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Borsuk–Sieklucki theorem in cohomological dimension theory

by

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Abstract. The Borsuk–Sieklucki theorem says that for every uncountable family $\{X_\alpha\}_{\alpha\in A}$ of n-dimensional closed subsets of an n-dimensional ANR-compactum, there exist $\alpha \neq \beta$ such that $\dim(X_\alpha \cap X_\beta) = n$. In this paper we show a cohomological version of that theorem:

Theorem. *Suppose a compactum X is $\mathrm{clc}_{\mathbb{Z}}^{n+1}$, where $n \geq 1$, and G is an Abelian group. Let $\{X_\alpha\}_{\alpha\in J}$ be an uncountable family of closed subsets of X. If $\dim_G X = \dim_G X_\alpha = n$ for all $\alpha \in J$, then $\dim_G(X_\alpha \cap X_\beta) = n$ for some $\alpha \neq \beta$.*

For G being a countable principal ideal domain the above result was proved by Choi and Kozlowski [C-K]. Independently, Dydak and Koyama [D-K] proved it for G being an arbitrary principal ideal domain and posed the question of validity of the Theorem for quasicyclic groups (see Problem 1 in [D-K]).

As applications of the Theorem we investigate equality of cohomological dimension and strong cohomological dimension, and give a characterization of cohomological dimension in terms of a special base.

1. Introduction. Borsuk [Bo] and Sieklucki [S] investigated dimension properties of ANR-compacta and proved the following:

1.1. Theorem. *Let $\{X_\alpha\}_{\alpha\in A}$ be an uncountable family of n-dimensional closed subsets of an n-dimensional ANR-compactum. Then there exist $\alpha \neq \beta$ in A such that $\dim(X_\alpha \cap X_\beta) = n$.*

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It is easy to see that Theorem 1.1 holds for X being the n-cube. Indeed, the interiors $\mathrm{int}(X_\alpha)$ must be nonempty, so there is a pair of indices $\alpha \neq \beta$ so that $\mathrm{int}(X_\alpha \cap X_\beta) \neq \emptyset$. Hence all n-dimensional manifolds and n-dimensional polyhedra have the same property. On the other hand, an n-dimensional compactum may admit an uncountable family $\{X_\alpha\}_{\alpha\in A}$ of n-dimensional (closed) subsets such that $\dim(X_\alpha \cap X_\beta) \leq n-1$ for all $\alpha \neq \beta$ in A. For example, the n-dimensional Menger compactum μ^n, $n \geq 1$, contains a copy of the product of the n-disk D^n and the Cantor set C. Thus, μ^n contains an uncountable family of pairwise disjoint n-dimensional closed subsets.

Choi and Kozlowski [C-K] generalized the Borsuk–Sieklucki theorem by using cohomological local connectivity and cohomological dimension based on Alexander–Spanier cohomology with compact supports and with coefficients in a countable principal ideal domain R. Their theorem states that if X is a clc^n locally compact separable metric space with $\dim_R X = n$ and $\{X_\lambda\}_{\lambda\in\Lambda}$ is an uncountable collection of closed subsets of X with $\dim_R X_\lambda = n$ for all λ, then there are two distinct indices $\mu, \lambda \in \Lambda$ such that $\dim_R(X_\mu \cap X_\lambda) = n$.

Independently, Dydak and Koyama [D-K] undertook an effort to generalize the Borsuk–Sieklucki theorem to cohomological dimension. Here is their result:

1.2. Theorem. *Suppose that a compactum X is clc^n_R, where $n \geq 1$ and R is a principal ideal domain. Let $\{X_\alpha\}_{\alpha\in A}$ be an uncountable family of closed subsets of X. If $\dim_R X = n$ and $\dim_R X_\alpha = n$ for each $\alpha \in A$, then there is a pair of indices $\alpha \neq \beta$ in A such that $\dim_R(X_\alpha \cap X_\beta) = n$.*

Thus, the result of [D-K] drops the assumption of R being countable and weakens the connectivity of X. However, the proof in [C-K] is more elegant. The virtue of the proof in [D-K] is that its idea can be applied to arbitrary groups, which is the goal of this paper. We prove a variant of Theorem 1.2 in which the principal ideal domain R is replaced by a group satisfying the descending chain condition. This allows us to deduce a generalization of the Borsuk–Sieklucki theorem for arbitrary groups if X is locally $(n+1)$-connected.

As an application, we investigate equality of cohomological dimension and strong cohomological dimension introduced by Kodama [K]. Also, we give a characterization of cohomological dimension in terms of existence of a special base.

2. Preliminaries. Let X be a compactum. The *cohomological dimension* of X with respect to the abelian group G, denoted by $\dim_G X$, is said to be less than or equal to n (notation: $\dim_G X \leq n$) provided that

$H^k(X, A; G) = 0$ for all $k \geq n+1$ and all closed subsets $A \subset X$, where H^* is the Čech cohomology theory (see [$\mathrm{Dr}_{1,2}$], [K], or [Ku]).

We are interested in the cohomological dimension where the coefficient group is the *quasicyclic group* $\mathbb{Q}_p$ (sometimes also denoted by $\mathbb{Z}(p^\infty)$) defined as follows:

Let p be a fixed prime number. The quasicyclic group $\mathbb{Q}_p$ is the union of all p^kth roots of unity, where k ranges over all positive integers. There is another way of describing $\mathbb{Q}_p$: for $n \in \mathbb{N}$ define $\mathbb{Z}_{1/p^n} = \{m/p^n : m$ is an integer modulo $p^n\}$. Then $\mathbb{Q}_p$ is the set $\bigcup_{n=1}^{\infty} \mathbb{Z}_{1/p^n}$ together with the abelian group operation $+ : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p$ defined by

$$\frac{m}{p^n} + \frac{r}{p^s} = \frac{p^{k-n}m + p^{k-s}r}{p^k}$$

where $k = \max\{n, s\}$ and the sum is modulo p^k. Observe that each subgroup $\mathbb{Z}_{1/p^n}$ can be generated by an element of the form m/p^n where m and p^n are relatively prime.

An alternative description of $\mathbb{Q}_p$ is as the quotient group $\mathbb{Q}/\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the group of all rational numbers m/n so that n is relatively prime to p. Under this description $\mathbb{Z}_{1/p^n}$ is the image of the group of all rational numbers m/p^k, where $k \leq n$.

2.1. Definition. An abelian group satisfies the *descending chain condition* (see [Hun], p. 374) if every decreasing chain $B_1 \supset B_2 \supset \ldots$ of subgroups of G stabilizes (i.e., there exists n so that $B_i = B_{i+1}$ for all $i > n$).

2.2. Lemma. (1) *The quasicyclic group $\mathbb{Q}_p$ satisfies the descending chain condition.*

(2) *If $G_1, \ldots, G_k$ satisfy the descending chain condition, then so does $G_1 \oplus \ldots \oplus G_k$.*

Proof. (1) is a consequence of the fact that all proper subgroups of $\mathbb{Q}_p$ are $\mathbb{Z}_{1/p^n}$ and $\mathbb{Z}_{1/p} \subset \mathbb{Z}_{1/p^2} \subset \ldots$ Indeed, if B is a nontrivial subgroup of $\mathbb{Q}_p$ and $0 \neq m/p^n \in B$, where m and p are relatively prime, then B contains $\mathbb{Z}_{1/p^n}$ (as B is generated by m/p^n). If B is a proper subgroup of $\mathbb{Q}_p$, then there is the smallest integer k so that B contains $\mathbb{Z}_{1/p^k}$ but does not contain $\mathbb{Z}_{1/p^{k+1}}$. If $m/p^n \in B - \mathbb{Z}_{1/p^k}$, where m and p are relatively prime, then n cannot be greater than k as B does not contain $\mathbb{Z}_{1/p^{k+1}}$, a contradiction. Thus $B = \mathbb{Z}_{1/p^k}$.

(2) is essentially Corollary 1.7 in [Hun], p. 374. ∎

It is easy to prove that if G and H are groups, G satisfies the descending chain condition and $f : G \to H$ is an epimorphism, then H also satisfies the descending chain condition. Hence if K is a subgroup of a group G satisfying the descending chain condition then both K and G/K satisfy the descending chain condition.

We need to generalize the concept of pro-epimorphisms of [D-K] to homomorphisms of groups.

2.3. DEFINITION. Suppose G is a group and $\{G_\alpha\}_{\alpha\in J}$ a family of groups. A homomorphism $f: G \to \prod_{\alpha\in J} G_\alpha$ is called a *pro-epimorphism* if, for any finite subset F of J, the composition

$$G \xrightarrow{f} \prod_{\alpha\in J} G_\alpha \xrightarrow{p_F} \prod_{\alpha\in F} G_\alpha$$

is an epimorphism.

The following result is a version of the Basic Lemma of [D-K] for groups satisfying the descending chain condition.

2.4. PROPOSITION. *Let G be a group satisfying the descending chain condition and let $f: G \to \prod_{\alpha\in J} G_\alpha$ be a pro-epimorphism, where $\{G_\alpha\}_{\alpha\in J}$ is a family of groups. If J is infinite, then there is $\beta \in J$ such that $G_\beta = 0$.*

Proof. Consider the subset $J' = \{\alpha \in J \mid G_\alpha \neq \{0\}\}$ of J. We plan to show that J' is finite. Suppose J' is infinite and pick an infinite sequence $\alpha_i \in J'$, $i \geq 1$, of points of J'. Define F_k, $k \geq 1$, as $\{\alpha_1, \ldots, \alpha_k\}$. Define A_k as the kernel of $p_{F_n} \circ f$ for $k \geq 1$. Since G satisfies the descending chain condition, there is $n \geq 2$ so that $A_n = A_{n-1}$. Pick $c \in G_{\alpha_n} - \{0\}$. Since $p_{F_n} \circ f$ is an epimorphism, there is $d \in G$ so that $p_{F_n} \circ f(d) = (0, \ldots, 0, c)$. That means $d \in A_{n-1} - A_n$, a contradiction. ■

The next result is a modification of the Pro Lemma in [D-K] with a finitely generated module replaced by a group satisfying the descending chain condition.

2.5. LEMMA. *Consider a commutative diagram of abelian groups*

$$(*) \qquad \begin{array}{ccc} M & \xrightarrow{f} & \prod_{\alpha\in J} M_\alpha \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \prod_{\alpha\in J} k_\alpha} \\ N & \xrightarrow{g} & \prod_{\alpha\in J} N_\alpha \end{array}$$

so that f is a pro-epimorphism and J is infinite. If $\mathrm{im}(h)$ *satisfies the descending chain condition, then there is $\alpha \in J$ such that $k_\alpha : M_\alpha \to N_\alpha$ is trivial.*

Proof. We can apply 2.4 to the homomorphism $u = g|\mathrm{im}(h) : \mathrm{im}(h) \to \prod_{\alpha\in J} \mathrm{im}(k_\alpha)$ by noticing that u is a pro-epimorphism since f is, and the diagram $(*)$ is commutative. ■

The following lemma is a version of Lemma 17.3 of [Br] for groups satisfying the descending chain condition.

2.6. Lemma. *Consider the commutative diagram of abelian groups*

$$\begin{tikzcd} & A_2 \arrow[r, "s"] \arrow[d, "f"] & A_3 \arrow[d, "k"] \\ B_1 \arrow[r, "i"] \arrow[d, "h"] & B_2 \arrow[r, "j"] \arrow[d, "g"] & B_3 \\ C_1 \arrow[r, "t"] & C_2 & \end{tikzcd}$$

in which the middle row is exact. If both $\operatorname{im}(h)$ *and* $\operatorname{im}(k)$ *satisfy the descending chain condition, then so does* $\operatorname{im}(g \circ f)$.

Proof. Suppose there is an infinite decreasing sequence $A_1 \supset A_2 \supset \ldots$ of subgroups of $\operatorname{im}(g \circ f) \subset C_2$. Since $\operatorname{im}(h)$ satisfies the descending chain condition, the chain $\operatorname{im}(t \circ h) \cap A_i$ stabilizes. Without loss of generality, we may assume $\operatorname{im}(t \circ h) \cap A_i = \operatorname{im}(t \circ h) \cap A_j$ for all i, j. Since $\operatorname{im}(k)$ satisfies the descending chain condition, the chain $(k \circ s)(g \circ f)^{-1}(A_i)$ stabilizes. Again, we may assume $(k \circ s)(g \circ f)^{-1}(A_i) = (k \circ s)(g \circ f)^{-1}(A_j)$ for all i, j. This is the same as $j(g^{-1})(A_i) = j(g^{-1})(A_j)$ for all i, j. Suppose $x \in A_i - A_j$ for some $j > i$. Pick $x' \in B_2$ so that $g(x') = x$. There is $y' \in g^{-1}(A_j)$ with $j(y') = j(x')$. Pick $z' \in B_1$ with $i(z') = x' - y'$. Notice that $t(h(z')) \in A_i$, so there is $z'' \in i^{-1}(g^{-1}(A_j))$ with $t(h(z')) = t(h(z''))$. This implies $g(i(z')) = g(x') - g(y') = g(i(z'')) \in A_j$. Thus $x = g(x') = (g(x') - g(y')) + g(y') \in A_j$, a contradiction. ■

2.7. Definition. A compactum X is said to be *cohomology locally n-connected* with respect to an abelian group G (X is n-clc_G for short) if for each point $x \in X$ and neighborhood N of x, there is a neighborhood $M \subset N$ of x such that the inclusion-induced homomorphism

$$i^n_{M,N} : H^n(N; G) \to H^n(M; G)$$

is trivial, where H^* is the reduced Čech cohomology theory.

X is said to be clc^n_G if it is k-clc_G for all $k \le n$.

2.8. Theorem. *Suppose G is an Abelian group satisfying the descending chain condition. If X is* clc^n_G, *and K and L are closed subsets of X with* $K \subset \operatorname{int}(L)$, *then the image of the inclusion-induced homomorphism* $i^k_{K,L} : H^k(L; G) \to H^k(K; G)$ *satisfies the descending chain condition for* $k = 0, \ldots, n$.

Proof. First consider the image of $i^0_{K,L}$. As X is clc^0_G and K is compact, K is contained in the union of finitely many components of L. This means that $H^0(L;G) \to H^0(K;G)$ factors through a finite direct sum of copies of G. By Lemma 2.2, $\mathrm{im}(i^0_{K,L})$ satisfies the descending chain condition.

Suppose 2.8 holds for all $n < m$, X is clc^m_G, and K and L are closed subsets of X with $K \subset \mathrm{int}(L)$. Consider the family $\mathcal{F}$ of all compact subsets C of the interior of L so that the image of the inclusion-induced homomorphism $i^m_{C',L} : H^m(L;G) \to H^m(C';G)$ satisfies the descending chain condition for some compact neighborhood C' of C in $\mathrm{int}(L)$. Since X is clc^m_G, each point $x \in \mathrm{int}(L)$ has a compact neighborhood $C_x \in \mathcal{F}$. If we prove that $\mathcal{F}$ is closed under taking unions of sets, we are done as K can be covered by a finite union of elements of $\mathcal{F}$. Suppose $C_1, C_2 \in \mathcal{F}$. Pick a neighborhood C'_j of C_j in $\mathrm{int}(L)$ so that the image of the inclusion-induced homomorphism $i^m_{C'_j,L} : H^m(L;G) \to H^m(C'_j;G)$ satisfies the descending chain condition for $j = 1,2$. Pick a neighborhood C''_j of C_j in $\mathrm{int}(C'_j)$ for $j = 1,2$. Applying the Meyer–Vietoris exact sequence and Lemma 2.6 to the following diagram shows that $\mathrm{im}(i^m_{C''_1\cup C''_2,L})$ satisfies the descending chain condition:

$$\begin{array}{ccccc}
 & & H^m(L;G) & \longrightarrow & H^m(L;G)\oplus H^m(L;G) \\
 & & \downarrow & & \\
H^{m-1}(C'_1\cap C'_2;G) & \longrightarrow & H^m(C'_1\cup C'_2;G) & \longrightarrow & H^m(C'_1;G)\oplus H^m(C'_2;G) \\
\downarrow & & \downarrow & & \\
H^{m-1}(C''_1\cap C''_2;G) & \longrightarrow & H^m(C''_1\cup C''_2;G) & &
\end{array}$$

■

The next result is an analog of Theorem 2.2 in [D-K] with a finitely generated module replaced by a group satisfying the descending chain condition.

2.9. THEOREM. *Suppose G is an Abelian group satisfying the descending chain condition. If X is clc^n_G and A_1, A_2, B_1, B_2 are closed subsets of X such that $A_1 \subset B_1$, $A_2 \subset B_2$, $A_1 \subset \mathrm{int}(A_2)$ and $B_1 \subset \mathrm{int}(B_2)$, then the image of $H^k(B_2,A_2;G) \to H^k(B_1,A_1;G)$ satisfies the descending chain condition for all $k \le n$.*

Proof. Use induction and apply Lemma 2.6 to the commutative diagram

$$\begin{array}{ccccc}
 & & H^n(B_2,A_2;G) & \longrightarrow & H^n(B_2;G) \\
 & & \downarrow & & \downarrow \\
H^{n-1}(A;G) & \longrightarrow & H^n(B,A;G) & \longrightarrow & H^n(B;G) \\
\downarrow & & \downarrow & & \\
H^{n-1}(A_1;G) & \longrightarrow & H^n(B_1,A_1;G) & &
\end{array}$$

where $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. ■

3. Borsuk–Sieklucki theorem. Using our previous results we are ready to prove the Borsuk–Sieklucki theorem for groups satisfying the descending chain condition in an analogous way to that of Dydak and Koyama [D-K] for rings.

3.1. THEOREM. *Suppose a compactum X is* clc_G^n, *where* $n \geq 1$, *and* G *is an Abelian group satisfying the descending chain condition. Let* $\{X_\alpha\}_{\alpha \in J}$ *be an uncountable family of closed subsets of* X. *If* $\dim_G X = \dim_G X_\alpha = n$ *for all* $\alpha \in J$, *then* $\dim_G(X_\alpha \cap X_\beta) = n$ *for some* $\alpha \neq \beta$.

Proof. Suppose that $\dim_G(X_\alpha \cap X_\beta) \leq n-1$ for each pair $\alpha \neq \beta$ in J. For each $\alpha \in J$, there is a closed subset $C_\alpha \subset X_\alpha$ such that $H^n(X_\alpha, C_\alpha; G) \neq 0$, because $\dim_G X_\alpha = n$. Let $\{N_i\}_{i=1}^\infty$ be a countable family of closed subsets of X such that for each closed subset K of X the family $\{N_i : K \subset \mathrm{int}(N_i)\}$ is a basis of neighborhoods of K in X. By the continuity of Čech cohomology, for each $\alpha \in J$, there is $N_{k(\alpha)} \subset \mathrm{int}(N_{h(\alpha)})$ such that the inclusion-induced homomorphism

$$H^n(X_\alpha, X_\alpha \cap N_{h(\alpha)}; G) \to H^n(X_\alpha, X_\alpha \cap N_{k(\alpha)}; G)$$

is not trivial.

Since J is uncountable, we may assume that $N_{k(\alpha)} = N$ for every $\alpha \in J$, and $N_{h(\alpha)} = M$ for every $\alpha \in J$. Namely, we have closed subsets M, N of X such that $N \subset \mathrm{int}(M)$ and for every $\alpha \in J$,

$$(1) \qquad H^n(X_\alpha, X_\alpha \cap M; G) \to H^n(X_\alpha, X_\alpha \cap N; G) \quad \text{is not trivial.}$$

On the other hand, since $\dim_G(X_\alpha \cap X_\beta) \leq n-1$ for all pairs $\alpha \neq \beta$ in J,

$$H^n\Big(\bigcup_{\alpha \in F} X_\alpha, \Big(\bigcup_{\alpha \in F} X_\alpha\Big) \cap M; G\Big) \to \bigoplus_{\alpha \in F} H^n(X_\alpha, X_\alpha \cap M; G)$$

is an epimorphism for all finite subsets $F \subset J$.

We note that the inclusion-induced homomorphism

$$H^n\Big(\Big(\bigcup_{\alpha \in F} X_\alpha\Big) \cup M, M; G\Big) \to H^n\Big(\bigcup_{\alpha \in F} X_\alpha, \Big(\bigcup_{\alpha \in F} X_\alpha\Big) \cap M; G\Big)$$

is an isomorphism by the excision axiom, and the inclusion-induced homomorphism

$$H^n(X, M; G) \to H^n\Big(\Big(\bigcup_{\alpha \in F} X_\alpha\Big) \cup M, M; G\Big)$$

is an epimorphism because $\dim_G X = n$. Hence, for all finite subsets $F \subset J$,

$$(2) \qquad H^n(X, M; G) \to \bigoplus_{\alpha \in F} H^n(X_\alpha, X_\alpha \cap M; G) \quad \text{is an epimorphism.}$$

Now we consider the following commutative diagram:

$$\begin{tikzcd} H^n(X,M;G) \arrow[r] \arrow[d] & \prod_{\alpha\in J} H^n(X_\alpha, X_\alpha\cap M;G) \arrow[d] \\ H^n(X,N;G) \arrow[r] & \prod_{\alpha\in J} H^n(X_\alpha, X_\alpha\cap N;G) \end{tikzcd}$$

Property (2) says that the upper homomorphism is a pro-epimorphism. By Theorem 2.8, the image of $H^n(X,M;G) \to H^n(X,N;G)$ satisfies the descending chain condition. Hence, by 2.5, there exists $\alpha \in J$ such that $H^n(X_\alpha, X_\alpha\cap M;G) \to H^n(X_\alpha, X_\alpha\cap N;G)$ is trivial. This contradicts (1). Therefore there are $\alpha \neq \beta$ in J such that $\dim_G(X_\alpha \cap X_\beta) = n$. ∎

3.2. Theorem. *Suppose a compactum X is* $\mathrm{clc}_{\mathbb{Z}}^{n+1}$ *and G is an Abelian group*, *where* $n \geq 1$. *Let $\{X_\alpha\}_{\alpha\in J}$ be an uncountable family of closed subsets of X. If* $\dim_G X = \dim_G X_\alpha = n$ *for all* $\alpha\in J$, *then* $\dim_G(X_\alpha\cap X_\beta) = n$ *for some* $\alpha\neq\beta$.

Proof. Bockstein's First Theorem (see [Ku]) says that there is a subset $\sigma(G)$ (we call it the *Bockstein basis* of G) of the family of groups consisting of $\mathbb{Q}$, $\mathbb{Z}/p$ (p prime), $\mathbb{Z}_{(p)}$ (p prime), and $\mathbb{Q}_p$ (p prime) such that $\dim_G Y = \max\{\dim_H Y : H \in \sigma(G)\}$ for all Y compact. Hence there is $R \in \sigma(G)$ such that $\dim_R X = n$, $\dim_R X_\alpha = n$ for uncountably many $\alpha \in J$, and R is either a countable PID or a group satisfying the descending chain condition. From $X \in \mathrm{clc}_{\mathbb{Z}}^{n+1}$ the Universal Coefficient Theorem implies $X \in \mathrm{clc}_R^n$. Therefore, by Theorem 3.1 and [D-K], there are $\alpha\neq\beta$ in J such that $\dim_R(X_\alpha\cap X_\beta) = n$. In particular, $\dim_G(X_\alpha\cap X_\beta) = n$. ∎

4. Applications. First we give applications of the cohomological dimension versions of the Borsuk–Sieklucki theorem to strong cohomological dimension introduced by Kodama [K] for compacta but which can be generalized to metrizable spaces without any problem.

4.1. Definition. X has *strong cohomological dimension at most n with respect to an abelian group G*, written $\mathrm{Ind}_G X \leq n$, provided that for any pair of a closed subset $A \subset X$ and an open subset U containing A, there exists an open subset V such that

$$A \subset V \subset U \quad \text{and} \quad \dim_G \partial V \leq n-1.$$

We define

$$\mathrm{Ind}_G X = \min\{n : \mathrm{Ind}_G X \leq n\}.$$

Kodama proved ([K], Lemma 38-9) that for every 2-dimensional compact ANR X and every nontrivial abelian group G, we have the equality $\dim_G X = \mathrm{Ind}_G X = \dim X = 2$, and asked ([K], Problem 38-10): *If X is* *a compact ANR, does the equality* $\dim_G X = \operatorname{Ind}_G X$ *hold for every abelian group* G*?*

This problem was affirmatively answered by Dydak and Koyama [D-K].

In this section we will improve some of the results of [D-K] related to the strong cohomological dimension.

4.2. Theorem. *Let* G *be an abelian group. Suppose that a compactum* X *is* $\mathrm{clc}_{\mathbb{Z}}^{n+1}$. *If* $\dim_G X = n$, *then* $\operatorname{Ind}_G X = n$.

Proof. By Lemma 4.2 of [D-K], it suffices to show that $\operatorname{Ind}_G X \le n$. For a given closed subset A of X, let

$$B(A,\varepsilon) = \{x \in X : d(a,A) \le \varepsilon\} \quad \text{and} \quad C_\varepsilon = \partial B(A,\varepsilon), \qquad \varepsilon > 0.$$

Since $C_\varepsilon \cap C_\delta = \emptyset$ for $\varepsilon \neq \delta$, Theorem 3.2 says that $\dim_G C_\varepsilon = n$ for only countably many $\varepsilon > 0$. Hence A has a neighborhood basis $\{U_i\}_{i\ge 1}$ such that $\dim_G \partial U_i \le n-1$ for every $i \ge 1$. It follows that $\operatorname{Ind}_G X \le n$. ∎

Remark. Theorem 4.2 improves Theorem 4.4 of [D-K].

Now we are ready to characterize $\dim_G$ by using bases, which corresponds to well known facts in usual dimension theory.

4.3. Theorem. *Suppose that* G *is an abelian group and a compactum* X *is* $\mathrm{clc}_{\mathbb{Z}}^{n+1}$. *Then the following statements are equivalent*:

(i) $\dim_G X \le n$,

(ii) X *has a countable base* $\mathcal{B}$ *such that* $\dim_G \partial U \le n-1$ *for all* $U \in \mathcal{B}$.

Proof. The implication (i)⇒(ii) follows from Theorem 4.2. To prove that (ii)⇒(i) we take a pair A, B of disjoint closed subsets of X. Then there exist elements $U_1, \ldots, U_k \in \mathcal{B}$ such that

$$A \subset \bigcup_{i=1}^{k} U_i \subset \bigcup_{i=1}^{k} \mathrm{cl}(U_i) \subset X \setminus B.$$

Then the boundary $\partial(\bigcup_{i=1}^k U_i)$ separates A and B and $\dim_G \partial(\bigcup_{i=1}^k U_i) \le n-1$. Hence, by Lemma 4.2 of [D-K], $\dim_G X \le n$. ∎

Remark. Theorem 4.3 improves Theorem 4.6 of [D-K].

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