On isomorphism classes of $C(2^m \oplus [0, \alpha])$ spaces

by

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Abstract. We provide a complete isomorphic classification of the Banach spaces of continuous functions on the compact spaces $2^m \oplus [0, \alpha]$, the topological sums of Cantor cubes $2^m$, with $m$ smaller than the first sequential cardinal, and intervals of ordinal numbers $[0, \alpha]$. In particular, we prove that it is relatively consistent with ZFC that the only isomorphism classes of $C(2^m \oplus [0, \alpha])$ spaces with $m \geq \aleph_0$ and $\alpha \geq \omega_1$ are the trivial ones. This result leads to some elementary questions on large cardinals.

1. Introduction and statement of the main results. Given a compact Hausdorff topological space $K$, $C(K)$ stands for the Banach space of all continuous real-valued functions on $K$, equipped with the supremum norm. For a fixed cardinal number $m$, $2^m$ denotes the product of $m$ copies of the two-point space $2$, provided with the product topology.

If $\alpha$ is an ordinal number, $[0, \alpha]$ denotes the interval of ordinals $\{\xi : 0 \leq \xi \leq \alpha\}$ endowed with the order topology. As usual, we denote by $\aleph_0$, $\aleph_1$, $\omega$ and $\omega_1$ the first infinite cardinal, the first uncountable cardinal, the first infinite ordinal and the first uncountable ordinal, respectively. The symbol $X \oplus Y$ will denote the Cartesian product of the Banach spaces $X$ and $Y$, i.e. the space of all pairs $(x, y)$, $x \in X$, $y \in Y$, with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. We write $X \sim Y$ when the Banach spaces $X$ and $Y$ are isomorphic. By $X \hookrightarrow Y$ we mean that the Banach space $Y$ contains a subspace isomorphic to the Banach space $X$. Other notations are standard and in conformity with [20].

This paper is concerned with the question of describing the isomorphism classes of $C(2^m \oplus [0, \alpha])$ spaces, where $2^m \oplus [0, \alpha]$ is the topological sum of $2^m$ and $[0, \alpha]$ for some cardinal $m$ and ordinal $\alpha$, that is, of the family of $C(2^m) \oplus C([0, \alpha])$ spaces. As we will see in the first three remarks below, the motivation for this work comes from some classical isomorphic
classifications of $C(K)$ spaces closely connected with $C(2^m) \oplus C([0,\alpha])$ spaces.

**Remark 1.1.** First of all, notice that this family of spaces includes the finite-dimensional spaces $\mathbb{R}^{2^m+\alpha}$ as well as the $C([0,\alpha])$ spaces with $\alpha \geq \omega$. The isomorphic classification of $C([0,\alpha])$ spaces was accomplished by Bessaga and Pełczyński [2] in the case where $\omega \leq \alpha < \omega_1$; by Semadeni [19] in the case where $\omega_1 < \alpha \leq \omega$; by Labbê [11] in the case where $\omega \leq \alpha < \omega_1$; and independently by Kislyakov [10] and Gul’ko and Os’kin [8] in the general case.

**Remark 1.2.** For $m = \aleph_0$ and $\alpha < \omega_1$, it follows from the classical Milyutin theorem [20, Theorem 21.5.10] about the isomorphic classification of $C(K)$ spaces, with $K$ compact metric uncountable, that every space of this family is isomorphic to $C(2^{\aleph_0})$.

**Remark 1.3.** Assume $m \geq \aleph_0$ and $\alpha < \omega_1$. Then $2^m$ is homeomorphic to the topological product $2^m \times 2^{\aleph_0}$. Hence $C(2^{\aleph_0})$ is isomorphic to a complemented subspace of $C(2^m)$. That is, there exists a Banach space $Y$ such that $C(2^m)$ is isomorphic to $Y \oplus C(2^{\aleph_0})$. Therefore, by the above mentioned Milyutin theorem we deduce

$$C(2^m) \oplus C([0,\alpha]) \sim Y \oplus C(2^{\aleph_0}) \oplus C([0,\alpha]) \sim Y \oplus C(2^{\aleph_0}) \sim C(2^m).$$

Consequently, if $m, n \geq \aleph_0$ and $\alpha, \beta < \omega_1$, then by [20, Corollary 8.2.7],

$$C(2^m) \oplus C([0,\alpha]) \sim C(2^n) \oplus C([0,\beta]) \text{ if and only if } m = n.$$

**Remark 1.4.** Now we turn to the cases where both $m$ and $\alpha$ are large: $m \geq \aleph_0$ and $\alpha \geq \omega_1$. Suppose that $n \geq \aleph_0$ and $\beta \geq \omega$. Then $C(2^m) \oplus C([0,\alpha]) \sim C(2^n) \oplus C([0,\beta])$ implies that $m = n$. Indeed, assume that $m < n$ and let $\Gamma$ and $\Lambda$ be two sets of the same cardinality as $\alpha$ and $\beta$, respectively. According to [17, Proposition 5.2],

$$\left( \sum_{2^m} \oplus L^1[0,1]^m \right)_1 \oplus l_1(\Gamma) \sim \left( \sum_{2^n} \oplus L^1[0,1]^n \right)_1 \oplus l_1(\Lambda).$$

Recall that given a Banach space $X$, the *dimension* of $X$ is the smallest cardinal $\delta$ for which there exists a subset of cardinality $\delta$ with linear span norm-dense in $X$. Pick a subspace $H$ of $L^1[0,1]^n$ which is isomorphic to a Hilbert space of dimension $n$ [18, Proposition 1.5]. Hence

$$H \hookrightarrow \left( \sum_{2^n} \oplus L^1[0,1]^n \right)_1 \oplus l_1(\Gamma).$$

Since $H$ contains no subspace isomorphic to $l_1$, by a standard gliding hump argument (see [3]) we infer that there exist a finite sum of $L^1[0,1]^m$ and
1 \leq p < \omega \text{ such that}
\begin{equation}
H \leftrightarrow L^1[0,1]^m \oplus \cdots \oplus L^1[0,1]^m \oplus \mathbb{R}^p,
\end{equation}
which is absurd, because the dimension of $L^1[0,1]^m$ is clearly $m$.

**Remark 1.5.** If $m \geq \aleph_0$ then $C(2^m) \oplus C([0,\alpha]) \sim C(2^m)$ for every $\alpha \geq \omega_1$. Indeed, suppose $C(2^m) \oplus C([0,\alpha]) \sim C(2^m)$ for some $\alpha \geq \omega_1$. Let $\Gamma$ be the set of isolated points of $[0,\alpha]$ and denote by $C_0(\Gamma)$ the classical Banach space of all functions defined on $\Gamma$ such that for every $\epsilon > 0$ the set $\{\gamma \in \Gamma : |f(\gamma)| \geq \epsilon\}$ is finite. Thus $C_0(\Gamma) \hookrightarrow C([0,\alpha]) \hookrightarrow C(2^m)$.

Now recall that a topological space $K$ is said to satisfy the **countable chain condition** (ccc) if every uncountable family of open subsets of $K$ contains two distinct sets with nonempty intersection. Since $C_0(\Gamma) \hookrightarrow C(2^m)$, it follows from [17, Theorem 4.5] that $2^m$ would not satisfy the ccc, which is absurd by [6, Theorem 2.3.17].

In order to present a complete isomorphic classification of $C(2^m) \oplus C([0,\alpha])$ spaces, we will state a more general result on isomorphic classification of some Banach spaces. To do this, we recall that a Banach space $X$ is said to have the **Mazur property** if every element of $X^{**}$, the bidual space of $X$, which is sequentially weak* continuous is weak* continuous and thus is an element of $X$. Such spaces were investigated in [5], [12] and also in [9] and [21] where they were called d-complete and $\mu$B-spaces, respectively. Section 2 is devoted to proving the following isomorphic classification theorem for $X \oplus C([0,\alpha])$ spaces:

**Theorem 1.6.** Let $X$ be a Banach space having the Mazur property and $\alpha, \beta \geq \omega_1$. If $X \oplus C([0,\alpha]) \sim X \oplus C([0,\beta])$ then $C([0,\alpha]) \sim C([0,\beta])$.

Before applying Theorem 1.6, we need to recall a concept which had its origins in the study of continuity of functions on large Cartesian products. Following Noble [14] and Antonovski˘ı–Chudnovski˘ı [1], we say that a cardinal $m$ is **sequential** if there exists a sequentially continuous but not continuous real-valued function on $2^m$. We recall that a function $f : 2^m \rightarrow \mathbb{R}$ is said to be sequentially continuous if $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to $k$ in $2^m$.

**Remark 1.7.** Important for us is a result due to Plebanek which states that $C(2^m)$ has the Mazur property for every nonsequential cardinal $m$ [15] (see also [16, Theorem 5.2.c]).

**Remark 1.8.** Mazur [13] showed that the first sequential cardinal $s$ is weakly inaccessible. Hence $\omega_1 < s$. Moreover, there are many weakly inaccessible cardinals before $s$ [4]. On the other hand, let $m_\mathbb{R}$ and $m_2$ denote the least real-valued measurable cardinal and two-valued measurable cardinal,
respectively [7]. It is well-known that \( s \leq m_R; s \leq 2^{\aleph_0} \) or \( s = m_2 \); and \( s = m_2 \) under Martin’s axiom [1], [7] and [13].

In particular, it is relatively consistent with ZFC that there exist no sequential cardinals [16]. Therefore, keeping in mind the above remarks, it is also consistent with ZFC that Corollary 1.9 completes the isomorphic classification of \( C(2^m) \oplus C([0, \alpha]) \) spaces.

**Corollary 1.9.** Suppose that \( m \) is a nonsequential cardinal and \( \alpha, \beta \geq \omega_1 \). If \( C(2^m) \oplus C([0, \alpha]) \sim C(2^m) \oplus C([0, \beta]) \) then \( C([0, \alpha]) \sim C([0, \beta]) \).

**Remark 1.10.** As another direct application of Theorem 1.6 we get the isomorphic classification of \( C(K) \oplus C([0, \alpha]) \) spaces where \( C(K) \) has the Mazur property and \( \alpha \geq \omega_1 \). This includes the cases where \( K \) is first-countable [16, Corollary 3.2], or the \( \omega_1 \) th Cantor derived set of \( K \) is empty [9, Theorem 4.1], or \( K \) is a Corson-compact [16, Corollary 3.4].

### 2. Proof of Theorem 1.6.

As in [2], \( C([0, \alpha]) \) will be denoted by \( R^\alpha \) and we set \( R^\alpha_0 = \{ f \in R^\alpha : f(\alpha) = 0 \} \). By [2, Lemma 1.2.1], \( R^\alpha \sim R^\alpha_0 \).

Since \( R^\alpha \) with \( \alpha \geq \omega_1 \) does not have the Mazur property [21, p. 49] and finite sums of Banach spaces with the Mazur property also have this property, it follows that Theorem 1.6 is an immediate consequence of Proposition 2.5 below.

A fundamental ingredient in the proof of Proposition 2.5 is Lemma 2.1, which generalizes the following result of Bessaga and Pełczyński [2, Lemma 2]:

\[ R^{\alpha \omega} \cong R^\alpha, \quad \forall \alpha \geq \omega. \]

**Lemma 2.1.** Let \( X \) be an infinite-dimensional Banach space and \( \alpha \geq \omega \). Then \( R^{\alpha \omega} \cong X \oplus R^\alpha \) implies that \( R^\alpha \cong X^n \) for some \( 1 \leq n < \omega \).

**Proof.** Assume that \( R^{\alpha \omega} \cong X \oplus R^\alpha_0 \) and consider the ordinal \( \lambda \) defined by

\[
\lambda = \min\{ \xi \leq \alpha : \exists m, 1 \leq m < \omega, \text{ with } R^\alpha_0 \cong X^m \oplus R^\xi_0 \}.
\]

Thus there exists \( m, 1 \leq m < \omega \), such that

\[ R^\alpha_0 \cong X^m \oplus R^\lambda_0. \]

We distinguish two cases:

**Case 1:** \( \lambda \) is finite. In this case, (1) yields \( R^\alpha_0 \cong X^m \oplus R^\lambda \cong X^{m+1} \), and we are done.

**Case 2:** \( \lambda \) is infinite. Then again by (1),

\[ R^{\alpha \omega} \cong R^{\alpha \omega} \cong X \oplus R^\alpha_0 \cong X^{m+1} \oplus R^\lambda_0. \]

Notice that if \( R^\lambda \cong X^{m+1} \oplus R^\xi_0 \) for some \( \xi < \lambda \), then by (1) we would have \( R^\alpha_0 \cong X^{2m+1} \oplus R^\xi_0 \),
which is absurd by the choice of \( \lambda \). Hence
\[
R^\lambda \hookrightarrow X^{m+1} \oplus R^\xi, \quad \forall \xi < \lambda.
\]
According to (2) there are operators \( \pi_1 : R^\lambda \to X^{m+1} \) and \( \pi_2 : R^\omega \to R^\lambda \), and \( a \in R_+ \), such that for every \( f \in R^\omega \),
\[
a \| f \| \leq \max\{\| \pi_1(f) \|, \| \pi_2(f) \|\} \leq \| f \|.
\]
Fix an integer \( N \) and \( \epsilon > 0 \) such that \( aN > 1 \) and \( 1 + \epsilon < aN \). For every \( 0 \leq \xi < \lambda \), write
\[
\Delta^1_\xi = (\lambda^N \xi, \lambda^N (\xi + 1)).
\]
Let \( Y_N \) be the subspace of \( R^\lambda \) given by
\[
\{ f \in R^\omega : f \text{ is constant on } \Delta^1_\xi \text{ for all } \xi \in [0, \lambda) \}, \quad \text{and} \quad f(\xi) = 0 \text{ for all } \xi \in [\lambda^{N+1}, \lambda^\omega]\}.
\]
Clearly, \( Y_N \) is isomorphic to \( R^\lambda \). Thus by (3), \( \pi_1 \) restricted to \( Y_N \) is not an isomorphism of \( Y_N \) into \( X^{m+1} \). So there exists \( f_1 \in Y_N \) such that \( \| f_1 \| = 1 \) and \( \| \pi_1(f_1) \| \leq \epsilon/2 \).
We may change \( f_1 \) to \( -f_1 \) and assume that there exists \( \xi_1 \in [0, \lambda) \) such that \( f_1(\gamma) = 1 \) for all \( \gamma \in (\lambda^N \xi_1, \lambda^N (\xi_1 + 1)) \).
Since \( \pi_2(f_1) \in R^\lambda \), there exists \( \lambda_1 < \lambda \) such that for every \( \gamma \in [\lambda_1 + 1, \lambda) \), we have \( |\pi_2(f_1)(\gamma)| \leq \epsilon/2 \).
For the second step, for every \( 0 \leq \xi < \lambda \), write
\[
\Delta^2_\xi = (\lambda^N \xi_1 + 1, \lambda^N (\xi_1 + 1)).
\]
Let \( Y_{N-1} \) be the subspace of \( R^\omega \) defined by
\[
\{ f \in R^\omega : f \text{ is constant on } \Delta^2_\xi \text{ for all } \xi \in [0, \lambda) \}, \quad \text{and} \quad f(\xi) = 0 \text{ for all } \xi \notin (\lambda^N \xi_1, \lambda^N (\xi_1 + 1))\}.
\]
Denote by \( P_\lambda \) the natural projection of \( R^\lambda \) onto \( R^\lambda \) and define the operator \( \pi_1 + P_\lambda \pi_2 : R^\omega \to X^{m+1} \oplus R^\lambda \) by
\[
(\pi_1 + P_\lambda \pi_2)(f) = (\pi_1(f), P_\lambda(\pi_2(f))), \quad \forall f \in R^\omega.
\]
Since \( Y_{N-1} \) is isomorphic to \( R^\lambda \), and since by (3), \( X^{m+1} \oplus R^\lambda \) contains no subspace isomorphic to \( R^\lambda \), it follows that \( \pi_1 + P_\lambda \pi_2 \) restricted to \( Y_{N-1} \) is not an isomorphism of \( Y_{N-1} \) into \( X^{m+1} \oplus R^\lambda \).
Hence there exists \( f_2 \in Y_{N-1} \) such that \( \| f_2 \| = 1 \), \( \| \pi_1(f_2) \| \leq \epsilon/2^2 \) and \( |\pi_2(f_2)(\gamma)| \leq \epsilon/2^2 \) for every \( \gamma \in [0, \lambda_1] \).
Since \( \pi_2(f_2) \in R^\lambda \), pick \( \lambda_2 \in [\lambda_1 + 1, \lambda) \) such that \( |\pi_2(f_2)(\gamma)| \leq \epsilon/2^2 \) also for all \( \gamma \in [\lambda_2 + 1, \lambda] \).
We may change \( f_2 \) to \( -f_2 \) and suppose that there exists \( \xi_2 \in [0, \lambda) \) such that \( f_2(\gamma) = 1 \) for all \( \gamma \in (\lambda^N \xi_1 + 1, \lambda^N (\xi_2 + 1)) \).
Repeating this procedure \( N \) times we will find
\[
\bullet \quad f_1, \ldots, f_N \in R^\omega,
\]
Let $f$ be the choice of $\epsilon > 0$ such that for every $1 \leq k \leq N$ and for every $\gamma$ belonging to $(\lambda^N \xi_1 + \lambda^{N-1} \xi_2 + \ldots + \lambda^{N-k+1} \xi_k, \lambda^N \xi_1 + \lambda^{N-1} \xi_2 + \ldots + \lambda^{N-k+1}(\xi_k + 1)]$ we have:

- $f_k(\gamma) = \|f_k\| = 1$,
- $\text{supp } f_2 \subset f_1^{-1}(1)$, $\text{supp } f_3 \subset f_2^{-1}(1), \ldots, \text{supp } f_k \subset f_{k-1}^{-1}(1)$,
- $\|\pi_1(f_k)\| \leq \epsilon / 2^k$,
- $|\pi_2(f_k)(\gamma)| \leq \epsilon / 2^k$, $\forall \gamma \in [\lambda_k + 1, \lambda]$,
- $|\pi_2(f_k)(\gamma)| \leq \epsilon / 2^k$, $\forall \gamma \in [0, \lambda_{k-1}], k > 1$.

Let $f = f_1 + \cdots + f_N$. Then it is obvious that $\|f\| = N$, $\|\pi_1(f)\| \leq \epsilon$, and $\|\pi_2(f)\| \leq 1 + \epsilon$. Finally, by (4) we conclude that $aN \leq 1 + \epsilon$, which is absurd by the choice of $\epsilon$. \]

To state the next lemmas, we need to recall some Banach spaces introduced in [8] and [10]. Let us recall that an ordinal $\alpha$ is said to be regular if the smallest ordinal cofinal with $\alpha$ is equal to $\alpha$. Otherwise $\alpha$ is said to be singular.

Let $X$ be a Banach space and $\alpha$ a regular ordinal. We denote by $X_\alpha$ the space of all $x^{**} \in X^{**}$ having the following property: for any limit ordinal $\beta < \alpha$ and transfinite sequence $(f_\xi)_{\xi < \beta}$ of continuous linear functionals on $X$ with $\sup \{\|f_\xi\| : \xi < \beta\} < \infty$ and $f_\xi(x) \xrightarrow{\xi \rightarrow \beta} 0$ for every $x \in X$, we have $x^{**}(f_\xi) \xrightarrow{\xi \rightarrow \beta} 0$.

From now on, if $X$ is a Banach space, then $cX$ denotes the canonical image of $X$ in $X^{**}$.

**Remark 2.2.** Clearly $cX \subset X_\alpha \subset X_{\omega_1}$ for every regular ordinal $\alpha \geq \omega_1$. Moreover, it may easily be shown that if $X \sim Y$, then

$$\frac{X_\alpha}{cX} \sim \frac{Y_\alpha}{cY}.$$ Observe also that if $X$ has the Mazur property, then $X_{\omega_1} = cX$.

**Lemma 2.3.** Let $X$ and $Y$ be Banach spaces and $\alpha$ be a regular ordinal. Then there exists an isomorphism $\Phi : X^{**} \oplus Y^{**} \rightarrow (X \oplus Y)^{**}$ satisfying

(i) $\Phi(cX \oplus cY) = c(X \oplus Y)$,
(ii) $\Phi(X_\alpha \oplus Y_\alpha) = (X \oplus Y)_\alpha$.

**Proof.** Let $T : (X \oplus Y)^* \rightarrow X^* \oplus Y^*$ be the isomorphism given by $T(z^*) = (z^*_X, z^*_Y)$ for $z^{**} \in (X \oplus Y)^*$. Then the isomorphism $T^* : (X^* \oplus Y^*)^* \rightarrow (X \oplus Y)^{**}$ is given by $(T^* z^{**})(w^*) = z^{**}(Tw^*)$ for $z^{**} \in (X^* \oplus Y^*)^*$ and $w^* \in (X \oplus Y)^*$.
Consider also the isomorphism $L : X^{**} \oplus Y^{**} \to (X \oplus Y)^*$ defined by $L(x^{**}, y^{**})(x^*, y^*) = x^{**}(x^*) + y^{**}(y^*)$ for $x^{**} \in X^{**}, y^{**} \in Y^{**}, x^* \in X^*$ and $y^* \in Y^*$.

Put $\Phi = T^*L$. Then $\Phi(x^{**}, y^{**})(w^*) = x^{**}(w^{|X}) + y^{**}(w^{|Y})$ for $x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $w^* \in (X \oplus Y)^*$. Now it is easy to see that (i) and (ii) hold.

The next lemma is a generalization of a result of Gul’ko and Os’kin [8] and independently of Kislyakov [10]. Let $\xi$ be any ordinal and $\alpha$ a regular ordinal. The cardinality of $\xi$ will be denoted by $\bar{\xi}$. Let $A_{\xi}^\alpha$ denote the subset of $[0, \xi]$ consisting of the nonisolated points that are not limit points for any set of cardinality smaller than $\bar{\xi}$.

**Lemma 2.4.** Let $\alpha$ be an uncountable regular ordinal and $\xi \in [\alpha, \alpha^2]$, with $\xi = \alpha \xi' + \delta$ and $\xi', \delta \leq \alpha$. Suppose that $X$ is a Banach space satisfying $X_\alpha = cX$. Then

$$\frac{(X \oplus \mathbb{R}\xi)_\alpha}{c(X \oplus \mathbb{R}\xi)} \sim C_0(A_{\xi}^\alpha).$$

**Proof.** Let $\Phi$ be as defined in Lemma 2.3. Then by using [10, Corollary 4.1], it can be easily checked that

$$\frac{(X \oplus \mathbb{R}\xi)_\alpha}{c(X \oplus \mathbb{R}\xi)} = \frac{\Phi(X_\alpha \oplus \mathbb{R}\xi)}{\Phi(cX \oplus c\mathbb{R}\xi)} \sim \frac{X_\alpha \oplus \mathbb{R}\xi}{cX \oplus c\mathbb{R}\xi} = \frac{cX \oplus \mathbb{R}\xi}{cX \oplus c\mathbb{R}\xi} \sim \frac{\mathbb{R}\xi}{c\mathbb{R}\xi} \sim C_0(A_{\xi}^\alpha).$$

**Proposition 2.5.** Suppose that $\omega_1 \leq \alpha \leq \beta$ and $X$ is a Banach space satisfying

- $\mathbb{R}\alpha \leftrightarrow X^n$ for every $1 \leq n < \omega$,
- $X_{\omega_1} = cX$.

Then $X \oplus \mathbb{R}\alpha \sim X \oplus \mathbb{R}\beta$ implies that $\mathbb{R}\alpha \sim \mathbb{R}\beta$.

**Proof.** First we will prove that $\bar{\alpha} = \bar{\beta}$. Suppose that $\alpha < \beta$. Then $\alpha^\omega < \beta$. Consequently,

$$\mathbb{R}\alpha^\omega \leftrightarrow \mathbb{R}\beta \leftrightarrow X \oplus \mathbb{R}\beta \sim X \oplus \mathbb{R}\alpha.$$

Therefore by Lemma 2.1, $\mathbb{R}\alpha \leftrightarrow X^n$ for some $1 \leq n < \omega$, contradicting our hypotheses.

Next let $\lambda$ be the first ordinal of cardinality $\bar{\alpha}$. There are two cases:

**Case 1:** $\lambda$ is a singular ordinal or $\lambda$ is a regular ordinal with $\lambda^2 \leq \alpha$. If $\alpha^\omega \leq \beta$, then (5) holds and again we obtain a contradiction. Thus $\beta < \alpha^\omega$ and by [10, Theorem 1], we conclude that $\mathbb{R}\alpha \sim \mathbb{R}\beta$.

**Case 2:** $\lambda$ is a regular ordinal with $\alpha < \lambda^2$. Thus $X_\lambda = cX$. Write $\alpha = \lambda \alpha' + \gamma$ with $\alpha', \gamma < \lambda$. If $\lambda^2 < \beta$, then $\mathbb{R}\lambda^2 \leftrightarrow \mathbb{R}\beta \leftrightarrow X \oplus \mathbb{R}\beta \sim X \oplus \mathbb{R}\alpha$
and according to [10, Lemmas 1.4 and 2.4] we deduce

\[ C_0(\Lambda_2^\lambda) \sim \frac{\mathbb{R}^2}{c\mathbb{R}^2} \leftrightarrow \frac{(X \oplus \mathbb{R}^\alpha)_\lambda}{c(X \oplus \mathbb{R}^\alpha)} \sim C_0(\Lambda_\alpha^\lambda). \]

Therefore by [10, Corollary 4.1], \( \bar{\lambda} \leq \alpha' \), which is absurd. So we may assume that \( \beta \leq \lambda^2 \). Write \( \beta = \lambda\beta' + \delta \), with \( \beta', \delta \leq \alpha \). Then Lemma 2.4 yields

\[ C_0(\Lambda_\alpha^\lambda) \sim \frac{(X \oplus \mathbb{R}^\alpha)_\lambda}{c(X \oplus \mathbb{R}^\alpha)} \sim \frac{(X \oplus \mathbb{R}^\beta)_\lambda}{c(X \oplus \mathbb{R}^\beta)} \sim C_0(\Lambda_\beta^\lambda). \]

Once again by [10, Corollary 4.1] we see that \( \alpha' = \beta' \) and by [10, Theorem 2] we conclude that \( \mathbb{R}^\alpha \sim \mathbb{R}^\beta \).

3. Some questions. Corollary 1.9 leads naturally to the following question.

**Question 3.1.** Assume that \( C(2^m) \) has the Mazur property. Does it follow that \( m \) is not sequential?

As pointed out by the referee, \( C(2^{m_2}) \) does not have the Mazur property. Moreover, he noticed that \( C(2^{m_2})_\lambda \neq cC(2^{m_2}) \) for every \( \omega_1 \leq \lambda < m_2 \). Indeed, let \( F \) be an \( m_2 \)-complete ultrafilter and \( x^{**} \) the weak*-limit along the ultrafilter \( F \) of \( \{c(p_\alpha) : \alpha \in m_2\} \subset C(2^{m_2})^{**} \), where \( p_\alpha : 2^{m_2} \rightarrow 2 \) is the \( \alpha \)th projection. Then \( x^{**} \in C(2^{m_2})_\lambda \setminus cC(2^{m_2}) \) for all \( \lambda < m_2 \).

However, we do not know the answer to the following question.

**Question 3.2.** Is it true that \( C(2^m)_{\omega_1} = cC(2^m) \) whenever \( \aleph_1 \leq m < m_2 \)?

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