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Separating by G_δ-sets in finite powers of ω_1

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Abstract. It is known that all subspaces of ω_1^2 have the property that every pair of disjoint closed sets can be separated by disjoint G_δ-sets (see [4]). It has been conjectured that all subspaces of ω_1^n also have this property for each $n < \omega$. We exhibit a subspace of $\{\langle\alpha,\beta,\gamma\rangle \in \omega_1^3 : \alpha \le \beta \le \gamma\}$ which does not have this property, thus disproving the conjecture. On the other hand, we prove that all subspaces of $\{\langle\alpha,\beta,\gamma\rangle \in \omega_1^3 : \alpha < \beta < \gamma\}$ have this property.

1. Introduction. A topological space X is said to be *subnormal* if every pair of disjoint closed sets can be separated by disjoint G_δ-sets. A *subshrinking* of an open cover $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ of X is an F_σ-cover $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of X such that $F_i \subseteq U_i$ for each $i \in \mathcal{I}$. A space X is said to be *subshrinking* if every open cover has a subshrinking. It is easy to see that every subshrinking space is subnormal. For these properties, see [2] or [7].

It is well known that ω_1^2 is normal but $\omega_1 \times (\omega_1 + 1)$ is not subnormal. Moreover it is known [5] that there is a nonnormal subspace of ω_1^2. For example, $X = A \times B$, where A and B are disjoint stationary sets in ω_1, is such a space. However, in [4] an unexpected result is proved that all subspaces of ω_1^2 are subshrinking, so subnormal. It has been conjectured that all subspaces of ω_1^n are subnormal for every $n < \omega$. In Section 4, we prove that this conjecture is false.

Theorem 1.1. *There exists a nonsubnormal subspace of ω_1^3.*

On the other hand, all subspaces of

$$\omega_1^n|_< = \{s \in \omega_1^n : s(i) < s(j) \text{ for each } i < j < n\}$$

are subnormal for an arbitrary $n < \omega$. We prove this in Section 5.

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Theorem 1.2. *Every subspace of $\omega_1^n|_<$ is subshrinking, so subnormal, for every $n < \omega$.*

To prove these theorems, we show some combinatorial lemmas in Section 3. We use the concept of trees of finite sequences and state the Pressing Down Lemma in terms of trees. The Pressing Down Lemma in a more general situation appears in [1].

2. Preliminaries. We identify an ordinal α with the set of all ordinals less than α. We do not distinguish natural numbers from finite ordinals. Hence a natural number n is the set $\{0, 1, \ldots, n-1\}$. A sequence s of finite length n is a function of domain n, so $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$. In particular, A^n denotes the set of all functions from $\{0, 1, \ldots, n-1\}$ into A.

For each sequence s, $\mathrm{lh}(s)$ denotes the length of s, and $\mathrm{ran}(s)$ denotes the set $\{s(i) : i < \mathrm{lh}(s)\}$. Let A be a set of sequences of ordinals. We use the following notations:

- $A|_< = \{s \in A : s(i) < s(j) \text{ for each } i < j < \mathrm{lh}(s)\}$.
- $A|_\leq = \{s \in A : s(i) \leq s(j) \text{ for each } i < j < \mathrm{lh}(s)\}$.
- For $n < \omega$, $\alpha^{\leq n}$ and $\alpha^{<n}$ denote the sets $\bigcup_{k\leq n} \alpha^k$ and $\bigcup_{k<n} \alpha^k$ respectively.

Throughout this paper, each ordinal α is considered to be a space with the order topology and each subset of α^n is considered to be a subspace of the product space.

A family $\mathcal{A} = \langle A_i : i \in \mathcal{I}\rangle$ of subsets of a space is called *σ-locally finite* (respectively *σ-discrete*) if $\mathcal{I}$ can be represented as $\bigcup_{j\in\mathcal{J}} \mathcal{I}_j$ for some $\mathcal{J}$ with $|\mathcal{J}| \leq \omega$ such that $\mathcal{A}{\restriction}\mathcal{I}_j = \langle A_i : i \in \mathcal{I}_j\rangle$ is locally finite (respectively discrete) for each $j \in \mathcal{J}$.

We will need the following two facts about σ-local finiteness and the subshrinking property. Their verification is routine.

Lemma 2.1. *Let X be a topological space and $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ a σ-locally finite, closed cover of X such that for each $i \in \mathcal{I}$, F_i is subshrinking. Then X is also subshrinking.*

Lemma 2.2. *Let X be a topological space, $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ a point finite family of open sets, and $\mathcal{G} = \langle G_i : i \in \mathcal{I}\rangle$ a family of G_δ-sets of X such that $G_i \subseteq U_i$ for each $i \in \mathcal{I}$. Then the union of $\mathcal{G}$ is also a G_δ-set.*

3. Trees and stationary sets

Definition 3.1. Let λ be a regular uncountable cardinal and $n < \omega$. A set $X \subseteq \lambda^n$ is *stationary* if $C^n \cap X \neq \emptyset$ for every closed unbounded (club) subset C of λ.

The empty sequence $\emptyset$ is considered to be the unique sequence of length 0, and $\lambda^0 = \{\emptyset\}$. Moreover $X \subseteq \lambda^0$ is stationary if and only if $\emptyset \in X$.

DEFINITION 3.2. A set T of sequences of ordinals less than λ is called a *tree of sequences on* λ if $s{\restriction}k \in T$ for each $s \in T$ and $k \leq \mathrm{lh}(s)$. We only use trees of finite sequences, so we will omit "of sequences" from now on.

Let λ be a regular uncountable cardinal and $n < \omega$. For a tree $T \subseteq \lambda^{\leq n}$ and $j \leq n$, $T \cap \lambda^j$ (respectively $T \cap \lambda^{<j}$, $T \cap \lambda^{\leq j}$) is denoted by $\mathrm{Lv}_j(T)$ (respectively $\mathrm{Lv}_{<j}(T)$, $\mathrm{Lv}_{\leq j}(T)$).

An *n-stationary tree* (respectively *n-cofinal tree*) *on* λ is a tree $T \subseteq \lambda^{\leq n}$ such that $\emptyset \in T$ and $\{\alpha < \lambda : s^\frown\langle\alpha\rangle \in T\}$ is stationary (respectively cofinal) in λ for each $s \in \mathrm{Lv}_{<n}(T)$.

Let $X \subseteq \lambda^n$. A function $f : X \to (\lambda \cup \{-\infty\})^n$ is called *regressive* if for each $s \in X$ and $k < n$, $f(s)(k) < s(k)$. ($-\infty$ is considered to be less than every ordinal.)

LEMMA 3.1. *Let λ be a regular uncountable cardinal and $n < \omega$.*

(1) *If T is an n-stationary tree on λ, then $\mathrm{Lv}_k(T)|_<$ is stationary in λ^k for all $k \leq n$.*

(2) *If $X \subseteq \lambda^n|_<$ is stationary, then there exists an n-stationary tree T on λ such that $\mathrm{Lv}_n(T) \subseteq X$.*

(3) (The Pressing Down Lemma) *If T is an n-stationary tree on λ and $f : \mathrm{Lv}_n(T) \to (\lambda \cup \{-\infty\})^n$ is a regressive function, then there exist an n-stationary subtree U of $T|_<$ and a function $g : \mathrm{Lv}_{<n}(U) \to \lambda \cup \{-\infty\}$ such that $f(s)(k) = g(s{\restriction}k)$ for each $s \in \mathrm{Lv}_n(U)$ and $k < n$.*

Proof. (1) is trivial.

(2) We define $X_k \subseteq \lambda^k|_<$ for each $k \leq n$ inductively. Put $X_n = X$ and

$$X_k = \{s \in \lambda^k|_< : \{\alpha < \lambda : s^\frown\langle\alpha\rangle \in X_{k+1}\} \text{ is stationary}\}$$

if $k < n$. We show that X_k is stationary by downward induction. First, X_n is stationary by the assumption. Assume that $k < n$ and X_{k+1} is stationary. For each $s \in \lambda^k|_< - X_k$, pick a club set C_s disjoint from $\{\alpha < \lambda : s^\frown\langle\alpha\rangle \in X_{k+1}\}$ and put

$$C = \{\alpha < \lambda : \alpha \in C_s \text{ for all } s \in \alpha^k|_< - X_k\}.$$

Note that C is a club subset of λ. If D is a club subset of λ, then there is an $s \in X_{k+1} \cap (C \cap D)^{k+1}$ since X_{k+1} is stationary, and such an s satisfies $s{\restriction}k \in X_k$ since $s(k) \in C$ and $s{\restriction}k \in s(k)^k|_<$. Hence, X_k is stationary.

Now $T = \{s \in \lambda^{\leq n}|_< : s{\restriction}k \in X_k \text{ for all } k \leq \mathrm{lh}(s)\}$ satisfies the required condition.

(3) Pick a regressive function $f_k : \mathrm{Lv}_k(T) \to (\lambda \cup \{-\infty\})^k$ for each $k \leq n$ inductively. Put $f_n = f$. Assume that $k < n$ and f_{k+1} is regressive. For each $s \in \mathrm{Lv}_k(T)$, $A_s = \{\alpha < \lambda : s^\frown\langle\alpha\rangle \in T\}$ is stationary and $f_{k+1}(s^\frown\langle\alpha\rangle)(k) < \alpha$

for each $\alpha \in A_s$. By the Pressing Down Lemma for λ, there are a stationary set $B_s \subseteq A_s$ and a $\xi_s \in \lambda \cup \{-\infty\}$ such that $f_{k+1}(s\hat{\ }\langle\alpha\rangle)(k) = \xi_s$ for all $\alpha \in B_s$. Since f_{k+1} is regressive, $|\{f_{k+1}(s\hat{\ }\langle\alpha\rangle){\restriction}k : \alpha \in B_s\}| < \lambda$. By the completeness of the club filter, there are a stationary set $N_s \subseteq B_s$ and $f_k(s) \in (\lambda \cup \{-\infty\})^k$ such that $f_{k+1}(s\hat{\ }\langle\alpha\rangle){\restriction}k = f_k(s)$ for all $\alpha \in N_s$. It is easily seen that f_k is regressive.

Put $U = \{s \in T|_< : s(k) \in N_{s\restriction k}$ for all $k < \mathrm{lh}(s)\}$. Then U is an n-stationary subtree of $T|_<$. Let $g(s) = \xi_s$ for each $s \in \mathrm{Lv}_{<n}(U)$. Inductively, $f_k(s)(i) = g(s{\restriction}i)$ for all $i < k \le n$ and $s \in \mathrm{Lv}_k(U)$. So $f(s)(k) = f_n(s)(k) = g(s{\restriction}k)$ for all $s \in \mathrm{Lv}_n(U)$ and $k < n$. ∎

Lemma 3.2. *Let λ be a regular uncountable cardinal, $n < \omega$, T an n-cofinal tree on λ, and $\mathcal{H} = \langle H_i : i \in \mathcal{I}\rangle$ a family of subsets of $\mathrm{Lv}_n(T)$ such that $\bigcup\mathcal{H} = \mathrm{Lv}_n(T)$. Then there exist an n-cofinal subtree U of T, $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0\rangle$ of elements of U satisfying the following conditions.*

(a) *For each $t \in \mathrm{Lv}_n(U)$, there is a unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t$.*
(b) *For each $i \in \mathcal{I}_0$ and $t \in \mathrm{Lv}_n(U)$, if $t_i \subseteq t$ then $t \in H_i$.*

Moreover, if $|\mathcal{I}| < \lambda$ then we can pick $\mathcal{I}_0$ as a singleton $\{i_0\}$ such that $t_{i_0} = \emptyset$.

Proof. By induction on n. If $n = 0$ then the statement is trivial. Assume that $n = n' + 1$. Put $T' = \mathrm{Lv}_{\le n'}(T)$ and $A_i(t) = \{\alpha < \lambda : t\hat{\ }\langle\alpha\rangle \in H_i\}$ for each $t \in \mathrm{Lv}_{n'}(T)$ and $i \in \mathcal{I}$. Then T' is an n'-cofinal tree. Define $\mathcal{H}' = \langle H_i' : i \in \mathcal{I}\rangle$ by

$$H_i' = \{t \in \mathrm{Lv}_{n'}(T) : A_i(t) \text{ is cofinal in } \lambda\}$$

for each $i \in \mathcal{I}$. Since $\mathrm{Lv}_{n'}(T') = \mathrm{Lv}_{n'}(T) = \bigcup\mathcal{H}' \cup (\mathrm{Lv}_{n'}(T) - \bigcup\mathcal{H}')$, there is an n'-cofinal subtree T'' of T' such that either $\mathrm{Lv}_{n'}(T'') \subseteq \bigcup\mathcal{H}'$ or $\mathrm{Lv}_{n'}(T'') \subseteq \mathrm{Lv}_{n'}(T) - \bigcup\mathcal{H}'$, by the "moreover" part of the inductive hypothesis. Moreover, if $|\mathcal{I}| < \lambda$ then $\mathrm{Lv}_{n'}(T) = \bigcup\mathcal{H}'$, so the latter case does not happen.

Case 1: $\mathrm{Lv}_{n'}(T'') \subseteq \bigcup\mathcal{H}'$. By the inductive hypothesis, there exist an n'-cofinal subtree U' of T'', $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0\rangle$ of elements of U' satisfying the following conditions:

(a′) For each $t \in \mathrm{Lv}_{n'}(U')$, there is a unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t$.
(b′) For each $i \in \mathcal{I}_0$ and $t \in \mathrm{Lv}_{n'}(U')$, if $t_i \subseteq t$ then $t \in H_i'$.

Moreover, if $|\mathcal{I}| < \lambda$ then we can pick $\mathcal{I}_0$ as a singleton $\{i_0\}$ such that $t_{i_0} = \emptyset$.

Put

$$U = U' \cup \{t\hat{\ }\langle\alpha\rangle : t \in \mathrm{Lv}_{n'}(U'),\ \alpha \in A_i(t) \text{ for all } i \in \mathcal{I}_0 \text{ such that } t_i \subseteq t\}.$$

It is easy to check that $U, \mathcal{I}_0$, and $\langle t_i : i \in \mathcal{I}_0\rangle$ satisfy the required conditions.

CASE 2: $\mathrm{Lv}_{n'}(T'') \subseteq \mathrm{Lv}_{n'}(T) - \bigcup \mathcal{H}'$. Fix a well ordering $\prec$ on $\mathrm{Lv}_{n'}(T'') \times \lambda$ of order type λ. For each $\langle t, \xi\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$, pick an $i(t,\xi) \in \mathcal{I}$ and an $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ inductively. Assume that $\langle t, \xi\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ and that $i(t',\xi') \in \mathcal{I}$ and $\alpha(t',\xi') \in A_{i(t',\xi')}(t')$ are defined for each $\langle t', \xi'\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ such that $\langle t', \xi'\rangle \prec \langle t, \xi\rangle$. The set $\bigcup_{i\in\mathcal{I}} A_i(t)$ is cofinal in λ since $t \in \mathrm{Lv}_{n'}(T)$. On the other hand,

$$\bigcup\{A_{i(t',\xi')}(t) : \langle t', \xi'\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda \text{ and } \langle t', \xi'\rangle \prec \langle t, \xi\rangle\}$$

is not cofinal in λ since $t \notin H'_i$ for every $i \in \mathcal{I}$. So we can pick an $i(t,\xi) \in \mathcal{I}$ and $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ such that $\xi \le \alpha(t,\xi)$ and $\alpha(t,\xi)$ does not belong to the set above.

Put $\mathcal{I}_0 = \{i(t,\xi) : \langle t,\xi\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda\}$ and

$$U = T'' \cup \{t^\frown\langle\alpha(t,\xi)\rangle : t \in \mathrm{Lv}_{n'}(T''),\ \xi < \lambda\}.$$

If $\langle t,\xi\rangle, \langle t',\xi'\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ and $\langle t', \xi'\rangle \prec \langle t, \xi\rangle$, then $\alpha(t,\xi) \in A_{i(t,\xi)}(t)$ but $\alpha(t,\xi) \notin A_{i(t',\xi')}(t)$, so $i(t,\xi) \ne i(t',\xi')$. Hence $\langle t,\xi\rangle \in \mathrm{Lv}_{n'}(T'') \times \lambda$ satisfying $i = i(t,\xi)$ is unique for each $i \in \mathcal{I}_0$. For each $i \in \mathcal{I}_0$, put $t_i = t^\frown\langle\alpha(t,\xi)\rangle$ where $\langle t,\xi\rangle$ is the element of $\mathrm{Lv}_{n'}(T'') \times \lambda$ such that $i = i(t,\xi)$.

It is easy to check that U, $\mathcal{I}_0$, and $\langle t_i : i \in \mathcal{I}_0\rangle$ satisfy the required conditions. ∎

DEFINITION 3.3. Let λ be a regular uncountable cardinal, $n < \omega$, $T \subseteq \lambda^{\le n}$ a tree, and $g : T \to \lambda$ a function. We say that $\langle \gamma, t\rangle$ is a *uniformly n-cofinal subtree of T closed under g* if $\gamma = \langle \gamma_\xi : \xi < \lambda\rangle : \lambda \to \lambda$ is a strictly increasing, continuous sequence, and $t = \langle t(s) : s \in \lambda^{\le n}|_<\rangle$ is a family of elements of T such that:

(i) $\mathrm{lh}(t(s)) = \mathrm{lh}(s)$, $t(s)\restriction k = t(s\restriction k)$ for each $s \in \lambda^{\le n}|_<$, $k \le \mathrm{lh}(s)$,
(ii) $g(t(s)) < \gamma_\xi$ for each $\xi < \lambda$ and $s \in \xi^{\le n}|_<$,
(iii) $\gamma_{s(k)} \le t(s)(k)$ for each $s \in \lambda^{\le n}|_<$ and $k < \mathrm{lh}(s)$.

LEMMA 3.3. *Let λ be a regular uncountable cardinal, $n < \omega$, T an n-cofinal tree on λ, $g : T \to \lambda$ a function, and $E \subseteq \lambda$ a club set. Then there exists a uniformly n-cofinal subtree $\langle \gamma, t\rangle$ of T closed under g such that $\gamma_\xi \in E$ for all $\xi < \lambda$.*

Proof. We define $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ for $s \in \xi^{\le n}|_<$ and $\gamma_\xi \in E$ by induction on $\xi < \lambda$. Assume that $\xi < \lambda$ and that $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ and $\gamma_\zeta \in E$ are defined for all $\zeta < \xi$ and $s \in \zeta^{\le n}|_<$.

First, we define $t(s) \in \mathrm{Lv}_{\mathrm{lh}(s)}(T)$ for $s \in \xi^{\le n}|_< - \bigcup_{\zeta<\xi}(\zeta^{\le n}|_<)$. In case ξ is a limit ordinal, such an s does not exist. In case $\xi = 0$, such an s is only $\emptyset$, and put $t(\emptyset) = \emptyset$. In case $\xi = \zeta + 1$ for some ζ, such an s has length $k+1$ for some $k < n$, $s(k) = \zeta$, and $s\restriction k \in \zeta^k|_<$; so $t(s\restriction k) \in \mathrm{Lv}_k(T)$ is defined and we can pick $t(s) \in \mathrm{Lv}_{k+1}(T)$ such that $t(s)\restriction k = t(s\restriction k)$ and $\gamma_\zeta \le t(s)(k)$ since T is an n-cofinal tree.

Now, $t(s)$ is defined for all $s \in \xi^{\leq n}|_<$. We define $\gamma_\xi \in E$. In case ξ is a limit ordinal, put $\gamma_\xi = \sup\{\gamma_\zeta : \zeta < \xi\}$; since E is a club set, $\gamma_\xi \in E$. In the other case, pick $\gamma_\xi \in E$ such that $\gamma_\xi > \gamma_\zeta$ for every $\zeta < \xi$ and $\gamma_\xi > g(t(s))$ for every $s \in \xi^{\leq n}|_<$.

It is easy to check that $\langle \gamma, t\rangle$ defined as above satisfies the required conditions. ■

4. Nonsubnormal subspaces of ω_1^3. In this section, we prove Theorem 1.1.

THEOREM 4.1. *Let X be a subspace of ω_1^3. If $X|_<$ is stationary in ω_1^3, $X_{0,1} = \{\langle \alpha, \beta\rangle \in \omega_1^2|_< : \langle \alpha, \alpha, \beta\rangle \in X\}$ and $X_{1,2} = \{\langle \alpha, \beta\rangle \in \omega_1^2|_< : \langle \alpha, \beta, \beta\rangle \in X\}$ are stationary in ω_1^2, and $X_{0,1,2} = \{\alpha \in \omega_1 : \langle \alpha, \alpha, \alpha\rangle \in X\}$ is not stationary in ω_1, then X is not subnormal.*

Proof. Pick a club subset C of ω_1 disjoint from $X_{0,1,2}$. Define

$$E = \{\langle \alpha, \beta, \gamma\rangle \in X : \alpha = \beta \text{ and } \gamma \in C\},$$
$$F = \{\langle \alpha, \beta, \gamma\rangle \in X : \beta = \gamma \text{ and } \alpha \in C\}.$$

These are disjoint closed sets. We show that they cannot be separated by disjoint G_δ-sets.

Assume that P_i and Q_i are open subsets of X such that $E \subseteq P_i$ and $F \subseteq Q_i$ for each $i < \omega$. It suffices to show that $\bigcap_{i<\omega} P_i \cap \bigcap_{i<\omega} Q_i \neq \emptyset$.

Since $X|_<, X_{0,1} \cap C^2$ and $X_{1,2} \cap C^2$ are stationary, there are a 3-stationary tree T and 2-stationary trees U, V on ω_1 such that $\mathrm{Lv}_3(T) \subseteq X|_<$, $\mathrm{Lv}_2(U) \subseteq X_{0,1} \cap C^2$, and $\mathrm{Lv}_2(V) \subseteq X_{1,2} \cap C^2$.

Let $i < \omega$. If $\langle \alpha, \beta\rangle \in \mathrm{Lv}_2(U)$, then it also belongs to $X_{0,1} \cap C^2$, so $\langle \alpha, \alpha, \beta\rangle \in E \subseteq P_i$. Since P_i is open, we can pick a regressive function $e_i : \mathrm{Lv}_2(U) \to (\omega_1 \cup \{-\infty\})^2$ satisfying

$$X \cap ((e_i(u)(0), u(0)]^2 \times (e_i(u)(1), u(1)]) \subseteq P_i$$

for each $u \in \mathrm{Lv}_2(U)$. In the same way, we can pick a regressive function $f_i : \mathrm{Lv}_2(V) \to (\omega_1 \cup \{-\infty\})^2$ with

$$X \cap ((f_i(v)(0), v(0)] \times (f_i(v)(1), v(1)]^2) \subseteq Q_i$$

for each $v \in \mathrm{Lv}_2(V)$. By the Pressing Down Lemma, there are 2-stationary subtrees U_i of U, V_i of V, and functions $g_i : \mathrm{Lv}_{<2}(U_i) \to \omega_1 \cup \{-\infty\}$, $h_i : \mathrm{Lv}_{<2}(V_i) \to \omega_1 \cup \{-\infty\}$ such that $e_i(u)(k) = g_i(u \restriction k)$, $f_i(v)(k) = h_i(v \restriction k)$ for every $u \in \mathrm{Lv}_2(U_i), v \in \mathrm{Lv}_2(V_i)$, and $k < 2$.

Pick $t \in \mathrm{Lv}_3(T), u_i \in \mathrm{Lv}_2(U_i)$, and $v_i \in \mathrm{Lv}_2(V_i)$ for $i < \omega$ such that:

(i) $g_i(\emptyset), h_i(\emptyset) < t(0)$ for all $i < \omega$,
(ii) $t(0) \leq v_i(0)$ for all $i < \omega$,
(iii) $t(0) \leq t(1)$ and $h_i(v_i \restriction 1) < t(1)$ for all $i < \omega$,

(iv) $t(1) \le u_i(0)$ for all $i < \omega$,
(v) $t(1) \le t(2)$ and $g_i(u_i{\restriction}1) < t(2)$ for all $i < \omega$,
(vi) $t(2) \le u_i(1), v_i(1)$ for all $i < \omega$.

It follows from $e_i(u_i)(0) = g_i(u_i{\restriction}0) = g_i(\emptyset) < t(0) \le t(1) \le u_i(0)$ and $e_i(u_i)(1) = g_i(u_i{\restriction}1) < t(2) \le u_i(1)$ that

$$t \in X \cap ((e_i(u_i)(0), u_i(0)]^2 \times (e_i(u_i)(1), u_i(1)]) \subseteq P_i.$$

Since $f_i(v_i)(0) = h_i(v_i{\restriction}0) = h_i(\emptyset) < t(0) \le v_i(0)$ and $f_i(v_i)(1) = h_i(v_i{\restriction}1) < t(1) \le t(2) \le v_i(1)$, we have

$$t \in X \cap ((f_i(v_i)(0), v_i(0)] \times (f_i(v_i)(1), v_i(1)]^2) \subseteq Q_i.$$

Hence, $t \in \bigcap_{i<\omega} P_i \cap \bigcap_{i<\omega} Q_i$. ∎

For instance, $X = \{\langle \alpha, \beta, \gamma\rangle \in \omega_1^3 : \alpha \le \beta < \gamma \text{ or } \alpha < \beta \le \gamma\}$ satisfies the assumption of the theorem above. So Theorem 1.1 holds.

5. Canonical subnormal subspaces of ω_1^n. The purpose of this section is to prove Theorem 1.2. We start with an easy fact.

FACT 5.1. *If $X \subseteq \omega_1$ is nonstationary in ω_1, then there is a pairwise disjoint family of clopen, bounded subsets of ω_1 which covers X.*

We show two ways of deriving the subshrinking property of some spaces from the properties of simpler spaces.

LEMMA 5.1. *Let $m \le n < \omega$ and $X \subseteq \omega_1^n|_{\le}$. If $X_m = \{s{\restriction}m : s \in X\}$ is not stationary in ω_1^m, then there exists a σ-discrete, closed cover $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of X such that for each $i \in \mathcal{I}$, there is a $k < m$ such that $\{s(k) : s \in F_i\}$ is bounded in ω_1.*

Proof. By induction on m. Fix an n. In case $m = 0$, if X_m is not stationary, then $X_m = \emptyset$, so $X = \emptyset$. The empty family satisfies the required condition.

Assume that $m < n$ and the statement holds for m. Let $X \subseteq \omega_1^n|_{\le}$ with $X_{m+1} = \{s{\restriction}m+1 : s \in X\}$ nonstationary. There is a club subset C of ω_1 such that $C^{m+1} \cap X_{m+1} = \emptyset$. Put $Y = \{s \in X : s{\restriction}m \in C^m\}$. Then Y is a closed subset of X. Moreover $\{s(m) : s \in Y\}$ is nonstationary in ω_1 since it is disjoint from C. Hence it is covered by a pairwise disjoint family of clopen bounded sets of ω_1 by Fact 5.1. By pulling back this family by the projection, we obtain a pairwise disjoint family $\mathcal{P} = \langle P_j : j \in \mathcal{J}\rangle$ of clopen subsets of X, covering Y, such that for each $j \in \mathcal{J}$, $\{s(m) : s \in P_j\}$ is bounded in ω_1. For each $j \in \mathcal{J}$, $Y \cap P_j$ is a G_δ-set because

$$Y \cap P_j = \bigcap_{\xi \in \mu - C} \bigcap_{k<m} \{s \in X : s(k) \ne \xi\} \cap P_j$$

where $\mu<\omega_1$ satisfies $\{s(m) : s\in P_j\}\subseteq\mu$. By Lemma 2.2, $Y=\bigcup_{j\in\mathcal{J}}(Y\cap P_j)$ is a G_δ-subset of X. Hence, there are a closed cover $\mathcal{E} = \langle E_i : i < \omega\rangle$ of X such that $E_0 = Y$ and $E_i \cap Y = \emptyset$ for every $i < \omega$ except 0. If $i \neq 0$, then $\{s{\restriction}m : s \in E_i\}$ is disjoint from C^m, so nonstationary, hence the inductive hypothesis can be applied to E_i. On the other hand, $\langle Y \cap P_j : j \in \mathcal{J}\rangle$ is a discrete, closed cover of E_0. In any case, there exists a σ-discrete, closed cover $\mathcal{F}_i = \langle F_{i,j} : j \in \mathcal{J}_i\rangle$ of E_i, for every $i < \omega$, such that for each $j \in \mathcal{J}_i$, there is a $k < m + 1$ such that $\{s(k) : s \in F_{i,j}\}$ is bounded in ω_1. Now $\mathcal{F} = \langle F_{i,j} : i < \omega,\ j \in \mathcal{J}_i\rangle$ is a σ-discrete, closed cover satisfying the required condition. Hence the statement also holds for $m + 1$. ∎

COROLLARY 5.1. *Let $n < \omega$ and X a nonstationary subset of ω_1^n. Then there exists a σ-discrete closed cover $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of X such that for each $i \in \mathcal{I}$, there is a $k < n$ such that $\{s(k) : s \in F_i\}$ is bounded in ω_1.*

For the next three lemmas, let T be a fixed n-cofinal tree on ω_1 with $n < \omega$, $g : T \to \omega_1$ a function such that $u(k) \leq g(u)$ for each $u \in T$ and $k < \mathrm{lh}(u)$, and $\langle\gamma, t\rangle$ a uniformly n-cofinal subtree of T closed under g. From these objects define $l(s, m, k)$, $r(s, m, k)$, $Z(s, m)$, and $Z(\widetilde{s})$ for $s, \widetilde{s} \in \omega_1^{\leq n}|_<$, $m \leq \mathrm{lh}(s)$, and $k < \mathrm{lh}(s)$ as follows:

- $l(s,m,k) = \begin{cases} g(t(s){\restriction}k) & \text{if } k \leq m,\\ \gamma_{s(k-1)+1} & \text{if } m < k,\end{cases}$
- $r(s,m,k) = \begin{cases} t(s)(k) & \text{if } k < m,\\ \gamma_{s(k)} & \text{if } m \leq k,\end{cases}$
- $Z(s,m) = \prod_{k<\mathrm{lh}(s)}(l(s,m,k), r(s,m,k)]$,
- $Z(\widetilde{s}) = \bigcup\{Z(s,\mathrm{lh}(\widetilde{s})) : s \in \omega_1^n|_<,\ \widetilde{s} \subseteq s\}$.

LEMMA 5.2. *If $s \in \omega_1^{\leq n}|_<$, $m \leq \mathrm{lh}(s)$, and $s(k)$ is a limit ordinal for every $m < k < \mathrm{lh}(s)$, then $\langle\gamma_{s(k)} : k < \mathrm{lh}(s)\rangle \in Z(s,m)$.*

Proof. Let (i), (ii), and (iii) be the conditions in Definition 3.3.

Let $k < \mathrm{lh}(s)$. Then $l(m,s,k) = g(t(s{\restriction}k)) < \gamma_{s(k)}$ by (i) and (ii) where $k \leq m$. If $m < k$ then $s(k-1) < s(k)$ and $s(k)$ is a limit ordinal, hence $s(k-1)+1 < s(k)$, $l(s,m,k) = \gamma_{s(k-1)+1} < \gamma_{s(k)}$.

We have $\gamma_{s(k)} \leq r(s,m,k)$ by (iii) where $k < m$, and it is trivial that $\gamma_{s(k)} \leq r(s,m,k)$ where $m \leq k$.

Therefore $\langle\gamma_{s(k)} : k < \mathrm{lh}(s)\rangle \in Z(s,m)$. ∎

LEMMA 5.3. *If $s \in \omega_1^{\leq n}|_<$ and $m \leq \mathrm{lh}(s)$, then*

$$Z(s,m) \subseteq \prod_{k<\mathrm{lh}(s)} (g(t(s){\restriction}k), t(s)(k)].$$

Proof. It suffices to show that $g(t(s){\restriction}k) \leq l(s,m,k)$ and $r(s,m,k) \leq t(s)(k)$ for each $k < \mathrm{lh}(s)$. Now, $g(t(s){\restriction}k) = l(s,m,k)$ if $k \leq m$, and

$r(s,m,k)=t(s)(k)$ if $k<m$ by definition. If $m<k$ then $g(t(s)\restriction k)=g(t(s\restriction k))<\gamma_{s(k-1)+1}=l(s,m,k)$ by (i) and (ii). And if $m\le k$ then $r(s,m,k)=\gamma_{s(k)}\le t(s)(k)$ by (iii). ∎

LEMMA 5.4. *If $\widetilde{s}\in\omega_1^{\le n}|_<$ and $X\subseteq\omega_1^n|_<$, then $X\cap Z(\widetilde{s})$ is an open F_σ-subset of X.*

Proof. For each limit ordinal $\xi<\omega_1$, let $\langle e(\xi,i):i<\omega\rangle:\omega\to\gamma_\xi$ be a strictly increasing, cofinal sequence. For each $i<\omega$ and $0<k<n$, put

$$E_{i,k}=\{x\in\omega_1^n|_<: x(k-1)\notin(e(\xi,i),\gamma_\xi]\text{ or }x(k)\notin(\gamma_\xi,\gamma_{\xi+1}]\text{ for every limit ordinal }\xi<\omega_1\},$$

$$F_{i,k}=\{x\in E_{i,k}: x(k-1)\le\gamma_\xi\text{ and }\gamma_{\xi+1}<x(k)\text{ for some }\xi<\omega_1\}.$$

CLAIM. *For each $i<\omega$ and $0<k<n$, $E_{i,k}$ and $F_{i,k}$ are closed in $\omega_1^n|_<$.*

Both $\{x\in\omega_1^n|_<: x(k-1)\in(e(\xi,i),\gamma_\xi]\}$ and $\{x\in\omega_1^n|_<: x(k)\in(\gamma_\xi,\gamma_{\xi+1}]\}$ are clopen for every $\xi<\omega_1$. So $E_{i,k}$ is closed. To see that $F_{i,k}$ is closed in $E_{i,k}$, let $x\in E_{i,k}$ and let $\xi_0<\omega_1$ be the least ordinal such that $x(k-1)\le\gamma_{\xi_0}$. If ξ_0 is a limit ordinal, then $\gamma_{\xi_0}=x(k-1)<x(k)$ and $x(k-1)=\gamma_{\xi_0}\in(e(\xi_0,i),\gamma_{\xi_0}]$, so $\gamma_{\xi_0+1}<x(k)$ and $x\in F_{i,k}$ because $x\in E_{i,k}$. Hence, if $x\in E_{i,k}-F_{i,k}$ then ξ_0 is not a limit ordinal and $x(k)\le\gamma_{\xi_0+1}$. If $\xi_0=0$ then $\{x\in E_{i,k}: x(k)\le\gamma_1\}$ is a neighborhood of x in $E_{i,k}$ disjoint from $F_{i,k}$. If $\xi_0=\xi+1$ then

$$\{x\in E_{i,k}: x(k-1)>\gamma_\xi\text{ and }x(k)\le\gamma_{\xi+2}\}$$

is a neighborhood of x in $E_{i,k}$ disjoint from $F_{i,k}$. So $F_{i,k}$ is closed in $E_{i,k}$.

Now we return to the proof of the lemma. It is trivial that $X\cap Z(\widetilde{s})$ is open. We prove that $X\cap Z(\widetilde{s})$ is F_σ. Put $\widetilde{m}=\mathrm{lh}(\widetilde{s})$ and

$$\widetilde{Z}=\{x\in X: g(t(\widetilde{s})\restriction k)<x(k)\text{ for every }k\in n\cap(\widetilde{m}+1),\text{ and }x(k)\le t(\widetilde{s})(k)\text{ for every }k<\widetilde{m}\}.$$

Then $\widetilde{Z}$ is closed in X. It suffices to show the following.

CLAIM. $X\cap Z(\widetilde{s})=\widetilde{Z}\cap\bigcup_{i<\omega}\bigcap_{\widetilde{m}<k<n}F_{i,k}$.

Assume that $x\in X\cap Z(\widetilde{s})$. There is an $s\in\omega_1^n|_<$ such that $\widetilde{s}\subseteq s$ and $x\in Z(s,\widetilde{m})$. That $x\in\widetilde{Z}$ is immediate from the definition. For each $\widetilde{m}<k<n$, let $\xi(k)<\omega_1$ be the least limit ordinal such that $x(k)\le\gamma_{\xi(k)+1}$. Since

$$(*)\qquad x(k-1)\le r(s,\widetilde{m},k-1)=\gamma_{s(k-1)}<\gamma_{s(k-1)+1}=l(s,\widetilde{m},k)<x(k)\le\gamma_{\xi(k)+1},$$

we have $s(k-1)<\xi(k)$ and $x(k-1)<\gamma_{\xi(k)}$. So there is an $i<\omega$ such that $x(k-1)\le e(\xi(k),i)$ for all $\widetilde{m}<k<n$. Let $\widetilde{m}<k<n$ and let $\xi<\omega_1$ be a limit ordinal. If $\xi<\xi(k)$ then $x(k)\notin(\gamma_\xi,\gamma_{\xi+1}]$ by the minimality of $\xi(k)$.

If $\xi = \xi(k)$ then since $x(k-1) \le e(\xi(k), i) = e(\xi, i)$, we have $x(k-1) \notin (e(\xi, i), \gamma_\xi]$. If $\xi > \xi(k)$ then $x(k) \le \gamma_{\xi(k)+1} < \gamma_\xi$ implies $x(k) \notin (\gamma_\xi, \gamma_{\xi+1}]$. So $x \in E_{i,k}$. From $(*)$, it follows that $x \in F_{i,k}$. So we have proved the $\subseteq$ inclusion of the claim.

Conversely, assume that $x \in \widetilde{Z} \cap \bigcup_{i<\omega} \bigcap_{\widetilde{m}<k<n} F_{i,k}$. Pick an $i < \omega$ such that $x \in \bigcap_{\widetilde{m}<k<n} F_{i,k}$. Let s be the sequence of length n such that $s{\restriction}\widetilde{m} = \widetilde{s}$ and for each $\widetilde{m} \le k < n$, $s(k) < \omega_1$ is the least ordinal satisfying $x(k) \le \gamma_{s(k)}$. If $\widetilde{m} < k < n$ then $x \in F_{i,k}$, so there is a $\xi' < \omega_1$ such that $x(k-1) \le \gamma_{\xi'}$ and $\gamma_{\xi'+1} < x(k)$; such a ξ' must satisfy $s(k-1) \le \xi'$, hence $\gamma_{s(k-1)+1} \le \gamma_{\xi'+1} < x(k) \le \gamma_{s(k)}$ and $s(k-1) < s(k)$. If $k < \widetilde{m} < n$ then $x \in \widetilde{Z}$ implies $\gamma_{\widetilde{s}(k)} \le t(\widetilde{s})(k) \le g(t(\widetilde{s})) < x(\widetilde{m}) \le \gamma_{s(\widetilde{m})}$, so $\widetilde{s}(k) < s(\widetilde{m})$. Hence $s \in \omega_1^n|_<$ and $x \in Z(s, \widetilde{m}) \subseteq Z(\widetilde{s})$. This proves the $\supseteq$ inclusion. ∎

LEMMA 5.5. *Let $n < \omega$ and $X \subseteq \omega_1^n|_<$. For each open cover $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ of X, there exists a family $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of open F_σ-sets of X such that*:

(i) $F_i \subseteq U_i$ *for every* $i \in \mathcal{I}$,
(ii) $X - \bigcup \mathcal{F}$ *is nonstationary.*

Proof. If X is not stationary, then $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$, where $F_i = \emptyset$ for each $i \in \mathcal{I}$, satisfies the required condition.

Assume that X is stationary. There is an n-stationary tree T'' on ω_1 such that $\mathrm{Lv}_n(T'') \subseteq X$ by Lemma 3.1(2). Pick a pairwise disjoint family $\mathcal{H} = \langle H_i : i \in \mathcal{I}\rangle$ such that $H_i \subseteq U_i$ for every $i \in \mathcal{I}$ and $\bigcup \mathcal{H} = \mathrm{Lv}_n(T'')$. Since $\mathcal{U}$ is an open cover of X, there is a regressive function $f : \mathrm{Lv}_n(T'') \to (\omega_1 \cup \{-\infty\})^n$ such that for each $i \in \mathcal{I}$ and $t \in H_i$, $X \cap \prod_{k<n}(f(t)(k), t(k)] \subseteq U_i$. By the Pressing Down Lemma, there are an n-stationary subtree T' of T'' and a function $g' : \mathrm{Lv}_{<n}(T') \to \omega_1$ such that $f(t)(k) = g'(t{\restriction}k)$ for every $t \in \mathrm{Lv}_n(T')$ and $k < n$. By Lemma 3.2, there exist an n-cofinal subtree T of T', $\mathcal{I}_0 \subseteq \mathcal{I}$, and a family $\langle t_i : i \in \mathcal{I}_0\rangle$ of elements of T satisfying (a), (b). Pick a function $g : T \to \omega_1$ such that $t(k) \le g(t)$ for every $t \in T$ and $k < \mathrm{lh}(t)$, and $g'(t) \le g(t)$ for every $t \in \mathrm{Lv}_{<n}(T)$. By Lemma 3.3, there exists a uniformly n-cofinal subtree $\langle \gamma, t\rangle$ of T closed under g.

Put $\mathcal{I}_1 = \{i \in \mathcal{I}_0 : t(s_i) = t_i \text{ for some } s_i \in \omega_1^{\le n}|_<\}$. For each $i \in \mathcal{I}_1$, there is a unique s_i witnessing $i \in \mathcal{I}_1$. Actually, if $s \in \omega_1^{\le n}|_<$ and $t(s) = t_i$, then $\mathrm{lh}(s) = \mathrm{lh}(t_i)$ and for each $k < \mathrm{lh}(t_i)$,

$$\gamma_{s(k)} \le t(s)(k) = t_i(k) \le g(t(s){\restriction}k+1) = g(t(s{\restriction}k+1)) < \gamma_{s(k)+1}.$$

Such an s is unique.

Apply Lemmas 5.2, 5.3, and 5.4 to T, g, and $\langle \gamma, t\rangle$. Put $F_i = X \cap Z(s_i)$ for each $i \in \mathcal{I}_1$ and $F_i = \emptyset$ for each $i \in \mathcal{I} - \mathcal{I}_1$. Let $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$.

By Lemma 5.4, F_i is an open F_σ-set for every $i \in \mathcal{I}$ in X. We show that conditions (i) and (ii) hold for $\mathcal{F}$.

For each $i \in \mathcal{I}_1$ and $s \in \omega_1^n|_<$ with $s_i \subseteq s$, we have $t(s) \in \mathrm{Lv}_n(T)$ and $t_i = t(s_i) \subseteq t(s)$, so $t(s) \in H_i$ by condition (b) of Lemma 3.2. Since $t(s) \in \mathrm{Lv}_n(T) \subseteq \mathrm{Lv}_n(T') \subseteq \mathrm{Lv}_n(T'')$, it follows that $f(t(s))(k) = g'(t(s){\restriction}k) \le g(t(s){\restriction}k)$ for each $k < n$, and $X \cap \prod_{k<n}(f(t(s))(k), t(s)(k)] \subseteq U_i$. By Lemma 5.3,

$$\begin{aligned} X \cap Z(s, \mathrm{lh}(s_i)) &\subseteq X \cap \prod_{k<n} (g(t(s){\restriction}k), t(s)(k)] \\ &\subseteq X \cap \prod_{k<n} (f(t(s))(k), t(s)(k)] \subseteq U_i. \end{aligned}$$

Hence (i) holds.

Now, $D = \{\gamma_\xi : \xi \text{ is a limit ordinal} < \omega_1\}$ is a club subset of ω_1. To see that (ii) holds, it suffices to show that $X \cap D^n \subseteq \bigcup \mathcal{F}$. Let $x \in X \cap D^n$, say $x = \langle \gamma_{s(k)} : k < n\rangle$ for some $s \in \omega_1^n|_<$. Then $s(k)$ is a limit ordinal for each $k < n$ and $t(s) \in \mathrm{Lv}_n(T)$. By Lemma 3.2(a), there is a unique $i \in \mathcal{I}_0$ such that $t_i \subseteq t(s)$. Since $t(s{\restriction}\mathrm{lh}(t_i)) = t(s){\restriction}\mathrm{lh}(t_i) = t_i$, we have $i \in \mathcal{I}_1$ and $s_i = s{\restriction}\mathrm{lh}(t_i) \subseteq s$. By Lemma 5.2, $x \in X \cap Z(s, \mathrm{lh}(s_i)) \subseteq X \cap Z(s_i) = F_i$. Hence $X \cap D^n \subseteq \bigcup \mathcal{F}$, so (ii) holds. ■

LEMMA 5.6. *Assume that $n < \omega$, $X \subseteq \omega_1^n$, and $X_{k,\alpha} = \{s\hat{\ }t : s \in \omega_1^k,\ t \in \omega_1^{n-(k+1)},\ s\hat{\ }\langle\alpha\rangle\hat{\ }t \in X\}$ is subshrinking for each $k < n$ and $\alpha < \omega_1$.*

(1) *If X is nonstationary, then X is subshrinking.*

(2) *If $X \subseteq \omega_1^n|_<$, then X is subshrinking.*

Proof. (1) Let $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ be a closed cover obtained by Corollary 5.1. By Lemma 2.1, it suffices to show that F_i is subshrinking for every $i \in \mathcal{I}$. Fix an $i \in \mathcal{I}$. We have $\{s(k) : s \in F_i\} \subseteq \mu$ for some $k < n$ and $\mu < \omega_1$. For $\alpha < \mu$, put $C_\alpha = \{s \in F_i : s(k) = \alpha\}$. Then $\langle C_\alpha : \alpha < \mu\rangle$ is a closed cover of F_i. By Lemma 2.1 again, it suffices to show that C_α is subshrinking for each $\alpha < \mu$. In $X_{k,\alpha}$, $\{s\hat{\ }t : s \in \omega_1^k,\ t \in \omega_1^{n-(k+1)},\ s\hat{\ }\langle\alpha\rangle\hat{\ }t \in C_\alpha\}$ is closed and homeomorphic to C_α, hence C_α is subshrinking.

(2) Let $\mathcal{U} = \langle U_i : i \in \mathcal{I}\rangle$ be an open cover of X. Pick a family $\mathcal{F} = \langle F_i : i \in \mathcal{I}\rangle$ of open F_σ-sets of X obtained by Lemma 5.5, and put $G = \bigcup \mathcal{F}$. For each $k < n$ and $\alpha < \omega_1$, $\{s\hat{\ }t : s \in \omega_1^k,\ t \in \omega_1^{n-(k+1)},\ s\hat{\ }\langle\alpha\rangle\hat{\ }t \in X - G\}$ is a closed subset of $X_{k,\alpha}$, so subshrinking. Since $X - G$ is nonstationary, we can apply (1) to $X - G$, so $X - G$ is subshrinking. Hence, there is a subshrinking $\mathcal{M} = \langle M_i : i \in \mathcal{I}\rangle$ of $\langle U_i \cap (X - G) : i \in \mathcal{I}\rangle$ in $X - G$. Since $X - G$ is closed in X, $\mathcal{M}$ is a family of F_σ-sets also in X. Finally, $\langle M_i \cup F_i : i \in \mathcal{I}\rangle$ is a subshrinking of $\mathcal{U}$ in X. Hence X is subshrinking. ■

Now we can prove Theorem 1.2.

Proof. Apply Lemma 5.6(2) inductively. Then the statement follows immediately. ■

Since all subspaces of ω_1^2 are subshrinking (see [4]), the following holds by Lemma 5.6(1).

Corollary 5.2. *All nonstationary subspaces of ω_1^3 are subshrinking.*

ω_1^3 in the corollary above cannot be changed to ω_1^4. Indeed, there is a nonsubnormal subspace X of ω_1^3 by Theorem 1.1. Therefore $\{0\} \times X$, which is homeomorphic to X, is a nonstationary and nonsubnormal subspace of ω_1^4.

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