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Urysohn universal spaces as metric groups of exponent 2

by

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Abstract. The aim of the paper is to prove that the bounded and unbounded Urysohn universal spaces have unique (up to isometric isomorphism) structures of metric groups of exponent 2. An algebraic-geometric characterization of *Boolean Urysohn spaces* (i.e. metric groups of exponent 2 which are metrically Urysohn spaces) is given.

The (unbounded) Urysohn universal space was introduced in [6]. This is the "most homogeneous" complete separable metric space. Cameron and Vershik [1] have proved that the (unbounded) rational Urysohn space can be endowed with a monothetic group structure and therefore the (unbounded) Urysohn space is isometric to an Abelian metric group. However, as they suggest, a group structure of the Urysohn space is not uniquely determined, i.e. there are at least two Abelian metric groups which are not isomorphic but both are isometric to the Urysohn space. (One of them is the above mentioned monothetic group of Cameron and Vershik and the other is a Boolean group as defined below.)

The purpose of this paper is to unify the algebraic structure of Urysohn spaces. Namely, we shall show that both the Urysohn spaces (bounded and unbounded) have unique (up to isometric isomorphism) structures of metric groups of exponent 2. In fact, we shall prove that such a structure is unique for every rational Urysohn space (see Definition 2). The reader is referred to [4, 5] for basic information on the Urysohn space.

In this paper $\mathbb{R}_+$ is the set of all nonnegative real numbers. By a *Boolean group* we mean a group of exponent 2, i.e. any group $(G, \cdot)$ such that $x^2 = e$ for all $x \in G$, where e is the neutral element of G. It is an easy exercise to show that every Boolean group is Abelian (and thus we will use the additive notation) and that a group is Boolean if and only if it is isomorphic (in the

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category of groups) to a vector space over the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (and therefore we may use the dimension $\dim_{\mathbb{Z}_2}$).

An Abelian group $(G, +)$ with a function $p\colon G \to \mathbb{R}_+$ is said to be a *metric group* if the function p satisfies the following three conditions (for any $x, y \in G$):

(D1) $p(x) = 0 \Leftrightarrow x = 0$,
(D2) $p(-x) = p(x)$,
(D3) $p(x + y) \leq p(x) + p(y)$.

The above conditions are equivalent to the fact that the function $D_p\colon G \times G \ni (x, y) \mapsto p(x - y) \in \mathbb{R}_+$ is an invariant metric on G. We shall always equip the metric group $(G, +, p)$ with the metric D_p and the topology induced by it. If the group G is Boolean, the condition (D2) may clearly be omitted.

Two Abelian metric groups $(G, +, p)$ and $(H, +, q)$ are (metrically) *isomorphic* if there is a bijection between G and H which is simultaneously a group isomorphism and an isometry. They are *topologically isomorphic* if there is a group isomorphism which is a homeomorphism.

Recall that a *Katětov map* on a metric space (X, d) is any function $f\colon X \to \mathbb{R}$ such that

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y) \quad \text{for all } x, y \in X$$

([3], [4, 5]). We shall say that a Katětov map $g : A \to \mathbb{R}$, where A is a subset of X, is *trivial in* X if there is $x \in X$ such that $g(a) = d(a, x)$ for each $a \in A$. The set of all Katětov maps on X is denoted by $E(X)$.

We begin with

1. Definition. An *Urysohn space* is a separable complete metric space X such that every separable metric space of diameter no greater than $\operatorname{diam} X$ is isometrically embeddable in X and each isometry between finite subsets of X is extendable to an isometry of X onto itself. An Urysohn space is *nontrivial* if it has more than one point.

For each $r \in [0, +\infty]$ there is a unique (up to isometry) Urysohn space of diameter r. We shall denote it by $\mathbb{U}_r$, and $\mathbb{U}$ will stand for the unbounded Urysohn space.

A fundamental result on Urysohn spaces says that a nonempty separable complete metric space (X, d) is Urysohn if and only if every Katětov map defined on a finite subset of X and upper bounded by $\operatorname{diam} X$ is trivial in X (this trivialization property is, in fact, equivalent to the finite injectivity of X, see [4, p. 386] or [5, Exercise 3]; Urysohn [6] has proved that a nonempty separable complete metric space is finitely injective iff it is Urysohn, cf. [5, Theorem 3.2]). If we modify this condition, we obtain the so-called rational Urysohn spaces:

2. Definition. A metric space (X,d) is said to be a *rational Urysohn space* if it satisfies the following conditions:

(QU1) the space X is nonempty and at most countable,
(QU2) the metric d is rational-valued, that is, $d\colon X\times X\to\mathbb{Q}$,
(QU3) every rational-valued Katětov map defined on a finite subset of X and upper bounded by $\operatorname{diam} X$ is trivial in X.

The following was proved by Urysohn [6] for $r=+\infty$; his proof works also for finite r.

3. Theorem (Urysohn). *For each $r\in[0,+\infty]$ there is a unique (up to isometry) rational Urysohn space $\mathbb{QU}_r$ of diameter r; its completion is isometric to $\mathbb{U}_r$.*

Now we shall prove our main result. All other results of the paper are consequences of it.

4. Theorem. *For any $r\in[0,+\infty]$ there is a Boolean metric group which is a rational Urysohn space of diameter r.*

Proof. We may assume that $r>0$. Put $\mathbb{Q}_r=[0,r]\cap\mathbb{Q}$. Let $\oplus$ denote the symmetric difference of sets, i.e. $A\oplus B=(A\setminus B)\cup(B\setminus A)$.

First assume that $\mathcal{G}$ is a finite family of sets and $p\colon\mathcal{G}\to\mathbb{Q}_r$ is a function such that $(\mathcal{G},\oplus,p)$ is a metric group. Let $f\colon\mathcal{G}\to\mathbb{Q}_r$ be a Katětov map. We shall show that there is a finite family $\mathcal{H}$ and a function $q\colon\mathcal{H}\to\mathbb{Q}_r$ such that $(\mathcal{H},\oplus,q)$ is a metric group, $(\mathcal{G},p)\subset(\mathcal{H},q)$ and f is trivial in $(\mathcal{H},D_q)$.

If there is $A\in\mathcal{G}$ such that $f(A)=0$, then $f(X)=p(X\oplus A)$ and it suffices to put $(\mathcal{H},q)=(\mathcal{G},p)$. So, we may assume that $f(X)>0$ for each $X\in\mathcal{G}$. Let U be any set such that $U\notin\mathcal{G}$. This implies that $U\oplus X\notin\mathcal{G}$ for each $X\in\mathcal{G}$. Let $\mathcal{H}=\mathcal{G}\cup\{U\oplus X\colon X\in\mathcal{G}\}$ and $q\colon\mathcal{H}\to\mathbb{R}$ be a function defined as follows: $q(X)=p(X)$ and $q(U\oplus X)=f(X)$ for $X\in\mathcal{G}$. It is easy to check that $(\mathcal{H},\oplus)$ is a group and q is a well defined $\mathbb{Q}_r$-valued function which extends p and satisfies the corresponding condition (D1). Therefore it suffices to show that q satisfies the triangle inequality (D3) (because then simply $f(X)=D_q(X,U)$ for each $X\in\mathcal{G}$). For any two elements X and Y of $\mathcal{G}$ we clearly have $q(X\oplus Y)\le q(X)+q(Y)$. Further, $q((U\oplus X)\oplus Y)=f(X\oplus Y)\le f(X)+p(X\oplus(X\oplus Y))=q(U\oplus X)+q(Y)$ and similarly $q((U\oplus X)\oplus(U\oplus Y))=p(X\oplus Y)\le f(X)+f(Y)=q(U\oplus X)+q(U\oplus Y)$, which finishes the proof of this part.

Having the above property, start with $\mathcal{G}_1=\{\emptyset\}$, $p_1\colon\mathcal{G}_1\to\{0\}$ and any sequence $(f_k^{(1)})_{k=1}^\infty$ consisting of all $\mathbb{Q}_r$-valued Katětov maps on $\mathcal{G}_1$, and using an easy induction argument construct a sequence $(\mathcal{G}_n,p_n,(f_k^{(n)})_{k=1}^\infty)_{n=1}^\infty$ such that for each $n\ge 1$:

$(1)_n$ $\mathcal{G}_n$ is a finite family of sets, p_n is a function from $\mathcal{G}_n$ to $\mathbb{Q}_r$ and $(\mathcal{G}_n, \oplus, p_n)$ is a metric group,
$(2)_n$ $\{f_k^{(n)} : k \geq 1\} = \{f : \mathcal{G}_n \to \mathbb{Q}_r \mid f \text{ is a Katětov map}\}$,
$(3)_n$ $(\mathcal{G}_{n-1}, p_{n-1}) \subset (\mathcal{G}_n, p_n)$ for $n \geq 2$,
$(4)_n$ $f_k^{(j)}$ is trivial in $(\mathcal{G}_n, D_{p_n})$ for each j and k such that $1 \leq j, k < n$.

Now it suffices to put $(\mathcal{G}, p) = \bigcup_{n=1}^{\infty} (\mathcal{G}_n, p_n)$ and to note that thanks to the conditions $(1)_n$–$(4)_n$, the triple $(\mathcal{G}, \oplus, p)$ is a metric group and simultaneously a rational Urysohn space of diameter r. (To see that $\operatorname{diam} \mathcal{G} = r$, observe that for each $s \in \mathbb{Q}_r$ the constant map s is a member of $\{f_k^{(1)} : k \geq 1\}$.) ∎

Since the completion of an Abelian [Boolean] metric group is an Abelian [Boolean] group as well, Theorems 3 and 4 imply that

5. Theorem. *For each $r \in [0, +\infty]$ there exists a Boolean metric group which is metrically an Urysohn space of diameter r.*

6. Remark. Cameron and Vershik [1], using a similar idea to that in the proof of Theorem 4, have shown that the countable Abelian group of exponent 2 acts regularly as an isometry group of the rational Urysohn space. However, they did not take note of the fact that this space is such a group itself.

7. Definition. A *Boolean Urysohn group* is a Boolean metric group which is metrically an Urysohn space.

Our next aim is to prove that each Urysohn space has a unique structure of a Boolean metric group. This immediately follows from the next result. Since its proof just uses the back-and-forth method and Huhunaišvili's theorem [2] (cf. the proof of Proposition 11), we omit it.

8. Theorem. *Let $(G, +, p)$ be a Boolean Urysohn group. Let $(B, +, q)$ be a separable Boolean metric group such that* $\operatorname{diam} B \leq \operatorname{diam} G$. *Let K be a compact subgroup of B and let $\psi \colon K \to G$ be an isometric homomorphism. There exists an extension $\Psi \colon B \to G$ of ψ which is an isometric homomorphism as well. What is more, if B is a Boolean Urysohn group of the same diameter as G, the extension Ψ may be constructed bijective.*

9. Corollary. *Any two Boolean Urysohn groups of the same diameter are isometrically isomorphic.*

10. Corollary. *If $\mathbb{U}_r$ is a nontrivial bounded Boolean Urysohn group and $\mathbb{U}$ is the unbounded one, then $\mathbb{U}_r$ and $\mathbb{U}$ are isomorphic as algebraic groups, but* ***not*** *isomorphic as topological groups.*

Proof. Since $\mathbb{U}_r$ and $\mathbb{U}$ are vector spaces over the field $\mathbb{Z}_2$ of the same cardinality, they are algebraically isomorphic. But there is no uniformly continuous map of $\mathbb{U}_r$ **onto** $\mathbb{U}$, because the existence of a geodesic segment from $[0,t]$ into $\mathbb{U}_r$ between any two points of $\mathbb{U}_r$ (with $t \leq r$ being the distance of these points) easily implies that the range of any such map has to be bounded. ∎

Another property of Boolean Urysohn groups is stated in the following

11. Proposition. *Let* $(\mathbb{U}_r,+,p)$ *be a Boolean Urysohn group and* $(B,+,q)$ *a separable Boolean metric group. Let* K *be a compact subgroup of* B *and* $\psi\colon K \to \mathbb{U}_r$ *a nonexpansive homomorphism. Then there is a nonexpansive homomorphism* $\Psi\colon B \to \mathbb{U}_r$ *which extends* ψ.

Proof. Thanks to the separability of B, it suffices to show that if $x \in B$, then there is a compact subgroup L of B and a nonexpansive homomorphism $\varphi\colon L \to \mathbb{U}_r$ such that $K \cup \{x\} \subset L$ and $\varphi|_K = \psi$. Put $L = K + \{0,x\}$ and

$$g\colon \psi(K) \ni b \mapsto \inf\{q(x+a)\colon a \in K,\ \psi(a) = b\} \in \mathbb{R}_+.$$

The set L is clearly a compact subgroup of B and the function g satisfies the condition $p(b_1+b_2) \leq g(b_1)+g(b_2)$ $(b_1,b_2 \in \psi(K))$. Now put

$$h\colon \psi(K) \ni b \mapsto \inf\{g(z)+p(b+z)\colon z \in \psi(K)\} \in \mathbb{R}_+.$$

It is easy to check that h is a Katětov map such that $h \leq g$. Since $\psi(K)$ is compact, we infer from Huhunaišvili's theorem [2] that there is $y \in \mathbb{U}_r$ such that $p(z+y) = \min(h(z),r)$ for each $z \in \psi(K)$. Now define $\varphi\colon L \to \mathbb{U}_r$ by $\varphi(a) = \psi(a)$ and $\varphi(a+x) = \psi(a)+y$ for $a \in K$. Clearly, φ is a well defined nonexpansive homomorphism. ∎

12. Example. Melleray [5] has proved that for the unbounded Urysohn space $(\mathbb{U},d)$ there is a function $\mathbb{U} \ni x \mapsto \varphi_x \in \mathrm{Iso}(\mathbb{U})$ such that each φ_x is an (isometric) involution, the map $x \mapsto \varphi_x(x)$ is constantly equal to some $a \in \mathbb{U}$, φ_a is the identity map on $\mathbb{U}$ and $d(\varphi_x(z),\varphi_y(z)) = d(x,y)$ for all $x,y,z \in \mathbb{U}$. His construction was complicated. But the existence of such a function is an immediate consequence of the fact that $\mathbb{U}$ may be endowed with the structure of a Boolean group: it suffices to put $\varphi_x(u) = u+x$.

Now we shall give a characterization of Boolean Urysohn groups. This is a variation of *finite injectivity* and we leave its proof as a simple exercise.

13. Theorem. *For a separable complete Boolean metric group* $(G,+,p)$ *the following conditions are equivalent:*

(i) *G is a Boolean Urysohn group*,
(ii) *for any triple* (H,K,ψ), *where* H *is a finite Boolean metric group of diameter no greater than* $\operatorname{diam} G$, K *is a subgroup of* H *and* $\psi\colon K \to G$ *is an isometric homomorphism*, *there is an isometric homomorphism* $\Psi\colon H \to G$ *which extends* ψ.

Theorem 8, Corollary 9 and Proposition 11 have their counterparts for rational Urysohn spaces (compact subgroups of separable groups should be replaced by finite subgroups of countable groups with $\mathbb{Q}$-valued metrics).

We end the paper with the following two problems.

QUESTION 1. Is a Boolean Urysohn group topologically isomorphic to its Cartesian square?

QUESTION 2. Given $k > 2$, is there a metric group (that is, a group with a left or right invariant metric) of exponent k which is isometric to an Urysohn space? If yes, is this group unique (up to isometric isomorphism)? Is it Abelian? (Our method of proof of Theorem 4 does not work for $k \neq 2$.)

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