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Flows near compact invariant sets. Part I

by

Pedro Teixeira (Porto)

In memoriam Vladimir I. Arnol'd

Abstract. It is proved that near a compact, invariant, proper subset of a C^0 flow on a locally compact, connected metric space, at least one, out of twenty eight relevant dynamical phenomena, will necessarily occur. Theorem 1 shows that the connectedness of the phase space implies the existence of a considerably deeper classification of topological flow behaviour in the vicinity of compact invariant sets than that described in the classical theorems of Ura–Kimura and Bhatia. The proposed classification brings to light, in a systematic way, the possibility of occurrence of *orbits of infinite height* arbitrarily near the compact invariant set in question, and this under relatively simple conditions. Singularities of C^∞ vector fields displaying this strange phenomenon occur in every dimension $n \geq 3$ (in this paper, a C^∞ flow on $\mathbb{S}^3$ exhibiting such an equilibrium is constructed). Near periodic orbits, the same phenomenon is observable in every dimension $n \geq 4$. As a corollary to the main result, an elegant characterization of the topological-dynamical Hausdorff structure of the set of all compact minimal sets of the flow is obtained (Theorem 2).

Contents

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1. Introduction. The present work establishes a natural classification of topological behaviour of C^0 flows near arbitrary compact invariant sets K, on locally compact connected metric spaces M (e.g. on 2nd countable, Hausdorff, connected manifolds). It can be seen as a considerably deep generalization of a classical topological-dynamical result of Ura and Kimura [urki] and Bhatia (see e.g. [bhsz, p. 114]), when the hypothesis of connectedness of the phase space is added. Ura–Kimura–Bhatia's Theorem states that if M is a locally compact (but not necessarily connected) metric space and K is as above, then at least one of the following four cases occurs:

I. K is an attractor (i.e. asymptotically stable).
II. K is a repeller (i.e. negatively asymptotically stable).
III. There exist $x, y \in M \setminus K$ such that $\emptyset \neq \alpha(x) \subset K$ and $\emptyset \neq \omega(y) \subset K$.
IV. Given any neighbourhood U of K, $U \setminus K$ contains an (entire) orbit, i.e. K is not an isolated invariant set.

This theorem originated around 1960 [urki], achieving its present form in Bhatia [bha1]. It was preceded by a related, though partially incorrect, result of Zubov [zubo], which stimulated a considerable amount of research. However, its roots are considerably older and may be traced to the first chapter of I. Bendixson's celebrated memoir [bend] (see also [cies]). Later, Ura–Kimura–Bhatia's Theorem was recognized as a fundamental tool in persistence theory, greatly simplifying the deduction of many important results (Hofbauer [hofb]; see also Garay and Hofbauer [gaho] for a recent historic perspective), new applications still being found (see e.g. Freedman, Ruan and Tang [free], Garay and Chua [gach], [gaho]). In the past, some effort was made to improve the classification by Bhatia [bha2], Saito [s1–s6], Ura [ura1, ura2] and others, but although interesting, these results seem quite fragmentary and no general dynamical picture emerges from them.

While valid for very general flows/phase spaces, and despite its importance, the above result has, in our opinion, an obvious serious limitation that hinders the possibility of a natural, substantial deepening of the classification it proposes: since the phase space is not assumed to be connected, a (nonvoid) compact invariant set $Q \subsetneq M$ may be open in M. This makes Q simultaneously an attractor and a repeller, while in fact Q neither attracts nor repels a single point outside itself. Actually as $M \setminus Q$ is closed, sufficiently near but outside Q the flow is vacuous!

Adding the assumption of connectedness of the phase space dramatically improves the possibility of partially describing the "dynamical landscape" in the vicinity of a compact invariant set. Natural considerations lead to the identification of twenty five possible relevant dynamical phenomena that fall under case IV of Ura–Kimura–Bhatia's Theorem. Moreover, all the twenty eight cases are distributed among five groups, two cases belonging to distinct groups being incompatible, i.e. cannot be simultaneously satisfied. A key role in the classification is played by compact invariant sets $\emptyset \neq K \subsetneq M$ that are either attractors or repellers or *isolated from minimal sets and stagnant.* By the latter we mean that for some neighbourhood U of K, $U \setminus K$ contains no minimal set of the flow and in addition condition III above is satisfied. Although the main result of this paper (Theorem 1, Section 4) goes much deeper, a flavour of some of its most important conclusions is given in

COROLLARY 1. *Let M be a locally compact, connected metric space with a C^0 flow and K a compact, invariant, proper subset of M. Then at least one of the following six conditions holds:*

I. *K is an attractor.*
II. *K is a repeller.*
III. *K is isolated from minimal sets and stagnant.*
IV. *There is a nonvoid, compact, connected invariant set $Q \subset \operatorname{bd} K$ and a sequence $\Lambda_n \subset M \setminus K$ of compact minimal sets of the flow such that the following three conditions hold:*

- *(Λ_n) converges to Q in the Hausdorff metric,*
- *all Λ_n's belong to the same one of the following three classes: equilibrium orbits, periodic orbits, compact aperiodic minimal sets,*
- *either all Λ_n's are attractors, or they are all repellers, or they are all isolated from minimal sets and stagnant.*

V. *For each sufficiently small open neighborhood U of K, the compact minimal sets contained in $U \setminus K$ form a nonvoid $\mathfrak{c}$-dense in itself set, i.e. any neighborhood of a compact minimal set $\Lambda \subset U \setminus K$ contains $\mathfrak{c}$ compact minimal sets ($\mathfrak{c}$ denotes the cardinality of the continuum).*
VI. *Orbits of infinite height will necessarily occur arbitrarily near but outside K, more precisely, given any neighborhood U of K, there is a sequence of orbits $\gamma_n \subset U \setminus K$ such that*

$$\operatorname{cl} \gamma_1 \supsetneq \operatorname{cl} \gamma_2 \supsetneq \cdots$$

and $(\operatorname{cl} \gamma_n)$ converges, in the Hausdorff metric, either to a compact, connected, invariant subset of $\operatorname{bd} K$ (and in this case, K is isolated from minimals) or to an isolated compact minimal set contained in $U \setminus K$.

Moreover, conditions **I** *to* **V** *are mutually exclusive. Conditions* **I**, **II** *and* **V** *each exclude* **VI**.

Therefore, if K is neither an attractor nor a repeller and conditions **III** and **IV** also fail, then "super-abundance" of compact minimal sets (case **V**)

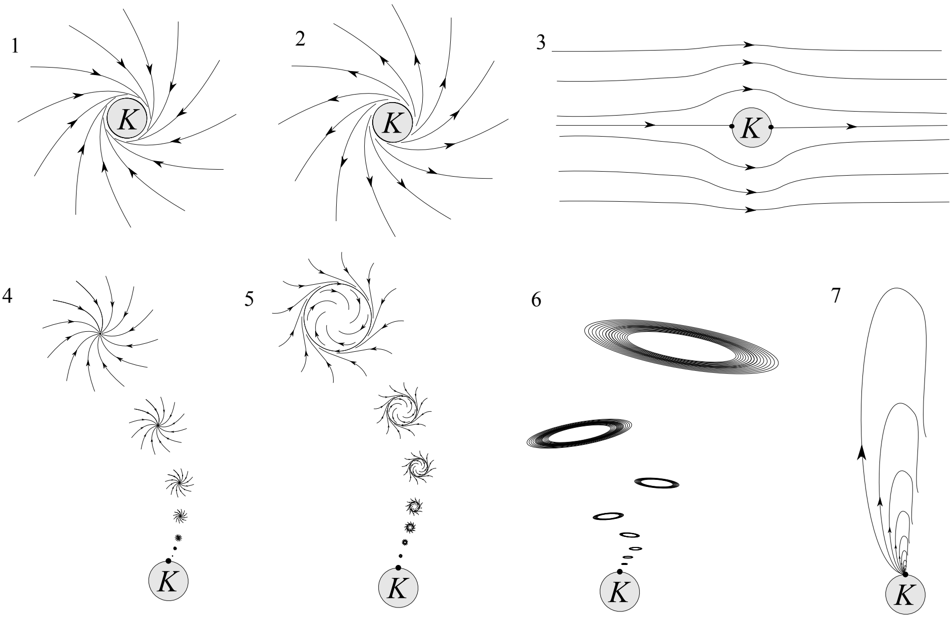

Fig. 1.1. Corollary 1, examples. 1: case **I**; 2: case **II**; 3: case **III**; 4&5: case **IV**; 6: case **V**; 7: case **VI**.

or an outstanding kind of limit behaviour (case **VI**) will emerge arbitrarily near (but outside) K. The possible occurrence of the latter disturbing dynamical phenomenon is not a mere theoretical speculation: in Section 8 a smooth (C^∞) flow on $\mathbb{S}^3$ exhibiting it is given. This is made possible by the existence of a smooth *flow without minimal sets* on a noncompact surface of infinite genus, smoothly embeddable in $\mathbb{R}^3$ (see Beniere and Meigniez [beni] and the pioneer work of Inaba [inab]). In a subsequent paper [teix], we shall actually show that our classification is both pertinent and non-redundant: each of the twenty eight cases it describes admits independent realizations by C^∞ flows on $\mathbb{R}^n$, for all $n \geq m$, where m is some integer ≤ 5 (see Section 8). With the obvious exceptions resulting from the closedness of the set of equilibria, the compact invariant set $K \subsetneq M$ may be taken as an equilibrium orbit or as a periodic orbit in all cases.

Among its many interesting consequences, the classification theorem has remarkable, and somewhat unexpected, implications in the topological-dynamical structure of the set $\mathrm{CMin}(M)$ of all compact minimal sets of the flow, endowed with the Hausdorff metric d_H. As an example, let $\mathfrak{A}$ be the set of compact minimal sets that are either attractors or repellers or isolated from minimal sets and stagnant. Then:

- *If a C^0 flow on a locally compact, connected metric space M has only countably many compact minimal sets and displays no orbits of infinite* *height, then* $\mathfrak{A}$ *is necessarily open dense in* $\mathrm{CMin}(M)$ (*in the Hausdorff metric*).

If the flow displays uncountably many compact minimal sets, then a preliminary result (Lemma 7, Section 6) permits establishing a topological decomposition of $\mathrm{CMin}(M)$, in a sense analogous to that of Cantor–Bendixson's Theorem for Polish spaces (Theorem 3, Section 5):

- *If* $\mathrm{CMin}(M)$ *is uncountable then all but a countable number of compact minimal sets of the flow have* $\mathfrak{c}$ *compact minimal sets in each of their neighbourhoods.*

Several other results are presented in Sections 4 and 5, culminating in a simple characterization of the topological-dynamical structure of the set of all compact minimal sets of a flow (Theorem 2, Section 5).

Despite its topological nature, the greatest interest of Theorem 1 lies in the context of $C^{r\geq 0}$ flows on C^r manifolds (1). Particularly noteworthy is, perhaps, its contribution to the understanding of what can happen, from the dynamical point of view, in two potential "nightmare" phenomena of differentiable dynamics: non-hyperbolic singularities and periodic orbits (see [teix]). To see how hopeless standard analytic methods may be in the study of the former, even in low dimensions, consider, for example, the case of complete smooth vector fields on $\mathbb{R}^2$ having the origin O as an isolated *flat* (2) singularity. It is not difficult to see that there are $\mathfrak{c}$ such vector fields X_i, $i \in \mathbb{R}$, that are pairwise topologically *non*equivalent (3) at O, and whose local topological behaviour at O cannot (with the possible exception of some very general dynamical properties, such as Lyapunov stability) be investigated by standard differential methods ($C^{r\geq 1}$ coordinate changes, blow-up desingularizations, etc.). This shows that already in $\mathbb{R}^2$, there are $\mathfrak{c}$ distinct possible topological smooth flow behaviours near an isolated singularity O, which are practically left in the dark by analytic methods, and in such cases there seems to be no much alternative to what can be learned from the topological-dynamical approach.

Finally, it is perhaps worth mentioning that some natural questions related to Ura–Kimura–Bhatia's Theorem have apparently remained unanswered until now:

(1) Unless otherwise stated, manifolds are always assumed to be smooth (C^∞), 2nd countable, Hausdorff, connected, boundaryless but *not* necessarily compact.

(2) The point O is a *flat* singularity of $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ if this vector field vanishes at O together with its derivatives of all orders.

(3) Two complete vector fields $X, Y \in \mathfrak{X}^\infty(\mathbb{R}^2)$ are *topologically equivalent at* O if there are open neighbourhoods U and V of O and a homeomorphism $\varphi : U \to V$, fixing O and carrying each maximal segment of X-orbit contained in U onto a maximal segment of Y-orbit contained in V, preserving time orientation.

1. In Ura–Kimura–Bhatia's Theorem, can condition IV be replaced by "given any neighbourhood U of K, there is a minimal set contained in $U \setminus K$" ?
2. If K is a compact invariant set isolated from minimal sets, is there necessarily an $x \in M \setminus K$ such that $\emptyset \neq \alpha(x) \subset K$ or $\emptyset \neq \omega(x) \subset K$?
3. Desbrow's conjecture [desb, p. 111]: if K is an unstable compact invariant set isolated from minimals, then there is an $x \in M \setminus K$ such that $\emptyset \neq \alpha(x) \subset K$.

All these questions have a negative answer, though counter-examples are not easy to find [4]. The main difficulty is that a flow failing to satisfy (any) one of these conditions must exhibit rather intricate "fractal-like structures" with respect to orbital limit relations (called K-α *shells*, K-ω *shells* and K-*trees*; see Fig. 1.2 and Section 3 for an accurate description). This difficulty may also, in part, account for the apparent stagnation in which the research around Ura–Kimura–Bhatia's Theorem has fallen along the years. The occurrence of any of the above mentioned three structures directly implies the existence of orbits of infinite height arbitrarily near but outside the compact invariant set K (their detection actually being a partial refinement of condition **VI** of Corollary 1). This is an immediate consequence of Theorem 1 (Section 4), which theoretically forecasts the possibility of occurrence of these strange dynamical phenomena. The existence of smooth flows exhibiting all these beautiful structures in every dimension $n \geq 3$ confirms that prediction. Again, the compact invariant set K can be taken as an equilibrium orbit. Raising the dimension of the phase space, it

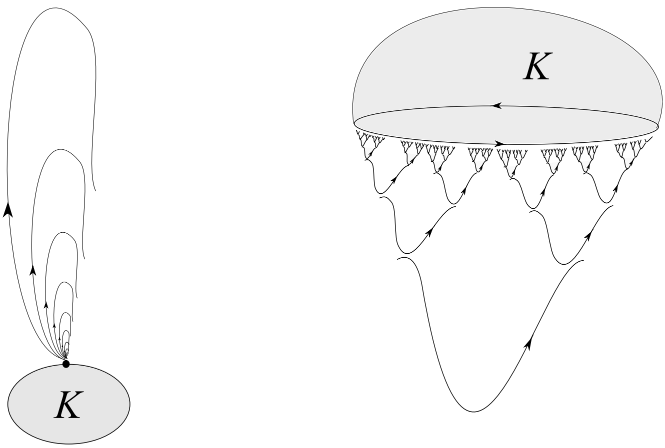

Fig. 1.2. Left: a K-ω shell. Right: a K-tree.

[4] Even if we require M to be a manifold and the flow smooth, the answer to these three questions remains negative (see [teix]).

may also be taken as a periodic orbit or as a compact aperiodic minimal set. Explicit constructions of subsmooth flows exhibiting K-α shells, K-ω shells and K-trees are given in Section 8, Example 1 negatively answering the above three questions. To the best of our knowledge, the only tool so far available for achieving this purpose are the above mentioned smooth flows without minimal sets (on noncompact manifolds), the first examples of which were only discovered near the end of the last century by T. Inaba [inab], answering a twenty years old question of P. Schweitzer [schw, p. 252].

2. Definitions and basic results. Let M be a metric space with a (global) C^0 flow $\theta : \mathbb{R} \times M \to M$ and K (always nonvoid) a compact, invariant, proper subset of M (we denote "the flow θ on M" by (M, θ)). A *minimal set* of (M, θ) is a nonvoid, closed, invariant subset of M that contains no proper subset satisfying these three conditions, i.e., an orbit closure that contains no smaller one. We reserve the term *periodic orbit* for orbits $\mathcal{O}(x)$ for which $\{t \in \mathbb{R} : \theta(t, x) = x\} = \lambda\mathbb{Z}$ for some $\lambda > 0$. In this case, the unique $\lambda > 0$ satisfying that identity is the *period of* $\mathcal{O}(x)$ (of the *periodic point* x). A minimal set that is neither an *equilibrium orbit* ([5]) nor a periodic orbit is called an *aperiodic minimal*, the standard compact example being the linear flows with irrational slope on $\mathbb{T}^2$.

Definition. Let M, θ, K as above and $x \in M$, $X \subset M$. We define

$$\begin{aligned}
\mathcal{N}_X &:= \text{the set of neighbourhoods of } X \text{ in } M,\\
\mathcal{O}(x) &:= \{\theta(t, x) : t \in \mathbb{R}\} = \text{the orbit of } x,\\
\mathcal{O}^+(x) &:= \{\theta(t, x) : t \geq 0\} = \text{the positive (half) orbit of } x,\\
\mathcal{O}(X) &:= \bigcup_{x \in X} \mathcal{O}(x) = \text{the orbital saturation of } X,\\
\mathcal{O}^+(X) &:= \bigcup_{x \in X} \mathcal{O}^+(x) = \text{the positive orbital saturation of } X,\\
\operatorname{Orb}(X) &:= \{\mathcal{O}(x) : \mathcal{O}(x) \subset X\} = \text{the set of orbits contained in } X.
\end{aligned}$$

$\mathcal{O}^-(x)$ and $\mathcal{O}^-(X)$ are the negative concepts corresponding to $\mathcal{O}^+(x)$ and $\mathcal{O}^+(X)$. When dealing with a unique flow θ we write x^t for $\theta(t, x)$ and set

$$\begin{aligned}
\omega(x) &:= \bigcap_{t>0} \operatorname{cl} \mathcal{O}^+(x^t) = \text{the } \omega\text{-limit set of } x,\\
\alpha(x) &:= \bigcap_{t<0} \operatorname{cl} \mathcal{O}^-(x^t) = \text{the } \alpha\text{-limit set of } x.
\end{aligned}$$

([5]) The orbit of an equilibrium point z, i.e. a singleton $\{z\} = \{\theta(t, z) : t \in \mathbb{R}\}$.

For any orbit $\gamma = \mathcal{O}(x)$, we define $\alpha(\gamma) := \alpha(x)$, $\omega(\gamma) := \omega(x)$. Moreover

$$\begin{aligned} B^+(K) &:= \{x \in M : \emptyset \neq \omega(x) \subset K\}, \\ B^-(K) &:= \{x \in M : \emptyset \neq \alpha(x) \subset K\}, \\ A^+(K) &:= \{x \in M : \emptyset \neq \omega(x) \cap K \neq \omega(x)\}, \\ A^-(K) &:= \{x \in M : \emptyset \neq \alpha(x) \cap K \neq \alpha(x)\}, \end{aligned}$$

i.e. $A^+(K)$ (resp. $A^-(K)$) is the set of points of M whose ω-limit (resp. α-limit) set intersects both K and $M \setminus K$. We say K is:

- (*Lyapunov*) *stable* if for any $U \in \mathcal{N}_K$ there is a $V \in \mathcal{N}_K$ such that $\mathcal{O}^+(V) \subset U$;
- *bi-stable with respect to* $N \subset M$ if for any $U \in \mathcal{N}_K$ there is a $V \in \mathcal{N}_K$ such that $\mathcal{O}(N \cap V) \subset U$, i.e. any point $x \in N$ sufficiently close to K has its orbit entirely contained in U;
- an *attractor* if it is stable and $B^+(K) \in \mathcal{N}_K$;
- a *repeller* if it is an attractor in the time-reversed flow $\phi(t,x) = \theta(-t,x)$;
- *stagnant* if there are points $x, y \in M \setminus K$ such that $\emptyset \neq \alpha(x) \subset K$ and $\emptyset \neq \omega(y) \subset K$;
- *isolated from minimal sets* if there is a $U \in \mathcal{N}_K$ such that $U \setminus K$ contains no minimal set of the flow.

(Note that the latter does not exclude the occurrence of minimal sets intersecting (but not completely contained in) $U \setminus K$.) We will use the abridged term *isolated from minimals*. If K is itself a minimal set, then we say that K is an *isolated minimal* (*set*). We define

$$\begin{aligned} \mathrm{C}(X) &:= \text{the set of nonvoid, compact subsets of } X, \\ \mathrm{Ci}(X) &:= \text{the set of nonvoid, compact, invariant subsets of } X, \\ \mathrm{Cc}(X) &:= \text{the set of nonvoid, compact, connected subsets of } X, \\ \mathrm{Cci}(X) &:= \mathrm{Ci}(X) \cap \mathrm{Cc}(X), \\ \mathrm{CMin}(X) &:= \text{the set of compact minimal sets contained in } X, \\ \mathrm{Eq}(X) &:= \text{the set of equilibrium orbits contained in } X, \\ \mathrm{Per}(X) &:= \text{the set of periodic orbits contained in } X, \\ \mathrm{Am}(X) &:= \text{the set of compact aperiodic minimal sets contained in } X. \end{aligned}$$

$\mathrm{C}(M)$ and its subsets are naturally endowed with the Hausdorff metric d_H. To emphasize that this metric is the one in question, we employ the terms d_H-*open/closed*, d_H-*near*, d_H-*converges* ($\xrightarrow{d_H}$), d_H-*isolated*, etc. (6) A set

(6) Metric concepts in $[C(M), d_H]$ are distinguished from the corresponding concepts in $[M, d]$ by the subscript $_H$, e.g., $B_H(X, \epsilon) := \{Y \in \mathrm{C}(M) : d_H(X, Y) < \epsilon\}$; analogously, closure, boundary and interior are denoted by cl_H, bd_H, int_H.

$\mathfrak{A} \subset \mathrm{C}(M)$ *d_H-accumulates in* $\mathfrak{B} \subset \mathrm{C}(M)$ if $(\mathrm{cl}_H \mathfrak{A}) \cap \mathfrak{B} \neq \emptyset$. A sequence $X_n \in \mathrm{C}(M)$ *d_H-accumulates in* $\mathfrak{B} \subset \mathrm{C}(M)$ if it has a subsequence d_H-converging to some $X \in \mathfrak{B}$. Working primarily in the Hausdorff metric, we shall deal essentially with equilibrium orbits rather than with equilibria. Note, however, that the set E of equilibria of the flow, endowed with the metric d of M, is isometric to the metric space $[\mathrm{Eq}(M), d_H]$ via the canonical map $e \mapsto \{e\}$. The following classical result, originally proved by W. Blaschke in the context of convex body theory, is of central importance to the present work:

THEOREM (see e.g. [bura, p. 253]). *If $[N, d]$ is a compact metric space, then so is* $[\mathrm{C}(N), d_H]$.

If N is a compact metric space and $\mathfrak{C}$ is a d_H-closed (and thus compact) subset of $\mathrm{C}(N)$, then the possibility of selecting from a giving sequence $\Lambda_n \in \mathfrak{C}$ a subsequence d_H-converging to some $\Lambda \in \mathfrak{C}$ will be referred to as *Blaschke's Principle.* Again in the metric space M, if $N \subset M$ is compact then the continuity of the flow implies that $\mathrm{Ci}(N)$ is d_H-closed in $\mathrm{C}(N)$ and thus compact; a simple argument shows that $\mathrm{Cc}(N)$ is also compact, hence $\mathrm{Cci}(N) = \mathrm{Ci}(N) \cap \mathrm{Cc}(N)$ is compact. Observe that while $\mathrm{Ci}(N)$, $\mathrm{Cc}(N)$, $\mathrm{Cci}(N)$ and $\mathrm{Eq}(N)$ are d_H-closed in $\mathrm{C}(N)$, thus compact, $\mathrm{CMin}(N)$, $\mathrm{Per}(N)$ and $\mathrm{Am}(N)$ in general are not. Note that $\mathrm{CMin}(N) = \mathrm{Eq}(N) \sqcup \mathrm{Per}(N) \sqcup \mathrm{Am}(N) \subset \mathrm{Cci}(N) \subset \mathrm{C}(N)$ ($\sqcup$ denotes disjoint union).

REMARK. The reader should keep in mind the following basic facts as they will often be implicitly used without mention. Suppose $N \subset M$ is compact. If $\mathcal{O}^+(x) \subset N$ then $\omega(x)$ and $\mathrm{cl}\, \mathcal{O}^+(x) = \mathcal{O}^+(x) \cup \omega(x)$ both belong to $\mathrm{Cci}(N)$ and in particular are nonvoid. The analogous facts hold for $\mathcal{O}^-(x)$, $\alpha(x)$ and $\mathrm{cl}\, \mathcal{O}^-(x)$ when $\mathcal{O}^-(x) \subset N$. Also $\gamma \in \mathrm{Orb}(N)$ implies $\mathrm{cl}\, \gamma = \gamma \cup \alpha(\gamma) \cup \omega(x) \in \mathrm{Cci}(N)$. If N is a nonvoid, compact invariant set then it contains at least one compact minimal set of the flow. If X is a minimal set and K is a closed invariant set, then either $X \subset K$ or $X \subset M \setminus K$, since the set of closed invariant sets is closed under intersections. If $N \subset M$ is invariant then $\mathrm{cl}\, N$, $\mathrm{bd}\, N$ and $\mathrm{int}\, N$ (respectively, the topological closure, boundary and interior of N) are also invariant.

DEFINITION. A set X is *countable* if $\#X \leq \aleph_0 = \#\mathbb{N}$, *denumerable* if $\#X = \aleph_0$, *uncountable* if $\#X > \aleph_0$. Moreover, $\mathfrak{c} = 2^{\aleph_0} = \#\mathbb{R}$ is the cardinality of the continuum.

DEFINITION. A set $\mathfrak{C} \subset \mathrm{C}(M)$ is *d_H-dense in itself* if every $\Lambda \in \mathfrak{C}$ is *non-d_H-isolated* in $\mathfrak{C}$, i.e. $\Lambda \in \mathrm{cl}_H(\mathfrak{C} \setminus \{\Lambda\})$ for all $\Lambda \in \mathfrak{C}$. A set $\Lambda \in \mathrm{C}(M)$ is a $\mathfrak{c}$-*condensation element* of $\mathfrak{C}$ if for every $\epsilon > 0$,

$$\#(B_H(\Lambda, \epsilon) \cap \mathfrak{C}) = \mathfrak{c}.$$

A set $\mathfrak{C} \subset \mathrm{C}(M)$ is $\mathfrak{c}$-*dense in itself* if every $\Lambda \in \mathfrak{C}$ is a $\mathfrak{c}$-condensation element of $\mathfrak{C}$.

(Note that in this paper, unless the context suggests otherwise, the term "$\mathfrak{c}$-dense in itself" always refers to the Hausdorff metric d_H, and the same applies to "$\mathfrak{c}$-condensation element").

Remark. If M is a locally compact, connected metric space, then M is necessarily separable (see e.g. [koba, p. 269]) and thus has at most $\mathfrak{c}$ points [levy, p. 223]. Therefore there are at most $\mathfrak{c}$ orbits in the flow (M,θ), and also at most $\mathfrak{c}$ minimal sets (distinct minimal sets are disjoint), thus if $\Lambda \in \mathfrak{C} \subset \mathrm{CMin}(M)$ then

$$\#(B_H(\Lambda,\epsilon) \cap \mathfrak{C}) \leq \mathfrak{c}.$$

In Section 6 (Corollary 9) we shall actually see that a set of compact minimal sets $\mathfrak{C} \subset \mathrm{CMin}(M)$ is $\mathfrak{c}$-dense in itself iff every neighbourhood $U \subset M$ of each $\Lambda \in \mathfrak{C}$ contains $\mathfrak{c}$ elements of $\mathfrak{C}$, showing that in this particularly important case, we may actually think in terms of the simpler metric d of M, instead of the Hausdorff metric d_H of $\mathrm{C}(M)$.

Definition. For each $\mathfrak{C} \subset 2^M$ (= the set of all subsets of M) and $A \subset M$, we set

$$\mathfrak{C}^* := \bigcup \mathfrak{C} = \bigcup_{\Gamma \in \mathfrak{C}} \Gamma, \qquad \mathfrak{C}(A) := \{X \in \mathfrak{C} : X \subset A\}.$$

Given any two nonvoid sets X, $Y \subset M$ and $\epsilon > 0$, we define

$$B(X,\epsilon) := \bigcup_{x \in X} B(x,\epsilon), \qquad B[X,\epsilon] := \bigcup_{x \in X} B[x,\epsilon],$$
$$|Y|_X := \sup\{\mathrm{dist}(y,X) : y \in Y\} \in [0,+\infty].$$

3. Special orbital structures. We will introduce three kinds of "orbital structures": X-*trees*, X-α *shells* and X-ω *shells.* The reason for considering these denumerable collections of orbits is that they capture essential features of the "dynamical complexity" of the flows in which they occur. In particular, their presence implies that arbitrarily near X there are orbits having limit sets of an outstanding kind.

Throughout this section, X is a compact, invariant, proper subset of a C^0 flow on a locally compact metric space M.

3.1. X**-trees.** Let $\mathrm{F} := \{0,1\}$ and $\mathrm{E}_0 := \{0\}$. Define

$$\mathrm{E}_n := \{0\} \times \mathrm{F}^n,\ n \geq 1, \qquad \mathcal{E} := \bigsqcup_{n \geq 0} \mathrm{E}_n, \qquad \mathrm{E}_\infty := \{0\} \times \mathrm{F}^{\mathbb{N}}$$

($\mathcal{E}$ and E_∞ are, respectively, the set of finite and the set of infinite sequences of 0's and 1's with first (left) digit 0). Since no risk of ambiguity arises, commas and brackets are omitted in the representation of both finite and infinite sequences of 0's and 1's, e.g. we write 01 and $00\ldots$ instead of $(0,1)$ and $(0,0,\ldots)$. If $a,b \in \mathcal{E}$, ab represents as usual the element of $\mathcal{E}$ obtained by adjoining b to the right end of a. For each $v \in \mathrm{E}_\infty$ ($v = 0c_1\ldots c_n\ldots$, $c_n \in \{0,1\}$ for all $n \geq 1$) define $v_0 := 0$ and $v_n := 0c_1\ldots c_n$ for all $n \geq 1$.

Definition. If $\gamma,\zeta \in \mathrm{Orb}(M)$, we denote $\zeta \subset \alpha(\gamma)$, $\zeta \subset \omega(\gamma)$ and $\zeta \subset \alpha(\gamma)\cup\omega(\gamma)$ by $\gamma \overset{0}{\succ} \zeta$, $\gamma \overset{1}{\succ} \zeta$ and $\gamma \succ \zeta$, respectively. Note that all these three relations are transitive, and $\gamma \overset{c}{\succ} \zeta$ and $\zeta \succ \xi$ implies $\gamma \overset{c}{\succ} \xi$, for $c \in \{0,1\}$.

Let U be a compact neighbourhood of X. An X-*tree* is a pair (Θ,ψ) where Θ is a collection of orbits contained in $U \setminus X$ and ψ is a surjective map

$$\psi : \mathcal{E} \to \Theta \subset \mathrm{Orb}(U\setminus X), \qquad a \mapsto \gamma_a,$$

such that for any $b \in \mathcal{E}$,

$$\begin{aligned} &\gamma_b \overset{0}{\succ} \gamma_{b0} \quad \text{and} \quad \gamma_{b0} \not\succ \gamma_b, \\ &\gamma_b \overset{1}{\succ} \gamma_{b1} \quad \text{and} \quad \gamma_{b1} \not\succ \gamma_b, \end{aligned} \tag{3.1}$$

and for every $v \in \mathrm{E}_\infty$,

$$|\mathrm{cl}\,\gamma_{v_n}|_X \to 0. \tag{3.2}$$

γ_0 is called the *first orbit* of the X-tree (see Fig. 3.1). Observe that (3.1) implies (because of the transitivity of $\succ$) that for every $v \in \mathrm{E}_\infty$, the sequence (γ_{v_n}) is injective, i.e. the γ_{v_n}'s are distinct and therefore Θ is denumerable

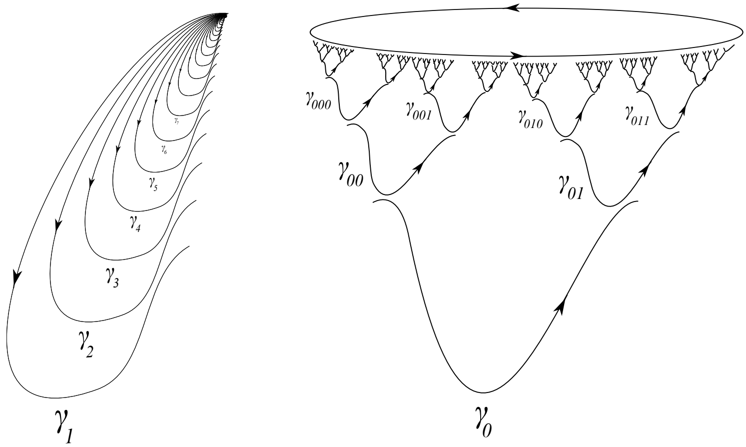

Fig. 3.1. Left: an X-ω shell with X an equilibrium orbit. Time-reversing the flow, an X-α shell is obtained. Right: an X-tree with X a periodic orbit.

(since $\mathcal{E}$ is). X-trees have significant dynamical properties, some of which we single out:

(i) Every $z \in \gamma \in \Theta$ belongs to $A^-(X) \cap A^+(X)$.

(ii) For each $v \in \mathrm{E}_\infty$ ($v = 0c_1 \dots c_n \dots$, $c_n \in \{0,1\}$ for all $n \geq 1$),

$$\gamma_{v_0} \overset{c_1}{\succ} \gamma_{v_1} \overset{c_2}{\succ} \cdots \tag{3.3}$$

and

$$q > p \;\Rightarrow\; \gamma_{v_q} \not\succ \gamma_{v_p}, \tag{3.4}$$

thus

$$\mathrm{cl}\,\gamma_{v_n} \supsetneq \mathrm{cl}\,\gamma_{v_{n+1}} \quad \text{for all } n \geq 0. \tag{3.5}$$

(iii) We have

$$\mathrm{cl}\,\gamma_{v_n} \xrightarrow{d_H} \Lambda_v := \bigcap_{n\geq 1} \mathrm{cl}\,\gamma_{v_n} \in \mathrm{Cci}(X). \tag{3.6}$$

Proof. (i) If $z \in \gamma_b \in \Theta$, $b \in \mathcal{E}$ then $\gamma_b \overset{0}{\succ} \gamma_{b0}$, $\gamma_b \overset{1}{\succ} \gamma_{b1}$ where $\gamma_{b0}, \gamma_{b1} \in \Theta \subset \mathrm{Orb}(U \setminus X)$, hence both the α-limit and ω-limit sets of z have points outside X. On the other hand, letting $k_n := \{0\}^n \in \mathrm{F}^n$ and $l_n := \{1\}^n \in \mathrm{F}^n$ it follows immediately from (3.1) that $\gamma_b \overset{0}{\succ} \gamma_{bk_n}$ and $\gamma_b \overset{1}{\succ} \gamma_{bl_n}$ for all $n \geq 1$; also (3.2) implies that both $|\gamma_{bk_n}|_X$ and $|\gamma_{bl_n}|_X$ tend to zero as $n \to \infty$, thus both the α-limit and ω-limit sets of z intersect X, since these two sets are closed.

(ii) (3.3) is trivial; (3.4) and (3.5) follow from (3.1) because $\succ$ is transitive.

(iii) $\mathrm{cl}\,\gamma_{v_n} \in \mathrm{Cci}(U)$ and $\mathrm{cl}\,\gamma_{v_{n+1}} \subset \mathrm{cl}\,\gamma_{v_n}$ for all $n \geq 0$, therefore by Lemma 8.1 (Section 6), $\mathrm{cl}\,\gamma_{v_n} \xrightarrow{d_H} \Lambda_v \in \mathrm{Cci}(U)$ since $\mathrm{Cci}(U)$ is compact (recall that $U \in \mathcal{N}_X$ is compact); on the other hand, $|\mathrm{cl}\,\gamma_{v_n}|_X \to 0$, hence $\Lambda_v \subset X$ and finally $\Lambda_v \in \mathrm{Cci}(X)$. ∎

Observe that if (Θ, ψ) is an X-tree, then given any $a \in \mathcal{E}$, letting

$$\Upsilon = \{\gamma_d : d = a \text{ or } d = ab,\ b \in \mathrm{F}^n,\ n \geq 1\}$$

and defining the surjective map

$$\begin{aligned} &\phi : \mathcal{E} \to \Upsilon, \\ &\quad 0 \mapsto \zeta_0 := \gamma_a = \psi(a), \\ &\quad 0b \mapsto \zeta_{0b} := \gamma_{ab} = \psi(ab) \text{ for each } b \in \bigsqcup_{n\geq 1} \mathrm{F}^n, \end{aligned}$$

we get an X-tree with first orbit γ_a, whose orbits are contained in Θ. We call (Υ, ϕ) a *sub-X-tree* of (Θ, ψ) and commit a safe abuse of language saying that (Υ, ϕ) is contained in (Θ, ψ). Note that $|\zeta_d|_X \leq |\mathrm{cl}\,\zeta_0|_X = |\zeta_0|_X = |\gamma_a|_X$

for all $d \in \mathcal{E}$, since $\zeta_d \subset \zeta_0 \cup \alpha(\zeta_0) \cup \omega(\zeta_0) = \operatorname{cl} \zeta_0$. Therefore, in virtue of (3.2), given an X-tree (Θ, ψ) and an $\epsilon > 0$, there is always a sub-X-tree of (Θ, ψ) with all its orbits contained in $B(X, \epsilon) \setminus X$.

3.2. X-α shells and X-ω shells. We will define X-ω shells; X-α shells are the time symmetric concept, more precisely, a sequence $(\gamma_n)_{n\geq 1}$ of orbits is an X-α *shell* if it is an X-ω shell in the time-reversed flow $\phi(t,x) = \theta(-t,x)$.

Let U be a compact neighbourhood of X. An X-ω *shell* is a sequence of orbits $\gamma_n \subset U \setminus X$ satisfying the following three conditions:

- $\gamma_n \subset B^-(X)$ for every $n \geq 1$,
- $\gamma_n \overset{1}{\succ} \gamma_{n+1}$ and $\gamma_{n+1} \not\succ \gamma_n$, for all $n \geq 1$,
- $|\operatorname{cl} \gamma_n|_X \to 0$.

These imply ([7]) that $\gamma_n \subset A^+(X)$ for every $n \geq 1$ and hence

$$\gamma_n \subset B^-(X) \cap A^+(X) \quad \text{for every } n \geq 1.$$

Also, the sequence (γ_n) is necessarily injective, i.e. the γ_n's are distinct (see Fig. 3.1). Again, as in the case of X-trees, it is easily seen that

$$\begin{aligned}
&\gamma_1 \overset{1}{\succ} \gamma_2 \overset{1}{\succ} \cdots, \\
&q > p \;\Rightarrow\; \gamma_q \not\succ \gamma_p, \\
&\operatorname{cl} \gamma_n \supsetneq \operatorname{cl} \gamma_{n+1} \quad \text{for all } n \geq 1, \\
&\operatorname{cl} \gamma_n \xrightarrow{d_H} \Lambda := \bigcap_{n\geq 1} \operatorname{cl} \gamma_n \in \operatorname{Cci}(X).
\end{aligned}$$

X-α shells have exactly the same properties, on interchanging α with ω, $+$ with $-$ and changing $\overset{1}{\succ}$ to $\overset{0}{\succ}$ everywhere. Obviously, if $(\gamma_n)_{n\geq 1}$ is an X-ω shell then any subsequence $(\gamma_{n_i})_{i\geq 1}$ is also an X-ω shell and we call it a *sub-X-ω shell* of $(\gamma_n)_{n\geq 1}$. Therefore, since $|\operatorname{cl} \gamma_n|_X \to 0$, given any $\epsilon > 0$, an X-ω shell always has a sub-X-ω shell with all its orbits contained in $B(X, \epsilon) \setminus X$. The analogous fact holds for X-α shells.

4. The main theorem. Corollaries. Let M be a locally compact, connected metric space with a C^0 flow. Consider the following six propositions where the variable X assumes values in the set $\operatorname{Ci}(M)$ of nonvoid, compact, invariant subsets of M:

1.X. *X is an attractor.*

2.X. *X is a repeller.*

([7]) Clearly $\gamma_n \overset{1}{\succ} \gamma_m$ for every $1 \leq n < m$, thus $\omega(\gamma_n) \cap X \neq \emptyset$ since $|\operatorname{cl} \gamma_m|_X \to 0$ and $\omega(\gamma_n)$ is closed. On the other hand, $\omega(\gamma_n) \not\subset X$ because $\gamma_{n+1} \subset \omega(\gamma_n)$ and $\gamma_{n+1} \subset M \setminus X$. Hence $\gamma_n \subset A^+(X)$.

3.X. *X is isolated from minimals and stagnant.*
4.X. *X is isolated from minimals and there is an X-α shell.*
5.X. *X is isolated from minimals and there is an X-ω shell.*
6.X. *X is isolated from minimals and there is an X-tree.*

Observe that, by the connectedness of M, if a proper subset $X \in \mathrm{Ci}(M)$ satisfies condition 1.X or 2.X, then it satisfies none of the remaining five conditions: if 1.X (resp. 2.X) holds, then the remaining five conditions contradict the stability (resp. negative stability, i.e. stability in the time-reversed flow) of X. *Isolated from minimals and stagnant* compact, invariant sets play an important role in the present work. In differentiable dynamics, typical, dynamically distinct examples are given by hyperbolic saddle, fake saddle and saddle-node equilibrium orbits and periodic orbits. Another instructive example is given by the orbit of the equilibrium $(0, \ldots, 0, 1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ in the compactification of the flow on $\mathbb{S}^n \setminus \{(0, \ldots, 0, 1)\}$ induced, via the inverse stereographic projection, by the constant vector field $\partial/\partial x_1$ on $\mathbb{R}^n$ (see Fig. 4.1, centre, for the case $n = 2$). A more subtle example is given by the unique (exceptional) minimal set of Denjoy's celebrated C^1 flow on $\mathbb{T}^2$.

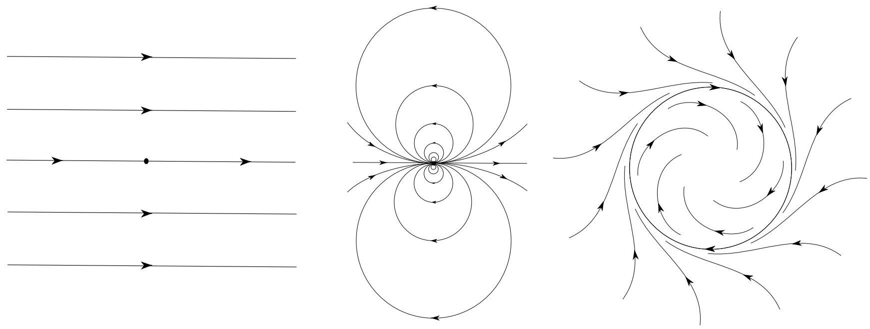

Fig. 4.1. Examples of isolated from minimals and stagnant compact invariant sets on $\mathbb{S}^2$. Left: fake saddle equilibrium orbit. Right: periodic orbit attracting on one side and repelling on the other (periodic orbit of saddle-node type).

Theorem 1. *Let M be a locally compact, connected metric space with a C^0 flow θ and K a compact, invariant proper subset of M. Then either*

1. *K is an attractor;*

or

2. *K is a repeller;*

or at least one of the following four conditions holds:

3. *K is isolated from minimals and stagnant;*
4. *K is isolated from minimals and there is a K-α shell;*

5. *K is isolated from minimals and there is a K-ω shell;*
6. *K is isolated from minimals and there is a K-tree;*

or at least one of the following eighteen conditions holds:

7.i $(1 \le i \le 6)$. *There is a sequence of equilibrium orbits $\{e_n\}$ contained in $M \setminus K$, d_H-converging to some equilibrium orbit contained in* $\operatorname{bd} K$*, such that condition $i.X$ is satisfied by all e_n;*
8.i $(1 \le i \le 6)$. *There is a sequence of periodic orbits $\gamma_n \subset M \setminus K$, d_H-converging to some (nonvoid) compact, connected invariant subset of* $\operatorname{bd} K$*, such that condition $i.X$ is satisfied by all γ_n;*
9.i $(1 \le i \le 6)$. *There is a sequence of compact aperiodic minimals $\Gamma_n \subset M \setminus K$, d_H-converging to some (nonvoid) compact, connected invariant subset of* $\operatorname{bd} K$*, such that condition $i.X$ is satisfied by all Γ_n;*

or

10. *There is an open neighbourhood U of K such that* $\operatorname{CMin}(U \setminus K)$ *is $\mathfrak{c}$-dense in itself and at least one of the following four conditions holds:*

10.1. $\operatorname{Eq}(U \setminus K)$ *is $\mathfrak{c}$-dense in itself and d_H-accumulates in* $\operatorname{Eq}(\operatorname{bd} K)$*;*
10.2. $\operatorname{Per}(U \setminus K)$ *is $\mathfrak{c}$-dense in itself and d_H-accumulates in* $\operatorname{Cci}(\operatorname{bd} K)$*;*
10.3. $\operatorname{Am}(U \setminus K)$ *is $\mathfrak{c}$-dense in itself and d_H-accumulates in* $\operatorname{Cci}(\operatorname{bd} K)$*;*
10.4. *There are $\mathfrak{c}$-dense in itself sets $P \subset \operatorname{Per}(U \setminus K)$ and $A \subset \operatorname{Am}(U \setminus K)$, d_H-open in* $\operatorname{Per}(M)$ *and in* $\operatorname{Am}(M)$, *respectively, and such that:*

- *both P and A d_H-accumulate in* $\operatorname{Cci}(\operatorname{bd} K)$,
- *K is bi-stable with respect to $P^* = \bigcup_{\gamma \in P} \gamma$ and $A^* = \bigcup_{\Gamma \in A} \Gamma$,*
- *for any sequence $\gamma_n \in P$,* $\operatorname{dist}(\gamma_n, K) \to 0$ *implies that* $\operatorname{period}(\gamma_n) \to +\infty$.

REMARK. (1) As $\operatorname{Eq}(M)$ is closed in M and since in **10** we may obviously assume that the neighbourhood $U \in \mathcal{N}_K$ has compact closure, denoting by $\mathscr{E}$ the set of equilibria contained in $U \setminus K$, we may give **10.1** the following stronger formulation:

10.1′. $\operatorname{cl} \mathscr{E}$ *is a compact and dense in itself set intersecting* $\operatorname{bd} K$. *Given any $z \in \operatorname{cl} \mathscr{E}$ and $\epsilon > 0$, there is an embedding h of Cantor's ternary set into $B(z, \epsilon) \cap \operatorname{cl} \mathscr{E}$, with $z \in \operatorname{im} h$* (see e.g. [levy, p. 227]).

GALLERY: THE 28 CASES OF THEOREM 1

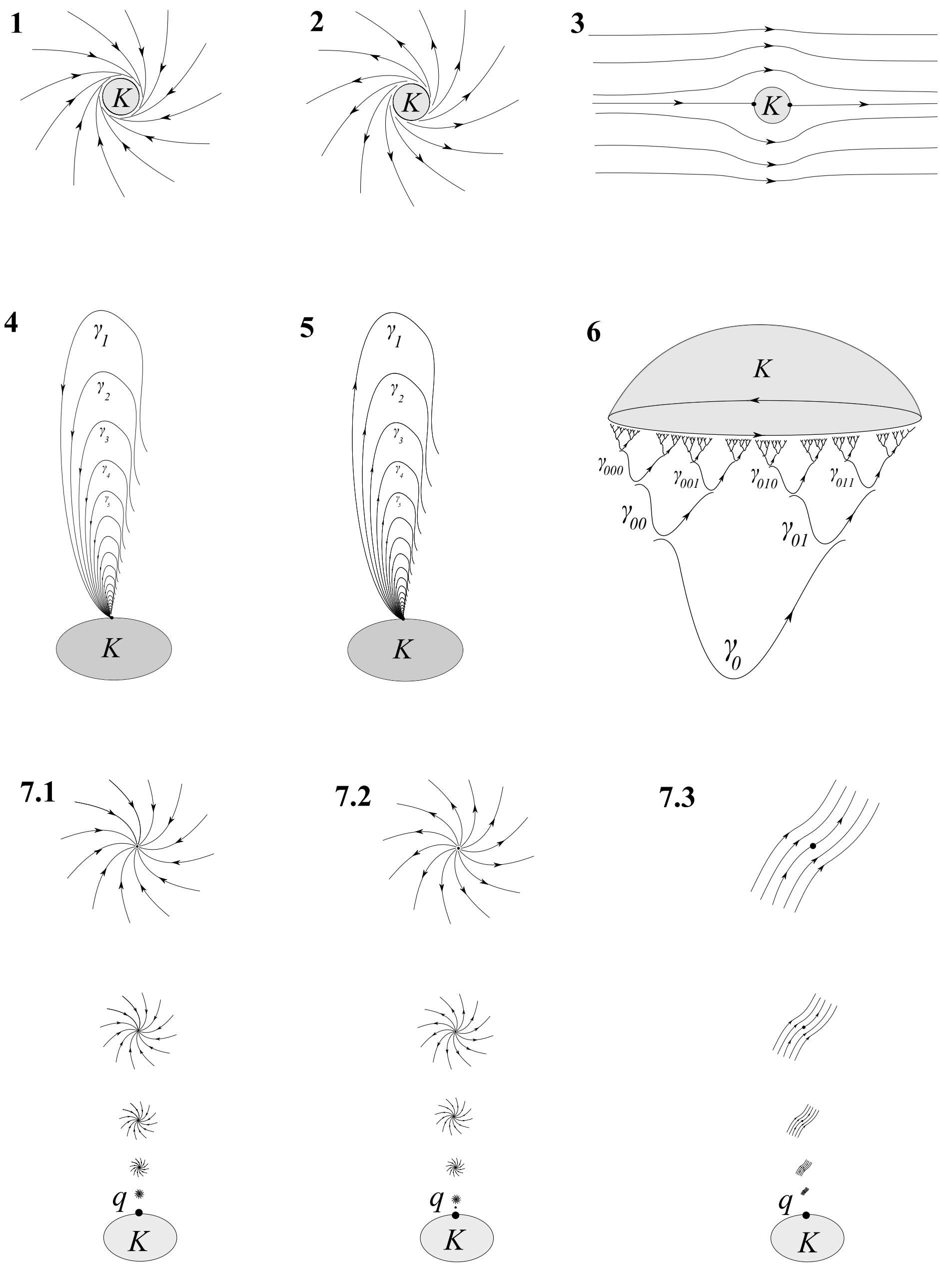

Fig. 4.2. Cases 1 to 7.3

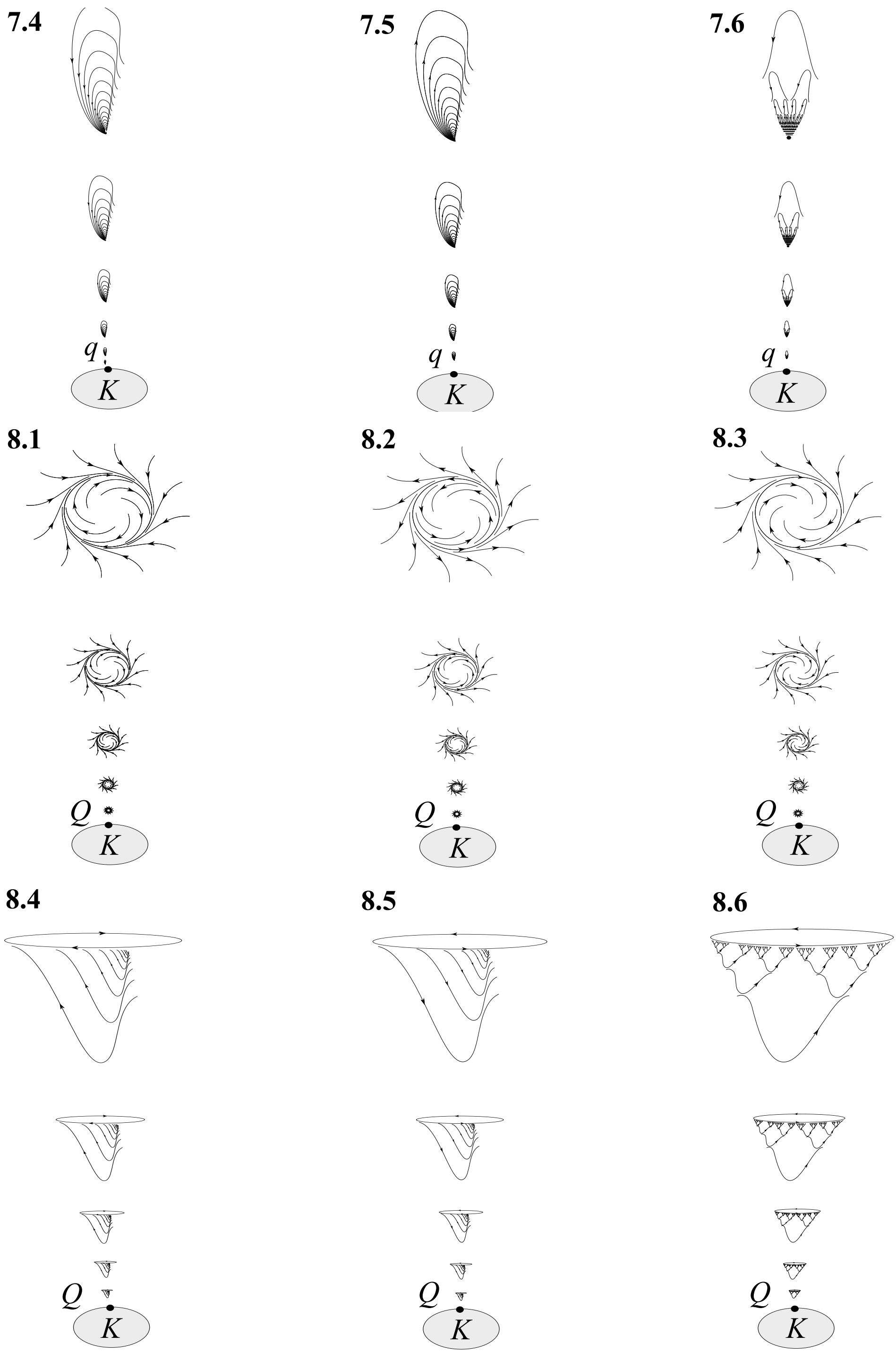

Fig. 4.3. Cases 7.4 to 8.6

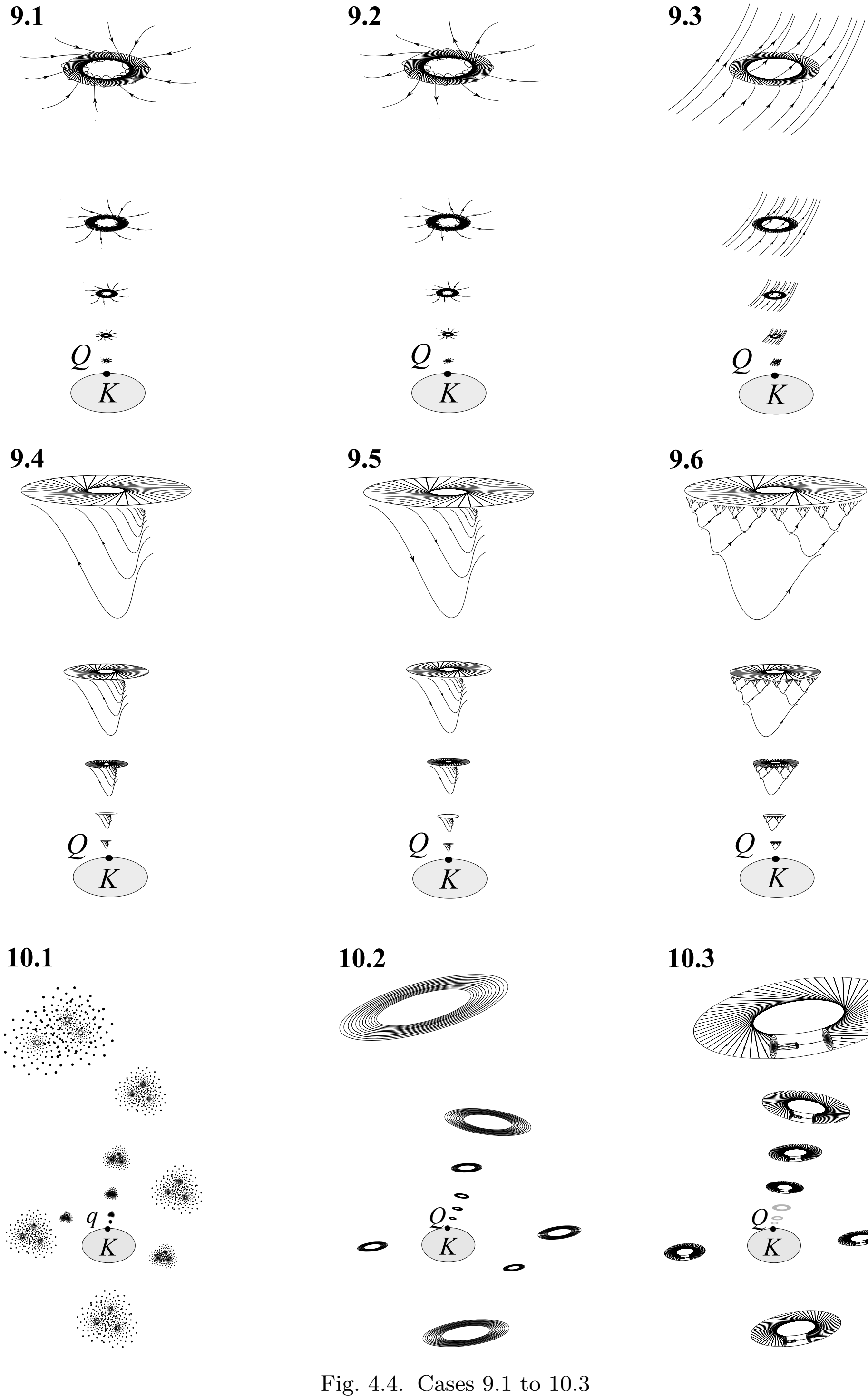

Fig. 4.4. Cases 9.1 to 10.3

10.4

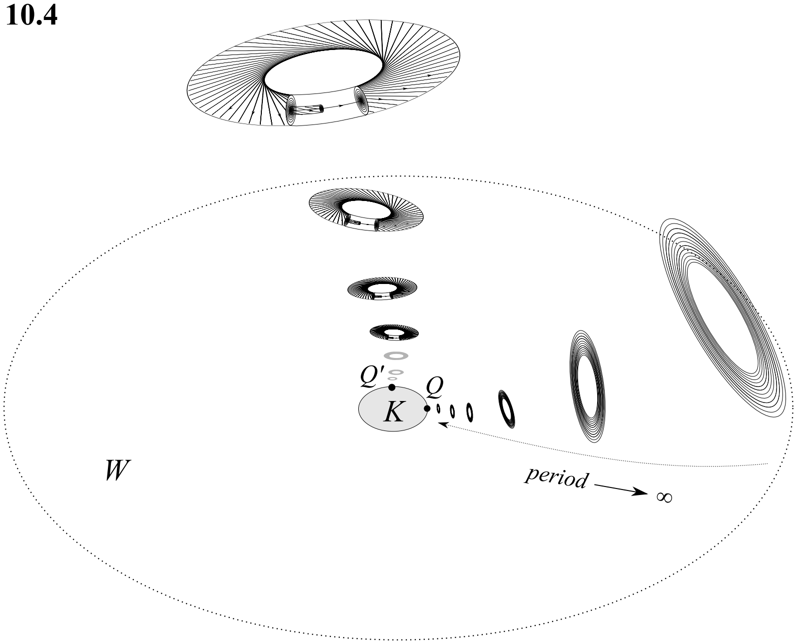

Fig. 4.5. Case 10.4

(2) Observe that $\gamma_n \in P$ and $\mathrm{dist}(\gamma_n, K) \to 0$ together actually imply that $|\gamma_n|_K \to 0$, since P^* is bi-stable with respect to K. On the other hand, by the 1st point of **10.4** there is a sequence $P \ni \gamma_n \xrightarrow{d_H} Q \in \mathrm{Cci}(\mathrm{bd}\, K)$. As P is $\mathfrak{c}$-dense in itself, the 3rd point can thus be replaced by the following condition:

- *given any $n \geq 1$, all periodic orbits $\gamma \in P$ contained in a sufficiently small neighbourhood V of K have period $> n$, and the set of these has cardinality $\mathfrak{c}$.*

Remark. The reader should keep in mind the following two elementary facts about locally compact metric spaces M:

- Any neighbourhood of a compact set $K \subset M$ contains a compact ball $B[K, \delta]$ for some $\delta > 0$.
- Every open/closed subset of M is also locally compact.

We now make a few brief comments concerning the applicability of Theorem 1 in the context of differentiable dynamics and also mention some of its consequences for the topological structure of the set $\mathrm{CMin}(M)$ of all compact minimal sets of the flow, endowed with the Hausdorff metric d_H.

Let M be a (2nd countable, Hausdorff) connected C^m $(0 \leq m \leq \infty)$ manifold (compact or not) with a C^r $(0 \leq r \leq m)$ flow. Due to its topological nature, Theorem 1 gives information not only about the possible behaviour of the flow near each of its compact, invariant, proper subsets (if there are any), but also it illuminates the behaviour of the flow *within* each closed, invariant, connected subset N, provided N contains a compact, invariant proper subset (this is always the case if N is, in addition, a nonminimal compact set). Moreover, if M is noncompact, then M has an *end-points compactification* $M^{\propto} = M \sqcup E(M)$ that, roughly speaking, captures the different possible ways of going to infinity on M. As is well known, besides being compact, $M^{\propto}$ is connected and metrizable (with the inclusion $M \to M^{\propto}$ defining a homeomorphism), and the flow ϕ on M (uniquely) extends to a C^0 flow θ on $M^{\propto}$ (C^r on M), with the end points $e \in E(M)$ becoming equilibria. At each end point $e \in E(M)$, not only the differentiable, but also the topological manifold structure may break (i.e. an end point may not even have a neighbourhood (in $M^{\propto}$) homeomorphic to $\mathbb{R}^n$). However as the extended flow is still continuous at these points, we may apply Theorem 1 to the equilibrium orbit $K = \{e\}$ of each end point $e \in E(M)$, therefore obtaining valuable insight into the possible behaviour of the original flow near each of its "points at infinity".

Theorem 1 has several interesting consequences. Below we give a selection of some simple corollaries. Part II of the present work will be devoted to the investigation of more subtle implications.

Remark. *All compact invariant sets considered are, by hypothesis, nonvoid.* All lemmas invoked in this section and the next are proved in Section 6.

Definition. Let M be a metric space. A set $\mathfrak{E} \subset 2^M$ has elements *arbitrarily near* $X \subset M$ if for any $\epsilon > 0$, $B(X, \epsilon)$ contains an element of $\mathfrak{E}$ (i.e. $\mathfrak{E}(B(X, \epsilon)) \neq \emptyset$). In this case we also say that X has elements of $\mathfrak{E}$ *arbitrarily near.* More restrictively, $\mathfrak{E}$ has elements *arbitrarily near* (*but*) *outside* X if for any $\epsilon > 0$, $\mathfrak{E}(B(X, \epsilon) \setminus X) \neq \emptyset$. We also use the expression that X has elements of $\mathfrak{E}$ *outside arbitrarily near.*

Observe that the last two concepts are defined using the metric of M and should not be confused with d_H-nearness. Also, note that the whole phase space M is always both an attractor and a repeller in any flow, hence from Theorem 1 we get the following immediate consequences (see also Corollary 1 in the introduction, Section 1):

Corollary 2. *Let M be a locally compact, connected metric space with a C^0 flow. Then every compact invariant set, isolated from minimals and having no orbits of infinite height arbitrarily near, is either an attractor or a repeller or stagnant.*

Corollary 3. *Let M be a locally compact, connected metric space with a C^0 flow and K a compact, invariant set, isolated from minimal sets. If K is neither an attractor, nor a repeller, nor stagnant, then orbits of infinite* *height occur arbitrarily near (but) outside K. Actually, there is a K-α shell or a K-ω shell or a K-tree in the flow.*

COROLLARY 4. *Let θ be a C^0 flow on a locally compact, connected metric space having only a finite number of compact minimal sets. Then any compact invariant set that is neither an attractor, nor a repeller, nor stagnant, has orbits of infinite height outside arbitrarily near.*

Again, suppose M is a locally compact, connected metric space with a C^0 flow θ. Let

$$\mathfrak{A} := \{X \in \mathrm{CMin}(M) : X \text{ satisfies one of conditions } 1.X \text{ to } 3.X\},$$

that is, $\mathfrak{A}$ is the set of compact minimal sets of the flow that are either attractors, or repellers, or isolated from minimals and stagnant. The next result shows that if the compact minimal sets belonging to $\mathfrak{A}$ are not d_H-dense in $\mathrm{CMin}(M)$, then "$\mathfrak{c}$-abundance" of minimal sets or orbits of infinite height will occur in the flow. In the above context, we have

COROLLARY 5. *Let M be a locally compact, connected metric space with a C^0 flow. If $\mathfrak{A}$ is not d_H-dense in $\mathrm{CMin}(M)$, then there is a nonvoid, $\mathfrak{c}$-dense in itself, d_H-open subset of $\mathrm{CMin}(M)$ or there are orbits of infinite height arbitrarily near every $Y \in \mathrm{CMin}(M) \setminus \mathrm{cl}_H\, \mathfrak{A} \neq \emptyset$.*

We prove a stronger "local" result. Corollary 5 then follows by letting $A = M$.

COROLLARY 6. *Let M be a locally compact, connected metric space with a C^0 flow and A an open subset of M. If the set*

$$\mathfrak{A}(A) = \{X \in \mathrm{CMin}(A) : X \textit{ satisfies one of conditions } 1.X \textit{ to } 3.X\}$$

is not d_H-dense in $\mathrm{CMin}(A)$, then at least one of the following two situations occurs:

(1) *there is a nonvoid, $\mathfrak{c}$-dense in itself, d_H-open subset of $\mathrm{CMin}(M)$ contained in $\mathrm{CMin}(A)$;*
(2) *there are orbits of infinite height arbitrarily near every set Y in $\mathrm{CMin}(A) \setminus \mathrm{cl}_H\, \mathfrak{A}(A) \neq \emptyset$.*

In particular, if there are only countably many compact minimal sets in A, then case (2) occurs.

Proof. By hypothesis, $A \subset M$ is open and $\varDelta := \mathrm{CMin}(A) \setminus \mathrm{cl}_H\, \mathfrak{A}(A) \neq \emptyset$, hence $\mathrm{CMin}(A)$ is a nonvoid, d_H-open subset of $\mathrm{CMin}(M)$ (Lemma 5, Section 6). The set

$$\varUpsilon := \{Y \in \varDelta : \text{there are orbits of infinite height arbitrarily near } Y\}$$

is clearly a d_H-closed subset of $\varDelta$. Suppose $\varTheta := \varDelta \setminus \varUpsilon \neq \emptyset$, i.e. assume there are compact minimal sets in $\varDelta$ having no orbits of infinite height arbitrarily near. Then Θ is a nonvoid, d_H-open subset of $\mathrm{CMin}(M)$. Let $K \in \Theta$. Since K is compact, there is an open $U \in \mathcal{N}_K$ with compact closure contained in A such that U contains no $Y \in \mathfrak{A}(A)$ (this follows from Lemma 4, as $K \in \mathrm{CMin}(A) \setminus \mathrm{cl}_H \mathfrak{A}(A)$), and also contains no orbit of infinite height (observe that these two facts together also imply that any $\Lambda \in \mathrm{CMin}(U)$ belongs to Θ). Hence none of the 24 cases **1** to **9.6** of Theorem 1 holds, thus by the same result, at least one of the four conditions **10.1** to **10.4** must be valid. But any of these implies the existence of $\mathfrak{c}$ compact minimal sets contained in every $B(K,\delta)$, $\delta > 0$, and thus, by Corollary 9 (Section 6), of $\mathfrak{c}$ compact minimal sets in every $B_H(K,\epsilon)$, $\epsilon > 0$, and for ϵ small enough these are contained in U and thus must belong to Θ. Therefore Θ is $\mathfrak{c}$-dense in itself. ∎

Obviously, every $X \in \mathfrak{A}(M)$ is d_H-isolated in $\mathrm{CMin}(M)$ (and thus $\{X\}$ is d_H-open in $\mathrm{CMin}(M)$), hence in the above context, we have

Corollary 7. *If θ is a C^0 flow on a locally compact, connected metric space M with only countably many compact minimal sets and displaying no orbits of infinite height, then the set $\mathfrak{A}(M)$ is d_H-open dense in* $\mathrm{CMin}(M)$.

Remark. Suppose N is a locally compact, connected metric space endowed with a C^0 flow ϕ, and M a connected, closed invariant subset of N, containing a compact invariant proper subset K. Then Theorem 1 applies to the subflow (M,θ) where $\theta := \phi|_{\mathbb{R}\times M}$, M being endowed with the metric of N. In this context, all definitions must be interpreted "within" (M,θ), i.e. as concerning this subflow (for example a nonvoid, compact invariant set $\Lambda \subset M$ may be an attractor in (M,θ) without being one in (N,ϕ)). The next result shows that if addition the phase space N is locally connected and is separated by the compact invariant set J, then a finer understanding of the flow behaviour near J is possible.

Corollary 8. *Let N be a locally compact, connected and locally connected metric space with a C^0 flow ϕ, and J a compact, invariant proper subset of N. Let D be a connected component of $N \setminus J$. Then Theorem* 1 *applies to $M := \mathrm{cl}\, D$, $\theta := \phi|_{\mathbb{R}\times M}$, $K := M \cap J$.*

Roughly speaking, this result means that *within the closure of each connected component D of $N \setminus J$*, at least one of the 28 phenomena described in Theorem 1 (see Section 6 for the full statement) occurs near the compact invariant set $(\mathrm{cl}\, D) \cap J$ (it being possible that within distinct components, different cases hold).

Proof of Corollary 8. Let D be a (connected) component of $N \setminus J$. Since N is locally connected and $N \setminus J$ is open, D is open in N, hence it cannot be closed as N is connected. On the other hand, D is closed in $N \setminus J$, hence $\emptyset \neq \mathrm{bd}\, D = (\mathrm{cl}\, D) \setminus D \subset J$. The invariance of D now follows from that of $N \setminus J$: the orbit of a point $z \in D$ cannot pass from D to a different component of $N \setminus J$ without intersecting $\operatorname{bd} D \subset J$, and this is impossible since $N \setminus J \supset D$ is invariant. Therefore $M := \operatorname{cl} D$ is a nonvoid, connected, closed (and hence locally compact) invariant subset of N, and $K := (\operatorname{cl} D) \cap J$ is a nonvoid, compact, invariant proper subset of M. Define the (sub)flow $\theta := \phi|_{\mathbb{R}\times M}$. Now endowed with the metric of N, M is a locally compact, connected metric space with a C^0 flow θ, and K is a compact, invariant (under θ) proper subset of M. Theorem 1 can thus be applied to these M, θ and K. ∎

EXAMPLE. Let ϕ be a C^r ($r \geq 0$) flow on $N = \mathbb{S}^n$ and $K \subset N$ an invariant, codimension one, compact, connected C^0 submanifold. As is well known, by the generalized Jordan–Brouwer Separation Theorem [al] ([8]), K separates the flow into three invariant regions, K, B and A, the last two being the connected components of $N \setminus K$, with common boundary K. Besides applying to N, ϕ, K, Theorem 1 also applies to $M = A \sqcup K$, $\theta = \phi|_{\mathbb{R}\times M}$, K, and to $M = B \sqcup K$, $\theta = \phi|_{\mathbb{R}\times M}$, K. Moreover, if K is not a minimal set, then it also applies to the (compact, connected, metric) phase space K, giving, in this case, information about the possible behaviour of the codimension one subflow $\theta = \phi|_{\mathbb{R}\times K}$ near any compact, invariant, proper subset of K (there is at least one). This is always the case if, for example, K is the image of a C^0 embedding $\mathbb{S}^{2m} \hookrightarrow \mathbb{S}^{2m+1}$, $n = 2m+1$ (since such a K must contain an equilibrium, even if ϕ is only C^0; this follows easily from the following corollary to the $\mathbb{S}^{2n}$-hairy ball theorem: a continuous map $\mathbb{S}^{2n} \to \mathbb{S}^{2n}$ that sends no point x to its antipode $-x$, has at least one fixed point).

5. Topological dynamics of $\mathrm{CMin}(M)$**.** Theorem 1 brings to light the importance of compact minimal sets to the characterization of the possible "dynamical landscapes" in the vicinity of a compact invariant proper subset of a flow. Obviously, there is a close relation between the dynamical behaviour of a flow near a compact minimal set X and the topological Hausdorff structure of $\mathrm{CMin}(M)$ near X. Actually, from Theorem 1 we easily obtain an elegant characterization of the set $\mathrm{CMin}(M)$ of all compact minimal sets of the flow, endowed with the Hausdorff metric.

Let M be a locally compact, connected metric space with a C^0 flow. Consider the following seven propositions, where the variable X now takes values in the set $\mathrm{CMin}(M)$ of all compact minimal subsets of the flow:

1.X. *X is an attractor.*
2.X. *X is a repeller.*
3.X. *X is an isolated minimal set and stagnant.*
4.X. *X is an isolated minimal set and there is an X-α shell.*

([8]) Alexander's term "immersed" means C^0-embedded. Recall that [al] is prior to Whitney's foundational papers on the theory of manifolds.

5.X. *X is an isolated minimal set and there is an X-ω shell.*
6.X. *X is an isolated minimal set and there is an X-tree.*
10.X. *There is an $\epsilon > 0$ such that the compact minimal sets contained in $B(X, \epsilon)$ form a $\mathfrak{c}$-dense in itself subset of* $\mathrm{CMin}(M)$.

By Corollary 9 (Section 6), the latter is equivalent to

10′.X. *There is an $\epsilon > 0$ such that every neighbourhood $U \subset M$ of each $Y \in \mathrm{CMin}(B(X, \epsilon))$ contains $\mathfrak{c}$ compact minimal sets.*

If X satisfies 10.X, then as X is itself a compact minimal set, every neighbourhood of X actually contains $\mathfrak{c}$ compact minimal sets.

Remark. Recall that the definition of the Hausdorff metric and Lemma 4 together imply that

- *X is an isolated compact minimal set iff it is d_H-isolated in* $\mathrm{CMin}(M)$.

(By definition (Section 2), X is an isolated minimal set if for some $U \in \mathcal{N}_X$, $U \setminus X$ contains no minimal set of the flow.)

Definition. Denote by $\mathfrak{M}_i$, $1 \le i \le 6$ or $i = 10$, the set of all X in $\mathrm{CMin}(M)$ satisfying condition $i.X$, and by $\mathfrak{M}_{1\text{-}6}$ the set of all $X \in \mathrm{CMin}(M)$ satisfying (at least) one of the six conditions 1.X to 6.X.

Theorem 2. *Let M be a locally compact, connected metric space with a C^0 flow. Then:*

(1) *$\mathfrak{M}_{1\text{-}6}$ is the set of isolated compact minimal sets and thus a countable, d_H-open subset of* $\mathrm{CMin}(M)$.
(2) *$\mathfrak{M}_{10}$ is a d_H-open and $\mathfrak{c}$-dense in itself subset of* $\mathrm{CMin}(M)$. *It is either empty or has cardinality $\mathfrak{c}$.*
(3) *$\mathfrak{M}_{1\text{-}6}$ is d_H-dense in* $\mathrm{CMin}(M) \setminus \mathfrak{M}_{10}$.

Proof. First, note the following trivial fact that will be implicitly used in (3) below: if X is a compact minimal set and Y_n is a sequence of compact minimal sets d_H-converging to $Q \in \mathrm{Cci}(X)$, then $Q = X$.

(1) Clearly every $X \in \mathfrak{M}_{1\text{-}6}$ is an isolated compact minimal set; on the other hand, by Theorem 1, any compact minimal set X satisfying none of the six conditions 1.X to 6.X is *not* an isolated compact minimal set ([9]), hence $\mathfrak{M}_{1\text{-}6}$ is the set of isolated compact minimal sets of the flow. By the remark preceding this theorem, every $X \in \mathfrak{M}_{1\text{-}6}$ is d_H-isolated in $\mathrm{CMin}(M)$, hence $\mathfrak{M}_{1\text{-}6}$ is d_H-open in $\mathrm{CMin}(M)$. Finally, $\mathfrak{M}_{1\text{-}6}$ is countable since it is a d_H-discrete and separable metric space (by Lemma 10, $\mathrm{C}(M) \supset \mathfrak{M}_{1\text{-}6}$ is d_H-separable).

([9]) Note that if $X \in \mathrm{CMin}(M)$ satisfies none of conditions 1.X to 6.X, then X is necessarily a proper subset of M, as M compact implies M is both an attractor and a repeller in the flow, therefore Theorem 1 can be applied.

(2) $10.X$ is clearly a d_H-open property in $\mathrm{CMin}(M)$, hence $\mathfrak{M}_{10}$ is a d_H-open and $\mathfrak{c}$-dense in itself subset of $\mathrm{CMin}(M)$. Since the phase space M is separable, there are at most $\mathfrak{c}$ compact minimal sets in the flow, therefore $\mathfrak{M}_{10}$ is either empty or has cardinality $\mathfrak{c}$, as it is $\mathfrak{c}$-dense in itself.

(3) This follows immediately from Theorem 1 once it is shown that $\mathfrak{M}_{10}$ is the set of minimals of the flow satisfying condition **10** of that theorem. Note that if K is a (compact) minimal set satisfying **10**, then there is a $U \in \mathcal{N}_K$ such that $\mathrm{CMin}(U)$ is $\mathfrak{c}$-dense in itself, implying that $K \in \mathfrak{M}_{10}$. On the other hand, if $K \in \mathfrak{M}_{10}$, then for some $\epsilon > 0$, $B(K, \epsilon)$ contains no isolated minimal set, hence none of the 24 conditions **1** to **9.6** of Theorem 1 holds, thus by the same theorem, K satisfies condition **10**. Therefore, if $X \in \mathrm{CMin}(M) \setminus \mathfrak{M}_{10}$, then X must satisfy (at least) one of conditions **1** to **9.6** of Theorem 1, and this obviously implies that $X \in \mathrm{cl}_H \mathfrak{M}_{1\text{-}6}$. ∎

Note, however, that $10.X$ is indeed a very strong condition, essentially due to its d_H-openness: even when $\mathrm{CMin}(M) \setminus \mathfrak{M}_{1\text{-}6}$ is nonvoid and $\mathfrak{c}$-dense in itself, it can happen that $\mathfrak{M}_{10}$ is empty, since it is still possible that $\mathfrak{M}_{1\text{-}6}$ is d_H-dense in the whole $\mathrm{CMin}(M)$ (simple examples of C^∞ flows exhibiting this phenomenon already occur on $\mathbb{S}^1$ and $\mathbb{S}^2$). However, the next result shows that a nonvoid $\mathfrak{c}$-dense in itself set of compact minimal sets always occurs whenever there are uncountably many compact minimal sets in the flow. More precisely, if $\mathrm{CMin}(M)$ is uncountable, then removing from this set a suitable countable (possibly empty) set we obtain a nonvoid $\mathfrak{c}$-dense in itself set of compact minimal sets. This decomposition theorem is analogous to the celebrated Cantor–Bendixson Theorem for separable, complete metric spaces (Polish spaces). Note, however, that although d_H-separable (since $\mathrm{C}(M) \supset \mathrm{CMin}(M)$ is, by Lemma 10), $\mathrm{CMin}(M)$ is in general neither d_H-complete nor d_H-locally compact. Also observe that since there are at most $\mathfrak{c}$ compact minimal sets in the flow (see Section 2), the above result implies that $\mathrm{CMin}(M)$ obeys, in a sense, the *Continuum Hypothesis*: its cardinality is either finite (possibly null), denumerable ($\aleph_0$), or the continuum $\mathfrak{c} = 2^{\aleph_0}$.

Lemma. *If $\mathfrak{M}$ is a separable metric space, then the set $\mathfrak{I}$ of all points having a countable neighbourhood is countable and open. If $\mathfrak{M}$ is uncountable, then $\mathfrak{M} \setminus \mathfrak{I}$ is dense in itself.*

Proof. By its very definition, $\mathfrak{I}$ has an open cover consisting of countable subsets. Since $\mathfrak{I}$ is separable, that cover has a countable subcover, thus $\mathfrak{I}$ is countable. If $\mathfrak{M}$ is uncountable, then the set $\mathfrak{D} := \mathfrak{M} \setminus \mathfrak{I}$ is dense in itself: obviously $\mathfrak{D}$ is nonvoid (since it is uncountable), and in fact, given any $x \in \mathfrak{D}$ and $\epsilon > 0$, $B(x, \epsilon) \cap \mathfrak{D}$ is uncountable since by definition of $\mathfrak{I}$, $B(x, \epsilon)$ is uncountable and so is $B(x, \epsilon) \cap \mathfrak{D} = B(x, \epsilon) \setminus \mathfrak{I}$ (as $\mathfrak{I}$ is countable). ∎

Theorem 3. *Let θ be a C^0 flow on a locally compact, separable metric space M, displaying uncountably many compact minimal sets. Then there is a countable (possibly empty) set $\mathfrak{I} \subset \mathrm{CMin}(M)$ such that:*

- *$\mathfrak{D} := \mathrm{CMin}(M) \setminus \mathfrak{I}$ is a $\mathfrak{c}$-dense in itself and d_H-closed subset of $\mathrm{CMin}(M)$, of cardinality $\mathfrak{c}$.*
- *$\mathfrak{I}$ is the set of all $X \in \mathrm{CMin}(M)$ having a neighbourhood containing only countably many compact minimal sets (possibly one), hence $\mathfrak{D}$ is the largest $\mathfrak{c}$-dense in itself subset of $\mathrm{CMin}(M)$.*

Therefore, if $\mathrm{CMin}(M)$ is uncountable, then all but a countable number of compact minimal sets of the flow have $\mathfrak{c}$ compact minimal sets in each of their neighbourhoods, or equivalently, $\mathrm{CMin}(M)$ is the union of a countable (possibly empty) set and a $\mathfrak{c}$-dense in itself set. The proof uses, in an essential way, a "Cantor's ternary set-like" construction that constitutes the core of the proof of Lemma 7.

Proof of Theorem 3. Suppose $\mathrm{CMin}(M)$ is uncountable. Let $\mathfrak{I}$ be the set of all $X \in \mathrm{CMin}(M)$ having a countable neighbourhood in the Hausdorff metric. By the previous lemma, $\mathfrak{I}$ is a countable, d_H-open subset and $\mathfrak{D} := \mathrm{CMin}(M) \setminus \mathfrak{I}$ a nonvoid d_H-dense in itself subset of $\mathrm{CMin}(M)$.

Claim. *$\mathfrak{D}$ is $\mathfrak{c}$-dense in itself.*

As $\mathfrak{D} \subset \mathrm{CMin}(M)$, in virtue of Corollary 9 (Section 6), we need only prove that given any $X \in \mathfrak{D}$ and $\epsilon > 0$, there are $\mathfrak{c}$ compact minimal sets $Y \in \mathfrak{D}$ contained in $B(X, \epsilon) \subset M$. Taking ϵ sufficiently small we may assume $B[X, \epsilon]$ is compact (X is compact and M is locally compact). Let $A := B(X, \epsilon)$, $\Lambda_0 := X$ and $\epsilon_0 := \epsilon/2$. Now since $\mathfrak{D}$ is d_H-dense in itself, we may carry the construction of the proof of Lemma 7 within $\mathfrak{D}(A) = \{Z \in \mathfrak{D} : Z \subset A\}$, i.e. we may select each Λ_a, $a \in \mathcal{F}$, in $\mathfrak{D}(A)$ rather than in $\mathrm{CMin}(A)$. As in the proof of Lemma 7, we get $\mathfrak{c}$ d_H-Cauchy sequences, d_H-converging to $\mathfrak{c}$ pairwise disjoint, nonvoid, compact invariant sets contained in A, thereby proving the existence of $\mathfrak{c}$ compact minimal sets contained in this open set (as each $K \in \mathrm{Ci}(A)$ contains at least one compact minimal set). Now since $\mathfrak{I}$ is countable, $\mathfrak{c}$ of these compact minimal sets $\Gamma \in \mathrm{CMin}(A)$ actually belong to $\mathfrak{D} = \mathrm{CMin}(M) \setminus \mathfrak{I}$. Therefore $\mathfrak{D}$ is $\mathfrak{c}$-dense in itself. Finally, $\mathfrak{D}$ is d_H-closed in $\mathrm{CMin}(M)$ since $\mathfrak{I}$ is d_H-open in the same set. ∎

It is easy to see that the set E of equilibria has the following analogous property, stronger than that expressed in Theorem 3:

- *If E is uncountable, then E is the union of a countable set and a perfect subset $\mathfrak{E}$ of M of cardinality $\mathfrak{c}$. For each $z \in \mathfrak{E}$ and $\epsilon > 0$ there is an* *embedding h of Cantor's ternary set into $B(z,\epsilon)\cap\mathfrak{E}$ with $z\in\operatorname{im}h$. Hence $\mathfrak{E}\subset E$ is a $\mathfrak{c}$-dense in itself closed subset of M* ([10]).

The question now arises whether the corresponding propositions analogous to Theorem 3, for the set $\mathrm{Per}(M)$ of all periodic orbits and for the set $\mathrm{Am}(M)$ of all compact aperiodic minimal sets of the flow, also hold:

1. *If* $\mathrm{Per}(M)$ *is uncountable then all but a countable number of periodic orbits of the flow have* $\mathfrak{c}$ *periodic orbits on each of their neighbourhoods.*
2. *If* $\mathrm{Am}(M)$ *is uncountable then all but a countable number of compact aperiodic minimal sets of the flow have* $\mathfrak{c}$ *compact aperiodic minimal sets on each of their neighbourhoods.*

It is, in a sense ([11]), useless to look for counterexamples to any of these two propositions within "standard" dynamical systems theory: both statements 1 and 2 are provable in ZFC set theory under the additional assumption of the Continuum Hypothesis CH. Hence each turns out to be either demonstrable in ZFC or independent of this standard axiomatic (due to Gödel's result [göde]). The proof that CH $\Rightarrow 1\wedge 2$ is simple and actually depends only on the fact $\mathrm{Per}(M)$ and $\mathrm{Am}(M)$ are separable metric spaces: 1 and 2 are particular cases of the following proposition, which is equivalent to the Continuum Hypothesis:

- $\mathfrak{c}$-Denseness Hypothesis ($\mathfrak{c}$DH): *If L is an uncountable separable metric space, then a $\mathfrak{c}$-dense in itself set is obtained by removing from L a suitable countable set* (*possibly empty*) ([12]).

As we could not locate a reference for the equivalence CH $\Leftrightarrow$ $\mathfrak{c}$DH, a short proof is given for the sake of completeness.

(CH $\Rightarrow$ $\mathfrak{c}$DH): Recall that a separable metric space L has at most $\mathfrak{c}$ points. Suppose L is uncountable. Let $\mathfrak{I}$ be the set of points of L having a countable neighbourhood. By the proof of the Lemma preceding Theorem 3, $\mathfrak{I}$ is a countable, open subset of L and $\mathfrak{D}:=L\setminus\mathfrak{I}$ is a set such that every neighbourhood U_z of each $z\in\mathfrak{D}$ contains uncountably many points of $\mathfrak{D}$.

([10]) This follows immediately from the following observation: since the phase space M is locally compact and separable it can be endowed with an equivalent boundedly compact metric (on which every closed bounded set is compact, see e.g. [brid, p. 157]), thus becoming a complete, separable metric space (Polish space). As E is closed in M, it is also a Polish space in this equivalent metric, and the proposition in question is well known to hold on such spaces (see e.g. [levy, Chap. VII.2]).

([11]) Namely working within Zermelo–Fraenkel Set Theory + Axiom of Choice (ZFC) and provided this standard axiomatic is consistent.

([12]) Here $X\subset M$ being $\mathfrak{c}$-*dense in itself* means, as for the d_H metric, that for every $x\in X$ and $\epsilon>0$, $B(x,\epsilon)\cap X$ has cardinality $\mathfrak{c}$.

As the cardinality of $L \supset \mathfrak{D}$ is at most $\mathfrak{c} = 2^{\aleph_0}$, the Continuum Hypothesis actually implies $\#(U_z \cap \mathfrak{D}) = \mathfrak{c}$. Hence $\mathfrak{D}$ is $\mathfrak{c}$-dense in itself.

($\neg$CH $\Rightarrow$ $\neg\mathfrak{c}$DH): Assume there is a cardinal $\aleph_0 < \beta < \mathfrak{c}$. Given a bijection between $\mathfrak{c} \supset \beta$ and $\mathbb{R}$, there is a set $S \subset \mathbb{R}$ with $\aleph_0 < \#S = \beta < \mathfrak{c}$. With the Euclidean metric inherited from $\mathbb{R}$, S is an uncountable separable metric space. Removing from S an arbitrary countable set $\mathfrak{I}$ we again obtain a set $\mathfrak{D} = S \setminus \mathfrak{I}$ of cardinality $\#(S \setminus \mathfrak{I}) = \#S = \beta$. Hence $\aleph_0 < \#\mathfrak{D} = \beta < \mathfrak{c}$ and therefore, since $\mathfrak{D}$ is nonvoid, it cannot be $\mathfrak{c}$-dense in itself.

6. Lemmas. Several lemmas, some of them dynamically interesting on their own, will be needed for the proof of Theorem 1. Due to the time-reversal symmetry of (global) flows, Lemmas 1 and 3 below admit analogous negative time formulations, which will be implicitly used later.

The following result gives an unusual characterization of attractors in terms of the behaviour of the negative orbits of points outside the compact invariant set in question. It illustrates a topological-dynamical phenomenon that plays a key role in the present work.

Lemma 1. *Let M be a locally compact metric space with a C^0 flow θ, and K a compact, invariant, proper subset of M. Then K is an attractor iff there is a neighbourhood U of K such that no point $z \in U \setminus K$ has its negative orbit $\mathcal{O}^-(z)$ entirely contained in U.*

Proof. ($\Rightarrow$) Suppose K is an attractor. Let U be a compact neighbourhood of K contained in $B^+(K)$ and $z \in U \setminus K$. Then $\mathcal{O}^-(z) \not\subset U$. Otherwise we would have $\emptyset \neq \alpha(z) \subset B^+(K)$, which implies $\alpha(z) \cap K \neq \emptyset$, contradicting the stability of K (as $\alpha(z)$ is closed invariant, if $y \in \alpha(z) \cap B^+(K)$, then $\emptyset \neq \omega(y) \subset \alpha(z) \cap K$).

($\Leftarrow$) Suppose that $U \in \mathcal{N}_K$ is such that

$$z \in U \setminus K \ \Rightarrow\ \mathcal{O}^-(z) \not\subset U. \tag{6.1}$$

Since condition (6.1) is hereditary under inclusion $U' \subset U$, we may without loss of generality assume that U is compact (as M is locally compact).

Claim I. *K is stable.*

Given $W \in \mathcal{N}_K$, let U_0 be a compact neighbourhood of K contained in $U \cap W$ and let $S = \operatorname{bd} U_0$. As (6.1) also holds for U_0, and S is a (nonvoid) compact, the continuity of the flow implies the existence of a finite $T < 0$ such that $\theta([T, 0] \times \{x\}) \not\subset U_0$ for every $x \in S$, which in its turn implies $\mathcal{O}^-(S) \cap U_0 \subset \theta([T, 0] \times S) =: \Theta$ (Fig. 6.1). Now Θ is a (nonvoid) compact set, disjoint and hence at a positive distance from the compact invariant set K. Taking $V \in \mathcal{N}_K$ disjoint from Θ, it follows that no point in V can leave U_0 in positive time, as its positive orbit cannot cross S. Thus $\mathcal{O}^+(V) \subset U_0 \subset W$, proving the stability of K.

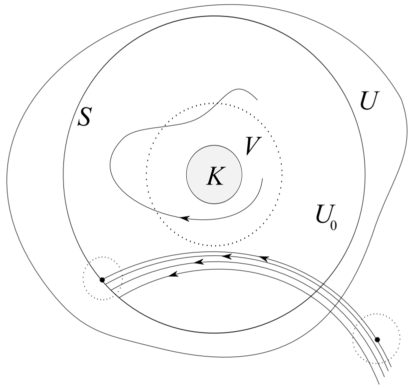

Fig. 6.1. Lemma 1: the existence of a compact $U \in \mathcal{N}_K$ for which $z \in U \setminus K \Rightarrow \mathcal{O}^-(z) \not\subset U$ implies the stability of K

Claim II. *$B^+(K)$ is a neighbourhood of K.*

As K is stable, there is a $V \in \mathcal{N}_K$ such that $\mathcal{O}^+(V) \subset U$. We claim that $V \subset B^+(K)$. Since U is compact, $\emptyset \neq \omega(x) \subset U$ for every $x \in V$ and no point $y \in U \setminus K$ may belong $\omega(x)$, as this would imply $\mathcal{O}(y) \subset \omega(x) \subset U$, contradicting the hypothesis $\mathcal{O}^-(y) \not\subset U$. Therefore $\emptyset \neq \omega(x) \subset K$ for every $x \in V$ and the claim is proved. ∎

The following result is essentially a version of Ura–Kimura–Bhatia's Theorem for locally compact, connected metric spaces (see e.g. [bhsz, p. 114]). Our proof is in the spirit of the present work. Note that the phase space of a C^0 flow being locally compact, a compact invariant set K is isolated from minimal sets if for every $U \in \mathcal{N}_K$, $U \setminus K$ contains no compact minimal set.

Lemma 2. *Let M be a locally compact, connected metric space with a C^0 flow θ, and K a compact, invariant, proper subset of M. Then either*

I. *K is an attractor, or*
II. *K is a repeller,*

or at least one of the following conditions holds:

III. *K is isolated from minimals and stagnant.*
IV. *Given any $U \in \mathcal{N}_K$, $U \setminus K$ contains an (entire) orbit.*

Proof. Suppose K is an attractor. Since $\emptyset \neq K \subsetneq M$ and M is connected, $U \setminus K \neq \emptyset$ for all $U \in \mathcal{N}_K$. Each of conditions II, III and IV then clearly implies the existence of a $z \in U \setminus K$ such that $\mathcal{O}^-(z) \subset U$ for every $U \in \mathcal{N}_K$. This contradicts the existence of a neighbourhood U in Lemma 1, thus none of them holds. Analogously if K is a repeller then none of conditions I, III and IV holds.

Suppose now that none of conditions I, II and IV holds. Then there is a compact $U \in \mathcal{N}_K$ such that $U \backslash K$ contains no (entire) orbit, in particular, K is isolated from minimals. Since K is neither an attractor nor a repeller, by Lemma 1, there are $x, y \in U \backslash K$ such that $\mathcal{O}^-(x) \subset U$ and $\mathcal{O}^+(y) \subset U$, hence $\emptyset \neq \alpha(x)$, $\omega(y) \subset U$. No point $z \in U \setminus K$ may belong to $\alpha(x)$ or $\omega(y)$, since otherwise the invariance of limit sets and of $M \backslash K$ would imply $\mathcal{O}(z) \subset U \backslash K$, contradicting the hypothesis about U. Thus $\emptyset \neq \alpha(x)$, $\omega(y) \subset K$, i.e. K is stagnant. Therefore, condition III holds, since K is also isolated from minimals. ∎

The next result is, in a sense, a counterpart to Butler–Mcgehee's Lemma (Butler and Waltman [butl, p. 259]). Since its proof involves three distinct flows, we shall indicate them by subscripts.

Lemma 3. *Let M be a locally compact metric space with a C^0 flow θ and K a compact invariant proper subset of M. If K is nonstagnant and $z \in A^+(K)$, then given any $U \in \mathcal{N}_K$ there is a $y \in \omega(z)$ such that $\mathcal{O}(y) \subset U \setminus K$ (see Fig.* 6.2).

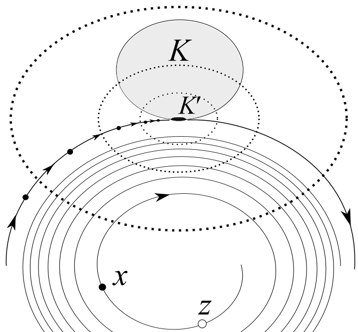

Fig. 6.2. Lemma 3

Proof. Using Carlson's globalization of local flows on metric spaces [carl, p. 198], we may remove the point z from M without changing the oriented phase portrait of θ elsewhere, i.e. there is a (global) C^0 flow φ on $M' = M \setminus \{z\}$ (with the metric of M), having the same oriented phase portrait as θ, except that the original orbit of z is broken into two distinct new orbits (z is not periodic), corresponding to its open half-orbits $t < 0$ and $t > 0$. The point z is suppressed. Note that being open in M, M' is a locally compact. Take $x \in \mathcal{O}^+_\theta(z)$ distinct from z. Let N be the closure of $\mathcal{O}_\varphi(x)$ in M', thus an invariant subset of φ. Now we focus on the subflow $(N, \phi = \varphi|_{\mathbb{R}\times N})$. Being an orbit closure, N is a locally compact and connected metric space. The fact that x belongs to the orbit of z in the original flow and $z \in A^+_\theta(K)$ implies that the compact invariant set $K' = N \cap K$ is nonvoid and $x \in A^+_\phi(K')$. Thus K' is neither an attractor nor a repeller in (N,ϕ) and it is also nonstagnant, since by hypothesis K is nonstagnant in θ. Applying Lemma 2 to (N,ϕ) and K', it follows that condition IV of that lemma holds. Since $K' = N \cap K$, the result easily follows on noting that all orbits in (N,ϕ), except those contained in $\mathcal{O}_\theta(z)$ (at most two), belong to the ω-limit set of z in (M,θ). ∎

Definition. Let M be a metric space with a C^0 flow. A sequence $A_n \subset M$ *approaches* $X \subset M$ if for every $\epsilon > 0$, $A_n \subset B(X,\epsilon)$ for all sufficiently large n, i.e. $|A_n|_X \to 0$.

In the next proof, the following elementary fact will be used:

- *In a metric space, a sequence converges to a point z if every subsequence contains a (sub)subsequence converging to z.*

Lemma 4. *Let M be a locally compact metric space with a C^0 flow. If a sequence $\Lambda_n \in \mathrm{Ci}(M)$ approaches a compact minimal set S then (Λ_n) actually d_H-converges to S, i.e.*

$$|\Lambda_n|_S \to 0 \;\Rightarrow\; \Lambda_n \xrightarrow{d_H} S.$$

Proof. Let U be a compact neighbourhood of S. Since $|\Lambda_n|_S \to 0$, given any subsequence (Λ_{n_i}), there is an $i_0 \geq 1$ such that $\Lambda_{n_i} \subset U$ for all $i > i_0$. Now $[\mathrm{Ci}(U), d_H]$ is a compact metric space by Blaschke's Theorem (Section 2), hence by Blaschke's Principle there is a (sub)subsequence $(\Lambda_{n_{i_k}})$ d_H-converging to some nonvoid, compact, invariant set $Q \subset U$. But $|\Lambda_{n_{i_k}}|_S \to 0$ implies $Q \subset S$, and since S is a minimal set, $Q = S$. Hence the above convergence criterion is satisfied. ∎

Therefore, $X \in \mathrm{CMin}(M)$ is an isolated minimal set iff X is d_H-isolated in $\mathrm{CMin}(M)$ (Lemma 4 establishes ($\Leftarrow$), and ($\Rightarrow$) follows from the definition of the Hausdorff metric).

Lemma 5. *Let M be a metric space. If $\mathfrak{C} \subset \mathrm{C}(M)$ and $A \subset M$ is open, then $\mathfrak{C}(A) := \{X \in \mathfrak{C} : X \subset A\}$ is d_H-open in $\mathfrak{C}$.*

Proof. Let $X \in \mathfrak{C}(A)$. Since X is compact and A is open, there is a $\lambda_X > 0$ such that $B(X,\lambda_X) \subset A$. But

$$Y \in \mathfrak{C} \cap B_H(X,\lambda_X) \;\Rightarrow\; Y \in \mathfrak{C}(B(X,\lambda_X)) \;\Rightarrow\; Y \in \mathfrak{C}(A),$$

therefore $\mathfrak{C}(A)$ is d_H-open in $\mathfrak{C}$. ∎

Lemma 6. *Given $Q \in \mathfrak{M} \subset \mathrm{CMin}(M)$, the set $\{\mathfrak{M}(B(Q,\delta)) : \delta > 0\}$ is a basis of neighbourhoods of Q in $[\mathfrak{M}, d_H]$.*

Proof. From Lemma 4 it follows that, given $Q \in \mathfrak{M}$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\mathfrak{B} := \mathfrak{M}(B(Q,\delta)) \subset \mathfrak{M} \cap B_H(Q,\epsilon)$, and by Lemma 5, $\mathfrak{B}$

is a d_H-open neighbourhood of Q in $\mathfrak{M}$. Since $\{\mathfrak{M} \cap B_H(Q,\epsilon) : \epsilon > 0\}$ is a basis of neighbourhoods of Q in $[\mathfrak{M}, d_H]$, the claim is proved. ∎

This immediately yields

Corollary 9. *Let M be a locally compact metric space with a C^0 flow. If $\mathfrak{M} \subset \mathrm{CMin}(M)$ and for every $X \in \mathfrak{M}$ and $\epsilon > 0$,*

$$\#\mathfrak{M}(B(X,\epsilon)) = \mathfrak{c} \quad (\textit{resp. } \mathfrak{M}(B(X,\epsilon) \setminus X) \neq \emptyset)$$

then $\mathfrak{M}$ is $\mathfrak{c}$-dense in itself (resp. d_H-dense in itself).

Hence a set of compact minimal sets $\mathfrak{M}$ is $\mathfrak{c}$-dense in itself (resp. d_H-dense in itself) iff every neighbourhood $U \subset M$ of each $X \in \mathfrak{M}$ contains $\mathfrak{c}$ elements of $\mathfrak{M}$ (resp. an element of $\mathfrak{M}$ distinct from X).

Lemma 7. *Let M be a locally compact metric space with a C^0 flow. If $\mathfrak{M}$ is a d_H-open and dense in itself subset of* $\mathrm{CMin}(M)$ *then $\mathfrak{M}$ is $\mathfrak{c}$-dense in itself.*

Before entering the proof, a few technical definitions will be needed. Let $\mathrm{F} := \{0,1\}$ and

$$\mathcal{F} := \bigsqcup_{n\in\mathbb{N}} \mathrm{F}^n.$$

If $a, b \in \mathcal{F}$, then ab represents, as usual, the element of $\mathcal{F}$ obtained by adjoining b to the right end of a ([13]). For any $n \in \mathbb{N}$ and $a \in \mathrm{F}^n$, $|a| := n$ (the *length* of a). We now define the operators $*, -$ on $\mathcal{F}$ (for every $b \in \mathcal{F}$, $c \in \mathrm{F}$):

$$0^* := 1, \quad 1^* := 0, \quad (bc)^* := bc^*,$$
$$0^- := 0, \quad 1^- := 0, \quad (bc)^- := b.$$

Proof of Lemma 7. As $\mathfrak{M}$ is d_H-open in $\mathrm{CMin}(M)$, given any $\Lambda_0 \in \mathfrak{M}$, take $\epsilon_0 > 0$ small enough so that $A := B(\Lambda_0, \epsilon_0)$ has compact closure and $\mathrm{CMin}(A) \subset \mathfrak{M}$ (use Lemma 6). We show that

$$\#\mathrm{CMin}(A) = \mathfrak{c},$$

which in virtue of Corollary 9 proves the lemma. The demonstration is based on a generalization of the idea lying behind the construction of Cantor's ternary set: we construct $\mathfrak{c}$ d_H-Cauchy sequences of compact minimal sets $\Lambda_{w_n} \in \mathrm{CMin}(A)$, d_H-converging to $\mathfrak{c}$ *pairwise disjoint* sets $\Lambda_w \in \mathrm{Cci}(A)$. The result then follows since each such Λ_w contains at least one compact minimal set of the flow. However, some care must be taken to ensure that these limit sets Λ_w are actually pairwise disjoint (which is obviously crucial).

([13]) Again, since no risk of ambiguity arises, commas and brackets are omitted in the representation of the elements of $\mathcal{F}$, thus we write 01 instead of $(0,1)$ and F^1 is naturally identified with F.

To achieve this, we must introduce the metric of M into the construction, as the d_H-metric permits determining if two compact sets are distinct, but not whether they are disjoint or not ([14]).

We proceed by induction over n: Suppose that, for a certain $n \in \mathbb{N}$, we have already defined $\Lambda_a \in \mathrm{CMin}(A)$ and numbers $\epsilon_m > 0$, for all $a \in \mathcal{F}$ with $1 \leq |a| \leq n$ and all integers $1 \leq m \leq n$, so that, for these a and m:

1. $\epsilon_m < \epsilon_0/2^m$,
2. $\Lambda_a \in B_H(\Lambda_{a^-}, \epsilon_{|a|-1})$,
3. $B[\Lambda_a, \epsilon_{|a|}\,] \cap B[\Lambda_{a^*}, \epsilon_{|a|}\,] = \emptyset$,
4. $B[\Lambda_a, \epsilon_{|a|}\,] \subset B(\Lambda_{a^-}, \epsilon_{|a|-1})$.

Since $\mathrm{CMin}(A)$ is d_H-dense in itself, this can clearly be done for $n = 1$ (note that Λ_0 and ϵ_0 were defined at the outset) and we only need to well define the induction step. Take, for each $b \in \mathcal{F}$ with $|b| = n$, two distinct compact minimal sets

$$\Lambda_{b0}, \Lambda_{b1} \in B_H(\Lambda_b, \epsilon_n)$$

(again, this is clearly possible, as $\mathrm{CMin}(A)$ is d_H-dense in itself) and

$$0 < \epsilon_{n+1} < \epsilon_0/2^{n+1}$$

small enough so that conditions 3 and 4 hold for all $a \in \mathcal{F}$ with $|a| = n+1$ (this is possible since Λ_a and Λ_{a^*} are disjoint and $\Lambda_a \subset B(\Lambda_{a^-}, \epsilon_{|a|-1})$). For each $w \in \mathrm{F}^{\mathbb{N}}$, let $w_n \in \mathrm{F}^n$ denote the sequence of the first n digits of w. Now 1, 2 and 4 together imply that Λ_{w_n} is a d_H-Cauchy sequence in the d_H-compact metric space $\mathrm{C} := \mathrm{Cci}(B[\Lambda_{w_1}, \epsilon_1]) \subset \mathrm{Cci}(B(\Lambda_0, \epsilon_0))$ and thus d_H-converges to some $\Lambda_w \in \mathrm{C}$. Moreover, if $v \in \mathrm{F}^{\mathbb{N}}$ is distinct from w, then $v_m = w_m^*$ for some $m \in \mathbb{N}$, and from 3 and 4 it follows that

$$\Lambda_v \cap \Lambda_w = \emptyset$$

where $\Lambda_v = \lim \Lambda_{v_n}$, as the sequences Λ_{v_n} and Λ_{w_n} are ultimately contained in disjoint compact "balls". We thus get $\mathfrak{c} = \#\mathrm{F}^{\mathbb{N}}$ disjoint (nonvoid) sets $\Lambda_v \in \mathrm{Cci}(B(\Lambda_0, \epsilon_0)) \subset \mathrm{Cci}(A)$, each containing a compact minimal set of the flow. Therefore, $\#\mathrm{CMin}(A) = \mathfrak{c}$, as $\#\mathrm{CMin}(M) \leq \mathfrak{c}$. ∎

Lemma 8 (Nested Compacts Lemma). *Let M be a metric space and $\mathfrak{C}$ a d_H-closed subset of $\mathrm{C}(M)$. If $K_n \in \mathfrak{C}$ and $K_n \supset K_{n+1}$ for all $n \geq 1$ then:*

(1) $K_n \xrightarrow{d_H} \bigcap_{n\geq 1} K_n \in \mathfrak{C}$,
(2) *every sequence $x_n \in K_n$ has a subsequence converging to some x in $\bigcap_{n\geq 1} K_n$,*

([14]) In this way, our construction differs from the standard one used to prove that a nonvoid, complete, separable metric space without isolated points has locally cardinality $\mathfrak{c}$ (see e.g. [levy, p. 227, 2.16]). There, only the original metric of the space enters into the construction.

(3) *every sequence* $\Lambda_n \subset K_n$ *where* $\Lambda_n \in \mathfrak{B}$, $\mathfrak{B}$ *a* d_H*-closed subset of* $\mathrm{C}(M)$, *has a subsequence* d_H*-converging to some* $X \in \mathfrak{B}(\bigcap_{n\geq 1} K_n)$.

The proof of Lemma 8, based on Blaschke's Theorem, is omitted since it presents no difficulty.

LEMMA 9. *Let M be a locally compact metric space with a C^0 flow. If Q is a compact aperiodic minimal set, then for any $m \geq 1$ there is an $\epsilon > 0$ such that*

$$\gamma \in \mathrm{Per}(M) \text{ and } \mathrm{dist}(\gamma, Q) < \epsilon \;\Rightarrow\; \mathrm{period}(\gamma) > m.$$

Proof. Suppose the contrary. Then there is an $m \geq 1$ and there are sequences $\gamma_n \in \mathrm{Per}(M)$ and $x_n \in \gamma_n$ such that $\mathrm{dist}(x_n, Q) \to 0$ and $0 < \mathrm{period}(x_n) = \mathrm{period}(\gamma_n) \leq m$. Take $\epsilon > 0$ sufficiently small so that $B[Q, \epsilon]$ is compact. Applying Lemma 8(2) (with $K_n := B[Q, \epsilon/n]$), we may suppose (taking a convergent subsequence) that $x_n \to p$ for some $p \in Q$. But $\mathrm{period}(x_n) \leq m$ implies that p is a periodic point or an equilibrium (because the periodic points with period $\leq T$ together with the equilibria form a closed set [bhsz, p. 18]), which is absurd since Q is an aperiodic minimal. ∎

Recall that every locally compact, connected metric space is separable.

LEMMA 10. *If M is a locally compact, separable metric space then so is $[\mathrm{C}(M), d_H]$, where $\mathrm{C}(M)$ is the set of all nonvoid compact subsets of M.*

Proof. It is known (Aleksandrov's one-point compactification) that $[M, d]$ is homeomorphic, via the inclusion map, to an open subset of a compact metric space $[M^*, d']$, $M^* \supset M$. By Blaschke's Theorem, $[\mathrm{C}(M^*), d'_H]$ is compact. Therefore $[\mathrm{C}(M), d'_H]$ is separable and also locally compact, since $\mathrm{C}(M)$ is d'_H-open in $\mathrm{C}(M^*)$ (as M is open in M^*, Lemma 5). The result now follows since d_H and d'_H are equivalent metrics on $\mathrm{C}(M)$ (as d and d' are equivalent on M). ∎

In the proof of the main theorem, the following well known cardinality principle will be systematically used:

CANTOR–DIRICHLET'S PRINCIPLE. *If $n_0 \in \mathbb{N}$ and $A = \bigcup_{1\leq n\leq n_0} A_n$ is infinite, then $\#A_n = \#A$ for some $1 \leq n \leq n_0$.*

7. Proof of Theorem 1. Synopsis. Assume neither **1** nor **2** hold.

(A) If K is isolated from minimal sets then we will prove that at least one of conditions **3** to **6** necessarily holds.

(B) If K is not isolated from minimals, then two possible cases are considered:

(B.1) If for every neighbourhood U of K, $U \setminus K$ contains an isolated compact minimal set X, then (by (A) above) X necessarily satisfies (at least) one of the six conditions $1.X$ to $6.X$, and it follows that (at least) one of the eighteen cases **7.1** to **9.6** occurs.

(B.2) If the contrary is true, then there is a neighbourhood U of K such that $\mathrm{CMin}(U \setminus K)$ is a d_H-open and dense in itself subset of $\mathrm{CMin}(M)$, d_H-accumulating in $\mathrm{Cci}(K)$. By Lemma 7, $\mathrm{CMin}(U \setminus K)$ is actually $\mathfrak{c}$-dense in itself, and it is proved that at least one of the four conditions **10.1** to **10.4** holds.

Proof. It is easily seen that condition **1** excludes the remaining 27 conditions and the same holds with condition **2** ([15]). Assume, throughout the remainder of this proof, that neither **1** nor **2** holds. In this situation we distinguish the two possible cases:

(A) *K is isolated from minimals, i.e. for some $U \in \mathcal{N}_K$,* $\mathrm{CMin}(U \setminus K) = \emptyset$.
(B) *For every $V \in \mathcal{N}_K$,* $\mathrm{CMin}(V \setminus K) \neq \emptyset$.

We recall an important elementary fact that will be implicitly used in several instances below: on a locally compact metric space, every sufficiently small neighbourhood of a compact set has compact closure, and thus may only contain compact minimal sets.

Case A. Since K is compact and M is locally compact, we may assume, without loss of generality, that U is compact. Then for any $z \in U$,

$$\mathcal{O}^+(z) \subset U \;\Rightarrow\; \omega(z) \cap K \neq \emptyset \;\Rightarrow\; z \in A^+(K) \sqcup B^+(K),$$
$$\mathcal{O}^-(z) \subset U \;\Rightarrow\; \alpha(z) \cap K \neq \emptyset \;\Rightarrow\; z \in A^-(K) \sqcup B^-(K).$$

since otherwise we would have $\mathrm{CMin}(U \setminus K) \neq \emptyset$. (As U is compact, $\mathcal{O}^+(z) \subset U$ implies $\emptyset \neq \omega(z) \subset U$. Being a nonvoid compact invariant set, $\omega(z)$ contains at least one minimal set, thus by (A), it cannot be contained in $U \setminus K$, hence $\omega(z) \cap K \neq \emptyset$. The case $\mathcal{O}^-(z) \subset U$ is analogous.)

Suppose now that condition **3** does not hold. Since K is isolated from minimals it follows that K is nonstagnant, therefore for every orbit $\mathcal{O}(z) \subset U \setminus K$, exactly one of the following three cases holds:

0. $z \in A^-(K) \cap B^+(K)$,
I. $z \in B^-(K) \cap A^+(K)$,
II. $z \in A^-(K) \cap A^+(K)$.

Accordingly, we say $\mathcal{O}(z)$ is an orbit of *type* 0, I or II. More generally, the fact that K is nonstagnant implies that orbits of type 0 and I cannot coexist in $U \setminus K$. This implies that exactly one of the following three conditions holds:

([15]) If K is an attractor, then by Lemma 1, there is an $U \in \mathcal{N}_K$ such that $z \in U \setminus K \;\Rightarrow\; \mathcal{O}^-(z) \not\subset U$. A time-symmetric fact holds if K is a repeller. This immediately implies that if **1** (resp. **2**) holds, then none of the remaining 27 conditions can be valid.

(1) there is an orbit $\mathcal{O}(x) \subset U \setminus K$ such that $(\mathrm{cl}\,\mathcal{O}(x)) \setminus K$ contains only orbits of type 0,
(2) there is an orbit $\mathcal{O}(y) \subset U \setminus K$ such that $(\mathrm{cl}\,\mathcal{O}(y)) \setminus K$ contains only orbits of type I,
(3) for every orbit $\mathcal{O}(z) \subset U \setminus K$, $(\mathrm{cl}\,\mathcal{O}(z)) \setminus K$ contains an orbit of type II.

We claim that:

- (1) implies there is a K-α shell,
- (2) implies there is a K-ω shell,
- (3) implies there is a K-tree.

Suppose there is an orbit $\mathcal{O}(y)$ satisfying condition (2). Since $\mathcal{O}(y)$ is of type I, by Lemma 3 (recall that K is, by hypothesis, nonstagnant), given any neighbourhood V of K, there is a $p \in \omega(y) \subset \mathrm{cl}\,\mathcal{O}(y)$ such that $\mathcal{O}(p) \subset V \setminus K$. Clearly $\mathcal{O}(p)$ is also of type I since $\mathcal{O}(p) \subset \mathrm{cl}\,\mathcal{O}(y)$. The existence of a K-ω shell with first orbit $\mathcal{O}(y)$ is now a straightforward inductive consequence of Lemma 3. Analogously, if $\mathcal{O}(x)$ is an orbit satisfying condition (1), then there is a K-α shell with first orbit $\mathcal{O}(x)$.

We now assume (3) holds. Recall that by hypothesis, K satisfies none of conditions **1**, **2** and **3**, therefore by Lemma 2, there is necessarily an orbit $\mathcal{O}(z) \subset U \setminus K$. By (3), $(\mathrm{cl}\,\mathcal{O}(z)) \setminus K$ contains an orbit γ_0 of type II. We will inductively define a map

$$\begin{aligned} \psi : \mathcal{E} &\to \mathrm{Orb}((\mathrm{cl}\,\gamma_0) \setminus K) \subset \mathrm{Orb}(U \setminus K) \subset \mathrm{Orb}(M \setminus K), \\ a &\mapsto \gamma_a, \end{aligned}$$

so that (Θ, ψ) is a K-tree, where $\Theta := \mathrm{im}\,\psi$. Adopt the following lexicographic order on $\mathcal{E}$:

$$0 < 00 < 01 < 000 < 001 < 010 < 011 < 0000 < 0001 < 0010 < \cdots .$$

Suppose $a \in \mathcal{E}$ is such that for all $\mathcal{E} \ni d < a$, γ_d is an already defined orbit of type II contained in $(\mathrm{cl}\,\gamma_0) \setminus K \subset U \setminus K$. We define γ_a: evidently, $a = bc$ for some $b \in \mathcal{E}$ and $c \in \{0, 1\}$; by Lemma 3 ([16]) there is an orbit $\zeta_{bc} \subset U \setminus K$ such that:

- $\gamma_b \overset{c}{\succ} \zeta_{bc}$,
- $0 < |\zeta_{bc}|_K < |\gamma_b|_K/2$,

hence $\zeta_{bc} \not\succ \gamma_b$ and $\zeta_{bc} \subset (\mathrm{cl}\,\gamma_b) \setminus K \subset (\mathrm{cl}\,\gamma_0) \setminus K$. By hypothesis (3), $(\mathrm{cl}\,\zeta_{bc}) \setminus K$ contains an orbit of type II and we identify γ_a with it. Clearly

([16]) Note that $K \in \mathrm{Ci}(M) \setminus \{M\}$, K is nonstagnant and by hypothesis, γ_b is of type II, i.e. $\gamma_b \subset A^-(K) \cap A^+(K)$.

$\gamma_b \overset{c}{\succ} \gamma_{bc}$ for every $b \in \mathcal{E}$, $c \in \{0,1\}$ since $\gamma_{bc} \subset \operatorname{cl}\zeta_{bc}$ and $\gamma_b \overset{c}{\succ} \zeta_{bc}$. Note that the inequality $|\gamma_{bc}|_K \le |\zeta_{bc}|_K < |\gamma_b|_K/2$ guarantees $\gamma_{bc} \not\succ \gamma_b$ for every $b \in \mathcal{E}$ and $c \in \{0,1\}$, and $|\operatorname{cl}\gamma_{v_n}|_K \to 0$ for every $v \in \mathrm{E}_\infty$. It is now immediate to verify that (Θ, ψ), where Θ is the inductively defined set $\{\gamma_a : a \in \mathcal{E}\}$, is indeed a K-tree.

We thus conclude that case (A) implies that at least one of conditions **3**, **4**, **5** or **6** necessarily holds, so if conditions **1** to **6** (i.e. $1.K$ to $6.K$) all fail then condition (B) holds (recall that we assumed, at the beginning of the proof, that both **1** and **2** are false). Note that since K is an arbitrary compact, invariant, proper subset of M, the above observation is true for all $X \in \mathrm{Ci}(M) \setminus \{M\}$, i.e. we have

Lemma 11. *If $X \subsetneq M$ is a compact invariant set and all conditions* $1.X$ to $6.X$ *fail, then arbitrarily near X there is always a compact minimal set disjoint from X, i.e., for any $\epsilon > 0$,* $\mathrm{CMin}(B(X,\epsilon) \setminus X) \neq \emptyset$.

Case (B). We distinguish two (sub)cases:

(B.1) For every $V \in \mathcal{N}_K$, $V \setminus K$ contains a compact minimal set X satisfying (at least) one of the six conditions $1.X$ to $6.X$.
(B.2) There is an open $U \in \mathcal{N}_K$ such that *no* $X \in \mathrm{CMin}(U \setminus K)$ satisfies any of the six conditions $1.X$ to $6.X$.

Case (B.1). We will show that in this case at least one of the eighteen cases **7.1** to **9.6** will necessarily hold. Take $\epsilon > 0$ such that $U := B[K,\epsilon]$ is compact. We may obviously define a sequence $\Lambda_n \in \mathrm{CMin}(M)$ such that for every $n \ge 1$,

- Λ_n satisfies at least one of the six conditions $1.X$ to $6.X$.
- $\Lambda_n \subset B[K,\epsilon/n] \setminus K \subset U \setminus K$.

Since $\Lambda_n \in \mathrm{CMin}(U) \subset \mathrm{Cci}(U)$ for all $n \ge 1$ and $|\Lambda_n|_K \to 0$, by Lemma 8(3), we may select from (Λ_n) a subsequence d_H-converging to some $Q \in \mathrm{Cci}(K)$. Obviously $Q \subset \operatorname{bd} K$, since $\Lambda_n \subset M \setminus K$, thus in fact $Q \in \mathrm{Cci}(\operatorname{bd} K)$. By Cantor–Dirichlet's Principle we may select from this subsequence another subsequence consisting of compact minimal sets all belonging to the same one of the following three classes: equilibrium orbits $\mathrm{Eq}(M)$, periodic orbits $\mathrm{Per}(M)$, compact aperiodic minimals $\mathrm{Am}(M)$. Finally, since each term of (Λ_n) satisfies at least one of the six conditions $1.X$ to $6.X$, using Cantor–Dirichlet's Principle again, we select from the last subsequence another subsequence such that (at least) one of the six conditions $1.X$ to $6.X$ is satisfied by all its terms, therefore obtaining a sequence of compact minimal sets contained in $M \setminus K$ satisfying at least one of the eighteen conditions **7.1** to **9.6**.

To complete the proof of Theorem 1 we show that in case (B.2), i.e. if

(1) $\mathrm{CMin}(V \setminus K) \neq \emptyset$ for all $V \in \mathcal{N}_K$ and
(2) for some open $U \in \mathcal{N}_K$, *no* $X \in \mathrm{CMin}(U \setminus K)$ satisfies any of the six conditions 1.X to 6.X,

then at least one of the four cases **10.1** to **10.4** necessarily holds.

CASE (B.2). We may obviously assume, without loss of generality, that U has compact closure. Observe that:

- $\mathrm{CMin}(U \setminus K)$ is d_H-open in $\mathrm{CMin}(M)$ since $U \setminus K$ is open and $\mathrm{CMin}(M) \subset \mathrm{C}(M)$ (Lemma 5).
- In virtue of Lemma 11, (2) implies that for any $X \in \mathrm{CMin}(U \setminus K)$ and $\epsilon > 0$, $\mathrm{CMin}(B(X, \epsilon) \setminus X) \neq \emptyset$, thus by Corollary 9, $\mathrm{CMin}(U \setminus K)$ is d_H-dense in itself. By Lemma 7, $\mathrm{CMin}(U \setminus K)$ is in fact $\mathfrak{c}$-dense in itself.
- Using Lemma 8(3) we infer from (1) that $\mathrm{CMin}(U \setminus K)$ d_H-accumulates in $\mathrm{Cci}(K)$, hence in $\mathrm{Cci}(\mathrm{bd}\, K)$.

Therefore, in case (B.2), there exists an open $U \in \mathcal{N}_K$ with compact closure such that

- $\mathrm{CMin}(U \setminus K)$ *is a* $\mathfrak{c}$*-dense in itself,* d_H*-open subset of* $\mathrm{CMin}(M)$*,* d_H*-accumulating in* $\mathrm{Cci}(\mathrm{bd}\, K)$.

Note that in particular, $\mathrm{Per}(U \setminus K) \sqcup \mathrm{Am}(U \setminus K)$ is a $\mathfrak{c}$-dense in itself, d_H-open subset of $\mathrm{CMin}(M)$ since $\mathrm{Eq}(M)$ is a d_H-closed subset of $\mathrm{CMin}(M)$. Now by the above remark concerning $\mathrm{CMin}(U \setminus K)$, there is a sequence $\Lambda_n \in \mathrm{CMin}(U \setminus K)$, d_H-accumulating in $\mathrm{Cci}(\mathrm{bd}\, K)$. By Cantor–Dirichlet's Principle we may suppose this sequence is such that all Λ_n belong to the same one of the following three classes: equilibrium orbits $\mathrm{Eq}(M)$, periodic orbits $\mathrm{Per}(M)$, compact aperiodic minimals $\mathrm{Am}(M)$.

Suppose now that conditions **10.1**, **10.2** and **10.3** all fail. We will show that condition **10.4** is necessarily true. The equality $\mathrm{CMin}(N) = \mathrm{Eq}(N) \sqcup \mathrm{Per}(N) \sqcup \mathrm{Am}(N)$, valid for all $N \subset M$, will be repeatedly used ($\sqcup$ denotes disjoint union). Three possible cases are distinguished:

1st case: There exists a sequence $\Lambda_n \in \mathrm{Am}(U \setminus K)$ d_H-accumulating in $\mathrm{Cci}(\mathrm{bd}\, K)$.

Since $\mathrm{Am}(U \setminus K)$ d_H-accumulates in $\mathrm{Cci}(\mathrm{bd}\, K)$ but **10.3** is false, given any open $V \in \mathcal{N}_K$, $\mathrm{Am}(V \setminus K)$ is not $\mathfrak{c}$-dense in itself. By Corollary 9, this implies that we may find $\Gamma \in \mathrm{Am}(V \setminus K)$ and $\epsilon > 0$ such that

$$B(\Gamma, \epsilon) \subset V \setminus K \quad \text{and} \quad \#\mathrm{Am}(B(\Gamma, \delta)) < \mathfrak{c}$$

Therefore it is easily seen that there are sequences $\Gamma_n \in \mathrm{Am}(U \setminus K)$ and $\delta_n > 0$ such that:

(1) $|\mathit{\Gamma}_n|_K \to 0$,
(2) $B[\mathit{\Gamma}_n, \delta_n] \subset U \setminus K$,
(3) $\#\mathrm{Am}(B(\mathit{\Gamma}_n, \delta_n)) < \mathfrak{c}$.

Again, by Lemma 8(3) (recall that $\mathit{\Gamma}_n \subset U \setminus K$ and $\mathrm{cl}\, U$ is compact), we may replace condition (1) above by

(1) $\mathit{\Gamma}_n \xrightarrow{d_H} Q$ for some $Q \in \mathrm{Cci}(\mathrm{bd}\, K)$,

Clearly,

(4) $\delta_n < d_H(\mathit{\Gamma}_n, Q)$

since $B[\mathit{\Gamma}_n, \delta_n] \cap Q = \emptyset$ (recall that $\emptyset \neq Q \subset K$). Taking a smaller δ_n if necessary, we may further require that

(5) $\mathrm{Eq}(B(\mathit{\Gamma}_n, \delta_n)) = \emptyset$,
(6) $\gamma \in \mathrm{Per}(B(\mathit{\Gamma}_n, \delta_n)) \Rightarrow \mathrm{period}(\gamma) > n$ (by Lemma 9).

$\mathrm{CMin}(U \setminus K)$ is nonvoid and $\mathfrak{c}$-dense in itself, thus so is $\mathrm{CMin}(B(\mathit{\Gamma}_n, \delta_n))$ since $B(\mathit{\Gamma}_n, \delta_n) \subset U \setminus K$ is open and $\mathit{\Gamma}_n \in \mathrm{CMin}(M)$; also by (5),

$$\mathrm{CMin}(B(\mathit{\Gamma}_n, \delta_n)) = \mathrm{Per}(B(\mathit{\Gamma}_n, \delta_n)) \sqcup \mathrm{Am}(B(\mathit{\Gamma}_n, \delta_n))$$

hence in virtue of (3), Cantor–Dirichlet's Principle implies

(7) $X \in \mathrm{CMin}(\mathit{\Gamma}_n, \delta_n)$ and $\epsilon > 0 \Rightarrow \#\mathrm{Per}(B(X, \epsilon)) = \mathfrak{c}$.

In particular, $P_n := \mathrm{Per}(B(\mathit{\Gamma}_n, \delta_n))$ is an d_H-open, $\mathfrak{c}$-dense in itself subset of $\mathrm{Per}(M)$, d_H-accumulating in $\mathit{\Gamma}_n$ (by (7) and Lemma 4). Let $P := \bigcup_{n \in \mathbb{N}} P_n$. Then since P d_H-accumulates in $\mathit{\Gamma}_n$ and $\mathit{\Gamma}_n \xrightarrow{d_H} Q \in \mathrm{Cci}(\mathrm{bd}\, K)$, it follows that

- $P \subset \mathrm{Per}(M \setminus K)$ *is a* $\mathfrak{c}$*-dense in itself,* d_H*-open subset of* $\mathrm{Per}(M)$, d_H*-accumulating in* $\mathrm{Cci}(\mathrm{bd}\, K)$.

Moreover,

- K *is bi-stable with respect to* $P^* = \bigcup_{\gamma \in P} \gamma$.

Indeed, P^* is a union of periodic orbits and hence invariant. Given any $V \in \mathcal{N}_K$ let $\lambda > 0$ be such that $B(K, \lambda) \subset V$. Since $Q \subset K$ and $d_H(\mathit{\Gamma}_n, Q) \to 0$ and $\delta_n < d_H(\mathit{\Gamma}_n, Q)$, there is an $n_0 \geq 1$ such that:

$$\begin{aligned} n > n_0 \ &\Rightarrow\ d_H(\mathit{\Gamma}_n, Q) < \lambda/2 \\ &\Rightarrow\ \mathit{\Gamma}_n \subset B(Q, \lambda/2) \subset B(K, \lambda/2) \text{ and } \delta_n < \lambda/2 \\ &\Rightarrow\ B(\mathit{\Gamma}_n, \delta_n) \subset B(K, \lambda) \\ &\Rightarrow\ P_n := \mathrm{Per}(B(\mathit{\Gamma}_n, \delta_n)) \subset \mathrm{Per}(B(K, \lambda)) \subset \mathrm{Per}(V). \end{aligned}$$

Since K is compact, by (2) there is a $0 < \delta < \lambda/2$ such that

$$B(K, \delta) \cap \bigcup_{1 \leq n \leq n_0} B[\mathit{\Gamma}_n, \delta_n] = \emptyset. \tag{7.1}$$

Therefore,

$$x \in B(K,\delta) \cap P^* \ \Rightarrow\ x \in P_n^* \text{ for some } n > n_0 \ \Rightarrow\ \mathcal{O}(x) \subset V$$

because $\mathcal{O}(x) \in P_n$ and $P_n \subset \mathrm{Per}(V)$. The bi-stability of K with respect to P^* is proved.

- *For any sequence* $\gamma_n \in P$, $\mathrm{dist}(\gamma_n, K) \to 0 \ \Rightarrow\ \mathrm{period}(\gamma_n) \to +\infty$.

Indeed, as $\mathrm{cl}\, U$ is compact, by (2), each $B[\Gamma_n, \delta_n]$ is a compact disjoint from K, hence given any $n_0 \geq 1$, there is a $\delta > 0$ satisfying the identity (7.1) above, therefore by (6),

$$\begin{aligned} \gamma \in P \text{ and } \gamma \cap B(K,\delta) \neq \emptyset \ &\Rightarrow\ \gamma \in P_n \text{ for some } n > n_0 \\ &\Rightarrow\ \mathrm{period}(\gamma) > n > n_0. \end{aligned}$$

2nd case: There exists a sequence $\Lambda_n \in \mathrm{Per}(U \setminus K)$ d_H-accumulating in $\mathrm{Cci}(\mathrm{bd}\, K)$.

Since by hypothesis, condition **10.2** is not true, an argument completely similar to that used in the 1st case proves that

- *there is a* $\mathfrak{c}$*-dense in itself set* $A \subset \mathrm{Am}(M \setminus K)$, d_H*-open in* $\mathrm{Am}(M)$, d_H*-accumulating in* $\mathrm{Cci}(\mathrm{bd}\, K)$ *and such that* K *is bi-stable with respect to* A^*.

3rd case: There exists a sequence $\Lambda_n \in \mathrm{Eq}(U \setminus K)$ d_H-accumulating in $\mathrm{Eq}(\mathrm{bd}\, K)$.

Since by hypothesis **10.1** is not true, reasoning as in the 1st and 2nd cases, there are sequences $z_n \in M$ and $\epsilon_n > 0$ such that

$$\{z_n\} \in \mathrm{Eq}(U \setminus K), \qquad \mathrm{dist}(z_n, K) \to 0, \qquad \#\mathrm{Eq}(B(z_n, \epsilon_n)) < \mathfrak{c}.$$

By the d_H-closedness of $\mathrm{Eq}(M)$ in conjunction with Lemma 8(2), we may suppose, taking a subsequence, that $\{z_n\} \xrightarrow{d_H} \{z\}$ for some $\{z\} \in \mathrm{Eq}(\mathrm{bd}\, K)$. Now $\mathrm{CMin}(U \setminus K)$ is $\mathfrak{c}$-dense in itself, $\{z_n\} \in \mathrm{CMin}(U \setminus K)$ for all $n \geq 1$, $\mathrm{CMin}(U \setminus K) = \mathrm{Eq}(U \setminus K) \sqcup \mathrm{Per}(U \setminus K) \sqcup \mathrm{Am}(U \setminus K)$ and moreover $\#\mathrm{Eq}(B(z_n, \epsilon_n)) < \mathfrak{c}$, so Cantor–Dirichlet's Principle implies that there is a subsequence (z_{n_i}) such that

$$\#\mathrm{Am}(B(z_{n_i}, \epsilon)) = \mathfrak{c}\ \forall i \geq 1,\ \epsilon > 0 \quad \text{or} \quad \#\mathrm{Per}(B(z_{n_i}, \epsilon)) = \mathfrak{c}\ \forall i \geq 1,\ \epsilon > 0.$$

Thus by Lemma 4,

$$\{z_{n_i}\} \in \mathrm{cl}_H \mathrm{Am}(U \setminus K)\ \forall i \geq 1 \quad \text{or} \quad \{z_{n_i}\} \in \mathrm{cl}_H\, \mathrm{Per}(U \setminus K)\ \forall i \geq 1,$$

and since $\{z_{n_i}\} \xrightarrow{d_H} \{z\} \in \mathrm{Eq}(\mathrm{bd}\, K) \subset \mathrm{Cci}(\mathrm{bd}\, K)$, either there is a sequence in $\mathrm{Am}(U \setminus K)$ d_H-accumulating in $\mathrm{Cci}(\mathrm{bd}\, K)$ or there is a sequence in $\mathrm{Per}(U \setminus K)$ d_H-accumulating in $\mathrm{Cci}(\mathrm{bd}\, K)$. Thus, the 3rd case implies the 1st or the 2nd. On the other hand, as we have seen, the 1st case implies the 2nd and vice versa, hence the 1st and 2nd cases always occur. Therefore if conditions **10.1**, **10.2** and **10.3** are false then **10.4** is true. The proof of Theorem 1 is complete. ■

8. Independent realizations. Examples. With Theorem 1 established, the question of whether all the 28 cases it describes are realizable naturally arises. Furthermore, we may doubt whether all these cases are mutually independent. Let (M, θ) be a C^0 flow on a compact, connected metric space, $\emptyset \neq K \subsetneq M$ a compact invariant set and Σ one of the 28 conditions of Theorem 1. We say that (M, θ), K is an *independent realization of* Σ if condition Σ is satisfied for this choice of M, θ and K but none of the remaining 27 conditions of Theorem 1 holds, for the same M, θ and K. Note that, due to the closedness of the set of equilibria, none of the seven cases **7.1** to **7.6** and **10.1** may occur with K a periodic orbit.

Definition. Let θ be a C^∞ flow on $M = \mathbb{R}^n$ having the origin O_n as an equilibrium. Then O_n is an *n-dimensional singularity of type* Σ if (M, θ), $K = \{O_n\}$ is an independent realization of condition Σ. Analogously, if the same is true for a C^∞ flow on $M = \mathbb{S}^1 \times \mathbb{B}^{n-1}$ having $K = \gamma = \mathbb{S}^1 \times \{O_{n-1}\}$ as a periodic orbit, we say that γ is an *n-dimensional periodic orbit of type* Σ.

We can give the following answer to the questions raised above: if Σ is (any) one of the 28 conditions of Theorem 1, then for some $m \leq 5$ there are n-dimensional singularities of type Σ, for all $n \geq m$. Analogously, if Σ is distinct from **7.1** to **7.6** and **10.1**, then for some $m \leq 5$ there are n-dimensional periodic orbits of type Σ, for all $n \geq m$. Obviously, we may smoothly transfer, via local charts and bump functions, these n-dimensional singularities and periodic orbits to any open sets of arbitrary n-manifolds. In the case of periodic orbits, we can make them coincide with any smoothly embedded circles, having trivial neighbourhood in the manifold, regardless of their homotopy class. Table 1 summarizes the author's present knowledge concerning the above answer.

Conjecture. *In each dimension $n \geq 3$, there are smooth singularities and periodic orbits of all types described in Theorem* 1 (*with the obvious seven exceptions in the case of periodic orbits*).

As indicated in Table 1, this has been established for $n \geq 5$, with most cases already occurring in lower dimensions. It is easily seen that the lower bounds m given in Table 1 cannot be reduced whenever $m \leq 3$.

Here, for the sake of brevity, we shall confine our attention to cases **4**, **5** and **6**, which are the simplest ones exhibiting less known and more interesting dynamical phenomena (orbits of infinite height). In order to keep the presentation both within moderate proportions and in harmony with

Table 1. Least known dimension m for each type is given in the $n \geq m$ columns.

C^∞ n-singularities	exist for all	C^∞ n-periodic orbits	exist for all
Type	$n \geq m$	*Type*	$n \geq m$
1, 2, 3	$n \geq 1$	1, 2, 3	$n \geq 2$
4, 5, 6	$n \geq 3$	4, 5, 6	$n \geq 4$
7.1–7.3	$n \geq 1$	–	–
7.4–7.6	$n \geq 3$	–	–
8.1–8.3	$n \geq 2$	8.1–8.3	$n \geq 2$
8.4–8.6	$n \geq 4$	8.4–8.6	$n \geq 4$
9.1–9.3	$n \geq 3$	9.1–9.3	$n \geq 3$
9.4–9.6	$n \geq 5$	9.4–9.6	$n \geq 5$
10.1	$n \geq 1$	–	–
10.2	$n \geq 2$	10.2	$n \geq 2$
10.3–10.4	$n \geq 3$	10.3–10.4	$n \geq 3$

the rest of the paper, we shall make some concessions concerning the class of differentiability of the flows constructed: our phase space M will be a compact, connected invariant subset of a smooth flow ϕ on $\mathbb{S}^3$, $\theta = \phi|_{\mathbb{R}\times M}$ and $K = \{p\}$, p an equilibrium of (M, ϕ). Hence, while being of class C^0, θ is a subflow of a C^∞ flow ϕ. We call independent realizations of this kind *subsmooth independent realizations*, and the manifold carrying the larger smooth flow ϕ its *ambient manifold*; moreover, ϕ is the *ambient flow.*

We shall first produce an example of a subsmooth independent realization for condition **6**, with K an equilibrium orbit ([17]).

Example 1 (Subsmooth independent realization of condition **6** with M an orbit closure of a C^∞ flow ζ^t on $\mathbb{S}^3 \subset \mathbb{R}^4$, $\theta = \zeta^t|_{\mathbb{R}\times M}$ and $K = \{(0,0,0,1)\} \subsetneq M$ an equilibrium orbit). Our point of departure is a beautiful example, due to Beniere and Meigniez [beni], of a smooth (C^∞) complete vector field υ generating a flow without minimal sets on a noncompact, orientable surface $\mathfrak{M}$ of infinite genus. The set $E(\mathfrak{M})$ of end points is homeomorphic to $\Delta := \{0\} \cup \{n^{-1} : n \in \mathbb{N}\} \subset \mathbb{R}$ and all end points are *flat* ([18]), except the non-

([17]) Vector fields on submanifolds $M \subset \mathbb{R}^n$ will always be represented in the usual abridged form $X : M \to \mathbb{R}^n$, i.e. instead of considering $\upsilon : M \to TM \subset T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, we work with $X = \pi_2 \circ \upsilon$, where π_2 is the projection onto the 2nd factor. When giving examples of flows generated by C^r ($r \geq 1$) vector fields on manifolds M, often the vector field in question is indicated as a subscript, e.g. we write $\omega_X(z)$ for the ω-limit set of z in the flow X^t generated by $X \in \mathfrak{X}^r(M)$. We use these notations freely (with the subscript indicating either the flow or the generating vector field) since no risk of ambiguity arises.

([18]) An end point $e \in E(\mathfrak{M})$ is *flat* if it has a neighbourhood homeomorphic to $\mathbb{R}^2$ in the end-points compactification $\mathfrak{M}^\propto = \mathfrak{M} \sqcup E(\mathfrak{M})$ of $\mathfrak{M}$. Richards [rich] calls such an end point *planar*. Beniere and Meigniez [beni] designate by M our surface $\mathfrak{M}$.

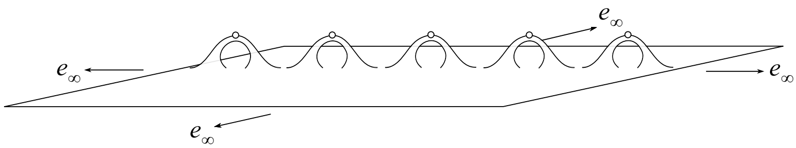

Fig. 8.1. The infinite genus smooth surface $S \subset \mathbb{R}^3$

isolated one. We shall first construct a smoothly embedded surface $S \subset \mathbb{R}^3$ that is smoothly diffeomorphic to $\mathfrak{M}$. Make the following smooth surgery within the ambient manifold $\mathbb{R}^3$: to the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ smoothly add denumerably many handles as shown in Fig. 8.1. From each handle remove one point e_n, $n \in \mathbb{N}$. We obtain a nonclosed, smooth surface $S \subset \mathbb{R}^3$. Just like $\mathfrak{M}$, S is orientable and of infinite genus, with all ends isolated and flat except one, e_∞, which is both nonisolated and nonflat. This implies that its end points set $E(S)$ is also homeomorphic to $\varDelta$ (see above), hence there is a homeomorphism $\xi : E(\mathfrak{M}) \to E(S)$ sending the unique nonflat end of $\mathfrak{M}$ to the unique nonflat end of S. By Kerekjarto's Theorem (see e.g. [rich, p. 262], [beni, p. 26]) the surfaces $\mathfrak{M}$ and S are homeomorphic ([19]), hence, as is well known, smoothly diffeomorphic. Let $f : \mathfrak{M} \to S$ be a smooth diffeomorphism defining an embedding $\mathfrak{M} \hookrightarrow \mathbb{R}^3 \supset S$ and inducing the smooth complete tangent vector field $X := f_* v$ on $S \subset \mathbb{R}^3$. Just as for v, the flow X^t has no minimal sets (f realizes a smooth flow conjugation).

Definition. Let θ be a C^0 flow on a metric space M. A point $x \in M$ is called a *limit point* of (M, θ) if x belongs to the α-limit set or to the ω-limit set of some point of M. In this case the orbit $\mathcal{O}(x)$ is called a *limit orbit* of the flow. We denote the set of limit points of the flow (M, θ) by $\varUpsilon_\theta$, and if the flow is given by a vector field v, by $\varUpsilon_v$.

From the inductively constructed tangentially orientable foliated atlas of $\mathfrak{M}$ corresponding to the vector field v (given in [beni]), it is easily seen that:

- each limit point $x \in \mathfrak{M}$ has nonvoid α-limit and ω-limit sets and both the positive and negative orbit of x accumulate in the unique nonisolated end of $\mathfrak{M}$ and in no other end of this surface.

Since a homeomorphism between noncompact surfaces uniquely extends to a homeomorphism between their respective end-points compactifications, it follows that in the flow X^t generated by $X \in \mathfrak{X}^\infty(S)$, both the α_X-limit and

([19]) We need not care about nonorientable ends since there are none: both $\mathfrak{M}$ and S are orientable.

ω_X-limit sets of each limit point of (S, X^t) are closed, unbounded ([20]) subsets of $\mathbb{R}^3$. Let U be an open normal tubular neighbourhood of S in $\mathbb{R}^3$ (indeed a trivial 1-dimensional vector bundle over S). Extend $X := f_*\upsilon \in \mathfrak{X}^\infty(S)$ to a nonsingular vector field $X_0 \in \mathfrak{X}^\infty(U)$ defining

$$X_0 : U \to \mathbb{R}^3, \quad z \mapsto X \circ \pi(z),$$

where $\pi : U \to S$ is the canonical smooth submersion (orthogonal projection of U over S). Let $p := (0,0,0,1)$, $O_4 := (0,0,0,0)$, $\varphi : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{p\}$ be the inverse stereographic projection ($\mathbb{R}^3$ identified with $\mathbb{R}^3 \times \{-1\}$). Then φ induces the smooth vector field $\varphi_* X_0$ on the open subset $\varphi(U)$ of $\mathbb{S}^3$.

By Kaplan's Smoothing Theorem [kapl, p. 157], there is a scalar function $\lambda \in C^\infty(\mathbb{S}^3, [0,1])$ with $\lambda^{-1}(0) = \mathbb{S}^3 \setminus \varphi(U)$ and such that

$$\begin{aligned} \zeta : \mathbb{R}^4 \supset \mathbb{S}^3 \to \mathbb{R}^4,& \\ s \mapsto O_4 & \quad \text{on } \mathbb{S}^3 \setminus \varphi(U), \\ s \mapsto \lambda\varphi_* X_0(s) & \quad \text{on } \varphi(U), \end{aligned}$$

defines a smooth vector field (generating the ambient flow) on $\mathbb{S}^3$ (the ambient manifold). The smoothly embedded surface $\varphi(S)$ is invariant under the flow ζ^t ($\varphi|_S$ realizes a smooth flow conjugation between the global flow (S, X^t) and $(\varphi(S), (\varphi_* X)^t)$; moreover $\operatorname{im} \lambda|_{\varphi(S)} \subset \,]0,1]$, thus $\lambda\varphi_* X = \zeta|_{\varphi(S)}$ is necessarily a complete vector field, topologically equivalent to $X \in \mathfrak{X}^\infty(S)$ via the smooth diffeomorphism φ).

Let $q \in \Upsilon_X$ and $z := \varphi(q)$. Recall that $\alpha_X(q)$, $\omega_X(q)$ and $\operatorname{cl} \mathcal{O}_X(q)$ are unbounded, closed subsets of $\mathbb{R}^3$, so for all such q and z,

$$\begin{aligned} &\alpha_\zeta(z) = \varphi(\alpha_X(q)) \sqcup \{p\}, \quad \omega_\zeta(z) = \varphi(\omega_X(q)) \sqcup \{p\}, \\ &\operatorname{cl} \mathcal{O}_\zeta(z) = \varphi(\operatorname{cl} \mathcal{O}_X(q)) \sqcup \{p\}. \end{aligned} \tag{8.1}$$

Let $M := \operatorname{cl} \mathcal{O}_\zeta(z)$, $\theta := \zeta^t|_{\mathbb{R}\times M}$ and $K := \{p\}$. Then θ is a C^0 flow on the compact, connected metric space $M \subset \mathbb{S}^3$ (with the Euclidean metric of $\mathbb{R}^4 \supset \mathbb{S}^3$) and K is a compact, invariant proper subset of M. Now with respect to the (sub)flow (M, θ), it is clear from (8.1) that every $y \in M \setminus K = M \setminus \{p\}$ belongs to $A_\theta^-(K) \cap A_\theta^+(K)$ as $M = \varphi(\operatorname{cl} \mathcal{O}_X(q)) \sqcup \{p\}$ and $\operatorname{cl} \mathcal{O}_X(q) = \mathcal{O}_X(q) \cup \alpha_X(q) \cup \omega_X(q) \subset \Upsilon_X$. Obviously K is isolated from minimals in (M, θ) and no $x \in M \setminus K$ has its α_θ-limit set or its ω_θ-limit set contained in K (i.e. equal to $\{p\}$). This immediately implies that, for this choice of M, θ and K, none of conditions of Theorem 1, except **6**, can hold. Therefore by Theorem 1 the above M, θ and K necessarily provide

([20]) Both these sets are closed subsets of S that do not accumulate in the isolated ends e_n, $n \in \mathbb{N}$, of this surface and the closure of S in $\mathbb{R}^3$ equals $S \sqcup \{e_n : n \in \mathbb{N}\}$. Their unboundedness also follows from the fact that a closed, bounded subset of $\mathbb{R}^3$ is compact and thus if it is a nonvoid, invariant subset of the flow X^t, then it must contain a minimal set of it. But X^t has no minimal sets.

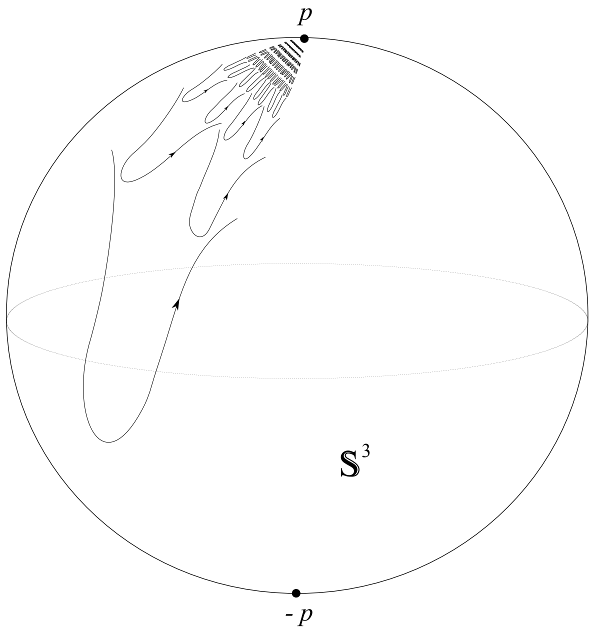

Fig. 8.2. K-tree in the smooth flow $(\mathbb{S}^3, \zeta^t)$, $K = \{p\}$

a subsmooth independent realization of condition **6**, with ambient manifold $\mathcal{M} = \mathbb{S}^3$ (see Fig. 8.2). This can be easily verified directly: the existence of a K-tree with $K = \{p\}$ and $\gamma_0 = \mathcal{O}_\theta(z)$ is now a straightforward inductive consequence of Lemma 3, since every $y \in M \setminus K$ belongs to $A^-_\theta(K) \cap A^+_\theta(K)$ and K is consequently nonstagnant in (M, θ).

The simplest way to get the analogous subsmooth independent realization of condition (6), with K a periodic orbit, is to form the product vector field (identifying $\mathbb{R}^2$ with $\mathbb{C}$),

$$\upsilon : \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{C} \times \mathbb{R}^4, \qquad (z_1, z_2) \mapsto (iz_1, \zeta(z_2)),$$

and then take $M := \operatorname{cl} \mathcal{O}_\upsilon(z_1, z_2)$, where $z_1 \in \mathbb{S}^1$, $z_2 \in \varphi(\varUpsilon_X) \subset \mathbb{S}^3$, $\theta := \upsilon^t|_{\mathbb{R} \times M}$ and $K := \mathbb{S}^1 \times \{p\}$ (a periodic orbit, since $p \in \operatorname{Sing}(\zeta)$).

We will now briefly indicate how to obtain a subsmooth independent realization of condition **5** with K an equilibrium orbit. For K a periodic orbit, we only have to proceed as above. The analogous subsmooth independent realizations of condition **4** are obtained by time-reversing those above.

Example 2. A subsmooth independent realization of condition **5** with K an equilibrium orbit and ambient manifold $\mathbb{S}^3$ is achieved through a simple (and obvious) modification of Beniere and Meigniez's construction: in their paper [beni, p. 23, bottom], the cut-and-paste operation that defines M_1 is performed *only* for each $p \in \mathbb{Z}^+_0$ (and not for all $p \in \mathbb{Z}$). Following, with this exception, their construction, we finally obtain a smooth orientable surface $\mathfrak{N}$ of infinite genus, again with its end points set homeomorphic to $\Delta := \{0\} \cup \{n^{-1} : n \in \mathbb{N}\} \subset \mathbb{R}$ and all ends flat except the nonisolated one. Again, by Kerekjarto's Theorem, this surface is smoothly diffeomorphic to the surface $\mathfrak{M}$ of Example 1 (i.e. to the original surface carrying a flow without minimal set constructed in [beni]) and hence to $S \subset \mathbb{R}^3$.

Now $\mathfrak{N}$ carries a smooth vector field υ whose flow is no longer without minimal sets, as there are points $x \in \mathfrak{N}$ with $\alpha(x) = \emptyset = \omega(x)$. However for each limit point $x \in \mathfrak{N}$ of the flow υ^t, it is easily seen that $\alpha(x) = \emptyset$, $\omega(x) \neq \emptyset$ and both the positive and negative orbits of x accumulate in the unique nonisolated end of $\mathfrak{N}$ and in no other end of this surface. As in Example 1, we again have a smooth diffeomorphism $f : \mathfrak{N} \to S \subset \mathbb{R}^3$ defining an embedding $\mathfrak{N} \hookrightarrow \mathbb{R}^3$ and inducing a complete tangent vector field $X := f_*\upsilon$ on $S \subset \mathbb{R}^3$. Then for each limit point x of (S, X^t), $\omega_X(x)$ is a closed, unbounded subset of $\mathbb{R}^3$ and $\lim_{t\to-\infty} \|X^t(x)\| = +\infty$ ($\|\cdot\|$ being the Euclidean norm on $\mathbb{R}^3$), i.e. the point x escapes to infinity on $\mathbb{R}^3$ as $t \to -\infty$. Then proceeding exactly as in Example 1, for each $q \in \Upsilon_X$, letting $z := \varphi(q)$ we have

$$\begin{aligned}&\alpha_\zeta(z) = \{p\}, \quad \omega_\zeta(z) = \varphi(\omega_X(q)) \sqcup \{p\},\\ &\operatorname{cl}\mathcal{O}_\zeta(z) = \varphi(\operatorname{cl}\mathcal{O}_X(q)) \sqcup \{p\}.\end{aligned}$$

For such a $z \in \varphi(\Upsilon_X) \subset \mathbb{S}^3$, letting $M := \operatorname{cl}\mathcal{O}_\zeta(z) \subset \mathbb{S}^3$, $\theta := \zeta^t|_{\mathbb{R}\times M}$ and $K := \{p\}$, we then have

$$y \in M \setminus K \;\Rightarrow\; y \in B_\theta^-(K) \cap A_\theta^+(K)$$

It is now immediate to verify that for these M, θ, K, none of the 28 conditions of Theorem 1, except of condition **5**, can hold and therefore (M, θ), K provide a subsmooth independent realization of condition **5** (see Example 1).

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References

[al] J. W. Alexander, *A proof and extension of the Jordan–Brouwer separation theorem*, Trans. Amer. Math. Soc. 23 (1922), 333–349.

[bend] I. Bendixson, *Sur les courbes définies par des équations différentielles*, Acta Math. 24 (1901), 1–88.

[beni] J.-C. Beniere and G. Meigniez, *Flows without minimal set*, Ergodic Theory Dynam. Systems 19 (1999), 21–30.

[bha1] N. Bhatia, *Dynamical systems*, in: G. Szegö and H. Kuhn (eds.), Mathematical Systems Theory and Economics I, Lecture Notes in Oper. Res. Math. Econom. 11, Springer, 1969, 1–10.

[bha2] N. Bhatia, *Attraction and nonsaddle sets in dynamical systems,* J. Differential Equations 8 (1970), 229–249.

[bhsz] N. Bhatia and G. Szegö, *Stability Theory of Dynamical Systems*, Springer, Berlin, 1970.

[brid] D. Bridges, *Foundations of Real and Abstract Analysis*, Grad. Texts in Math. 174, Springer, 1998.

[bura] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Grad. Stud. Math. 33, Amer. Math. Soc., Providence, RI, 2001.

[butl] G. Butler and P. Waltman, *Persistence in dynamical systems*, J. Differential Equations 63 (1986), 255–263.

[carl] D. Carlson, *A generalization of Vinograd's theorem for dynamical systems*, J. Differential Equations 11 (1972), 193–201.

[cies] K. Ciesielski, *The Poincaré–Bendixson Theorem: from Poincaré to the XXIst century*, Cent. Eur. J. Math. 10 (2012), 2110–2128.

[desb] D. Desbrow, *On unstable invariant sets*, Funkcial. Ekvac. 13 (1970), 109–126.

[free] H. Freedman, S. G. Ruan and M. Tang, *Uniform persistence and flows near a closed positively invariant set*, J. Dynam. Differential Equations 6 (1994), 583–600.

[gach] B. Garay and L. Chua, *Isles of Eden and the ZUK theorem in $\mathbb{R}^d$*, Int. J. Bifur. Chaos Appl. Sci. Engrg. 18 (2008), 2951–2963.

[gaho] B. Garay and J. Hofbauer, *Robust permanence for ecological differential equations, minimax, and discretizations*, SIAM J. Math. Anal. 34 (2003), 1007–1039.

[göde] K. Gödel, *The consistency of the axiom of choice and of the generalized continuum-hypothesis*, Proc. Nat. Acad. Sci. USA 24 (1938), 556–557.

[hofb] J. Hofbauer, *A unified approach to persistence*, in: Evolution and Control in Biological Systems (Laxenburg, 1987), Acta Appl. Math. 14 (1989), 11–22.

[inab] T. Inaba, *An example of a flow on a non-compact surface without minimal set*, Ergodic Theory Dynam. Systems 19 (1999), 31–33.

[kapl] W. Kaplan, *The singularities of continuous functions and vector fields*, Michigan Math. J. 9 (1962), 151–159.

[koba] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry Vol. I*, Interscience Publ., 1963.

[levy] A. Levy, *Basic Set Theory*, Springer, Berlin, 1979.

[rich] I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963), 259–269.

[s1] T. Saito, *On the flow outside an isolated minimal set*, in: Proc. USA-Japan Seminar on Differential and Functional Equations (Minneapolis, 1967), Benjamin, New York, 1967, 301–311.

[s2] T. Saito, *Isolated minimal sets*, Funkcial. Ekvac. 11 (1968), 155–167.

[s3] T. Saito, *On a compact invariant set isolated from minimal sets*, Funkcial. Ekvac. 12 (1969), 193–203.

[s4] T. Saito, *A supplement to the paper: "On a compact invariant set isolated from minimal sets"*, Funkcial. Ekvac. 13 (1970), 127–129.

[s5] T. Saito, *On the flow outside a nonsaddle compact invariant set*, Keio Engrg. Rep. 29 (1976), no. 6, 61–71.

[s6] T. Saito, *On the flow outside a compact invariant set*, Keio Math. Sem. Rep. No. 2 (1977), 69–74.

[schw] P. Schweitzer (ed.), *Differential Topology, Foliations and Gelfand–Fuks Cohomology* (Rio de Janeiro, 1976), Springer, 1976.

[teix] P. Teixeira, *Non-hyperbolic singularities and periodic orbits: an exact topological classification*, arXiv (in preparation).

[ura2] T. Ura, *Sur le courant extérieur à une région invariante*, Funkcial. Ekvac. 9 (1966), 171–179.

[ura1] T. Ura, *On the flow outside a closed invariant set; stability, relative stability and saddle sets*, Contributions Differential Equations 3 (1964), 249–294.

[urki] T. Ura et I. Kimura, *Sur le courant extérieur à une région invariante: Théorème de Bendixson*, Comment. Math. Univ. St. Pauli 8 (1960), 23–39.

[zubo] V. Zubov, *The Methods of A. M. Lyapunov and their Applications*, Izdat. Leningrad. Univ., Moscow, 1957 (in Russian).

Pedro Teixeira
Centro de Matemática da Universidade do Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal
E-mail: pedro.teixeira@fc.up.pt

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