) \cap A = U$ and $k(U) \cap k(U') \neq \emptyset$ implies that $U \cap U' \neq \emptyset$ for all U, U' in τA. Note that dense subspaces are always K_1-embedded. We omit the definition of κ-metrizable spaces (as we will not need it again), but observe that all Tychonoff cubes are κ-metrizable.

THEOREM 3. *Let Y be a compact κ-metrizable space, and let X be a K_1-embedded subspace of Y which has calibre $(\omega_1, \omega, \omega)$. Then every point of X with a σ-cushioned local pairbase is a point of first countability.*

Since $C_p(X)$ is a dense subspace of $\mathbb{R}^X$, which is homeomorphic to $(0,1)^X$, and $(0,1)^X$ is dense in the κ-metrizable space I^X, we immediately deduce:

COROLLARY 4. *If $C_p(X)$ is a σ-m_3-space and has calibre $(\omega_1, \omega, \omega)$ then X is countable.*

A space X is said to be *functionally countable* if every continuous real-valued function on X has countable image.

PROPOSITION 5. *If $C_p(X)$ is a σ-m_3-space then X is functionally countable.*

Proof. Suppose, for a contradiction, $C_p(X)$ is a σ-m_3-space and $f : X \to \mathbb{R}$ is continuous with uncountable image $M = f(X)$. Then $f^{\#} : C_p(M) \to C_p(X)$ is an embedding where $f^{\#}(g) = g \circ f$. But $C_p(M)$ is now a σ-m_3-space (inherited from $C_p(X)$) and has calibre $(\omega_1, \omega, \omega)$ ($C_p(M)$ is cosmic, as M is separable metric, hence separable). This contradicts Corollary 4. ∎

We now show that if a $C_p(X)$ is a σ-m_3-space and X is uncountable, then X contains many 'supersequences', homeomorphs of $A(\kappa)$, the one-point compactification of an uncountable discrete space of size κ.

PROPOSITION 6. *If $C_p(X)$ is a σ-m_3-space then for every uncountable subset S of X, there is an uncountable $T \subseteq S$ and an $x \in X$ such that for every open U containing x, the set $T \setminus U$ is finite.*

Proof. Let $\mathcal{P} = \bigcup_n \mathcal{P}_n$ be a σ-cushioned local pairbase at $\mathbf{0}$ of basic open sets. Let S be an uncountable subset of X. For each s in S pick $P^s \in \mathcal{P}$ such that $P_2^s \subseteq B(\mathbf{0}, \{s\}, 1/2)$. Write $P_1^s = B(\mathbf{0}, F^s, 1/m_s)$, and note that without loss of generality we can assume $s \in F^s$. Tidying, and applying the Δ-system lemma, we can find an uncountable subset T_0 of S, a finite

subset R of X, and natural numbers m and n such that: the map $t \mapsto F^t$ is injective on T_0; for all t in T_0 the element of the pairbase P^t is in $\mathcal{P}_n$ and $P_1^t = B(\mathbf{0}, F^s, 1/m)$; and the collection $\{F^t : t \in T_0\}$ forms a Δ-system with root R.

Take any open set U containing R. We show that for all but finitely many t in T_0 the point t is in U. If not, then there would be an infinite family $\{F^{t_r} : r \in \mathbb{N}\}$ such that t_r is not in U for every r in $\mathbb{N}$. Pick a continuous function g on X such that g is 1 outside U and zero on R. Then $g(t_r) = 1$ for every r, and so

$$g \notin \overline{B(\mathbf{0}, \{t_r\}, 1/2)} \supset \overline{P}_{2^{t_r}} \qquad \text{for all } r.$$

As $\mathcal{P}_n$ is cushioned it follows that g is not in the closure of $\bigcup_r B(\mathbf{0}, F^{t_r}, 1/m)$. But this last statement is false, giving our desired contradiction. Indeed, take any basic neighborhood $B(g, E, \epsilon)$ of g. Because the sets $F^{t_r} \setminus R$ are pairwise disjoint, there is an r_0 such that $E \cap F^{t_{r_0}} \subseteq R$, and hence $B(g, E, \epsilon) \cap B(\mathbf{0}, F^{t_{r_0}}, 1/m) \neq \emptyset$.

If R were empty, then we could take $U = \emptyset$ and get an immediate contradiction. Hence we can write $R = \{x_1, \ldots, x_k\}$. Now pick disjoint open sets $U_1, \ldots, U_k$ with $x_i \in U_i$ for $1 \leq i \leq k$. Then there is a fixed i and an uncountable subset T of T_0 such that $T \subseteq U_i$. Now $x = x_i$ and T are as required. ∎

It follows that for many spaces X the function space $C_p(X)$ is not a σ-m_3-space. As a specific example, let $L(\omega_1)$ be the space with underlying set $\omega_1 \cup \{*\}$, with topology in which all points in ω_1 are isolated, and basic open neighborhoods of $*$ have the form $\{*\} \cup (\omega_1 \setminus C)$ where C is countable. Then $C_p(L(\omega_1))$ is not a σ-m_3-space.

More generally, if $C_p(X)$ is a σ-m_3-space and X has any closed-hereditary property $\mathcal{P}$ not possessed by $A(\omega_1)$, then X is countable. For example, for $\mathcal{P}$ we can take 'countable pseudocharacter'.

It is well known that a compact space is functionally countable if and only if it is scattered. Thus from Proposition 5, we deduce:

COROLLARY 7. *If K is compact and $C_p(K)$ is a σ-m_3-space, then K is scattered.*

Next is a necessary condition for a space X to have $C_p(X)$ a σ-m_1-space via certain standard basic open sets. We prove later (Proposition 12) that this condition is also sufficient. Recall that a subset C of a space X is said to be *functionally closed* if there is a continuous function $f : X \to \mathbb{R}$ such that $C = f^{-1}([a, b])$ for some $a, b \in \mathbb{R}$. Further, if C is functionally closed, then for any $a < b$ in $\mathbb{R}$ there is an f as in the definition.

PROPOSITION 8. *Let X be a space containing a point $*$. If $C_p(X)$ is a σ-m_1-space with a σ-closure-preserving local base $\mathcal{B}$ at $\mathbf{0}$, where, for a fixed*

countable subset Q of $(0, \infty)$, the elements of $\mathcal{B}$ all have the form $B(\mathbf{0}, F, r)$ for some finite subset F of X and r from Q, then there is a family $\mathcal{F} = \bigcup_{s \in \mathbb{N}} \mathcal{F}_s$ cofinal in $([X]^{<\omega}, \subseteq)$ such that for every n and every functionally closed neighborhood, C, of $*$ there is a set E, which is finite and disjoint from C, such that for all $F \in \mathcal{F}_n$ either $F \subseteq C$ or $F \cap E \neq \emptyset$.

Proof. We can suppose that we have $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ a local base of $\mathbf{0}$ such that for each n there is a fixed $r = r_n$ in Q and $m = m_n$, the family $\mathcal{B}_n$ is closure-preserving, and each element B in $\mathcal{B}$ has the form $B = B(\mathbf{0}, F_B, r)$ where $|F_B| = m$.

For each n in $\mathbb{N}$, let $\mathcal{F}_n = \{F_B : B \in \mathcal{B}_n\}$. Let $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$. Since $\mathcal{B}$ is a local base at $\mathbf{0}$, the collection $\mathcal{F}$ is cofinal in the collection of finite subsets of X.

Further, take any $\mathcal{F}_n$, recall that we have a fixed $r = r_n$ from Q, and pick any functionally closed neighborhood C of $*$, say $C = f^{-1}([-r, r])$ where f is in $C(X)$. Then for any $B \in \mathcal{B}_n$, the function f is not in $\overline{B}$ if and only if $F_B \setminus C \neq \emptyset$. Hence there is an open neighborhood of f, say $B(f, E', \epsilon)$, such that $B(f, E', \epsilon)$ is disjoint from $\bigcup\{B : B \in \mathcal{B}_n \text{ and } F_B \setminus C \neq \emptyset\}$. It is easy to verify that if $B(f, E', \epsilon) \cap B(\mathbf{0}, F, r) = \emptyset$ then there exists $x \in (E' \cap F) \setminus C$. Setting $E = E' \setminus C$, we see that $E \cap F \neq \emptyset$ for all $F \in \mathcal{F}_n$ such that $F \setminus C \neq \emptyset$. Thus E and $\mathcal{F}_n$ are as required. $\blacksquare$

A similar proof to that of Proposition 8 yields a necessary condition for a space X to have $C_p(X)$ a σ-m_3-space. The given necessary condition is technical, but we will see later (Proposition 13) that it is also sufficient.

PROPOSITION 9. *Let X be a space containing a point $*$. If $C_p(X)$ is a σ-m_3-space, then $[X]^{<\omega}$ can be written as a union of families $\mathcal{F}_{s,i,j}$ where $s, i, j \in \mathbb{N}$ and every F in $\mathcal{F}_{s,i,j}$ has size no more than j, and there is an expander G on $[X]^{<\omega}$ with the property that for any s, i, j from $\mathbb{N}$ and functionally closed neighborhood $C = f^{-1}[-1/j, 1/j]$ of $*$, there is a set E, finite and disjoint from the closed neighborhood $C' = f^{-1}[-1/i, 1/i]$ of $*$, such that, for all $F \in \mathcal{F}_{s,i,j}$, either $F \subseteq C$ or $G(F) \cap E \neq \emptyset$.*

Proof. Suppose that $C_p(X)$ has a local σ-cushioned pairbase at $\mathbf{0}$. Applying Lemma 1 to the local base $\{B(\mathbf{0}, F, 1/|F|) : F \in [K]^{<\omega}\}$, we can find an expander G on $[X]^{<\omega}$ such that

$$\mathcal{P} = \{(B(\mathbf{0}, G(F), 1/|G(F)|), B(\mathbf{0}, F, 1/|F|)) : F \in [X]^{<\omega}\}$$

is a σ-cushioned pairbase; fix a sequence $\{\mathcal{Q}_s : s \in \mathbb{N}\}$ of cushioned families such that $\mathcal{P} = \bigcup_{s \in \mathbb{N}} \mathcal{Q}_s$. Fix s, i and j in $\mathbb{N}$. Let

$$\mathcal{P}_{s,i,j} = \{P = (B(\mathbf{0}, G(F), 1/i), B(\mathbf{0}, F, 1/j)) : P \in \mathcal{Q}_s\}$$

and

$$\mathcal{F}_{s,i,j} = \{F : (B(\mathbf{0}, G(F), 1/i), B(\mathbf{0}, F, 1/j)) \in \mathcal{P}_{s,i,j}\}.$$

Note that every member F of $\mathcal{F}_{s,i,j}$ has size no more than j. Evidently the union of the families $\mathcal{F}_{s,i,j}$ is $[X]^{<\omega}$. Since $\mathcal{P}_{s,i,j}$ is empty when $i < j$, we suppose $i \geq j$.

Pick any functionally closed neighborhood $C = f^{-1}[-1/j, 1/j]$ of $*$. Let $C' = f^{-1}[-1/i, 1/i]$. Then for any $(P_1, P_2) = (B(\mathbf{0}, G(F), 1/i), B(\mathbf{0}, F, 1/j))$ in $\mathcal{P}_{s,i,j}$, the function f is outside $\bar{P}_2$ if and only if $F \setminus C \neq \emptyset$. Since $\mathcal{P}_{s,i,j}$ is cushioned, there is an open neighborhood of f, say $B(f, E', \epsilon)$, such that $B(f, E', \epsilon)$ is disjoint from $\bigcup\{P_1 : (P_1, P_2) = (B(\mathbf{0}, G(F), 1/i), B(\mathbf{0}, F, 1/j)) \in \mathcal{P}_{s,i,j}$ and $F \setminus C \neq \emptyset\}$. It is easy to see that the set $E = E' \setminus C'$ is such that $E \cap G(F) \neq \emptyset$ provided $F \setminus C \neq \emptyset$. Thus the expander G and families $\mathcal{F}_{s,i,j}$ satisfy the condition in the proposition. ■

3. Restrictions on X when $C_p(X)$ is an m_3-space. We turn now to the properties a space X must possess if the function space $C_p(X)$ is an m_3-space. The key result limits the number of isolated points.

PROPOSITION 10. *If $C_p(X)$ is an m_3-space, then X does not contain an uncountable set of isolated points.*

Proof. Suppose $S \subseteq X$ with $|S| = \omega_1$ is a set of isolated points. Suppose, for a contradiction, $C_p(X)$ is an m_3-space and $\mathcal{P}$ is a cushioned pairbase at $\mathbf{0}$. Without loss of generality

$$\mathcal{P} = \{(B(\mathbf{0}, G(F), 1/|G(F)|), B(\mathbf{0}, F, 1/|F|)) : F \text{ is finite and non-empty}\},$$

where G is an expander on $[X]^{<\omega}$. Below we will find sets F_n contained in S such that $|F_n| = n$, $G(F_n)$ is disjoint from F_r, and F_n disjoint from $G(F_r)$ for all $r < n$.

For each n define a continuous $f_n : X \to [0, 2/|F_n|]$ such that $f_n = 0$ outside F_n, and f_n equals $2/|F_n|$ on F_n. Let $f = \sum_n f_n$. Then:

(1) f is well defined (the sets F_n are disjoint),
(2) f is continuous,
(3) f is not in $\overline{B(\mathbf{0}, F_n, 1/|F_n|)}$ for any n (indeed

$$B(f, F_n, 1/|F_n|) \cap B(\mathbf{0}, F_n, 1/|F_n|) = \emptyset,$$

since $f(x) = f_n(x) = 2/|F_n|$ for every x in F_n), but
(4) f is in $\overline{\bigcup_n B(\mathbf{0}, G(F_n), 1/|G(F_n)|)}$.

To verify (4), take any basic neighborhood $B(f, F, \epsilon)$ of f. Then, as the family $\{F_n : n \in \mathbb{N}\}$ is disjoint, there is an N such that $F \cap F_N = \emptyset$. Pick a continuous g such that $g = f$ on $F \setminus G(F_N)$ and zero on $G(F_N)$. Then $g \in C_p(X)$, $g \in B(\mathbf{0}, G(F_N), 1/|G(F_N)|)$ (since $g = 0$ on $G(F_N)$), $g \in B(f, F \setminus G(F_N), \epsilon)$ (as $g = f$ on $F \setminus G(F_N)$), and $g \in B(f, G(F_N) \cap F, \epsilon)$ (since both g and f are zero on $G(F_N) \cap F$, which is a subset of $X \setminus \bigcup_n F_n$).

Hence

$$g \in B(f, F, \epsilon) \cap B(\mathbf{0}, G(F_N), 1/|G(F_N)|).$$

Thus we have found a subfamily $\{(P_1^n, P_2^n) : n \in \mathbb{N}\}$ of $\mathcal{P}$ and f in $C_p(X)$ such that $f \in \overline{\bigcup_n P_1^n}$ but not in any $\overline{P}_{2^n}$, contradicting the assumption that $\mathcal{P}$ is cushioned.

Call a sequence $(F_1, \dots, F_n)$ of finite subsets of S *good* if $|F_i| = i$, and $G(F_i)$ is disjoint from F_j for all distinct $i, j \leq n$. Inductively construct a 'candidate F_n' family $\mathcal{CF}_n$, sets S_n, T_n, and 'failed F_r' families $\mathcal{FF}_r^n$ for $r \leq n$ such that:

(a) $S_1, \dots, S_n$ and T_n are uncountable, pairwise disjoint subsets of S,
(b) $\mathcal{CF}_n$ is an uncountable collection of pairwise disjoint n-element subsets of S_n,
(c) $\mathcal{FF}_r^n$ is a finite subfamily of $\mathcal{CF}_r$, and
(d) if $F_i \in \mathcal{CF}_i \setminus \bigcup_{j=1}^n \mathcal{FF}_i^j$ for $i = 1, \dots, k < n$ are such that $(F_1, \dots, F_k)$ is good, then for all but finitely many F_{k+1} in $\mathcal{CF}_{k+1} \setminus \bigcup_{j=1}^n \mathcal{FF}_{k+1}^j$ the sequence $(F_1, \dots, F_k, F_{k+1})$ is also good.

If this is true, it is easy to find an infinite good sequence as desired.

To start the induction ($n = 1$), split S into two uncountable and disjoint subsets S_1 and T_1. Take a Δ-system $\mathcal{S}_1$ in the family $\{G(\{s\}) : s \in S_1\}$, say with root R_1, and let $\mathcal{CF}_1 = \{\{s\} : G(\{s\}) \in \mathcal{S}_1\}$. Let $\mathcal{FF}_1^1 = \{F \in \mathcal{CF}_1 : R_1 \cap F \neq \emptyset\}$.

Assume we have constructed the promised sets for some n in $\mathbb{N}$. Split T_n into disjoint uncountable subsets S_{n+1} and T_{n+1}. Choose an uncountable disjoint family $\mathcal{CF}_{n+1}$ of subsets of cardinality $n+1$ in S_{n+1}. We can assume, without loss of generality, that $\{G(B) : B \in \mathcal{CF}_{n+1}\}$ is a Δ-system with root R_{n+1}. Let $\mathcal{FF}_r^{n+1} = \{F \in \mathcal{CF}_r : F \cap R_{n+1} \neq \emptyset\}$. Then everything works—(a), (b), (c) and (d) hold for $n + 1$. ∎

COROLLARY 11. *If K is compact and $C_p(K)$ is an m_3-space, then K is separable.*

To see this, recall that compact scattered spaces have a dense set of isolated points, and apply Corollary 7 and Proposition 10. For example, $C_p(A(\omega_1))$ is not an m_3-space, although it is a σ-m_1-space (see Example 15).

4. Spaces X for which $C_p(X)$ is a σ-m_1-space. In this section we give sufficient conditions for spaces X to have $C_p(X)$ a σ-m_1-space, and derive concrete examples.

PROPOSITION 12. *Let X be a space. Then $C_p(K)$ is a σ-m_1-space, provided there is a point $*$ in X and a family $\mathcal{F} = \bigcup_{s \in \mathbb{N}} \mathcal{F}_s$ cofinal in $([X]^{<\omega}, \subseteq)$ such that for all s in $\mathbb{N}$ and functionally closed neighborhoods C of $*$, there*

is a set E, finite and disjoint from C, such that, for every $F \in \mathcal{F}_s$, either $F \subseteq C$ or $F \cap E \neq \emptyset$.

Proof. According to Lemma 2 it suffices to show $C_p(X; \{*\})$ is a $\sigma\text{-}m_1$-space.

Let $\mathcal{B}_{r,s} = \{B(\mathbf{0}, F, 1/r) : F \in \mathcal{F}_s\}$ for $r \in \mathbb{N}$. Then $\mathcal{B} = \bigcup_{r,s} \mathcal{B}_{r,s}$ is a local base at $\mathbf{0}$. We need to show each $\mathcal{B}_{r,s}$ is closure-preserving.

Suppose $f \notin \overline{B(\mathbf{0}, F_\lambda, 1/r)}$ where $B(\mathbf{0}, F_\lambda, 1/r)$ is from $\mathcal{B}_{r,s}$ for all λ in Λ. As $f \in C_p(X; \{*\})$, the set $C = \{x : |f(x)| \leq 1/r\}$ is a functionally closed neighborhood of $*$. So for every F_λ, we have $F_\lambda \not\subseteq C$. Then the finite set E disjoint from C given by the hypothesis is such that $E \cap F_\lambda \neq \emptyset$ for all λ. For each $x \in E$, let $\epsilon_x = |f(x)| - 1/r > 0$. Let $\epsilon = \min\{\epsilon_x : x \in E\}$.

Then $B(f, E, \epsilon) \cap B(\mathbf{0}, F_\lambda, 1/r) = \emptyset$ for all λ (because if $x \in E \cap F_\lambda \neq \emptyset$ then $(-1/r, 1/r) \cap (f(x) - \epsilon_x, f(x) + \epsilon_x) = \emptyset$). Thus the elements of $\mathcal{B}_{r,s}$ indeed form a closure-preserving family. ∎

PROPOSITION 13. *Let X be a space. Then $C_p(X)$ is a $\sigma\text{-}m_3$-space, provided there exists a point $*$ of X and a family $\mathcal{F} = \bigcup_{s,i,j \in \mathbb{N}} \mathcal{F}_{s,i,j}$ cofinal in $([X]^{<\omega}, \subseteq)$ where every F in $\mathcal{F}_{s,i,j}$ has size no more than j, and an expander G on $\mathcal{F}$ such that for every s, i and j in $\mathbb{N}$ and functionally closed neighborhood $C = f^{-1}[-1/j, 1/j]$ of $*$, there is a finite set E disjoint from $C' = f^{-1}[-1/i, 1/i]$ such that for all $F \in \mathcal{F}_s$ either $F \subseteq C$ or $G(F) \cap E \neq \emptyset$.*

Proof. Again, by Lemma 2 it suffices to show $C_p(X; \{*\})$ is a $\sigma\text{-}m_3$-space.

Let $\mathcal{P}_{s,i,j} = \{(B(\mathbf{0}, G(F), 1/i), B(\mathbf{0}, F, 1/j)) : F \in \mathcal{F}_{s,i,j}\}$. Then $\mathcal{P} = \bigcup_{s,i,j} \mathcal{P}_{s,i,j}$ is a local pairbase at $\mathbf{0}$ (here we use the fact that every F in $\mathcal{F}_{s,i,j}$ has size at most j). We need to show that a fixed $\mathcal{P}_{s,i,j}$ is cushioned.

Suppose that a continuous function f is not in $\overline{B(\mathbf{0}, F_\lambda, 1/j)}$ for each λ in Λ. Let $C = f^{-1}[-1/j, 1/j]$ and $C' = f^{-1}[-1/i, 1/i]$. Since f is continuous and $f(*) = 0$, the sets C and C' are functionally closed neighborhoods of $*$. Note that for every λ in Λ we have $F_\lambda \not\subseteq C$.

Then the finite E disjoint from C' given by the hypothesis is such that $E \cap G(F_\lambda) \neq \emptyset$ for all λ. For each $x \in E$, let $\epsilon_x = |f(x)| - 1/i > 0$. Let $\epsilon = \min\{\epsilon_x : x \in E\}$.

Then $B(f, E, \epsilon) \cap B(\mathbf{0}, G(F_\lambda), 1/i) = \emptyset$ for all λ (because if $x \in E \cap G(F_\lambda) \neq \emptyset$ then $(-1/i, 1/i) \cap (f(x) - \epsilon_x, f(x) + \epsilon_x) = \emptyset$). Thus $\mathcal{P}_{s,i,j}$ is indeed cushioned. ∎

Now we present some examples of spaces X with $C_p(X)$ a $\sigma\text{-}m_1$-space.

For any free filter p on a set X, write $X(p)$ for the space with underlying set $X \cup \{*\}$, and topology where the points of X are isolated and neighborhoods of $*$ are $\{*\} \cup U$ for U in p. The supersequences $A(\kappa)$ and $L(\omega_1)$ are both of this form. For the space $A(\kappa)$, take $X = \kappa$ and p to be the filter generated by the family $\{(\kappa \setminus F) : F \text{ is a finite subset of } \kappa\}$. For the

space $L(\omega_1)$, take $X = \omega_1$ and p to be the filter generated by the family $\{(\omega_1 \setminus C) : C \text{ is a countable subset of } \omega_1\}$.

From Proposition 12 we deduce:

COROLLARY 14. *If X is a set and p is a filter on X, then $C_p(X(p))$ is a σ-m_1-space if there is a family $\mathcal{F} = \bigcup_{s \in \mathbb{N}} \mathcal{F}_s$ cofinal in $([X]^{<\omega}, \subseteq)$ such that for all s in $\mathbb{N}$ and $U \in p$ there is a finite set E disjoint from U such that, for every $F \in \mathcal{F}_s$, either $F \subseteq U$ or $F \cap E \neq \emptyset$.*

EXAMPLE 15. *$C_p(A(\omega_1))$ and $C_p(A(\omega_1) \oplus \omega)$ are σ-m_1-spaces.*

Proof. It is immediate that $A(\omega_1)$ satisfies the conditions of Proposition 12.

We know that $C_p(A(\omega_1) \oplus \omega)$ is homeomorphic to $C_p(A(\omega_1)) \times C_p(\omega)$. Also, $C_p(\omega)$ has a countable base, hence it is a σ-m_1-space. Since the σ-m_1-property is finitely productive, $C_p(A(\omega_1) \oplus \omega)$ is a σ-m_1-space. ∎

EXAMPLE 16. *For the set $X = \omega_1 \times \omega$, denote by p the filter generated by the family $\{(\omega_1 \setminus F) \times \omega : F \text{ is a finite subset of } \omega_1\}$. Then $C_p(X(p))$ is a σ-m_1-space.*

Proof. To see this, for each s in $\mathbb{N}$ let $\mathcal{F}_s = \{F \times [0, s] : F \subseteq \omega_1 \text{ is a finite set}\}$. Then the family $\mathcal{F} = \bigcup_s \{\mathcal{F}_s : s \in \mathbb{N}\}$ is easily seen to be cofinal in $[X]^{<\omega}$. Further, given any $s \in \mathbb{N}$ and a basic element $(\omega_1 \setminus G) \times \omega$ of the filter p, let $E = G \times \{0\}$. It is straightforward to check that this satisfies the conditions of Proposition 12. ∎

The following theorem presents a condition on a compact space K which implies that $C_p(K)$ is a σ-m_1-space. Let K be a compact scattered space. Recall the Cantor–Bendixson process: let $K^{(0)}$ be the set of isolated elements in K, and inductively let $K^{(\beta)}$ be the set of isolated elements in $K \setminus \bigcup_{\gamma < \beta} K^{(\gamma)}$. Also write $K^{(\leq \beta)}$ for $\bigcup_{\gamma \leq \beta} K^{(\gamma)}$ and $K^{(\geq \beta)}$ for $\bigcup_{\gamma \geq \beta} K^{(\gamma)}$. This process terminates for some minimal ordinal $\alpha + 1$, called the *scattered height* of K, when $K^{(\alpha+1)} = \emptyset$.

Note that, since K is compact, the scattered height is not a limit and the 'top level', $K^{(\alpha)}$, is finite. If $K^{(\alpha)} = \{x_1, \ldots, x_n\}$, then there are clopen sets $U_1, \ldots, U_n$ partitioning K so that x_i is in U_i. Thus $C_p(K)$ factors into $C_p(U_1) \times \cdots \times C_p(U_n)$, which is a σ-m_3-space if and only if each $C_p(U_i)$ is a σ-m_3-space. Thus, for compact spaces K, to determine if $C_p(K)$ is, or is not, a σ-m_3-space it is sufficient to only consider compact scattered spaces with exactly one point in the top level.

Observe that the hypothesis about K in Theorem 17 below is the same as saying that K is a compact hereditarily paracompact space of finite scattered height, or equivalently, that K is a compact monotonically normal space of finite scattered height (see [7]).

THEOREM 17. *Given a compact space K of finite scattered height $a+1$, assume that for every $b < a$, in the subspace $K^{(\leq b)}$, there exists a point-finite family $\mathcal{U}_b = \{U_x : x \in K^{(b)}\}$ of open subsets of K such that $x \in U_x \subseteq K^{(\leq b)}$ for all $b < a$ and $x \in K^{(b)}$. Then $C_p(K)$ is a σ-m_1-space.*

Proof. As discussed above we can assume, without loss of generality, that $K^{(a)} = \{*\}$, and then it is sufficient to prove that $C_p(K; \{*\})$ is a σ-m_1-space.

Let $\mathcal{F}$ be the collection of all finite subsets F of K such that for any $x \in K^{(b)}$ and $y \in K^{(b')}$ where $b < b'$, if $x \in U_y \in \mathcal{U}_{b'}$, then $x \in F$ implies that $y \in F$. Since a is finite and each family $\mathcal{U}_b$, for $b < a$, is point-finite, the family $\mathcal{F}$ is cofinal in the collection of all finite subsets of K. For each $n \in \mathbb{N}$, choose a map $\phi_n : \{0, 1, \ldots, a\} \to [1/(n+1), 1/n]$ such that $\phi_n(a) = 1/(n+1)$, $\phi_n(0) = 1/n$ and $\phi_n(b) > \phi_n(b')$ when $b < b'$. Then let $\mathcal{B}_n = \{A(\mathbf{0}, F, \phi_n) : F \in \mathcal{F}\}$, where $A(\mathbf{0}, F, \phi_n) = \bigcap_{0 \leq b \leq a} B(\mathbf{0}, F \cap K^{(b)}, \phi_n(b))$. By cofinality of $\mathcal{F}$ it is easy to see that $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a local base at $\mathbf{0}$. To complete the proof, we will show $\mathcal{B}_n$ is closure-preserving for a fixed n.

Let $f \notin \overline{A(\mathbf{0}, F_\lambda, \phi_n)}$ for each $\lambda \in \Lambda$. For $b = 1, \ldots, a$ set $V_{a-b} = \{x : |f(x)| < \phi_n(a-b)\}$ and $\hat{V}_{a-b} = \{x : |f(x)| \leq \phi_n(a-b)\}$. For $b = 1, \ldots, a$ we will define certain finite subsets E_{a-b} of $K^{(a-b)}$. Then we have open sets $W_{a-b} = \bigcup\{U_x : x \in E_{a-b} \text{ and } U_x \in \mathcal{U}_{a-b}\}$ and $W_{\geq a-b} = \bigcup\{W_{a-j} : j = 1, \ldots, b\}$, a real number $\epsilon_{a-b} = \min\{|f(x)| - \phi_n(a-b) : \text{for } x \in E_{a-b}\}$ and a set $C_{a-b} = B(f, E_{a-b}, \epsilon_{a-b})$. These sets will satisfy, for $b = 1, \ldots, a$:

(1) $K^{(\geq a-b)} \subseteq V_{a-(b+1)} \cup W_{\geq a-b}$,
(2) ϵ_{a-b} is strictly positive (so C_{a-b} is an open neighborhood of f), and
(3) for each $\lambda \in \Lambda$, if

$$f \notin \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-b)}, \phi_n(a-b))} \quad \text{and}$$
$$f \in \bigcap_{1 \leq j \leq b-1} \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-j)}, \phi_n(a-j))},$$

then $C_{a-b} \cap B(\mathbf{0}, F_\lambda \cap K^{(a-b)}, \phi_n(a-b)) = \emptyset$.

Assuming the above sets exist and have the claimed properties we finish the proof that $\mathcal{B}_n$ is closure-preserving as follows. Let $C = \bigcap_{b=1}^{a} C_{a-b}$. Then C is an open neighborhood of f. Since $f \notin \overline{A(\mathbf{0}, F_\lambda, \phi_n)}$, there is a minimal b_0 such that $f \in \bigcap_{1 \leq j < b_0} \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-j)}, \phi_n(a-j))}$ but $f \notin \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-b_0)}, \phi_n(a-b_0))}$ for each $\lambda \in \Lambda$. Then $C_{a-b_0} \cap B(\mathbf{0}, F_\lambda \cap K^{(a-b_0)}, \phi_n(a-b_0)) = \emptyset$ implies that $C \cap A(\mathbf{0}, F_\lambda, \phi_n) = \emptyset$ for each $\lambda \in \Lambda$, as required.

It remains to show that, for $b = 1, \ldots, a$, the sets E_{a-b} exist and have the required characteristics. We do so first when $b = 1$, and then explain how to get from a given b to $b+1$.

To start suppose $b = 1$. Let

$$E_{a-1} = \{x \in K^{(a-1)} : |f(x)| > \phi_n(a-1)\}.$$

Since f is continuous at $*$, the subset E_{a-1} of $K^{(a-1)}$ is finite and $\epsilon_{a-1} > 0$. By definition of C_{a-1}, it follows that for each $\lambda \in \Lambda$, we have $C_{a-1} \cap B(\mathbf{0}, F_\lambda \cap K^{(a-1)}, \phi_n(a-1)) = \emptyset$ if $f \notin \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-1)}, \phi_n(a-1))}$. Since $\phi_n(a-2) > \phi_n(a-1)$, we see that $\hat{V}_{a-1} \subset V_{a-2}$. Therefore, $K^{(\geq a-1)} \subseteq V_{a-2} \cup W_{a-1}$.

Suppose now that we have constructed E_{a-j} (and hence W_{a-j}, $W_{\geq a-j}$, ϵ_{a-j} and C_{a-j}) for all $j = 1, \ldots, b$, where $b < a$, satisfying the given conditions. We construct $E_{a-(b+1)} = E_{a-b-1}$ and verify that the conditions continue to hold. Since K is compact and $K^{(\geq a-b)} \subseteq V_{a-(b+1)} \cup W_{\geq a-b}$, the set $K^{(a-b-1)} \setminus (V_{a-b-1} \cup W_{\geq a-b})$ is finite. Then

$$E_{a-b-1} = \{x \in K^{(a-b-1)} \setminus (V_{a-b-1} \cup W_{\geq a-b}) : |f(x)| > \phi_n(a-b-1)\}$$

is a finite subset of $K^{(a-b-1)}$ and $\epsilon_{a-b-1} > 0$. Since f is continuous and $\phi_n(a-b-2) > \phi_n(a-b-1)$, the set V_{a-b-2} is open and contains $\hat{V}_{a-b-1}$. Therefore $K^{(\geq a-b)} \subseteq V_{a-b-2} \cup W_{\geq a-b}$. By the definition of E_{a-b-1}, we can see that $K^{(a-b-1)} \subseteq \hat{V}_{a-b-1} \cup W_{\geq a-b} \cup W_{a-b-1} \subseteq V_{a-b-2} \cup W_{\geq a-b-1}$. Hence $K^{(\geq a-b-1)} \subseteq V_{a-b-2} \cup W_{\geq a-b-1}$.

Take $\lambda \in \Lambda$. Suppose that $f \notin \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-b-1)}, \phi_n(a-b-1))}$ and

$$(\dagger) \qquad\qquad f \in \bigcap_{1 \leq j \leq b} \overline{B(\mathbf{0}, F_\lambda \cap K^{(a-j)}, \phi_n(a-j))}.$$

Then there exists $x_0 \in F_\lambda \cap K^{(a-b-1)}$ such that $|f(x_0)| > \phi_n(a-b-1)$. Next we will show x_0 is in E_{a-b-1}, which will guarantee that $C_{a-b-1} \cap B(\mathbf{0}, F_\lambda \cap K^{(a-b-1)}, \phi_n(a-b-1)) = \emptyset$. Let $G_{a-j} = \{x : x \in K^{(a-j)}, x_0 \in U_x \text{ and } U_x \in \mathcal{U}_{a-j}\}$ for $j = 1, \ldots, b$. By the definition of F_λ, $G_{a-j} \subseteq F_\lambda$ for $j = 1, \ldots, b$. Because of $(\dagger)$ we have $G_{a-j} \subseteq \hat{V}_{a-j}$, hence $G_{a-j} \cap E_{a-j} = \emptyset$ for $j = 1, \ldots, b$. Therefore, $x_0 \notin (\bigcup\{W_{a-j} : j = 1, \ldots, b\})$. By the definition of E_{a-b-1} clearly $x_0 \in E_{a-b-1}$. ∎

Thus, for example, $C_p(K)$ is a $\sigma\text{-}m_1$-space where K is the one-point compactification of an uncountable disjoint sum of $A(\omega_1)$'s.

Theorem 17 applies to compacta of finite scattered height, but finite height is not a necessary condition.

EXAMPLE 18. *There is a compact space, K, of scattered height ω_1, such that $C_p(K)$ is a $\sigma\text{-}m_1$-space (but not an m_3-space).*

Proof. Let $X = \bigoplus_{\omega \leq \alpha < \omega_1} (\alpha + 1) \times \{\alpha\}$ (here $\alpha + 1$ has its ordinal topology). Let $K = X \cup \{*\}$ be the one-point compactification of X. Note that K is compact and has scattered height ω_1. By Proposition 10, we can

conclude that $C_p(K)$ is not an m_3-space because K has uncountably many isolated points. Next, we show $C_p(K; \{*\})$ is a σ-m_1-space. Since $C_p(K) = C_p(K; \{*\}) \times \mathbb{R}$, we are then done.

Fix, for each countably infinite α, a bijection $\phi_\alpha : \mathbb{N} \to (\alpha + 1) \times \{\alpha\}$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$, where $\mathcal{B}_n = \{B(\mathbf{0}, F, 1/n) : F = \bigcup_{\alpha \in G} \phi_\alpha([1, n]) \times \{\alpha\}$ where $G \subseteq \omega_1 \setminus \omega$ and $|G| = n\}$.

It is easy to see that $\mathcal{B}$ is a local base at $\mathbf{0}$. We check that each $\mathcal{B}_n$ is closure-preserving.

Suppose $f \notin \overline{B(\mathbf{0}, F_\lambda, 1/n)}$ where $B(\mathbf{0}, F_\lambda, 1/n)$ is in $\mathcal{B}_n$ for all $\lambda \in \Lambda$. There is an $x_\lambda \in F_\lambda = \bigcup_{\alpha \in G_\lambda} \phi_\alpha([1, n]) \times \{\alpha\}$ such that $|f(x_\lambda)| > 1/n$. As f is continuous at $*$, and $f(*) = 0$, there is a finite set G contained in $\omega_1 \setminus \omega$ such that $\{x : |f(x)| > 1/n\} \subseteq \bigcup_{\alpha \in G}(\alpha + 1) \times \{\alpha\}$.

Let $E = \bigcup_{\alpha \in G} \phi_\alpha([1, n]) \times \{\alpha\}$, and let $\epsilon = \min\{|f(x)| - 1/n : x \in E$ and $|f(x)| > 1/n\}$. Then for every λ, we see that $B(f, E, \epsilon) \cap B(\mathbf{0}, F_\lambda, 1/n) = \emptyset$ (because $x_\lambda \in E \cap F_\lambda$, and by choice of ϵ), as required. ∎

5. Spaces X for which $C_p(X)$ has the m_1-property. We give sufficient conditions on a space X to ensure that $C_p(X)$ has the m_1-property, and give concrete compact examples.

PROPOSITION 19. *Suppose that X is a space with a countable dense set D of isolated points. Assume that there exists a point $*$ in $X \setminus D$ and a family $\mathcal{F}$ cofinal in $([X]^{<\omega}, \subseteq)$ such that for any closed neighborhood C of the point $*$, we can find a finite $E \subseteq X \setminus C$ such that $E \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ which is not contained in C. Then $C_p(X)$ is an m_1-space.*

Proof. By Lemma 2 it is sufficient to show that $C_p(X; \{*\})$ is an m_1-space. Let ϕ be a bijection between $\mathbb{N}$ and the countable dense set of isolated points D. Let $\mathcal{B} = \{B(\mathbf{0}, \{\phi(i) : 1 \leq i \leq |F|\} \cup F, 1/(2|F|)) : F \in \mathcal{F}\}$. Clearly $\mathcal{B}$ is a local base at $\mathbf{0}$. To complete the argument, we will show that $\mathcal{B}$ is closure-preserving.

To this end, suppose that

$$f \notin \overline{B(\mathbf{0}, \{\phi(i) : 1 \leq i \leq |F_\lambda|\} \cup F_\lambda, 1/(2|F_\lambda|))},$$

where F_λ is in $\mathcal{F}$ for every λ in Λ. Since f is continuous, there exist natural numbers N_1 and N_2 such that $f(\phi(N_1)) > 1/(2N_2)$. Let $N = \max\{N_1, N_2\}$, $\epsilon_N = f(\phi(N_1)) - 1/(2N_2)$ and $U_N = B(f, \{\phi(N_1)\}, \epsilon_N)$. Then we have $B_N \cap B(\mathbf{0}, \{\phi(i) : 1 \leq i \leq |F_\lambda|\} \cup F_\lambda, 1/(2|F_\lambda|)) = \emptyset$ for any λ with $|F_\lambda| \geq N$.

Fix any $1 \leq i < N$. Then $C_i = \{x : |f(x)| \leq 1/(2i)\}$ is a closed neighborhood of $*$. By hypothesis, there is a finite $E_i \subseteq K \setminus C_i$ such that for any $F \in \mathcal{F}$, we have $E_i \cap F \neq \emptyset$ or $F \subseteq C_i$. Let $\epsilon_i = \min\{|f(x) - 1/(2i)| : x \in E_i\}$ and $U_i = B(f, E_i, \epsilon_i)$. Then $U_i \cap B(\mathbf{0}, \{\phi(i) : 1 \leq i \leq |F_\lambda|\} \cup F_\lambda, 1/(2|F_\lambda|)) = \emptyset$ for any λ with $|F_\lambda| = i$.

Let $U = \bigcap_{i=1}^{N} U_i$, which is an open neighborhood of f. We have $U \cap B(\mathbf{0}, \{\phi(i) : 1 \le i \le |F_\lambda|\} \cup F_\lambda, 1/(2|F_\lambda|)) = \emptyset$ for any $\lambda \in \Lambda$. Therefore, $\mathcal{B}$ is closure-preserving. ∎

The following example of an uncountable compact space K such that $C_p(K)$ is an m_1-space answers negatively the questions raised by Dow, Ramírez Martínez and Tkachuk [3], and by Tkachuk [11]. It is based on Mrowka's space Ψ constructed as follows. Let $\mathcal{A}$ be a maximal almost disjoint family on $\mathbb{N} = \{1, 2, \dots\}$. Then $\Psi = \Psi(\mathcal{A})$ is a space with underlying set $\mathbb{N} \cup \mathcal{A}$, in which the points in $\mathbb{N}$ are isolated, and a basic neighborhood of A in $\mathcal{A}$ is $\{A\} \cup (A \setminus F)$ where F is a finite subset of $\mathbb{N}$.

EXAMPLE 20. *Let K be the one-point compactification of Ψ. Then $C_p(K)$ is an m_1-space.*

Proof. Let $*$ be the 'point at infinity' of the one-point compactification of Ψ. Let $\mathcal{F} = \{F : F = G \cup [1, |G|] \text{ for some finite } G \subseteq \mathcal{A}\}$. Let C be a closed neighborhood of $*$. If $K = C$ then there is nothing to prove. Otherwise, we can pick a point $N \in \mathbb{N} \setminus C$ because $\mathbb{N}$ is dense in K. Then the set $\mathcal{A} \setminus C$ is easily seen to be finite so the set $E = (\mathcal{A} \cup [1, N]) \setminus C$ is finite as well. Given any $F \in \mathcal{F}$ with $F \setminus C \ne \emptyset$, if $|F| \ge 2N$ then $[1, N] \subset F$ so that $N \in F \cap E$. If $|F| < 2N$ and $F \cap (\mathcal{A} \setminus C) = \emptyset$ then the non-empty set $F \setminus C$ is contained in $[1, N] \setminus C \subset E$ so $E \cap F \ne \emptyset$. Therefore $E \cap F \ne \emptyset$ for any $F \in \mathcal{F}$ which is not contained in C, i.e., we can apply Proposition 19 to see that $C_p(K)$ is an m_1-space. ∎

The space in Example 20 has finite scattered height; the following example shows that this is not necessary for compact spaces K such that $C_p(K)$ is an m_1-space.

EXAMPLE 21. *There exists a compact space K of scattered height ω_1 such that $C_p(K)$ is an m_1-space.*

Proof. Let $X = \bigoplus_{\omega \le \alpha < \omega_1} (\alpha + 1) \times \{\alpha\}$ (here $\alpha + 1$ has its ordinal topology). Let $K' = X \cup \{*\}$ be the one-point compactification of X. By the theory of compactifications of $\mathbb{N}$ (see [8]), since K' is compact and has weight $\le \aleph_1$, there is a compact space K such that $K = \mathbb{N} \cup K'$, and the points of $\mathbb{N}$ are isolated and $\mathbb{N}$ is dense in K. It suffices to show $C_p(K; \{*\})$ is an m_1-space.

Fix, for each countably infinite ordinal α, a bijection $\phi_\alpha : \mathbb{N} \to (\alpha + 1)$. Let

$$\mathcal{B} = \Big\{ B(\mathbf{0}, F, 1/|F|) : F = [1, |G|] \cup \bigcup_{\alpha \in G} \phi_\alpha([1, |G|]) \times \{\alpha\} \Big\},$$

where G is a finite subset of $\omega_1 \setminus \omega$. Clearly $\mathcal{B}$ is a local base at $\mathbf{0}$. We will show that $\mathcal{B}$ is closure-preserving.

Suppose that $f \notin \overline{B(\mathbf{0}, F_\lambda, 1/|F_\lambda|)}$, where $B(\mathbf{0}, F_\lambda, 1/|F_\lambda|)$ is in $\mathcal{B}$ for all $\lambda \in \Lambda$. For each λ in Λ, there is an $x_\lambda \in F_\lambda = [1, |G_\lambda|] \cup \bigcup_{\alpha \in G_\lambda} \phi_\alpha([1, |G_\lambda|]) \times \{\alpha\}$ such that $|f(x_\lambda)| > 1/|F_\lambda|$.

Pick λ_0 such that $|F_{\lambda_0}|$ is minimal. By continuity of f at x_{λ_0}, and density of $\mathbb{N}$, there is a $k \in \mathbb{N}$ such that $|f(k)| > 1/|F_{\lambda_0}|$. Let $\epsilon_0 = |f(k)| - 1/|F_{\lambda_0}|$ and $E_0 = \{k\}$. Take any λ with $|G_\lambda| \geq k$. Then $k \in [1, |G_\lambda|] \subseteq F_\lambda$. So $B(f, E_0, \epsilon_0) \cap B(\mathbf{0}, F_\lambda, 1/|F_\lambda|) = \emptyset$ (because $k \in E_0 \cap F_\lambda$, and $|F_\lambda| \geq |F_{\lambda_0}|$).

Fix $1 \leq i < k$. Let $E_i' = \{\alpha \in \omega_1 : |f(\beta, \alpha)| > 1/(i + i^2)$ for some (β, α) in $(\alpha + 1) \times \{\alpha\}\}$. Observe that E_i' is finite and let

$$E_i = \Big(\bigcup_{\alpha \in E_i'} \phi_\alpha([1, i]) \times \{\alpha\} \Big) \cup [1, i].$$

Note that $\{x_\lambda : |G_\lambda| = i\} \subseteq E_i$. Define $\epsilon_i = \min\{|f(x)| - 1/(i + i^2) : x \in E_i$ and $|f(x)| > 1/(i + i^2)\}$. Take any λ such that $|G_\lambda| = i$. Then $B(f, E_i, \epsilon_i) \cap B(\mathbf{0}, F_\lambda, 1/|F_\lambda|) = \emptyset$ because $x_\lambda \in E_i \cap F_\lambda$.

Finally let $B = \bigcap_{0 \leq i < k} B(f, E_i, \epsilon_i)$. Then B is an open neighborhood of f disjoint from $B(\mathbf{0}, F_\lambda, 1/|F_\lambda|)$ for every λ in Λ, as required. ∎

6. Impact of the m_1-property in topological groups. First countable topological groups are metrizable, and hence hereditarily paracompact. Further, *separable* subspaces of groups with the σ-m_3-property are stratifiable, and hence hereditarily paracompact. We prove below (Theorem 23) that monotonically normal topological groups are hereditarily paracompact. In contrast, although 'first countable' and 'monotonically normal' both naturally imply the m_3-property, Example 20 shows that a locally convex topological vector space can be an m_1-space but not even normal (see [11, S.390] for a proof of non-normality).

A space X is *Maltsev* if there is a continuous map $M : X^3 \to X$ such that $M(x, y, y) = x = M(y, y, x)$; such a map M is called a *Maltsev operator* for X. Observe that if there is a retraction, r, say, of a topological group G onto a space X, then the map $M(x, y, z) = r(xy^{-1}z)$ is a Maltsev operator. In particular, topological groups are Maltsev spaces.

LEMMA 22. *A stationary subset of an uncountable regular cardinal cannot be K_1-embedded in a Maltsev space.*

Proof. Let S be a stationary subset of an uncountable regular cardinal κ. For α in S, let $\alpha_+ = \min\{\alpha' \in S : \alpha < \alpha'\}$. Write L for the limit points in S, and I for the isolated points. Note that L is stationary.

Let S be a subspace of a Maltsev space X, and let $M : X^3 \to X$ be a Maltsev operator for X. We suppose, for a contradiction, that $k : \tau S \to \tau X$ is a K_1-operator. Define $m : S^3 \to X$ to be $M|(S^3)$. For α in I, let $U_\alpha = m^{-1}(k(\{\alpha\}))$. Then for every α in I:

(1) the set U_α is an open subset of S^3 (because $\{\alpha\}$ is open in S, k is a K_1-operator, and m is continuous),

(2) $\{(\alpha, \beta, \beta) : \beta \in S\} \cup \{(\beta, \beta, \alpha) : \beta \in S\} \subseteq U_\alpha$ (because m is the restriction of the Maltsev operator M), and

(3) $\{U_\alpha : \alpha \in I\}$ is a family of pairwise disjoint sets (since $\{\{\alpha\} : \alpha \in I\}$ is a pairwise disjoint family, and k is a K_1-operator).

From (1), (2), the definition of the topology on S^3, and the Pressing Down Lemma, for each α in I, there is a $\beta_\alpha > \alpha$ such that $\{\alpha\} \times (\beta_\alpha, \kappa)^2 \subseteq U_\alpha$. For each λ in L its successor λ_+ in the set S is isolated and $(\lambda, \lambda, \lambda_+)$ is in U_{λ_+}. So there is a λ_- in S, with $\lambda_- < \lambda$, such that $(\lambda_-, \lambda]^2 \times \{\lambda_+\}$ is contained in U_{λ_+}. By stationarity of L, and the Pressing Down Lemma, there is a cofinal $L' \subseteq L$ and λ_0 in S such that, for all λ in L', $\lambda_- = \lambda_0$.

Fix any $\alpha \in I \cap (\lambda_0, \kappa)$, and λ_1 in L' such that $\lambda_1 > \beta_\alpha$. Set $\alpha' = (\lambda_1)_+$. Then

$$\emptyset \neq \{\alpha\} \times (\beta_\alpha, \lambda_1] \times \{\alpha'\} \subseteq (\{\alpha\} \times (\beta_\alpha, \kappa)^2) \cap ((\lambda_0, \lambda_1]^2 \times \{\alpha'\}) \subseteq U_\alpha \cap U_{\alpha'}.$$

This contradicts (3) above. ∎

THEOREM 23. *Monotonically normal Maltsev spaces are hereditarily paracompact.*

Proof. Every subspace of a monotonically normal Maltsev space X is K_1-embedded (because X is monotonically normal), and so (because X is Maltsev, by Lemma 22) contains no stationary subset of an uncountable regular cardinal. Therefore the claim follows from the Rudin and Balogh theorem [1] characterizing paracompact spaces in the class of monotonically normal ones as those not containing a closed subspace homeomorphic to a stationary subset of an uncountable regular cardinal. ∎

COROLLARY 24. *Every locally compact subspace of a monotonically normal Maltsev space is metrizable.*

Proof. Let A be a subspace of X which is a monotonically normal Maltsev space. First suppose A is compact. Then as X is monotonically normal, the subspace A is K_1-embedded in X, and hence—as a compact K_1-embedded subspace of a Maltsev space—is κ-metrizable (see [10]). But hereditarily normal κ-metrizable spaces are metrizable. Now suppose A is locally compact. We have just seen that A is locally metrizable, while from the preceding result we know A is paracompact. Hence A is metrizable. ∎

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