

On fractals which are not so terrible

by

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Abstract. The notion of NST domain and the closely related notion of ball condition, both topological in nature and quite useful within the theory of function spaces, are compared with each other (and with the older concept of porosity) and also with other notions of interest, like those of d -set and of interior regular domain, which have a measure-theoretical nature. Also, after extending the idea of NST (not so terrible) to a larger class of sets, the property is studied in the context of anisotropic self-affine fractals.

1. Introduction. In [3, pp. 143–144] Frazier and Jawerth introduced the concept of NST (not so terrible) domain in order to characterize a property of certain functions belonging to Triebel–Lizorkin spaces. This follows a long trend, not to be discussed here, of imposing conditions on (or near the boundary of) domains in order to guarantee certain properties for function spaces on those domains. The interested reader may want to consult [7], [8] and [5], as well as the references given there, for the older material.

One of our points here is that the main idea underlying the concept of NST domain has been repeatedly introduced after the work [3] (and even before it, under the name of porosity—cf. Section 3 below), maybe in disguised form. So, for example, under mild hypotheses on a (fractal-to-be) set Γ , saying that Γ satisfies the so-called ball condition considered in [9, p. 142] is equivalent to saying that Γ^c is a NST domain. We shall detail this below, which will eventually lead us to define what should be meant by a not so terrible (NST) fractal.

Afterwards we give a direct proof that d -sets in \mathbb{R}^n (see [5, pp. 28–32] or [9, p. 5]) are not so terrible fractals (or, what is the same, satisfy the ball condition) when $d < n$.

Next we study a subclass of the family of self-affine fractals and give sufficient conditions for an element of the subclass to be NST. Finally, we

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exhibit an example of a self-affine fractal which is really terrible (that is, which is not NST), though it might not look like that at first glance.

2. Preliminary discussion

DEFINITION 2.1 ([3, pp. 143–144]). A domain (i.e., a non-empty open set) Ω of \mathbb{R}^n is said to be NST (not so terrible), or to be in NST (thinking about NST as a denomination for the class of all NST domains in \mathbb{R}^n , for some fixed $n \in \mathbb{N}$) if there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and $\overline{Q} \cap \partial\Omega \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \partial\Omega = \emptyset$.

Of course, we will rule out, because of lack of interest, the case when $\Omega = \mathbb{R}^n$, that is, we shall only consider proper domains.

In the above definition, $\overset{\circ}{P}$, \overline{Q} and $\partial\Omega$ stand, respectively, for the interior of P , the closure of Q and the boundary of Ω , as usual. A *dyadic cube* Q in \mathbb{R}^n is a cube

$$Q_{\nu k} \equiv \{(x_1, \dots, x_n) : 2^{-\nu}k_i \leq x_i < 2^{-\nu}(k_i + 1) \text{ for } i = 1, \dots, n\}$$

for some $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. In particular, for each such ν , the family $\{Q_{\nu k} : k \in \mathbb{Z}^n\}$ constitutes a tessellation (without intersections) of \mathbb{R}^n .

In this paper, the word *cube*, even if not dyadic, is always used in the sense of *cube with sides parallel to the axes*.

The simplest examples of proper NST domains are the domains Ω where $\partial\Omega$ has just a finite set of points. On the other hand, if $\partial\Omega$ is “too crowded” then surely Ω is not NST. However, NST domains do not have a clear relationship with the cardinality of their boundaries: for example, as will become clear along this paper, if $\partial\Omega$ is the Cantor set, then Ω is NST, though the Cantor set is uncountable; nevertheless, there are domains Ω not in NST for which $\partial\Omega$ is countable. We give an example in \mathbb{R} :

Let $\partial\Omega \equiv \{0\} \cup \{1/m : m \in \mathbb{N}\}$. Then $\Omega \notin \text{NST}$. In fact, assume that such an Ω were in NST. Consider the sequence of dyadic cubes $Q_{\nu 1}$, $\nu \in \mathbb{N}$, which clearly intersect $\partial\Omega$, and a corresponding sequence of dyadic cubes $P(\nu) \subset Q_{\nu 1}$ with $\ell(P(\nu)) = 2^{-\mu}\ell(Q_{\nu 1})$ and $\overset{\circ}{P}(\nu) \cap \partial\Omega = \emptyset$. It is clear from the structure of $\partial\Omega$ that $\ell(P(\nu))$ must not exceed $1/2^{\nu-1} - 1/(2^{\nu-1} + 1)$, that is, $2^{-\nu+1}/(2^{\nu-1} + 1)$, while $\ell(Q_{\nu 1}) = 2^{-\nu}$. Therefore $0 < 2^{-\mu} = \ell(P(\nu))/\ell(Q_{\nu 1}) \leq 2/(2^{\nu-1} + 1) \rightarrow 0$ as ν goes to infinity, which is absurd!

3. NST and the ball condition

DEFINITION 3.1. A non-empty subset Γ of \mathbb{R}^n such that $\overset{\circ}{\Gamma} = \emptyset$ is said to satisfy the *ball condition* if

$$\exists \eta \in]0, 1[: \forall x \in \Gamma, \forall r \in]0, 1[, \exists y \in \mathbb{R}^n : \\ B(y, \eta r) \subset B(x, r) \text{ and } B(y, \eta r) \cap \overline{\Gamma} = \emptyset,$$

where $B(z, s)$ stands for the closed (Euclidean) ball centred at z with radius s .

This is a slight modification of the definition presented in [9, p. 142], where Γ was assumed to be Borel with zero Lebesgue measure. We decided to strip the definition off of any allusion to measure theory (and, since $|\Gamma| = 0 \Rightarrow \overset{\circ}{\Gamma} = \emptyset$, we are retaining all the sets considered to satisfy the ball condition according the definition given in [9]). Observe that the assumption $\overset{\circ}{\Gamma} = \emptyset$ is quite natural, because if $\overset{\circ}{\Gamma} \neq \emptyset$ then the statement with quantifiers in the definition cannot be fulfilled. Actually, it can even be proved that this statement also implies that $|\overline{\Gamma}| = 0$. This is an immediate consequence of the fact that such a $\overline{\Gamma}$ has Hausdorff dimension strictly less than n , which in turn follows from the fact that $\overline{\Gamma}$ has strong porosity bounded away from 0 at all of its points, in the sense explained in [6, p. 156].

We also mention that a set satisfying the ball condition is a particular kind of strongly porous set (cf. [6, p. 156]) and that, according to [6, p. 158], the notion of porosity goes back, at least in its weakest forms, to a 1920 work of Denjoy.

The following result can be proved by straightforward arguments:

PROPOSITION 3.2. *Let Γ be a non-empty closed subset of \mathbb{R}^n with $\overset{\circ}{\Gamma} = \emptyset$. Then Γ satisfies the ball condition if, and only if, $\Gamma^c \in \text{NST}$.*

REMARK 3.3. Actually, the above statement is valid even without the assumption that Γ is closed, as long as we replace “ $\Gamma^c \in \text{NST}$ ” by

“there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and $\overline{Q} \cap \overline{\Gamma} \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \overline{\Gamma} = \emptyset$ ”,

where we can, alternatively, write $\partial(\Gamma^c)$ or $\partial\Gamma$ for $\overline{\Gamma}$.

The similarity between the quotation written above (with the possibility of replacing $\overline{\Gamma}$ by $\partial\Gamma$) and the definition of a NST domain prompts the following extension of Definition 2.1:

DEFINITION 3.4. Let Γ be a non-empty subset of \mathbb{R}^n such that either $\overset{\circ}{\Gamma} = \Gamma$ or $\overset{\circ}{\Gamma} = \emptyset$. The set Γ is said to be *NST* (not so terrible), or to be in NST, if there exists $\mu \in \mathbb{N}$ with the property that for any dyadic cube Q with side-length $\ell(Q) \leq 1$ and $\overline{Q} \cap \partial\Gamma \neq \emptyset$, there exists a dyadic cube $P \subset Q$ with $\ell(P) = 2^{-\mu}\ell(Q)$ such that $\overset{\circ}{P} \cap \partial\Gamma = \emptyset$.

REMARK 3.5. (i) Of course, in the case of $\overset{\circ}{\Gamma} = \emptyset$ we already know that $\partial\Gamma$ in the above definition can be replaced by $\partial(\Gamma^c)$ or $\overline{\Gamma}$. Also in that case, the remark above shows that $\Gamma \in \text{NST}$ if, and only if, Γ satisfies the ball condition.

(ii) With the above definition it is easy to see that a proper domain belongs to NST if, and only if, its boundary does; or, equivalently, if, and only if, its boundary satisfies the ball condition.

We do not want to go into details, but we exhibit the relationship between the concept of NST set just introduced and the concept of open set condition as given in [10] (see there for the definition): it is not difficult to see that a non-empty subset Γ of \mathbb{R}^n with $\overset{\circ}{\Gamma} = \emptyset$ belongs to NST if, and only if, it satisfies that open set condition (except possibly the assumption that Γ is Borel, made in [10]). Be aware, however, that the notion of open set condition used in [10] is different from the notion with the same name which we are going to consider later in this paper.

4. NST and d -sets. We recall the notion of d -set [5, pp. 28–32]:

DEFINITION 4.1. Let Γ be a non-empty closed subset of \mathbb{R}^n and $d \in]0, n]$. Γ is said to be a d -set if

$$(1) \quad \exists c_1, c_2 > 0 : \forall \gamma \in \Gamma, \forall 0 < r \leq 1, \quad c_1 r^d \leq \mathcal{H}^d(B(\gamma, r) \cap \Gamma) \leq c_2 r^d,$$

where \mathcal{H}^d stands for the d -dimensional Hausdorff measure on \mathbb{R}^n .

REMARK 4.2. (i) A d -set has Hausdorff dimension equal to d .

(ii) It does not matter whether we use the Euclidean balls above or balls for the infinity norm (i.e., cubes), or substitute $0 < r \leq 1$ by $0 < r \leq r_0$, for some fixed $r_0 > 0$, in the sense that these modifications do not change the class of sets to which the definition applies.

This concept of d -set has been found very convenient when dealing with function spaces on closed sets (see, for example, [5] and [9]). Be aware that the expression “ d -set” has also been used with a different meaning in Fractal Geometry (cf., for example, [1]).

The following result can be derived from an assertion presented (without proof) in [4, Prop. 2]. We give a direct proof below.

PROPOSITION 4.3. *Let Γ be a d -set with $d < n$. Then $\Gamma \in \text{NST}$.*

Proof. From the properties of Hausdorff measure, it is clear that $\overset{\circ}{\Gamma} = \emptyset$.

Let Q_ν be a dyadic cube with side-length $2^{-\nu} \leq 1$ (that is, with $\nu \in \mathbb{N}_0$) and with $\overline{Q}_\nu \cap \Gamma \neq \emptyset$. For each $\varrho \in \mathbb{N}$ consider the tessellation of Q_ν by dyadic subcubes q with side-length $2^{-(\nu+\varrho)}$. Clearly, there are $2^{2\varrho}$ such subcubes.

Pick up only the subcubes q of that tessellation such that $\overline{q} \cap \Gamma \neq \emptyset$ and put them in some order: $q_1, \dots, q_{n_\varrho}$. In particular, n_ϱ stands for the number of such smaller cubes. Next form a subsequence $(q_{\sigma(r)})_r$ using the following criteria (where cq stands for the cube with the same centre of q but with c times its side-length):

- (i) q_1 is the first element of the subsequence;

(ii) if $q_{\sigma(r)}$ is an element of the subsequence and

$$J_r \equiv \{j \in \mathbb{N} : \forall s = 1, \dots, r, 3^{\dot{q}_{\sigma(s)}} \cap q_{\sigma(r)+j} = \emptyset\} \neq \emptyset,$$

then $q_{\sigma(r+1)} \equiv q_{\sigma(r)+\min J_r}$;

(iii) the subsequence terminates when $J_r = \emptyset$.

Denote by m_ϱ the number of cubes in this subsequence. It is clear that, for each cube q of the original sequence which remains in the subsequence, we may be throwing away as many cubes as the number of cubes of the tessellation which are neighbours of q (but no more than that). Since the number of such neighbours is at most $3^n - 1$, we can write

$$(2) \quad n_\varrho \leq m_\varrho + m_\varrho(3^n - 1) = 3^n m_\varrho.$$

Recall that each $\bar{q}_{\sigma(r)}$ contains an element of Γ . Fix one such element (call it x_r) in each $\bar{q}_{\sigma(r)}$ and denote by K_r the closed cube centred at x_r and with side-length $2^{-(\nu+\varrho+1)}$. Observe that $K_r \subset 2^{\dot{q}_{\sigma(r)}}$ and the m_ϱ cubes $2^{\dot{q}_{\sigma(r)}}$ are pairwise disjoint.

Fix also a point in $\bar{Q}_\nu \cap \Gamma$ and consider the closed cube K centred at that point and with side-length $3 \times 2^{-\nu}$, so that, in particular, $\bigcup_{r=1}^{m_\varrho} 2^{\dot{q}_{\sigma(r)}} \subset K$.

Now we use the hypothesis that Γ is a d -set in order to write

$$\begin{aligned} c_2(3 \times 2^{-\nu-1})^d &\geq \mathcal{H}^d(K \cap \Gamma) \\ &\geq \mathcal{H}^d\left(\bigcup_{r=1}^{m_\varrho} 2^{\dot{q}_{\sigma(r)}} \cap \Gamma\right) = \sum_{r=1}^{m_\varrho} \mathcal{H}^d(2^{\dot{q}_{\sigma(r)}} \cap \Gamma) \\ &\geq \sum_{r=1}^{m_\varrho} \mathcal{H}^d(K_r \cap \Gamma) \geq \sum_{r=1}^{m_\varrho} c_1 2^{-(\nu+\varrho+2)d} \\ &= m_\varrho c_1 2^{-\nu d} 2^{-(\varrho+2)d}, \end{aligned}$$

from which it follows, by (2), that

$$(3) \quad n_\varrho \leq c_3 2^{\varrho d},$$

where $c_3 > 0$ depends only on n, d and Γ .

Since there are exactly $2^{\varrho n}$ dyadic cubes of side-length $2^{-(\nu+\varrho)}$ in Q_ν , the number of such cubes q with $\bar{q} \cap \Gamma = \emptyset$ is $2^{\varrho n} - n_\varrho$. Using (3) we can then guarantee that the number of such cubes is bounded below by $2^{\varrho n} - c_3 2^{\varrho d}$. As this goes to infinity with ϱ (here we have used the hypothesis $d < n$), we can be sure that there is a value ϱ_0 of ϱ (depending only on n, d and Γ) such that there is at least one dyadic subcube q of side-length $2^{-\varrho_0} \ell(Q_\nu)$ in Q_ν such that $\dot{q} \cap \Gamma \neq \emptyset$.

This completes the proof that $\Gamma \in \text{NST}$ (with $\mu = \varrho_0$ in Definition 3.4). ■

REMARK 4.4. Even if one considers only not so terrible sets Γ which are closed, there are some which are not d -sets, for any d . Take, for example, $\Gamma = \{0\} \cup \{2^{-m} : m \in \mathbb{N}_0\}$ in \mathbb{R} . Of course, no open NST set can be a d -set with $d < n$ (even if we omit the assumption of being closed in the definition of d -set, as we will do whenever necessary). Observe also that there are open NST sets which are not open n -sets (consider, for example, a domain with an outward cusp, as in Fig. 5 of [11, p. 657]). On the positive side, it is not difficult to see that, given a NST domain Ω , the domain $(\partial\Omega)^c$ is an open n -set.

Clearly, an n -set might not be a NST set in the sense of Definition 3.4: a simple example is given by a closed cube in \mathbb{R}^3 , just because it is neither an open set nor has empty interior. However, the difference between the two concepts is deeper: if one tries to apply (1) to open instead of closed sets, one might find open sets which are n -sets in this modified sense but which do not belong to NST: a simple example is given by the open Γ in \mathbb{R} such that $\Gamma^c = \partial\Gamma = \{0\} \cup \{1/m : m \in \mathbb{N}\}$ (recall that we have already proved that such a Γ is not NST—see Section 2). We give a more complicated example in two dimensions later on. As for n -sets (closed or not) Γ such that $\overset{\circ}{\Gamma} = \emptyset$, they cannot be in NST: on the one hand, being n -sets, they must satisfy $|\Gamma| > 0$; on the other hand, if this is true then Γ cannot satisfy the ball condition (see comments after Definition 3.1); since $\overset{\circ}{\Gamma} = \emptyset$, it follows then by Remark 3.5(i) that $\Gamma \notin \text{NST}$.

We also mention that a domain Ω in \mathbb{R}^n is an open n -set if, and only if, it is an interior regular domain in the sense of [11], except that we do not assume boundedness of Ω and the equality $\Omega = \overline{\overset{\circ}{\Omega}}$, required there. More explicitly, if, and only if, there exists a positive c with the property that for any $x \in \partial\Omega$ and any cube Q centred at x with side-length at most 1, $|\Omega \cap Q| \geq c|Q|$.

5. NST and self-affine fractals. Let $N \geq 2$ be a natural number and let A_1, \dots, A_N be affine contractions on \mathbb{R}^n . That is, for each $l = 1, \dots, N$, A_l is an affine map, i.e.,

$$(4) \quad A_l x = T_l x + b_l, \quad \forall x \in \mathbb{R}^n,$$

where $b_l \in \mathbb{R}^n$ and T_l is a linear transformation on \mathbb{R}^n , and A_l is a contraction, i.e.,

$$\exists 0 < c_l < 1 : \quad |A_l x - A_l y| \leq c_l |x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Of course, in view of (4), the last property is equivalent to the existence of $0 < c_l < 1$ such that $|T_l x| \leq c_l |x|$ for all $x \in \mathbb{R}^n$, or just to the statement that $\|T_l\| < 1$.

DEFINITION 5.1. Given N affine contractions A_1, \dots, A_N , the unique non-empty compact subset Γ of \mathbb{R}^n satisfying the equality

$$(5) \quad \Gamma = \bigcup_{l=1}^N A_l \Gamma$$

is called the *self-affine fractal* generated by those N maps.

REMARK 5.2. (i) The word “fractal” is being used loosely here—we could as well have written “set” instead.

(ii) This definition includes a proposition, namely that there is one and only one non-empty compact set Γ satisfying (5). This is indeed the case: see [2, p. 114], where it is also shown that Γ can be obtained as

$$(6) \quad \Gamma = \bigcap_{k=1}^{\infty} A^k E,$$

for any non-empty compact subset E of \mathbb{R}^n such that $A_l E \subset E$ for $l = 1, \dots, N$. In (6), $A^k E$ is defined in the following way:

$$(7) \quad A^0 E = E; \quad A E = \bigcup_{l=1}^N A_l E; \quad A^k E = A(A^{k-1} E), \quad \forall k \in \mathbb{N}.$$

(iii) For examples and pictures, the interested reader is referred to [2].

The case when the affine contractions A_1, \dots, A_N are similarities, that is, satisfy

$$(8) \quad |A_l x - A_l y| = c_l |x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $0 < c_l < 1$, $l = 1, \dots, N$, is well studied, mainly if those affine contractions are also supposed to satisfy the *open set condition*, which means that there exists a non-empty open set \mathcal{U} in \mathbb{R}^n such that

$$(9) \quad A \mathcal{U} \subset \mathcal{U}, \quad A_l \mathcal{U} \cap A_h \mathcal{U} = \emptyset \quad \text{for } l \neq h$$

(with A as defined in (7)). Thus, the following is known (see [6, p. 67]):

PROPOSITION 5.3. *Let A_1, \dots, A_N be similarities satisfying the open set condition. Then the self-affine fractal generated by these contractions is a d -set, where d is the unique positive number satisfying*

$$\sum_{l=1}^N c_l^d = 1$$

(c_l as in (8)).

In the case when $d < n$ we can immediately conclude, on the basis of Proposition 4.3, that a self-affine fractal as in the preceding proposition is NST.

Our next aim is to consider a class of self-affine fractals of sufficient interest (and where the contractions are not necessarily similarities) and establish easily verifiable conditions which guarantee that a particular element of that class is NST. Since we shall also show that not all elements of such a class are NST, what follows can also be considered as a contribution to understand how far the class of self-affine fractals is from the class NST.

Let Q be the closed unit cube in \mathbb{R}^n :

$$Q \equiv \{x \equiv (x_j)_{j=1}^n : 0 \leq x_j \leq 1, \forall j = 1, \dots, n\}.$$

Our class of self-affine fractals should contain only those which are generated by surjective (\Leftrightarrow injective in this context) affine contractions A_l , $l = 1, \dots, N$, such that

$$(10) \quad A_l Q \subset Q, \quad l = 1, \dots, N,$$

$$(11) \quad A_l \overset{\circ}{Q} \cap A_m \overset{\circ}{Q} = \emptyset \quad \text{if } l \neq m; l, m = 1, \dots, N,$$

and

$$(12) \quad \sum_{l=1}^N |A_l Q| < 1,$$

where here $|\cdot|$ stands for Lebesgue measure in \mathbb{R}^n .

According to our Remark 5.2(ii) and (10), the self-affine fractal generated by a family of A_l as above is Γ given by

$$\Gamma = \bigcap_{k=1}^{\infty} A^k Q,$$

where A^k has the same meaning as in (7). In particular,

$$A^k Q = \bigcup_{1 \leq l_1, \dots, l_k \leq N} A_{l_1} \dots A_{l_k} Q, \quad k \in \mathbb{N}.$$

Note that the hypotheses on A_l also imply that Γ satisfies the open set condition (9) with $\mathcal{U} = \overset{\circ}{Q}$.

LEMMA 5.4. *Each self-affine fractal of the class introduced above has zero Lebesgue measure.*

Proof. We have

$$\begin{aligned} 0 \leq |\Gamma| &\leq |A^k Q| \leq \sum_{1 \leq l_1, \dots, l_k \leq N} |A_{l_1} \dots A_{l_k} Q| \\ &= \sum_{1 \leq l_1, \dots, l_k \leq N} |\det T_{l_1}| \dots |\det T_{l_k}| \\ &= \left(\sum_{1 \leq l_1 \leq N} |\det T_{l_1}| \right) \dots \left(\sum_{1 \leq l_k \leq N} |\det T_{l_k}| \right) = r^k \end{aligned}$$

where $r \equiv \sum_{l=1}^N |A_l Q| < 1$ by (12). The conclusion follows immediately by letting k go to ∞ . ■

REMARK 5.5. As a consequence, such sets Γ have empty interior. Therefore, a set Γ of that class is NST if, and only if, it satisfies the ball condition (cf. Remark 3.5(i)).

The above class is still too large for what we intend to do. We restrict further attention to the so-called anisotropic fractals, in the sense of [9, p. 14] (though here n can be any natural number). For these, the affine contractions A_l have the form

$$(13) \quad A_l : (x_j)_{j=1}^n \mapsto (\eta_j^{(l)} r_j^{(l)} x_j)_{j=1}^n + (b_j^{(l)})_{j=1}^n,$$

where $\eta_j^{(l)} \in \{-1, 1\}$, $r_j^{(l)} \in]0, 1[$ and $b_j^{(l)}$, $j = 1, \dots, n$, are given constants for each $l = 1, \dots, N$.

THEOREM 5.6. *Let Γ be an anisotropic self-affine fractal. Let A_l , $l = 1, \dots, N$, denote the affine contractions (13) which generate Γ . Assume that for every $i \in \{1, \dots, n\}$, every $l \in \{1, \dots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_l Q$, there exist $y \equiv (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap \bigcup_{\ell=1}^N A_\ell Q = \emptyset.$$

Then Γ is NST.

Proof. Step 1. Observe that there is a $\mu > 0$ such that the side-length of any q as above can be taken to be $\geq \mu$. Also, given that q is open, $q \subset Q$ is equivalent to $q \subset \overset{\circ}{Q}$.

According to Remark 5.5, we must prove that Γ satisfies the ball condition.

Consider any

$$x \equiv (x_j)_{j=1}^n \in \Gamma = \bigcap_{k=1}^{\infty} A^k Q = \bigcap_{k=1}^{\infty} \bigcup_{1 \leq l_1, \dots, l_k \leq N} A_{l_1} \dots A_{l_k} Q$$

and any closed ball $B(x, r)$ with $r \in]0, 1[$. Note that $x \in A^k Q$ for all $k \in \mathbb{N}$, and for each k considered, x belongs to some $A_{l_1} \dots A_{l_k} Q$, where $1 \leq l_1, \dots, l_k \leq N$.

Step 2. Let

$$\begin{aligned} \alpha &\equiv \min\{r_j^{(l)} : j \in \{1, \dots, n\}, l \in \{1, \dots, N\}\}, \\ \beta &\equiv \max\{r_j^{(l)} : j \in \{1, \dots, n\}, l \in \{1, \dots, N\}\}, \\ \theta &\equiv (\alpha\beta)/(\sqrt{n}\alpha + \beta). \end{aligned}$$

Note that $\theta < \alpha$, whence $r_j^{(l)} > \theta r$ for all $j \in \{1, \dots, n\}$ and $l \in \{1, \dots, N\}$.

On the other hand, the sequence $(r_j^{(l_1)} \dots r_j^{(l_m)})_{m \in \mathbb{N}}$, viewed as a sequence of functions of $(l_s)_{s \in \mathbb{N}}$ and j , tends to zero uniformly on the set $\{(l_s)_{s \in \mathbb{N}}, j) : (l_s)_{s \in \mathbb{N}} \subset \{1, \dots, N\}, j \in \{1, \dots, n\}\}$, which follows from the inequalities $0 < r_j^{(l_1)} \dots r_j^{(l_m)} \leq \beta^m$ and $\beta < 1$. Therefore it is possible to find $k \in \mathbb{N}$ such that $r_j^{(l_1)} \dots r_j^{(l_k)} \leq \theta r$ for any such j and $(l_s)_{s \in \mathbb{N}}$.

Fix now l_1, \dots, l_k in such a way that for the given x we have

$$(14) \quad x \in A_{l_1} \dots A_{l_k} Q.$$

Since, for each j , $r_j^{(l_1)} > \theta r$ and $r_j^{(l_1)} \dots r_j^{(l_k)} \leq \theta r$, there is a first natural $k_j \in [2, k]$ such that the product $r_j^{(l_1)} \dots r_j^{(l_{k_j})}$ becomes $\leq \theta r$. Let $k_0 \equiv \min_{1 \leq j \leq n} k_j$, so that $r_j^{(l_1)} \dots r_j^{(l_{k_0-1})} > \theta r$ for all $j \in \{1, \dots, n\}$. Among all j 's such that $r_j^{(l_1)} \dots r_j^{(l_{k_0})} \leq \theta r$ choose one—call it i —such that the product is minimum. For convenience, we shall denote k_0 again by k . So, in what follows the $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$ are chosen such that

$$(15) \quad r_i^{(l_1)} \dots r_i^{(l_k)} \leq \theta r$$

and

$$(16) \quad \forall j \in \{1, \dots, n\}, \quad r_j^{(l_1)} \dots r_j^{(l_{k-1})} > \theta r, \quad r_j^{(l_1)} \dots r_j^{(l_k)} \geq r_i^{(l_1)} \dots r_i^{(l_k)}.$$

Step 3. Note that (14) still holds for this new k (this is an easy consequence of (10)), so that there exists $x' \equiv (x'_j)_{j=1}^n \in A_{l_k} Q$ such that $x = A_{l_1} \dots A_{l_{k-1}} x'$. From the hypothesis, there exists $y' = (x'_1, \dots, x'_{i-1}, y'_i, x'_{i+1}, \dots, x'_n)$ and an open cube $q' \subset \overset{\circ}{Q}$ such that $y' \in \overline{q'}$ and

$$(17) \quad q' \cap \bigcup_{l=1}^N A_l Q = \emptyset,$$

from which it follows that

$$(18) \quad A_{l_1} \dots A_{l_{k-1}} q' \subset A_{l_1} \dots A_{l_{k-1}} \overset{\circ}{Q}$$

and

$$(19) \quad (A_{l_1} \dots A_{l_{k-1}} q') \cap \bigcup_{l=1}^N A_{l_1} \dots A_{l_{k-1}} A_l Q = \emptyset.$$

As a consequence,

$$(20) \quad (A_{l_1} \dots A_{l_{k-1}} q') \cap \Gamma = \emptyset.$$

In fact, if $z \in A_{l_1} \dots A_{l_{k-1}} q'$ then, by (18), $z \in A_{l_1} \dots A_{l_{k-1}} \overset{\circ}{Q}$, and therefore $z \notin A_{h_1} \dots A_{h_{k-1}} \overset{\circ}{Q}$ for any $(h_1, \dots, h_{k-1}) \neq (l_1, \dots, l_{k-1})$ —this is an easy consequence of (11).

Step 4. Consider $y \equiv A_{l_1} \dots A_{l_{k-1}} y'$. Note that $q' = y' + q''$, where $q'' \equiv -y' + q'$ is also an open cube. Define

$$q''' \equiv \{r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} z_j)_{j=1}^n : (z_j)_{j=1}^n \in q''\},$$

which is also an open cube, now with side-length $r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1}$ times that of q' . The open cube $q \equiv y + q'''$ has the same side-length as q''' . We claim that

$$(21) \quad y \in \bar{q}$$

and

$$(22) \quad q \subset A_{l_1} \dots A_{l_{k-1}} q'.$$

Assertion (21) is clear, as $y' \in \bar{q}' \Rightarrow 0 \in \overline{-y' + q'} \Rightarrow 0 \in \overline{q''} \Rightarrow y \in \overline{y + q''}$. For (22), observe that, given $z \in q$, $z = y + z'''$ for some $z''' \in q'''$, so $z = A_{l_1} \dots A_{l_{k-1}} y' + r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} z_j)_{j=1}^n$ for some $(z_j)_{j=1}^n \in q''$, that is,

$$\begin{aligned} z &= (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} y'_j)_{j=1}^n + c_{l_1, \dots, l_{k-1}} \\ &\quad + (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} \lambda_j z_j)_{j=1}^n, \end{aligned}$$

where $\lambda_j \equiv r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} / (r_j^{(l_1)} \dots r_j^{(l_{k-1})})$ and $c_{l_1, \dots, l_{k-1}}$ is the independent vector of the affine map $A_{l_1} \dots A_{l_{k-1}}$ (as the b_l in (4)). That is, any $z \in q$ can be written as

$$\begin{aligned} z &= (\eta_j^{(l_1)} \dots \eta_j^{(l_{k-1})} r_j^{(l_1)} \dots r_j^{(l_{k-1})} (y'_j + \lambda_j z_j))_{j=1}^n + c_{l_1, \dots, l_{k-1}} \\ &= A_{l_1} \dots A_{l_{k-1}} (y' + (\lambda_j z_j)_{j=1}^n) \end{aligned}$$

for some $z'' \equiv (z_j)_{j=1}^n \in q''$. Since $0 < \lambda_j \leq 1$, $j = 1, \dots, n$ (cf. (16) and the definitions of α and β , given at the beginning of Step 2), $z'' \in q''$ and $0 \in \bar{q}''$, it follows that $(\lambda_j z_j)_{j=1}^n$ is also in q'' (recall that we are only considering cubes with sides parallel to the axes) and (22) now follows easily.

As a consequence of (20) and (22), we have

$$(23) \quad q \cap \Gamma = \emptyset.$$

Step 5. Observe now that, given any $z \in \bar{q}$,

$$\begin{aligned} |z - x| &\leq |z - y| + |y - x| \\ &\leq \sqrt{n} r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} + r_i^{(l_1)} \dots r_i^{(l_{k-1})} |y'_i - x'_i| \\ &\leq r_i^{(l_1)} \dots r_i^{(l_{k-1})} (\sqrt{n} \alpha \beta^{-1} + 1) \\ &\leq r_i^{(l_1)} \dots r_i^{(l_{k-1})} r_i^{(l_k)} \alpha^{-1} (\sqrt{n} \alpha \beta^{-1} + 1) \\ &\leq \theta r \alpha^{-1} (\sqrt{n} \alpha \beta^{-1} + 1) = r \end{aligned}$$

(we have used (15) in the last inequality), so that

$$\bar{q} \subset B(x, r).$$

Finally, let c stand for the centre of q . Then the ball $B(c, \ell(q)/2)$ is in \bar{q} , therefore also in $B(x, r)$, and has radius $r_i^{(l_1)} \dots r_i^{(l_{k-1})} \alpha \beta^{-1} \ell(q)/2$, which, by (16) and the first phrase in Step 1, is strictly bounded below by $\theta r \alpha \beta^{-1} \mu/2$. Hence

$$B(c, \theta \alpha \beta^{-1} \mu r/2) \subset q \subset B(x, r)$$

and, due to (23),

$$B(c, \theta \alpha \beta^{-1} \mu r/2) \cap \Gamma = \emptyset.$$

This accomplishes the proof that Γ satisfies the ball condition, with $\eta = \theta \alpha \beta^{-1} \mu/2$ in Definition 3.1. ■

COROLLARY 5.7. *Let Γ be an anisotropic self-affine fractal. Let $A_l, l = 1, \dots, N$, denote the affine contractions (13) which generate Γ . Assume that there is $k_0 \in \mathbb{N}$ such that for every $i \in \{1, \dots, n\}$, every $l_1, \dots, l_{k_0} \in \{1, \dots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_{l_1} \dots A_{l_{k_0}} Q$, there exist $y \equiv (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap A^{k_0} Q = \emptyset.$$

Then Γ is NST.

Proof. Let Γ' be the self-affine fractal generated by the family $\{A_{l_1 \dots l_{k_0}} : l_1, \dots, l_{k_0} \in \{1, \dots, N\}\}$ of affine contractions $A_{l_1 \dots l_{k_0}} \equiv A_{l_1} \dots A_{l_{k_0}}$. It is not difficult to see that Γ' satisfies the conditions of Theorem 5.6, so it is NST. Now just note that $\Gamma = \Gamma'$. ■

In the following examples of anisotropic self-affine fractals in \mathbb{R}^2 , the affine contractions A_l are represented by means of the images $A_l Q$, which, in turn, are represented by the shaded rectangles in the corresponding figure.

EXAMPLE 1. The anisotropic fractal generated by the affine contractions represented in Figure 1 is NST.

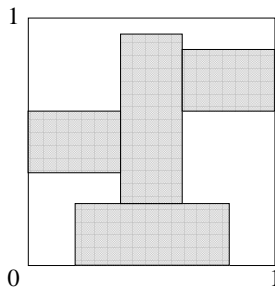


Fig. 1

EXAMPLE 2. An anisotropic fractal to which the corollary (with $k_0 = 3$)—but not the theorem—can be applied is generated by the affine contractions represented in Figure 2.

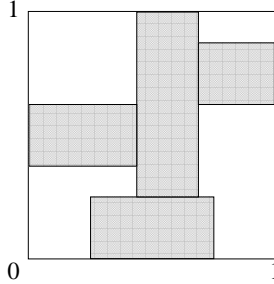


Fig. 2

We can weaken the hypotheses of Theorem 5.6 if we deal with a special subclass of anisotropic fractals. We start with some preparations:

With the same notations as in (13), given A_l , $l = 1, \dots, N$, if $i \in \{1, \dots, n\}$ is such that $r_i^{(l)} = \min_{1 \leq j \leq n} r_j^{(l)}$, we say that the i -axis is an *axis of strongest contraction* (for the affine contraction A_l).

DEFINITION 5.8. An anisotropic fractal is called a *CASC-fractal* if the affine contractions of a generating family $\{A_l : l \in \{1, \dots, N\}\}$ have a Common Axis of Strongest Contraction, that is, if there is $i \in \{1, \dots, n\}$ such that for all $l = 1, \dots, N$, $\min_{1 \leq j \leq n} r_j^{(l)} = r_i^{(l)}$.

THEOREM 5.9. Let Γ be a *CASC-fractal*. Let A_l , $l = 1, \dots, N$, denote the affine contractions (13) which generate Γ and let the i -axis be a common axis of strongest contraction. Assume that, for every $l \in \{1, \dots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_l Q$, there exist $y \equiv (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and

$$q \cap \bigcup_{\ell=1}^N A_\ell Q = \emptyset.$$

Then Γ is NST.

Proof. Since the proof is very similar to the proof of Theorem 5.6, we only point out the necessary modifications.

Step 1 is the same.

Step 2: To begin with, θ is now defined as $\alpha/(\sqrt{n} + 1)$. The parameter α can be defined more simply as $\min\{r_i^{(l)} : l \in \{1, \dots, N\}\}$, because of the role played by the number i , which is here given from the very beginning. As to β , it is enough now to define it by $\max\{r_i^{(l)} : l \in \{1, \dots, N\}\}$. Since i is given *a priori*, in this step we can fix $j = i$, so that there is no point in considering

k_0 —or, if one prefers, just take $k_0 = k_i$ —and one should suppress the part where the i is chosen. We again use the letter k to denote k_0 (that is, k_i), and the last part of Step 2 should now read: in what follows the $k \in \mathbb{N}$ is chosen such that

$$(24) \quad r_i^{(l_1)} \dots r_i^{(l_k)} \leq \theta r$$

and

$$(25) \quad r_i^{(l_1)} \dots r_i^{(l_{k-1})} > \theta r.$$

Step 3 is the same.

Step 4: Suppress the factor $\alpha\beta^{-1}$ in all considerations. In particular, λ_j now reads $r_i^{(l_1)} \dots r_i^{(l_{k-1})} / (r_j^{(l_1)} \dots r_j^{(l_{k-1})})$. The justification that $0 < \lambda_j \leq 1$ must be accordingly modified (but it is trivial).

Step 5: Suppress the factor $\alpha\beta^{-1}$ in all considerations. ■

COROLLARY 5.10. *Let Γ be a CASC-fractal. Let A_l , $l = 1, \dots, N$, denote the affine contractions (13) which generate Γ and let the i -axis be a common axis of strongest contraction. Assume that there is $k_0 \in \mathbb{N}$ such that for every $l_1, \dots, l_{k_0} \in \{1, \dots, N\}$ and every $x \equiv (x_j)_{j=1}^n \in A_{l_1} \dots A_{l_{k_0}} Q$, there exist $y \equiv (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ and an open cube $q \subset Q$ such that $y \in \bar{q}$ and*

$$q \cap A^{k_0} Q = \emptyset.$$

Then Γ is NST.

This follows just as Corollary 5.7, with Theorem 5.9 replacing Theorem 5.6.

EXAMPLE 3. A CASC-fractal to which Theorem 5.9—but not Theorem 5.6 nor Corollary 5.7—can be applied is generated by the affine contractions represented in Figure 3.

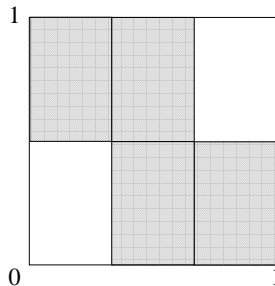


Fig. 3

EXAMPLE 4. A CASC-fractal to which Corollary 5.10—but not Theorems 5.6 and 5.9 nor Corollary 5.7—can be applied is generated by the affine contractions represented in Figure 4.

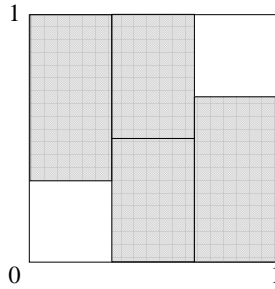


Fig. 4

Of course, Theorems 5.6, 5.9 and Corollaries 5.7, 5.10 only give sufficient conditions for an anisotropic fractal to be NST:

EXAMPLE 5. The 2-dimensional Cantor set generated by the similarities represented in Figure 5 is a CASC-fractal to which none of those results apply; nevertheless, it is NST (cf. Proposition 5.3 and comments afterwards).

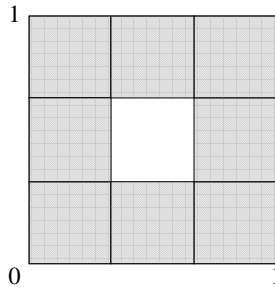


Fig. 5

Some conditions must, however, be imposed in order for an anisotropic (even if CASC-) fractal to be NST, as the following example shows.

EXAMPLE 6. Consider the anisotropic 2-dimensional Cantor-like set T generated by the affine contractions represented in Figure 6.

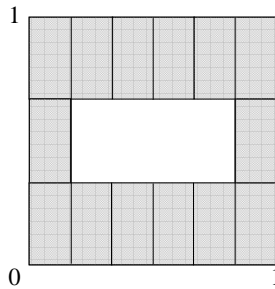


Fig. 6

Clearly, this is even a CASC-fractal. We shall see, however, that it is not NST:

First observe that, denoting by R the free space open rectangle in the middle of Q in Figure 6 (that is, $R = \overset{\circ}{Q} \setminus \bigcup_{l=1}^{14} A_l Q$), we have

$$\overset{\circ}{Q} \setminus \Gamma = R \cup \bigcup_{k=1}^{\infty} \bigcup_{1 \leq l_1, \dots, l_k \leq 14} A_{l_1} \dots A_{l_k} R.$$

Calling $A_{l_1} \dots A_{l_k} R$ *free space rectangles of the k th generation* and using the usual convention to denote coordinates in dimension 2, note that, below the line $y = 3^{-(k+1)}$ in Figure 6 there can be no free space rectangles of generation k popping in. On the other hand, the basis of each free space rectangle of generation k has length $(2/3)6^{-k}$, so that the higher the generation, the smaller the length of the bases of its free space rectangles.

Consider then the sequence $((1/3, 3^{-(k+1)}))_{k \in \mathbb{N}}$ of points of Γ and the sequence of closed balls B_k of radius $3^{-(k+1)}$ centred at those points. Observe that each such ball is contained in Q and stays below the line $y = 3^{-k}$. Therefore, the only points of that ball which are not in Γ must be in free space rectangles of k th or higher generation. Since these are pairwise disjoint, it is not possible to find a closed ball contained in $B_k \setminus \Gamma$ and with radius greater than $(1/3)6^{-k}$. And since the ratio $(1/3)6^{-k}/3^{-(k+1)}$, that is, 2^{-k} , tends to 0 as k goes to infinity, it is not possible to find $\eta \in]0, 1[$ such that, for any $k \in \mathbb{N}$ and for some $y_k \in \mathbb{R}^2$, $B(y_k, \eta 3^{-(k+1)}) \subset B_k$ and $B(y_k, \eta 3^{-(k+1)}) \cap \overline{\Gamma} = \emptyset$.

This shows that such a Γ does not satisfy the ball condition. That Γ is not NST follows now from Remark 5.5.

REMARK 5.11. (i) This also serves as an example of a fractal satisfying the open set condition (see (9)) but which is not in NST, so that in this way we complement what has been said after Remark 3.5.

(ii) The Γ of Example 6 is not a d -set. This is clear for $d < n$, due to Proposition 4.3; for $d = n$ it is also clear from the fact that $|\Gamma| = 0$ (Lemma 5.4).

(iii) Consider the open set Γ^c , with Γ the fractal in Example 6. From Proposition 3.2 it follows that $\Gamma^c \notin \text{NST}$. Since $|\Gamma| = 0$, Γ^c is an open n -set and therefore we have found another example (less trivial than the one exhibited after Remark 4.4) of an open n -set which is not NST.

EXAMPLE 7. From Theorem 5.9 it is clear that the modification of Figure 6 given in Figure 7 yields, in the usual way, a representation of a CASC-fractal which is NST.

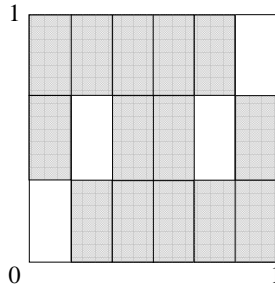


Fig. 7

REMARK 5.12. This together with Example 6 shows that, for anisotropic fractals, the property of being NST or not has nothing to do with the so-called affine dimension of fractals, as defined in [9]. The fractals in Examples 6 and 7 have the same affine dimension (just because the number of shaded rectangles of the same size is the same, irrespective of their positions)—namely $(2+2\log_2 7)/(1+2\log_2 3)$ —while one is NST and the other is not (the results proved above show that the relative position of the shaded rectangles can make a difference).

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