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An invariant of tangle cobordisms via subquotients of arc rings

by

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Abstract. We construct an explicit categorification of the action of tangles on tensor powers of the fundamental representation of quantum sl(2).

1. Introduction. The extension of the Jones polynomial of links [6] to tangles is governed, from the algebraic viewpoint, by the quantum group $U_q(\mathrm{sl}(2))$ and its representation theory. In one possible extension, to n points on the plane there is assigned $V^{\otimes n}$, the nth tensor power of the fundamental two-dimensional representation of $U_q(\mathrm{sl}(2))$, and to an (m,n)-tangle T a homomorphism $f(T)$ of representations $V^{\otimes n} \to V^{\otimes m}$. An (m,n)-tangle is a tangle in $\mathbb{R}^2 \times [0,1]$ with m top and n bottom endpoints. Alternatively, it is possible to restrict to tangles with even number of top and bottom endpoints, assign to $2n$ points on the plane the space $\mathrm{Inv}(V^{\otimes 2n})$ of $U_q(\mathrm{sl}(2))$-invariants in $V^{\otimes 2n}$, and to a $(2m,2n)$-tangle T the map

$$f_{\mathrm{inv}} : \mathrm{Inv}(V^{\otimes 2n}) \to \mathrm{Inv}(V^{\otimes 2m}),$$

the restriction of f to the subspace of invariants. When the tangle is a link, $f(T) = f_{\mathrm{inv}}(T)$ is the endomorphism of a one-dimensional vector space, given by multiplication by the Jones polynomial of T.

A categorification of the Jones polynomial [7] was extended to tangles and tangle cobordisms in [8], [9]. The space $\mathrm{Inv}(V^{\otimes 2n})$, interpreted as a free $\mathbb{Z}[q,q^{-1}]$-module of rank equal to the nth Catalan number, becomes the Grothendieck group of the triangulated category $\mathcal{K}_n$, which is the category of bounded complexes of finitely generated graded modules over a certain graded ring H^n, up to chain homotopies of complexes. The invariant $\mathcal{F}(T)$ of a tangle T is an appropriate exact functor $\mathcal{K}_n \to \mathcal{K}_m$, which on

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the Grothendieck group gives the map $f_{\mathrm{inv}}(T)$. The invariant of a tangle cobordism is a natural transformation of functors.

Bar-Natan [1] suggested a more general and geometric categorification of the Jones polynomial and its extension to tangles and their cobordisms (see also [12]). From the algebraic viewpoint, he considers the universal deformation of the original construction, with homology defined over a bigger ground ring, and tangle invariants taking values in a category similar but richer than $\mathcal{K}_n$. The Grothendieck group of his category is, again, naturally isomorphic to a $\mathbb{Z}[q,q^{-1}]$-lattice in $\mathrm{Inv}(V^{\otimes n})$.

In each of these two examples, generalization of the link homology to tangles utilizes a categorification of the space of invariants $\mathrm{Inv}(V^{\otimes 2n})$. The categorification [10] of the quantum $\mathrm{sl}(m)$ invariant of links and tangles, when specialized to $m = 2$, uses categories of complexes of matrix factorizations with potentials $\sum \pm x_i^3$, which contain proper subcategories equivalent to $\mathcal{K}_n$ (over $\mathbb{Q}$). The Grothendieck groups of these categories of matrix factorizations have not been computed.

A categorification of the entire tensor product $V^{\otimes n}$ was investigated in [2]. One first forms a suitable direct sum

$$\mathcal{O}^n = \bigoplus_{k=0}^{n} \mathcal{O}^{n-k,k}$$

of parabolic blocks of the highest weight category for $\mathrm{sl}(n)$. The category $\mathcal{O}^{n-k,k}$ is a full subcategory of a regular block of $\mathcal{O}$ for $\mathrm{sl}(n)$ consisting of modules which are locally finite as $\mathrm{sl}(n-k)\times\mathrm{sl}(k)$-modules. The Grothendieck group of $\mathcal{O}^{k,n-k}$ is free abelian of rank $\binom{n}{k}$ and, after tensoring with $\mathbb{C}$ over $\mathbb{Z}$, can be naturally identified with the weight $n-2k$ subspace in $V^{\otimes n}$, i.e.

$$K(\mathcal{O}^{n-k,k}) \otimes_{\mathbb{Z}} \mathbb{C} \cong V^{\otimes n}(n-2k).$$

The action of the Temperley–Lieb algebra on $V^{\otimes n}$ lifts to exact endofunctors (called translation across the wall) in $\mathcal{O}^n$.

The next and major development in this direction was due to Stroppel [13], [14], who considered a graded version of $\mathcal{O}^n$ and of the translation functors, and showed that they produce an invariant of tangles and tangle cobordisms. In this extension the invariant of an (m,n)-tangle T is a functor $D^b(\mathcal{O}^n) \to D^b(\mathcal{O}^m)$ between the derived categories.

The $U_q(\mathrm{sl}(2))$-invariants $\mathrm{Inv}(V^{\otimes 2n})$ form a subspace of the weight zero space $V^{\otimes 2n}(0)$ of $V^{\otimes 2n}$. One would expect that the inclusion

$$\mathrm{Inv}(V^{\otimes 2n}) \subset V^{\otimes 2n}(0)$$

can be lifted to the level of categories, to some relation between $\mathcal{K}_n$ and $D^b(\mathcal{O}^{n,n})$. That this is indeed the case was conjectured by Stroppel [14] and recently proved by her in [15]. Namely, the category $\mathcal{O}^{n-k,k}$ is equivalent to the category of finite-dimensional modules over a certain finite-dimensional basic $\mathbb{C}$-algebra $A_{n-k,k}$, described by Braden [3] via generators and relations. Stroppel showed that the ring H^n which controls the categorification $\mathcal{K}_n$ of the invariant space is isomorphic to a subring of $A_{n,n}$. More precisely,

$$H^n \otimes_{\mathbb{Z}} \mathbb{C} \cong eA_{n,n}e,$$

where e is an idempotent such that $A_{n,n}e$ is the largest direct summand of $A_{n,n}$ which is both a projective and an injective $A_{n,n}$-module; see [15].

Our paper was motivated by the problem of categorifying $V^{\otimes n}$ and the linear maps $f(T)$ directly, in a down-to-earth way, avoiding the highly sophisticated machinery of highest weight categories and their graded versions. We define a collection of graded rings $A^{n-k,k}$, consider the categories of finitely generated graded $A^{n-k,k}$-modules, and identify their Grothendieck groups with $\mathbb{Z}[q,q^{-1}]$-lattices in the weight spaces of $V^{\otimes n}$. Next, we form product rings

$$A^n := \prod_{k=0}^{n} A^{n-k,k},$$

to an (m,n)-tangle T assign a complex $\mathcal{F}(T)$ of graded (A^m, A^n)-bimodules, and to a tangle cobordism a homomorphism of complexes.

During our work on this project, Stroppel's paper [15] came out, where she defines a ring $\mathcal{K}^n$ isomorphic to $A^{n,n} \otimes \mathbb{C}$ and shows that $\mathcal{K}^n$ is isomorphic to the Braden algebra $A_{n,n}$ (see [3]). Furthermore, Stroppel announced the theorem that the inclusion of subrings $H^n \otimes_{\mathbb{Z}} \mathbb{C} \subset A_{n,n}$ extends to bimodules and bimodule homomorphisms in the two theories, allowing her to directly relate tangle and tangle cobordism invariants of [8], [9] to those of [13, 14].

Our constructions and results have a nonempty intersection with Stroppel [15], and, as we expect, will be easily surpassed by her announced work. We decided to publish this paper, nevertheless, since our work, which was done independently, is a basis for the paper [4].

2. Arc ring H^n. We first recall the definition of H^n from [8]. Let $\mathcal{A}$ be a free graded abelian group of rank 2 spanned by $\mathbf{1}$ in degree -1 and X in degree 1. Define the unit map $\iota : \mathbb{Z} \to \mathcal{A}$ and the trace map $\epsilon : \mathcal{A} \to \mathbb{Z}$ by

$$\iota(1) = \mathbf{1}, \quad \varepsilon(\mathbf{1}) = 0, \quad \varepsilon(X) = 1.$$

Define multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by

$$\mathbf{1}^2 = \mathbf{1}, \quad \mathbf{1}X = X\mathbf{1} = X, \quad X^2 = 0, \tag{1}$$

and comultiplication Δ by

$$\Delta : \mathcal{A} \to \mathcal{A}^{\otimes 2}, \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes X + X \otimes \mathbf{1}, \quad \Delta(X) = X \otimes X. \tag{2}$$

Assign to $\mathcal{A}$ a 2-dimensional TQFT $\mathcal{F}$ which associates $\mathcal{A}^{\otimes k}$ to a disjoint union of k circles. To the elementary cobordisms S_0^1, S_1^0, S_2^1 and S_1^2, depicted in Figure 1, $\mathcal{F}$ associates maps ι, ε, m and Δ respectively. The map $\mathcal{F}(S)$ between tensor powers of $\mathcal{A}$ induced by a cobordism S is homogeneous of degree minus the Euler characteristic of S,

$$\deg(\mathcal{F}(S)) = -\chi(S). \tag{3}$$

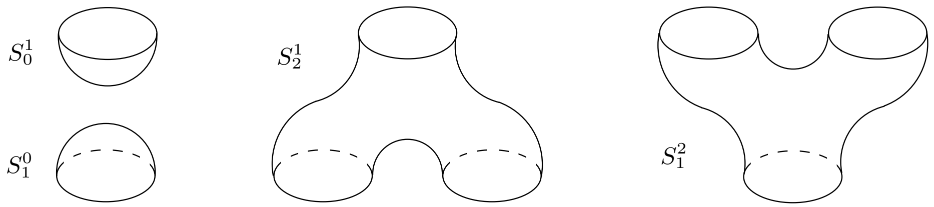

Fig. 1. Elementary cobordisms

Let B^n be the set of crossingless matchings of $2n$ points. Figure 2 shows the set B^3. For $a, b \in B^n$ denote by $W(b)$ the reflection of b about the horizontal axis, and by $W(b)a$ the closed 1-manifold obtained by closing $W(b)$ and a along their boundaries (see Figure 3).

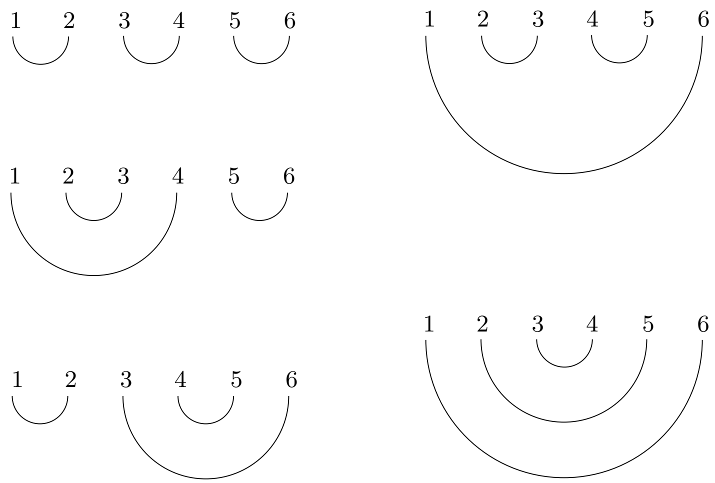

Fig. 2. Crossingless matchings of six points

The graded abelian group $\mathcal{F}(W(b)a)$ is isomorphic to $\mathcal{A}^{\otimes I}$, where I is the set of circles in $W(b)a$. The symbol $\{n\}$ denotes shifting the grading up by n. For $a, b \in B^n$ let

$$_b(H^n)_a := \mathcal{F}(W(b)a)\{n\},$$

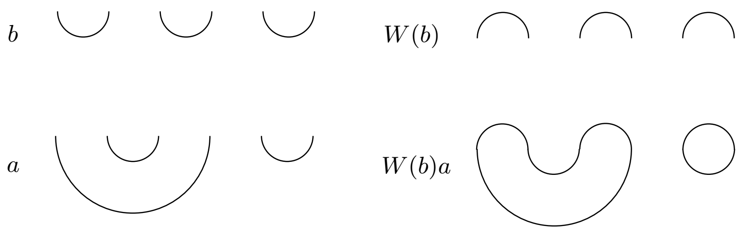

Fig. 3. Gluing in B^3

and define H^n as the direct sum

$$H^n := \bigoplus_{a,b\in B^n} {}_b(H^n)_a.$$

Multiplication maps in H^n are defined as follows. We set $xy = 0$ if $x \in {}_b(H^n)_a$, $y \in {}_c(H^n)_d$ and $c \neq a$. The multiplication maps

$${}_b(H^n)_a \otimes {}_a(H^n)_c \to {}_b(H^n)_c$$

are given by homomorphisms of abelian groups

$$\mathcal{F}(W(b)a) \otimes \mathcal{F}(W(a)c) \to \mathcal{F}(W(b)c)$$

which are induced by "minimal" cobordisms from $W(b)aW(a)c$ to $W(b)c$ (see Figure 4).

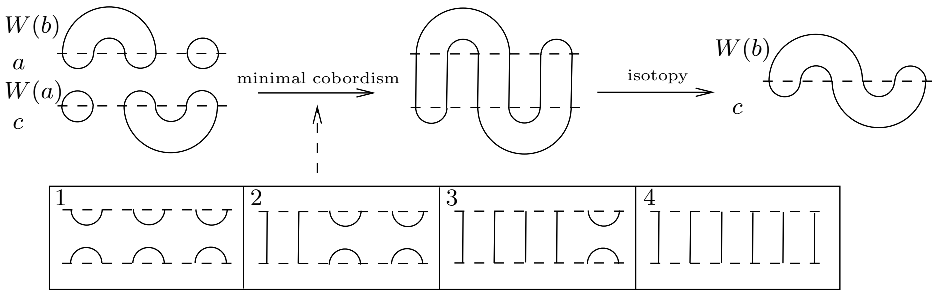

Fig. 4. Multiplication in H^n

The element $1_a := \mathbf{1}^{\otimes n} \in \mathcal{A}^{\otimes n} \cong {}_a(H^n)_a$ is an idempotent in H^n. The sum $\sum_a 1_a$ is the unit element of H^n. See [8] for details.

3. Subquotients of H^n. For each $n \geq 0$ and $0 \leq k \leq n$, define $B^{n-k,k}$ to be the subset of B^n consisting of diagrams with no matchings among the first $n-k$ points and among the last k points. Figure 5 shows $B^{1,2}$ (compare with Figure 2). We put two "platforms", one on the first $n-k$ points and

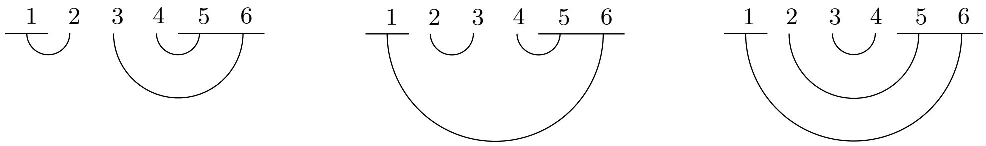

Fig. 5. The three elements in $B^{1,2}$

one on the last k points, to indicate that these endpoints are special. Define

$$\widetilde{A}^{n-k,k} := \bigoplus_{a,b\in B^{n-k,k}} \mathcal{F}(W(b)a)\{n\}. \tag{4}$$

Then $\widetilde{A}^{n-k,k}$ sits inside H^n as a graded subring which inherits its multiplication from H^n (the inclusion takes $1 \in \widetilde{A}^{n-k,k}$ to an idempotent of H^n).

For $a, b \in B^{n-k,k}$ the circles of $W(b)a$ fall into three different types (see Figure 6):

- Type I: Circles that are disjoint from platforms.
- Type II: Circles that intersect at least one platform and intersect each platform at most once.
- Type III: Circles that intersect one of the platforms at least twice.

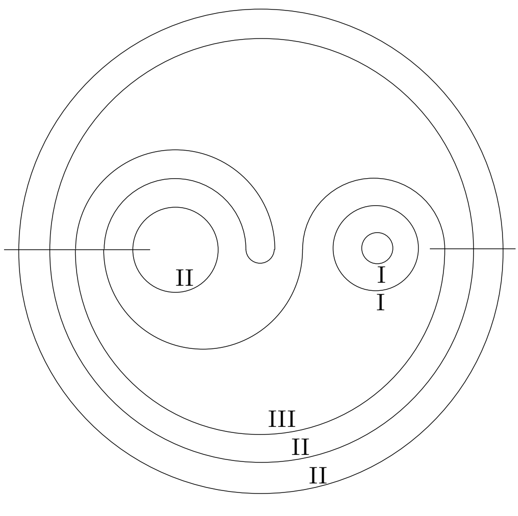

Fig. 6. Three types of circles in $\mathcal{F}(W(b)a)$

We call an intersection point between a circle and a platform a "mark". Next, we introduce an ideal $I^{n-k,k} \subset \widetilde{A}^{n-k,k}$. If $W(b)a$ contains at least one type III circle (see Figure 7), set

$${}_b(I^{n-k,k})_a = \mathcal{F}(W(b)a).$$

If $W(b)a$ contains only circles of types I and II, we write $\mathcal{F}(W(b)a) =$

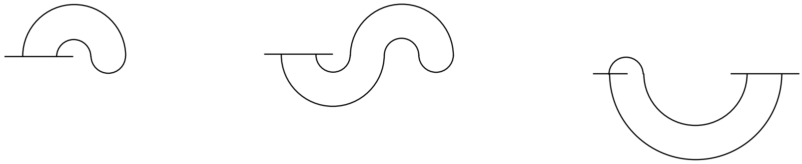

Fig. 7. Examples of portions of type III circles

$\mathcal{A}^{\otimes i} \otimes \mathcal{A}^{\otimes j}$, where type II circles correspond to the first i tensor factors, and define ${}_b(I^{n-k,k})_a$ as the span of

$$y_1 \otimes \cdots \otimes y_{t-1} \otimes X \otimes y_{t+1} \otimes \cdots \otimes y_{i+j} \in \mathcal{A}^{\otimes i} \otimes \mathcal{A}^{\otimes j} \cong \mathcal{F}(W(b)a),$$

where $1 \leq t \leq i$ and $y_s \in \{\mathbf{1}, X\}$. By taking the direct sum over all $a, b \in B^{n-k,k}$ we get a subgroup of $\widetilde{A}^{n-k,k}$,

$$I^{n-k,k} := \bigoplus_{a,b \in B^{n-k,k}} {}_b(I^{n-k,k})_a.$$

Lemma 1. *$I^{n-k,k}$ is a two-sided graded ideal of the ring $\widetilde{A}^{n-k,k}$.*

Proof. To prove it is a left ideal, it suffices to show that $uv \in {}_c(I^{n-k,k})_b$ whenever $u \in \mathcal{F}(W(c)a)$ and $v \in {}_a(I^{n-k,k})_b$. Without loss of generality, we can assume that both u and v are tensor products:

$$u = u_1 \otimes \cdots \otimes u_s \in \mathcal{A}^{\otimes s} \cong \mathcal{F}(W(c)a),$$
$$v = v_1 \otimes \cdots \otimes v_t \in \mathcal{A}^{\otimes t} \cong \mathcal{F}(W(a)b),$$

where $u_i,\ v_j \in \{\mathbf{1}, X\}$. We can visualize u and v as sets of circles with labels $\mathbf{1}$ or X, as in Figure 8.

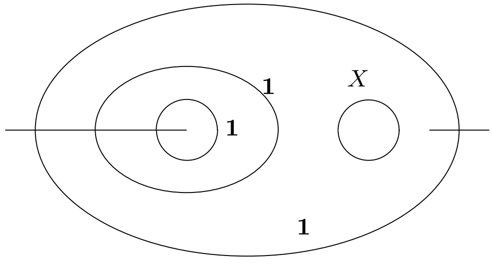

Fig. 8. Visualization of a tensor product element in $\mathcal{F}(W(a)b)$

Case 1: ${}_a(I^{n-k,k})_b \neq \mathcal{F}(W(a)b)$. In this case, v contains a marked circle C with label X. Pick a mark p on C. Denote by $M \in \{\mathbf{1}, X\}$ the label of the circle containing p in $W(c)b$. It follows from equations (1) and (2) that after multiplying v by u, either M will remain X or $uv = 0$. In either case uv will belong to ${}_c(I^{n-k,k})_b$. See Figure 17 for a similar example.

Case 2: ${}_a(I^{n-k,k})_b = \mathcal{F}(W(a)b)$. In this case v contains a circle connecting two points p_1 and p_2 on the same platform. If C is labelled by X, it follows from the previous case that either the labels of circles containing p_1 and p_2 will remain X or $uv = 0$. So uv will belong to ${}_c(I^{n-k,k})_b$. Now assume that the label on C is $\mathbf{1}$. If, during the process of multiplying u and v, a splitting of C takes place it follows from equation (2) that either the circle containing p_1 or the circle containing p_2 will have label X. Otherwise, a sequence of merging with C will keep p_1 and p_2 connected by a single arc. In this case ${}_c(I^{n-k,k})_b$ equals $\mathcal{F}(W(c)b)$, and therefore contains uv.

Similar arguments show that $I^{n-k,k}$ is a right ideal, and the lemma follows. ■

The ring $A^{n-k,k}$ is defined as the quotient of $\widetilde{A}^{n-k,k}$ by the ideal $I^{n-k,k}$:

$$A^{n-k,k} := \widetilde{A}^{n-k,k}/I^{n-k,k}. \tag{5}$$

It naturally decomposes into a direct sum of graded abelian groups:

$$A^{n-k,k} = \bigoplus_{a,b\in B^{n-k,k}} {}_a(A^{n-k,k})_b,$$

where ${}_a(A^{n-k,k})_b = \mathcal{F}(W(a)b)/{}_a(I^{n-k,k})_b\{n\}$. The abelian group ${}_a(A^{n-k,k})_b$ is 0 if and only if $W(b)a$ contains a type III circle. Otherwise, ${}_a(A^{n-k,k})_b$ is a free abelian group of rank 2^{c_1} where c_1 is the number of type I circles in $W(b)a$. Assuming that $\mathcal{F}(W(a)b) \cong \mathcal{A}^{\otimes m}$ in which type II circles correspond to the first i tensor factors, ${}_a(A^{n-k,k})_b$ has a basis of the form

$$\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes a_{i+1} \otimes \cdots \otimes a_m$$

where $a_s \in \{\mathbf{1}, X\}$ for all $i+1 \leq s \leq m$.

The element $1_a := \mathbf{1}^{\otimes n} \in {}_a(A^{n-k,k})_a$ is a minimal idempotent in $A^{n-k,k}$. The sum $1 := \sum_{a\in B^{n-k,k}} 1_a$ is the unit element of $A^{n-k,k}$.

The relations among the three rings H^n, $\widetilde{A}^{n-k,k}$, and $A^{n-k,k}$ are described in the following diagram:

$$H^n \xleftarrow{\text{inclusion of subrings}} \widetilde{A}^{n-k,k} \xrightarrow{\text{quotient by } I^{n-k,k}} A^{n-k,k}.$$

We now write down explicitly the rings $A^{n-k,k}$ in simplest cases.

• The ring $A^{0,n}$ is isomorphic to $\mathbb{Z}$, since $B^{0,n}$ contains only one diagram, and the functor $\mathcal{F}$ applied to its closure produces $\mathbb{Z}$ (see Figure 9).

• The vertical reflection induces an isomorphism $A^{n-k,k} \cong A^{k,n-k}$. Namely, reflecting a diagram in $B^{n-k,k}$ about a vertical axis produces a diagram in $B^{k,n-k}$, leading to an isomorphism of sets $B^{n-k,k} \cong B^{k,n-k}$. This isomorphism induces an isomorphism of rings $\widetilde{A}^{n-k,k} \cong \widetilde{A}^{k,n-k}$ and of

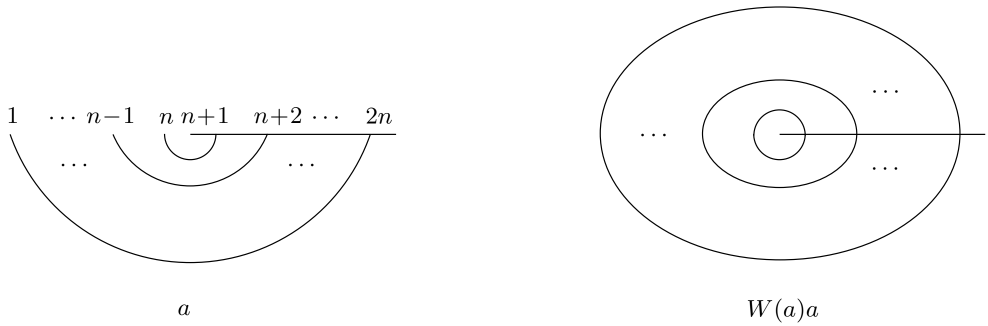

Fig. 9. The only element in $B^{0,n}$ and its closure

the quotient rings $A^{n-k,k} \cong A^{k,n-k}$. In particular,

$$A^{n,0} \cong A^{0,n} \cong \mathbb{Z}.$$

• The set $B^{1,1}$ contains two diagrams which we denote by a and b respectively (see Figure 10).

Fig. 10. The set $B^{1,1}$

From Figure 11 we can see that

$$\begin{aligned} {}_a(A^{1,1})_a &= \mathcal{A}\{1\}, \quad {}_b(A^{1,1})_a = \mathbb{Z}\{1\}, \\ {}_a(A^{1,1})_b &= \mathbb{Z}\{1\}, \quad {}_b(A^{1,1})_b = \mathbb{Z}, \end{aligned}$$

where $\{1\}$ denotes shifting the grading up by 1. The grading shifts in above formulas follow from the definition of $A^{n-k,k}$. For example, ${}_b(A^{1,1})_b = \mathcal{F}(W(b)b)/{}_b(I^{1,1})_b\{2\}$ generated by $\mathbf{1}\otimes\mathbf{1}\{2\}$ which sits in degree 0. The ring

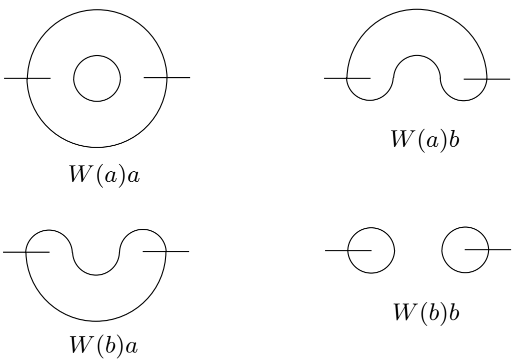

Fig. 11. The ring $A^{1,1}$

$A^{1,1}$ has a simple quiver description, as the path ring of the graph

$$\overset{a}{\circ} \ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \ \overset{b}{\circ}$$

with the defining relation $\alpha\beta = 0$. Paths α and β correspond to generators of ${}_b(A^{1,1})_a$ and ${}_a(A^{1,1})_b$ respectively.

• The ring $A^{1,n-1}$ has a simple quiver description as well. Denote by Q_n the quotient of the path ring

$$\overset{1}{\circ} \rightleftarrows \overset{2}{\circ} \rightleftarrows \overset{3}{\circ} \rightleftarrows \dots \rightleftarrows \overset{n-1}{\circ} \rightleftarrows \overset{n}{\circ}$$

by the defining relations $(1|2|1) = 0$, $(i|i+1|i) = (i|i-1|i)$ for $1 < i < n$, $(i|i+1|i+2) = 0$ for $i < n-1$, and $(i|i-1|i-2) = 0$ for $i \geq 2$, where $(i|j|k)$ denotes the path which starts at i, goes to j and then to k. See [11] for a more detailed discussion of this algebra.

The rings Q_n and $A^{1,n-1}$ are isomorphic. The isomorphism takes the idempotent 1_{a_i}, for a_i shown in Figure 12, to the minimal idempotent (i), which is the length zero path that starts and ends at i. The path $(j|i)$ with $j = i \pm 1$ corresponds to a generator of $\mathcal{F}(W(a_j)a_i) \cong \mathbb{Z}$.

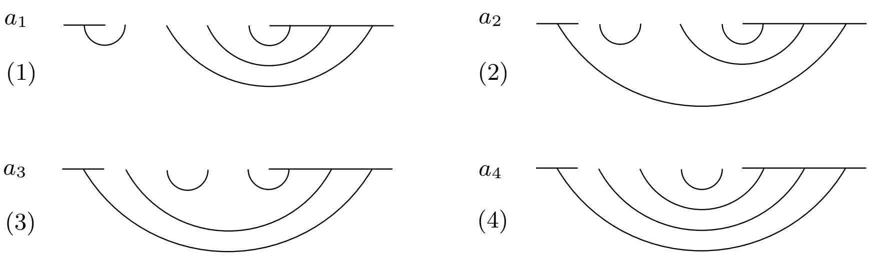

Fig. 12. Elements in $B^{1,3}$

Figure 12 depicts all a_i for $A^{1,3}$. The diagram $W(a_3)a_1$ has a type III circle so ${}_{a_3}(A^{1,3})_{a_1} = 0$. More generally, ${}_{a_i}(A^{1,3})_{a_j} = 0$ iff $|i-j| > 1$. See Figure 13 for an example of multiplication in $A^{1,3}$, and note that Q_4 is given by

$$\overset{1}{\circ} \rightleftarrows \overset{2}{\circ} \rightleftarrows \overset{3}{\circ} \rightleftarrows \overset{4}{\circ}$$

with the relations $(1|2|1) = 0$, $(2|1|2) = (2|3|2)$, $(3|2|3) = (3|4|3)$, and $(1|2|3) = (2|3|4) = (4|3|2) = (3|2|1) = 0$.

Now we list some basic properties of the rings $A^{n-k,k}$.

• The rings $A^{n-k,k}$ are indecomposable (0 and 1 are the only central idempotents in $A^{n-k,k}$).

• $B^{n-k,k}$ is the union of two disjoint subsets B_1 and B_2 as follows. Any two elements in $B^{k,n-k}$ can be connected by a sequence of elementary

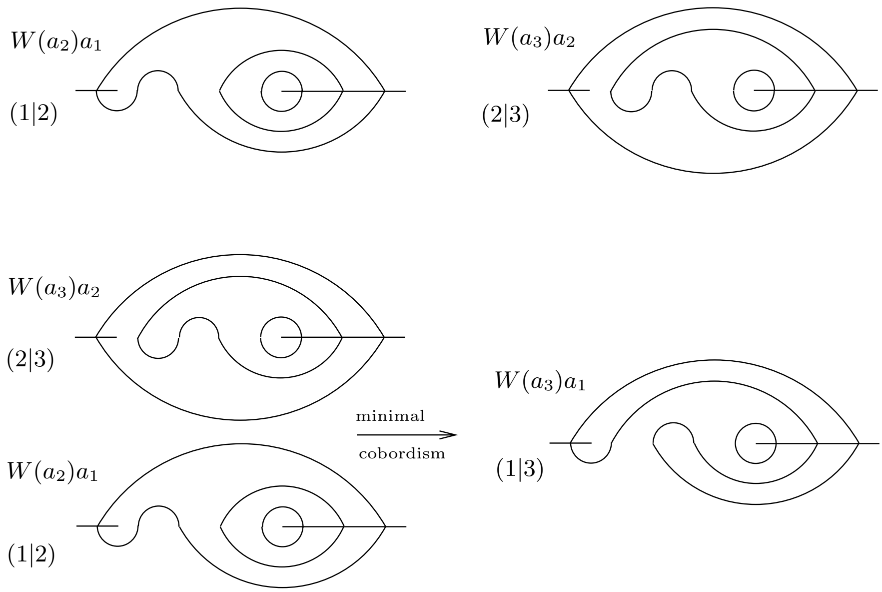

Fig. 13. Quiver description of $A^{1,3}$

changes as in Figure 14. Pick an element a in $B^{n-k,k}$ and put it in B_1. For any $b \in B^{n-k,k}$, if a and b can be connected in an even number of steps we put b in B_1, otherwise put it in B_2. By taking the sum of all minimal idempotents in each subset we get the idempotents

$$e_1 = \sum_{a \in B_1} 1_a, \qquad e_2 = 1 - e_1 = \sum_{a \in B_2} 1_a$$

such that all homogeneous elements in $e_i A^{n-k,k} e_j$ for i, j in the same subset have even degrees, and all homogeneous elements in $e_i A^{n-k,k} e_j$ for $i \neq j$ have odd degrees.

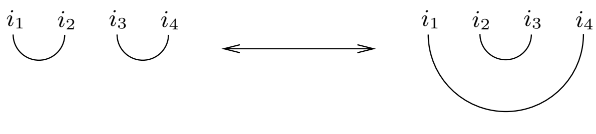

Fig. 14. The elementary change in $B^{n-k,k}$

• The degree 0 part of $A^{n-k,k}$ is the product of rings $\mathbb{Z}$, one for each element of $B^{n-k,k}$,

$$(A^{n-k,k})^0 \cong \prod_{a \in B^{n-k,k}} \mathbb{Z}1_a.$$

4. Flat tangles and bimodules. Denote by $\widehat{B}_n^m$ the space of flat tangles with m top endpoints and n bottom endpoints (note that in [8],

$\widehat{B}^m_n$ denotes the space of flat tangles with $2m$ top endpoints and $2n$ bottom endpoints). Recall that a *flat* (m,n)-*tangle* T is a proper, smooth embedding of $(n+m)/2$ arcs and a finite number of circles into $\mathbb{R}\times[0,1]$ such that:

- The boundary points of arcs map to
$$\{1,\dots,n\}\times\{0\},\{1,\dots,m\}\times\{1\}.$$
- Near the endpoints, the arcs are perpendicular to the boundary of $\mathbb{R}\times[0,1]$.

We impose these conditions to ensure that the concatenation of two such embeddings is still a smooth embedding. Flat tangles constitute a category with objects nonnegative integers, and morphisms from n to m being the isotopy classes of flat (m,n)-tangles.

To a tangle $T\in\widehat{B}^m_n$ we would like to assign an $(A^{m-k-l,k+l},A^{n-k,k})$-bimodule for all k in the range $\max(0,(n-m)/2)\le k\le\min(n,(m+n)/2)$ and $l=(m-n)/2$. First, define a graded $(\widetilde{A}^{m-k-l,k+l},\widetilde{A}^{n-k,k})$-bimodule $\widetilde{\mathcal{F}}(T)$ by
$$\widetilde{\mathcal{F}}(T)=\bigoplus_{b,c}{}_c\widetilde{\mathcal{F}}(T)_b,$$
where b ranges over elements of $B^{n-k,k}$, c over elements of $B^{m-k-l,k+l}$, and
$$(6)\qquad {}_c\widetilde{\mathcal{F}}(T)_b:=\mathcal{F}(W(c)Tb)\{n\}.$$
Note that $W(c)Tb$ is not a union of circles. We close it in the obvious way and still call it $W(c)Tb$ (see Figure 15), then apply the functor $\mathcal{F}$. See Figure 16 for an example where $T\in\widehat{B}^1_3$, $k=2$ and $l=-1$, $b_i\in B^{1,2}$ and $c\in B^{0,1}$. The left action $\widetilde{A}^{m-k-l,k+l}\times\widetilde{\mathcal{F}}(T)\to\widetilde{\mathcal{F}}(T)$ comes from the maps
$$\mathcal{F}(W(a)c)\otimes{}_c\widetilde{\mathcal{F}}(T)_b\to{}_a\widetilde{\mathcal{F}}(T)_b.$$

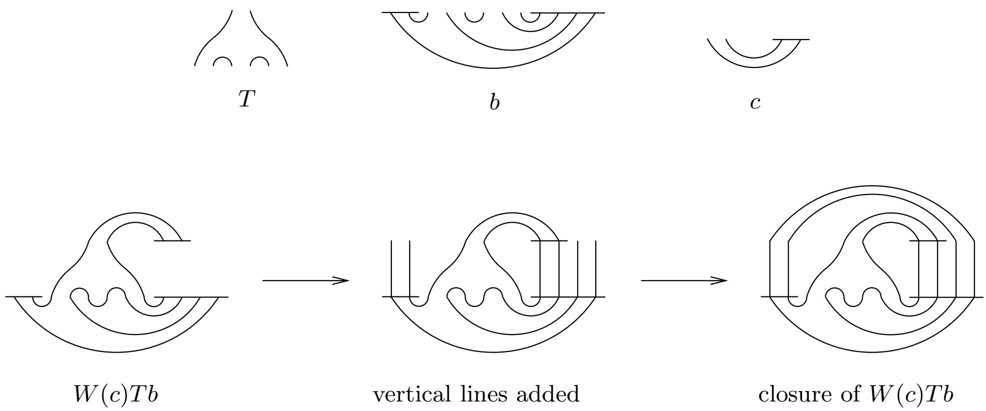

Fig. 15. Closing $W(c)Tb$

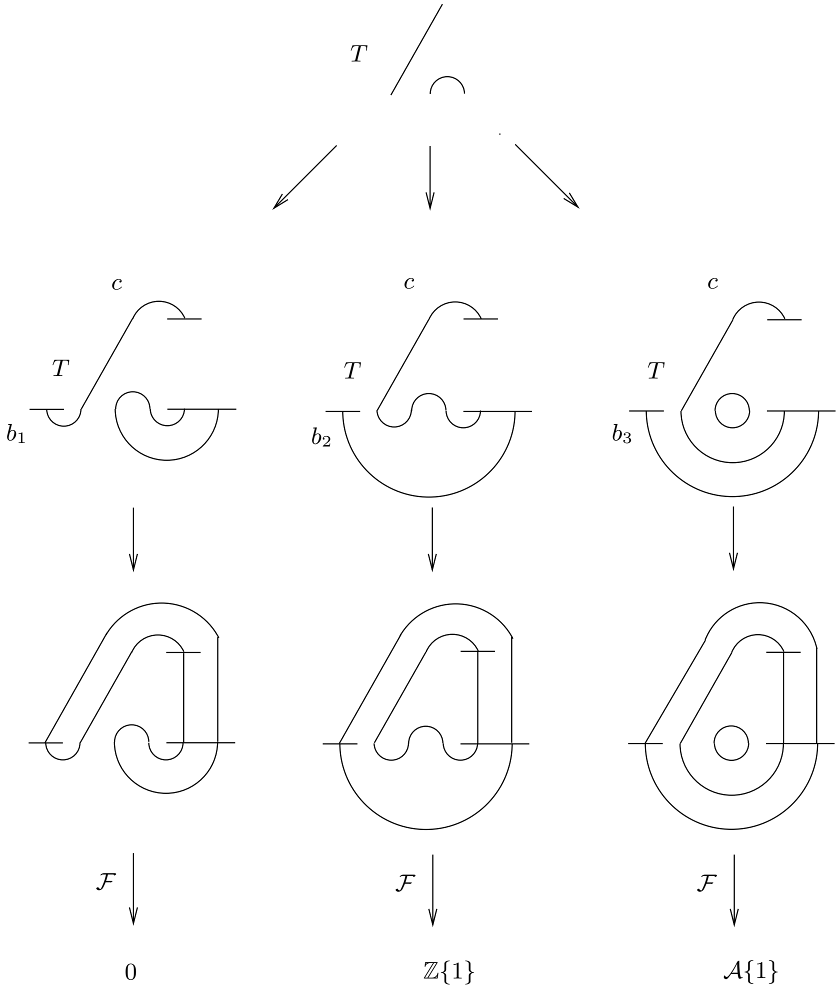

Fig. 16. Closures of a flat tangle T with fixed sizes of platforms

Likewise, the right action $\widetilde{\mathcal{F}}(T) \times \widetilde{A}^{n-k,k} \to \widetilde{\mathcal{F}}(T)$ comes from the maps

$$ {}_c\widetilde{\mathcal{F}}(T)_b \otimes \mathcal{F}(W(b)a) \to {}_c\widetilde{\mathcal{F}}(T)_a. $$

Both maps are induced by the obvious "minimal cobordism" (see Figure 4).

Similarly to the definition of ${}_b(I^{n-k,k})_a$, we define a subgroup ${}_bI(T)_a$ of ${}_b\widetilde{\mathcal{F}}(T)_a$ as follows: If $W(b)Ta$ contains a type III arc, set ${}_bI(T)_a = {}_b\widetilde{\mathcal{F}}(T)_a$. Otherwise, assuming that $\mathcal{F}(W(b)Ta) \cong \mathcal{A}^{\otimes r}$ in which type II circles correspond to the first i tensor factors, ${}_bI(T)_a$ is spanned by the elements

$$ u_1 \otimes \cdots \otimes a_{j-1} \otimes X \otimes u_{j+1} \otimes \cdots \otimes u_r \in \mathcal{F}(W(b)Ta) \cong \mathcal{A}^{\otimes r}, $$

where $1 \le j \le i$ and $u_s \in \{\mathbf{1}, X\}$ for each $1 \le s \le r$, $s \ne j$. By taking the direct sum we get the subgroup

$$I(T) := \bigoplus_{a \in B^{m-k-l,k+l},\, b \in B^{n-k,k}} {}_aI(T)_b.$$

LEMMA 2. *$I(T)$ is a subbimodule of $\widetilde{\mathcal{F}}(T)$.*

The proof is similar to that of Lemma 1 and we omit it. See Figure 17 for an example. The distinguished circle C is thickened.

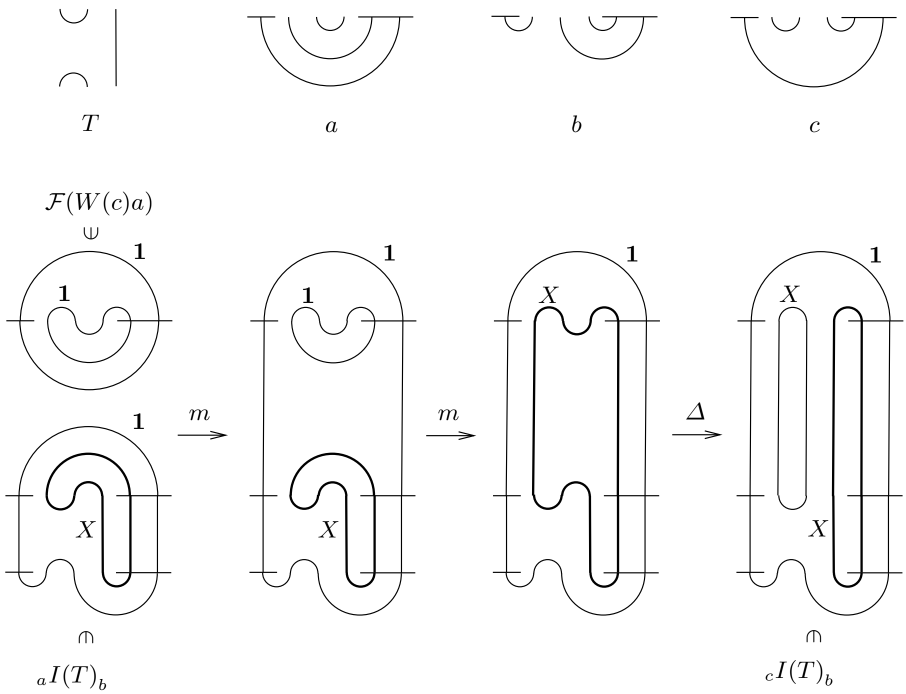

Fig. 17. Invariance of $I(T)$ under left action

Define $\mathcal{F}(T)$ to be the quotient bimodule of $\widetilde{\mathcal{F}}(T)$ over $I(T)$:

$$\mathcal{F}(T) := \widetilde{\mathcal{F}}(T)/I(T).$$

LEMMA 3. *The action of $I^{n-k,k}$ on $\mathcal{F}(T)$ is trivial.*

The proof is similar to that of Lemma 1.

It follows from the previous lemma that the $(\widetilde{A}^{m-k-l,k+l}, \widetilde{A}^{n-k,k})$-bimodule structure on $\mathcal{F}(T)$ descends to an $(A^{m-k-l,k+l}, A^{n-k,k})$-bimodule structure.

By taking the direct product over all $0 \le k \le n$, we collect the rings $A^{n-k,k}$ together into a graded ring A^n:

$$A^n := \prod_{0 \le k \le n} A^{n-k,k}.$$

As a graded abelian group, A^n is the direct sum of $A^{n-k,k}$ over $0 \le k \le n$. Similarly, for a flat tangle T, by taking the direct sum over all

$$\max(0,(n-m)/2) \le k \le \min(n,(n+m)/2)$$

we collect the $(A^{m-k-l,k+l}, A^{n-k,k})$-bimodules $\mathcal{F}(T)$ into an (A^m, A^n)-bimodule (which we still call $\mathcal{F}(T)$):

$$\mathcal{F}(T) := \bigoplus_{\max(0,(n-m)/2) \le k \le \min(n,(n+m)/2)} \mathcal{F}(T).$$

Note that we use the same notation $\mathcal{F}(T)$ for both the (A^m, A^n)-bimodule and individual $(A^{m-k-l,k+l}, A^{n-k,k})$-bimodules.

Proposition 1. *Let $T_1, T_2 \in \widehat{B}_n^m$ and S a cobordism between T_1 and T_2. Then S induces a degree $(n+m)/2 - \chi(S)$ homomorphism of (A^m, A^n)-bimodules*

$$\mathcal{F}(S) : \mathcal{F}(T_1) \to \mathcal{F}(T_2),$$

where $\chi(S)$ is the Euler characteristic of S.

Proof. We only need to prove the proposition for each $\max(0,(n-m)/2) \le k \le \min(n,(n+m)/2)$. We have

$$\widetilde{\mathcal{F}}(T_1) = \bigoplus_{a,b} \mathcal{F}(W(b)T_1a)\{n\} \quad \text{and} \quad \widetilde{\mathcal{F}}(T_2) = \bigoplus_{a,b} \mathcal{F}(W(b)T_2a)\{n\}$$

where the sums are over $a \in B^{n-k,k}$ and $b \in B^{m-k-l,k+l}$. The surface S induces a cobordism $S' = \mathrm{Id}_{W(b)}\, S\, \mathrm{Id}_a$ from $W(b)T_1a$ to $W(b)T_2a$ defined as the "vertical" composition of the identity cobordism from a to a, the cobordism S from T_1 to T_2, and the identity cobordism from $W(b)$ to $W(b)$. Observe that S' induces a homogeneous map of graded abelian groups $\mathcal{F}(W(b)T_1a) \to \mathcal{F}(W(b)T_2a)$. Summing over all a and b we get a homomorphism of $(\widetilde{A}^{m-k-l,k+l}, \widetilde{A}^{n-k,k})$-bimodules

$$\widetilde{\mathcal{F}}(S) : \widetilde{\mathcal{F}}(T_1) \to \widetilde{\mathcal{F}}(T_2).$$

Split S into the composition of elementary cobordisms

$$S = S_1 \circ \cdots \circ S_j.$$

The effect of each elementary cobordism is just an application of ι, ε, m or Δ. We only need to show that $\widetilde{\mathcal{F}}(S)$ takes $I(T_1)$ into $I(T_2)$. This follows from an argument similar to that in Lemma 1. The grading assertion follows from (3). Finally, $\widetilde{\mathcal{F}}(S)$ is independent of the presentation of S as the product of elementary cobordisms since $\mathcal{F}$ is a functor. ∎

Proposition 2. *Isotopic (rel boundary) surfaces induce equal bimodule maps.*

Proposition 3. *Let $T_1, T_2, T_3 \in \widehat{B}^m_n$ and let S_1, S_2 be cobordisms from T_1 to T_2 and from T_2 to T_3 respectively. Then $\mathcal{F}(S_2)\mathcal{F}(S_1) = \mathcal{F}(S_2 \circ S_1)$.*

Proposition 4. *For $T_1 \in \widehat{B}^s_n$, $T_2 \in \widehat{B}^m_s$ there is a canonical isomorphism of (A^m, A^n)-bimodules*

$$\mathcal{F}(T_2T_1) \cong \mathcal{F}(T_2) \otimes_{A^s} \mathcal{F}(T_1).$$

The proofs of the above propositions are similar to those in [8, Section 2.7].

5. Tangles, complexes of bimodules and tangle cobordisms. First we recall the definition of tangles. An *unoriented (m, n)-tangle L* is a proper, smooth embedding of $(n + m)/2$ arcs and a finite number of circles into $\mathbb{R}^2 \times [0, 1]$ such that:

- The boundary points of arcs map to
$$\{1, \dots, n\} \times \{0\} \times \{0\}, \{1, \dots, m\} \times \{0\} \times \{1\}.$$
- Near the endpoints, the arcs are perpendicular to boundary planes.

An *oriented (m, n)-tangle* comes with an orientation of each connected component. The unoriented tangles constitute a category with objects nonnegative integers, and morphisms isotopy classes of (m, n)-tangles. The composition of morphisms is defined as the concatenation of tangles. Likewise, the oriented tangles form a category with objects finite sequences of plus and minus signs, indicating orientations of the tangle near the endpoints.

A plane diagram of a tangle is a generic projection of the tangle onto $\mathbb{R} \times [0, 1]$. Two diagrams are called *isotopic* if they can be transformed into each other through generic projections. Two plane diagrams represent isotopic tangles if and only if they can be connected by a chain of diagram isotopies and Reidemeister moves $R1$, $R2$, and $R3$.

To each diagram D we associate integers $x(D)$ and $y(D)$ which count the numbers of negative and positive crossings of D respectively (see Figure 18).

$x(D)$ $y(D)$

Fig. 18. Negative and positive crossings

Fix a diagram D with s crossings of an oriented (m,n)-tangle L. We inductively define the complex of (A^m, A^n)-bimodules $\mathcal{F}(D)$ associated to D as follows. If D is crossingless, $\mathcal{F}(D)$ is the complex with the only nontrivial term in cohomological degree zero, which is given by the construction of the previous section.

If the diagram contains one crossing, consider the complex $\overline{\mathcal{F}}(D)$ of (A^m, A^n)-bimodules

$$0 \to \mathcal{F}(D(0)) \xrightarrow{\partial} \mathcal{F}(D(1))\{-1\} \to 0$$

where $D(i)$, $i = 0,1$, denotes the i-smoothing of the crossing (they are flat (m,n)-tangles), ∂ is induced by the obvious "saddle" cobordism (see Figure 19), and $\mathcal{F}(D(0))$ sits in the cohomological degree zero.

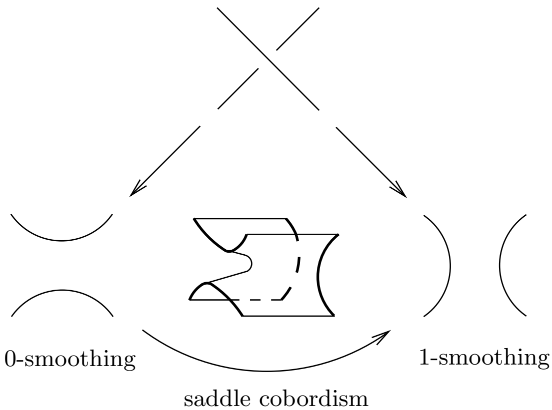

Fig. 19. Two smoothings of a crossing

Inductively, to a diagram with $t+1$ crossings we associate the total complex $\overline{\mathcal{F}}(D)$ of the bicomplex

$$0 \to \mathcal{F}(D(c_0)) \xrightarrow{\partial} \mathcal{F}(D(c_1))\{-1\} \to 0$$

where $D(c_i)$, $i = 0,1$, denotes the i-smoothing of a crossing c of D. Finally, define $\mathcal{F}(D)$ to be $\overline{\mathcal{F}}(D)$ shifted by $[x(D)]\{2x(D) - y(D)\}$.

Theorem 1. *If D_1 and D_2 are diagrams of an oriented (m,n)-tangle L, the complexes $\mathcal{F}(D_1)$ and $\mathcal{F}(D_2)$ of graded (A^m, A^n)-bimodules are chain homotopy equivalent.*

Since isotopies of tangles do not involve platforms, the proof of the theorem is essentially the same as in [8]. It follows from the above theorem that the isomorphism class of the complex $\mathcal{F}(D)$ is an invariant of L, denoted by $\mathcal{F}(L)$.

For a graded ring R denote by $\mathcal{K}(R)$ the category of bounded complexes of graded A-bimodules up to homotopies of complexes. Objects of $\mathcal{K}(R)$

are bounded complexes of graded A-bimodules and morphisms of $\mathcal{K}(R)$ are grading-preserving morphisms of complexes quotient by null-homotopic ones. We call $M \in \mathcal{K}(R)$ *invertible* if there exists $N \in \mathcal{K}(R)$ such that $N \otimes_R M \cong R$ and $M \otimes_R N \cong R$ in $\mathcal{K}(R)$. Here R denotes the complex $(0 \to R \to 0)$ with R in cohomological degree zero. For example, if L is any n-stranded braid, $\mathcal{F}(L) \in \mathcal{K}(A^n)$ is invertible. If M is invertible then

$$\mathrm{Hom}_{\mathcal{K}(R)}(M, M) \cong Z_0(R),$$

where $Z_0(R)$ is the degree zero component of the center of R (see [8]). Furthermore, we have

$$\mathrm{Aut}_{\mathcal{K}(R)}(M) \cong Z_0^*(R),$$

where $\mathrm{Aut}_{\mathcal{K}(R)}(M)$ is the group of automorphisms of M in $\mathcal{K}(R)$, and $Z_0^*(R)$ is the group of invertible elements in $Z_0(R)$.

For the ring $A^{n-k,k}$ we have

PROPOSITION 5. *The only invertible degree zero central elements in $A^{n-k,k}$ are ± 1:*

$$Z_0^*(A^{n-k,k}) \cong \{\pm 1\}.$$

Proof. Degree zero elements of $A^{n-k,k}$ have the form

$$v = \sum_{a \in B^{n-k,k}} v_a 1_a,$$

where $v_a \in \mathbb{Z}$. For any $a, b \in B^{n-k,k}$ such that ${}_a(A^{n-k,k})_b \neq 0$, pick nonzero $x \in \mathcal{F}(W(b)a)$. Then $vx = v_a x$ and $xv = v_b x$. If v is central we get $v_a = v_b$. We can connect any pair $c, d \in B^{n-k,k}$ by a sequence $c = c_0, c_1, \ldots, c_m = d$ such that $W(c_i)c_{i+1}$ contains no type III circles. This is equivalent to ${}_{c_i}(A^{n-k,k})_{c_{i+1}} \neq 0$, so $v_{c_i} = v_{c_{i+1}}$ and $v_c = v_d$. Since $v_a = m$ for all $a \in B^{n-k,k}$ and some integer m, we have $v = m \sum 1_a = m1 = m$. The proposition follows. ∎

From here on we assume familiarity with [9], where to an oriented tangle cobordism was associated a homomorphism of complexes of graded (H^m, H^n)-bimodules, in a consistent way so as to produce a projective 2-functor from the 2-category of tangle cobordisms to the 2-category of natural transformations between exact functors between homotopy categories of complexes of graded H^n-modules. The construction there extends without difficulty to our framework. A tangle cobordism can be presented by a movie S, which is a sequence of Reidemeister moves and critical point moves. To each consequent pair of tangle diagrams D_1, D_2 in a movie there is associated a natural homomorphism $\mathcal{F}(D_1) \to \mathcal{F}(D_2)$ between the corresponding complexes. In the case of a Reidemeister move, the homomorphism is an isomorphism in the homotopy category, while for the critical point moves the homomorphism is induced by either the unit, counit, multiplication, or comultiplication map on the ring $\mathcal{A}$.

The composition of these homomorphisms gives us a homomorphism $\mathcal{F}(S) : \mathcal{F}(D) \to \mathcal{F}(D')$ where D and D' are the first and the last frame in the movie S. The same argument as in [9] shows that $\mathcal{F}(S) = \pm\mathcal{F}(\tilde{S})$, where $\tilde{S}$ is any movie between D and D' representing the same cobordism as S. Proposition 5 above is a necessary ingredient in this argument.

The choice of sign in the equation $\mathcal{F}(S) = \pm\mathcal{F}(\tilde{S})$ does not depend on the sizes of platforms, since the rings $A^{n-k,k}$ are subquotients of H^n, our bimodules $\mathcal{F}(D)$ are subquotients of the bimodules in [8], and our bimodule homomorphism are induced by those in [8, 9] via subquotient maps. Therefore, the sign is always the same as in the invariant constructed in [9] and does not depend on the choice of k between 0 and n.

We can summarize the properties of our construction as follows.

Proposition 6. *The complexes $\mathcal{F}(T)$ of bimodules and homomorphisms $\pm\mathcal{F}(S)$ assigned to diagrams of tangle cobordisms assemble into a projective 2-functor from the 2-category of oriented tangle cobordisms to the 2-category of natural transformations between exact functors between homotopy categories of complexes of graded A^n-modules.*

Our invariant of tangles and tangle cobordisms carries the same amount of information as the one in [8, 9]. Indeed, since the invariants coming from the rings A^n are subquotients of the invariants built from H^n, we do not gain new information. On the other hand, the ring $A^{n,n}$ contains H^n as a subring, since H^n is isomorphic to the direct sum of $\mathcal{F}(W(b)a)$ over all pairs a, b of diagrams in $B^{n,n}$ which contain n parallel arcs connecting n points on the left platform and n points on the right platform. The inclusion $H^n \subset A^{n,n} \subset A^{2n}$ extends to bimodules and bimodule homomorphisms in the two descriptions of tangle and tangle cobordism invariants. Therefore, the second construction, via A^n, contains at least as much information as the original one, and our claim follows.

6. The Grothendieck group of A^n. The disjoint union of the sets $B^{n-k,k}$, as k ranges from 0 to n, can be naturally identified with the set J_n of length n sequences of 1's and -1's. An element $a \in B^{n-k,k}$ consists of n arcs with $2n$ endpoints, n of which lie on platforms and the other n directly between the platforms. We call the endpoints of the second type *free* endpoints. To each free endpoint we assign 1 or -1 as follows (see Figure 20 for an example). First, assign 1 to the left endpoint of each arc and -1 to the right endpoint. We get a sequence of length $2n$ with n ones and n minus ones. Remove the first $n-k$ and the last k terms in the sequence (notice that the first $n-k$ terms are all ones, and the last k are all minus ones).

The result is a sequence of length n with k ones and $n-k$ minus ones. We denote this sequence by $s(a)$.

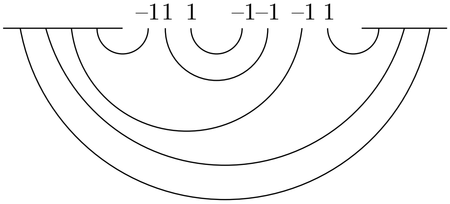

Fig. 20. Converting a to a sequence of 1's and -1's

Let V^n be the free $\mathbb{Z}[q,q^{-1}]$-module of rank 2^n with the basis v_s, over all sequences $s \in J_n$. For a sequence $s=(s_1,\dots,s_n)$ we write $v_s = v_{s_1}\otimes\cdots\otimes v_{s_n}$ and identify V^n with the nth tensor power of the rank 2 module V^1. Define the weight $w(s) = s_1+\cdots+s_n$ and $V^n(m)$ to be the subspace of V_n spanned by the vectors v_s with s of weight m. Then

$$V^n = \bigoplus_{k=0}^{n} V^n(2k-n).$$

To each $a \in B^{n-k,k}$ we associated $s(a) \in J_n$ of weight $2k-n$. We will also denote $v_{s(a)}$ simply by v_a.

To each $a \in B^{n-k,k}$ we associate an element $p_a \in V^n$ as follows. Convert each arc in a disjoint from the platforms into

$$v_1 \otimes v_{-1} + q v_{-1} \otimes v_1,$$

the indices placed in appropriate positions in the n-fold tensor product. An arc with one end on the left platform and one free end is converted into v_{-1}, in the corresponding position in the tensor product. An arc with one end on the right platform and one free end contributes v_1 to the tensor product. For example, for a in Figure 20,

$$\begin{aligned} p_a = {} & v_{-1}\otimes v_1\otimes v_1\otimes v_{-1}\otimes v_{-1}\otimes v_{-1}\otimes v_1 \\ & + q v_{-1}\otimes v_1\otimes v_{-1}\otimes v_1\otimes v_{-1}\otimes v_{-1}\otimes v_1 \\ & + q v_{-1}\otimes v_{-1}\otimes v_1\otimes v_{-1}\otimes v_1\otimes v_{-1}\otimes v_1 \\ & + q^2 v_{-1}\otimes v_{-1}\otimes v_{-1}\otimes v_1\otimes v_1\otimes v_{-1}\otimes v_1. \end{aligned}$$

Notice that

$$p_a = v_a + \text{lower order terms},$$

with respect to the order induced by the relation $1 > -1$ ($v_s > v_t$ if $s_i > t_i$ for the first i where the sequences differ). Hence, $\{p_a\}$, over all $a \in \bigsqcup_{k=0}^{n} B^{n-k,k}$, is a basis of the free $\mathbb{Z}[q,q^{-1}]$-module V^n.

The projective Grothendieck group $K_p(A^n\text{-gmod})$ of the category of finitely generated graded projective A^n-modules has generators $[P]$, where P

is a projective object of A^n-gmod, and relations $[P_1] = [P_2] + [P_3]$ whenever $P_1 \cong P_2 \oplus P_3$. The grading shift functor induces a $\mathbb{Z}[q, q^{-1}]$-module structure on $K_p(A^n\text{-gmod})$. An argument similar to the one in [8, Proposition 2] shows that $P_a\{i\}$ are the only projective indecomposable graded A^n-modules and that $K_p(A^n\text{-gmod})$ is a free $\mathbb{Z}[q, q^{-1}]$-module of rank 2^n with a basis $[P_a]$, $a \in \bigsqcup_{k=0}^{n} B^{n-k,k}$.

Consider the isomorphism of $\mathbb{Z}[q, q^{-1}]$-modules

$$K_p(A^n\text{-gmod}) \cong V^n \tag{7}$$

that takes $[P_a]$ to p_a. For each (m, n)-tangle T the complex of bimodules $\mathcal{F}(T)$ consists of right projective bimodules, and the tensor product with $\mathcal{F}(T)$ is an exact functor from the category $\mathcal{K}(A^n\text{-gmod})$ to $\mathcal{K}(A^m\text{-gmod})$. Here $\mathcal{K}(\mathcal{W})$ denotes the category of bounded complexes of objects of an abelian category $\mathcal{W}$ up to chain homotopies.

This functor takes a projective object of A^n-gmod to a complex of projective objects of A^m-gmod, and hence induces a homomorphism $[\mathcal{F}(T)]$ of $\mathbb{Z}[q, q^{-1}]$-modules

$$K_p(A^n\text{-gmod}) \to K_p(A^m\text{-gmod}).$$

It is easy to compute these maps directly and check that under the isomorphism (7) they give the standard actions of the category of tangles on tensor powers

$$V_1^{\otimes n} \cong V^n \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$$

of the fundamental representation of quantum $\mathfrak{sl}_2$.

Under this isomorphism the basis of V^n given by images of indecomposable projective modules $[P_a]$ goes to the Lusztig dual canonical basis of $V_1^{\otimes n}$, after changing q to $-q^{-1}$ (the latter basis was explicitly computed in [5]).

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