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Representations of the Kauffman bracket skein algebra of the punctured torus

by

Jea-Pil Cho and **Răzvan Gelca** (Lubbock, TX)

Abstract. We describe the action of the Kauffman bracket skein algebra on some vector spaces that arise as relative Kauffman bracket skein modules of tangles in the punctured torus. We show how this action determines the Reshetikhin–Turaev representation of the punctured torus. We rephrase our results to describe the quantum group quantization of the moduli space of flat SU(2)-connections on the punctured torus with fixed trace of the holonomy around the boundary.

1. Introduction. Skein modules and skein algebras [19], [23] have been studied intensively, especially in connection with the Witten–Reshetikhin–Turaev theory [2], [17], [8] and the Jones polynomial [12]. The skein algebras of surfaces and their action of skein modules of handlebodies have been studied in [6], [10], and [11] in relation to the quantization of moduli spaces of connections on surfaces, which is part of the Witten–Reshetikhin–Turaev theory [26]. Extensive studies have been performed in the case of the torus.

In this note we go one step further and examine the punctured torus. We do this because the basic data of the modular functor of the Reshetikhin–Turaev topological quantum field theory is contained entirely in the punctured torus and in the sphere with four punctures [8].

We describe below the representations of the Kauffman bracket skein algebra that are relevant to the quantization problem. Then we show how the Reshetikhin–Turaev representations of the mapping class group of the punctured torus, which form the basic data on the punctured torus, can be recovered explicitly from these representations.

2. The Kauffman bracket skein algebra of the punctured torus. In what follows we will use the notations and results from [14], with t standing for A. Let $\Sigma_{1,1}$ be the *punctured torus*, that is, the torus with

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one open disk removed. For a curve that is a smooth embedding of a circle in $\Sigma_{1,1} \times [0,1]$, a *framing* is a smooth vector field along the curve always orthogonal to it. We are concerned with families of such curves, called *framed links*. The framing can always be made parallel to $\Sigma_{1,1}$, the *blackboard framing*. As such we will represent framed links by curves, the framing being self-understood.

It is standard [19] to define the Kauffman bracket skein module $K_t(\Sigma_{1,1} \times [0,1])$ as the quotient of the free $\mathbb{C}[t,t^{-1}]$-module with basis the set of isotopy classes of framed links in $\Sigma_{1,1} \times [0,1]$ by the Kauffman bracket skein relation shown in Figure 1. The links that show up in this figure are supposed to be equal except in an embedded ball. Also, every trivial link component is replaced by a factor of $-t^2 - t^{-2}$.

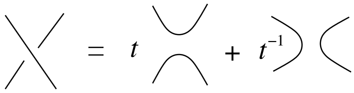

Fig. 1

The topological operation of gluing one cylinder over the punctured torus on top of another induces a multiplication on $K_t(\Sigma_{1,1} \times [0,1])$, which turns it into an algebra. This is the *Kauffman bracket skein algebra of the punctured torus*.

Simple closed curves on the punctured torus are defined by pairs (p,q) of coprime integers with $p \geq 0$, with q/p being the slope of the curve, together with a curve ∂ that is parallel to the boundary. As shown in [3], the Kauffman bracket skein algebra of the punctured torus is generated by the curves $(1,0)$, $(0,1)$, $(1,1)$, depicted in Figure 2. Let ∂ denote the boundary curve.

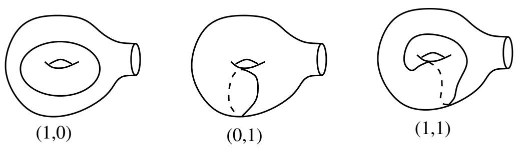

Fig. 2

As a $\mathbb{C}[t,t^{-1}]$-module, $K_t(\Sigma_{1,1} \times [0,1])$ is free with basis $(p,q)_T\partial^k$, $p,k \geq 0$, $q \in \mathbb{Z}$, where ∂ is a curve parallel to the boundary and $(p,q)_T = T_n(p/n, q/n)$, with $n = \gcd(p,q)$ and $T_n(x)$ the Chebyshev polynomial of the first kind ($T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$, $T_0(x) = 2$, $T_1(x) = x$).

That multiplication of curves on the punctured torus is significantly different from that on the torus is illustrated by the second of the following formulas.

Proposition 2.1. *In* $K_t(\Sigma_{1,1} \times [0,1])$,

$$(2,0)_T(0,1)_T = t^2(2,1)_T + t^{-2}(2,-1)_T,$$
$$(2,1)_T(0,1)_T = t^2(2,2)_T + t^{-2}(2,0)_T + \partial + t^2 + t^{-2}.$$

3. The representations. First, fix a positive integer $r > 1$ and let $t = \exp\frac{\pi i}{2r}$. Let also $[n] = \sin(n\pi/r)/\sin(\pi/r)$ be the quantized integer. The vector spaces on which we represent the Kauffman bracket skein algebra of the punctured torus are parametrized by the integers n with $0 \le 2n \le r-2$. These are the vector spaces $V_{r,n}$ which will be defined in what follows.

Consider a solid torus with $2n$ disjoint marked points on the boundary, which will be numbered by $1, \ldots, 2n$. For computational purposes, we can consider these points to lie, in increasing order, on a diameter of the puncturing disk. Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the set of isotopy classes of framed tangles with ends the $2n$ marked points. Such a tangle consists of several embedded circles together with n embedded arcs whose ends are the $2n$ points. The framing consists of a normal continuous vector field on each of the components of the tangle, with the convention that the vectors that belong to the endpoints lie on the specified diameter of the puncturing disk. We will always draw our tangles in the blackboard framing, that is, so that the vector field is parallel to the plane of the paper. An example for $n = 2$ is shown in Figure 3(a).

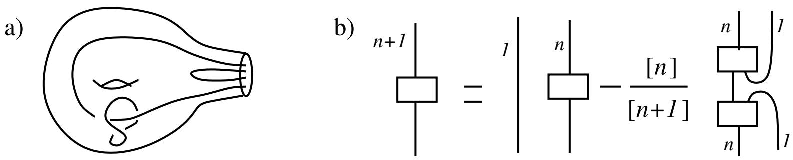

Fig. 3

We define the *Kauffman bracket skein module* of the solid torus with $2n$ points on the boundary, $K_t(S^2 \times \mathbb{D}^2, 2n)$, to be the quotient of this free module by the Kauffman bracket skein relations.

The topological operation of gluing the cylinder over the punctured torus $\Sigma_{1,1} \times [0,1]$ to the complement of the puncturing disk in the boundary of the solid torus gives rise to an action of $K_t(\Sigma_{1,1} \times [0,1])$ on $K_t(S^1 \times \mathbb{D}^2, 2n)$.

For $k < n$, we will define a family of inclusions of $K_t(S^1 \times \mathbb{D}^2, 2k)$ into $K_t(S^1 \times \mathbb{D}^2, 2n)$. To this end, let δ be the data consisting of a function $f : \{1, \ldots, 2k\} \to \{1, \ldots, 2n\}$ such that $f(i) - f(i-1)$ is odd for $i = 2, 3, \ldots, 2k$, and a pairing of the $2n - 2k$ numbers in the complement of $\operatorname{Im} f$ such that if (p,q) is a pair then p and q belong to the same interval $(f(i-1), f(i))$ and for any two pairs (p,q) and (r,s), $(r-p)(r-q)(s-p)(s-q) > 0$. For each such δ we define the inclusion

$$i_\delta : K_t(S^1 \times \mathbb{D}^2, 2k) \to K_t(S^1 \times \mathbb{D}^2, 2n)$$

by identifying the $2k$ boundary points of $K_t(S^1 \times \mathbb{D}^2, 2k)$ with the boundary points of $K_t(S^2 \times \mathbb{D}^2, 2n)$ indexed by $f(i)$, $i = 1, \dots, 2k$, for each pair (p, q), connecting these points by an arc isotopic to the segment $[p, q]$ ([1]).

Next we factor $K_t(S^1 \times \mathbb{D}^2, 2n)$ by the skein relation $f^{r-1} = 0$, where f^n, $n \geq 1$, are the Jones–Wenzl idempotents [25] defined recursively in Figure 3(b), with f^1 being just one strand. The result of this factorization is a finite-dimensional space $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$ called the *reduced Kauffman bracket skein module*. This space was first considered in [22]. The inclusions i_δ factor to maps between reduced Kauffman bracket skein modules. The action of $K_t(\Sigma_{1,1} \times [0, 1])$ factors to an action of the same algebra on the reduced Kauffman bracket skein module.

For $n \leq m \leq r - 2 - n$, define the skein $v_{2n,m}$ as shown in Figure 4(a). Here, there are m respectively $2n$ parallel strands, as specified, with the corresponding Jones–Wenzl idempotents placed on them. At the trivalent vertex there is the Kauffman triad described in Figure 4(b) (see [14]).

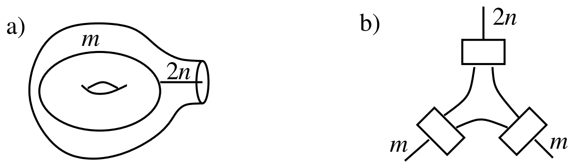

Fig. 4

Lemma 3.1. *The vector space $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$ is finite-dimensional with basis $i_\delta(v_{2k,m})$, where $0 \leq k \leq n$, $k \leq m \leq r - 2 - k$, and δ ranges over all possible sets of data defined above.*

Proof. There is a planar projection of the solid torus onto an annulus such that the projections of all these elements have no crossings. To see that they form a basis, note first that any skein in $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$ can be written as a linear combination of skeins of the form $i_\delta(\sigma)$, where σ is a skein in some $K_{t,r}(S^1 \times \mathbb{D}^2, 2k)$ which in a tubular neighborhood of the puncturing disk is just an $2k$-strand decorated by the $2k$th Jones–Wenzl idempotent. Consequently, the set $i_\delta(v_{2k,m})$, indexed by $0 \leq k \leq m$, $k \leq m \leq r - 2$, and δ, spans $K_{t,r}(S^1 \times \mathbb{D}^2, 2n)$. Next, we can apply the arguments from [16], to reduce everything to the case where the trivalent vertex has an admissible coloring, namely to $k \leq m \leq r - 2 - k$, and then conclude that these elements form a basis. ∎

[1] If in a diagram a skein σ has no crossings, then $i_\delta(\sigma)$ has no crossings either.

As a corollary of the lemma we deduce that i_δ factors to an inclusion of reduced Kauffman bracket skein modules.

Because ∂ is in the center of $K_t(\Sigma_{1,1} \times [0,1])$, the eigenspaces of ∂ are invariant subspaces of the above representation. The eigenvalues of ∂ are $-t^{4k+2} - t^{-4k-2}$, $k \leq n$. The eigenspace of the eigenvalue $-t^{4k+2} - t^{-4k-2}$ has a basis $i_\delta(v_{2k}, m)$ for all δ and m, $k \leq m \leq r-2-k$. Among these we distinguish the ones with $k = n$. These are the spaces of interest to us since they correspond to quantizations of moduli spaces of flat connections on the punctured torus (see §4 below). Let therefore $V_{r,n}$ be the eigenspace of ∂ corresponding to the eigenvalue $-t^{4n+2} - t^{-4n-2}$, $n = 0, 1, \ldots, r-2$. A basis of this vector space consists of the vectors $v_{2n,m}$, $n \leq m \leq r-2-n$.

THEOREM 3.2. *Let n be an integer such that $0 \leq n \leq (r-2)/2$. The representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$ is given by*

$$(1,0)v_{2n,m} = v_{2n,m+1} + \frac{[m-n][m+n+1]}{[m][m+1]} v_{2n,m-1},$$
$$(0,1)v_{2n,m} = (-t^{2m+2} - t^{-2m-2})v_{2n,m},$$
$$(1,1)v_{2n,m} = (-t^{-2m-3})v_{2n,m+1} + (-t^{2m+1}) \frac{[m-n][m+n+1]}{[m][m+1]} v_{2n,m-1},$$

where $n \leq m \leq r-2-n$, with the convention that $v_{2n,n-1} = v_{2n,r-1-n} = 0$.

Proof. The second relation is standard (see [16]). To compute $(1,0)v_{2n,m}$, we proceed as in Figure 5, with the notation from [14] where

$$\Delta_n = (-1)^{n+1}[n+1]$$

and $\theta(m,n,p)$ is the quantum invariant of the trivalent graph of the letter θ with the edges colored by m, n, p. Here for the second equality we use the Recoupling Theorem [14, Ch. 7] for one of the strands colored by m.

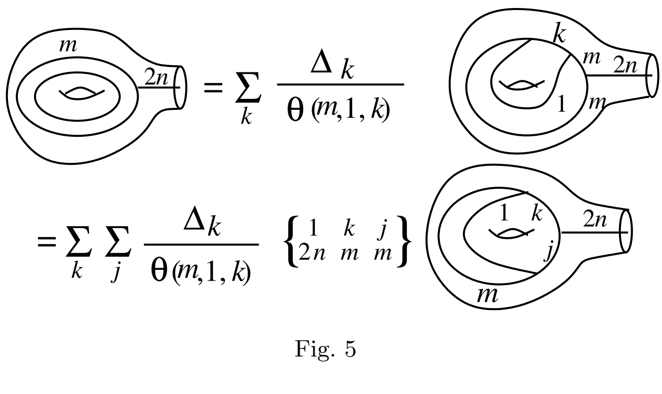

Fig. 5

Next, by [14, Ch. 5, Lemma 7], this is nonzero if and only if $j = k$, and in this case it is equal to the skein in Figure 6. Also, since the triple $(1, m, k)$

$$\sum_k \frac{\Delta_k}{\theta(m,1,k)} \begin{Bmatrix} 1 & k & j \\ 2n & m & m \end{Bmatrix} \frac{\theta(m,1,k)}{\Delta_k}$$

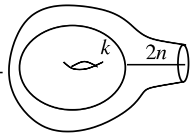

Fig. 6

is admissible only if $k = m - 1$ or $k = m + 1$, it follows that the sum has only two terms. Substituting the values of the two quantum $6j$-symbols we obtain the formula from the statement.

The action of $(1,1)$ is computed using

$$(1,1) = \frac{1}{t^2 - t^{-2}}[t(1,0)(0,1) - t^{-1}(0,1)(1,0)]. \blacksquare$$

REMARK 3.3. A proof of the theorem given in [4] is based only on the definition of Jones–Wenzl idempotents and Kauffman triads.

In the case where $n = 0$, we obtain the representation of the Kauffman bracket skein algebra of the torus on the reduced skein module of the punctured torus from [6].

REMARK 3.4. After this result was announced at Knots in Washington XXXII, it was also announced by J. Marché and T. Paul in [18].

COROLLARY 3.5. *The representation of* $K_t(\Sigma_{1,1} \times [0,1])$ *on* $V_{r,n}$ *is irreducible.*

Proof. We prove this by showing that every nonzero vector is cyclic. Let $v = \sum_m c_m v_{2n,m}$ be a vector in $V_{r,n}$. Because $v_{2n,m}$ are eigenvectors of $(0,1)$ with distinct eigenvalues, using linear combinations of elements of the form $(0,1)^n v$ we can produce any basis vector that appears with nonzero coefficient in the expansion of v. Next, by applying $(1,0)$ repeatedly, we can further generate from this vector a vector that contains any other basis vector with nonzero coefficient. And again using $(0,1)$ we can separate that basis vector. Consequently, from v we can produce every basis vector. This proves that v is cyclic, and so the representation is irreducible. ■

4. The action of the mapping class group. In this section we will show how the Reshetikhin–Turaev representation of the mapping class group of the punctured torus, which appears in Reshetikhin–Turaev theory [20], can be computed from the representation of the Kauffman bracket skein algebra described in Theorem 3.2. The Reshetikhin–Turaev representation of the punctured torus is part of the basic data of the modular functor (see [24], [8], [7]). It is known that the representation of the skein algebra of the punctured torus determines this representation, see for example [21]; here we show, with explicit formulas, how this was done in [9] for the torus.

The mapping class group of the punctured torus, with fixed boundary, is generated by the maps S, T, and T_1, where S exchanges the curves $(1,0)$ and $(0,1)$, T is the positive Dehn twist along the curve $(0,1)$, and T_1 is the positive twist of the boundary. The Reshetikhin–Turaev representation of the mapping class group associates to S, T, and T_1 unitary operators on $V_{r,n}$, referred to as the S-matrix, the T-matrix, and the T_1-matrix. These unitary operators interpolate the action of the mapping class group on $K_t(\Sigma_{1,1} \times [0,1])$. Below we show how they can be derived from the representation described in the previous section.

4.1. The S-matrix. For a number λ and a sequence $(x_l)_{l\geq 1}$ define the sequence $P_n(\lambda,(x_l)_{l\geq 1})$ recursively by

$$P_{n+1}(\lambda,(x_l)_{l\geq 1}) = \lambda P_n(\lambda,(x_l)_{l\geq 1}) - x_n P_{n-1}(\lambda,(x_l)_{l\geq 1}),$$
$$P_0(\lambda,(x_l)_{l\geq 1}) = 1, \qquad P_1(\lambda,(x_l)_{l\geq 1}) = \lambda.$$

Let $S = (a_{jk})$, for $0 \leq j,k \leq r-2n-2$, be the S-matrix. The equations

$$(1,0)Sv_{2n,n+j} = S(0,1)v_{2n,n+j}, \qquad (0,1)Sv_{2n,n+j} = S(1,0)v_{2n,n+j}$$

yield respectively the recursive relations

$$a_{j-1,k} = (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j,k} - \frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]}a_{j+1,k},$$
$$a_{j,k-1} = (-t^{2n+2j+2} - t^{-2n-2j-2})a_{j,k} - \frac{[k+1][2n+k+2]}{[n+k+1][n+k+2]}a_{j+1,k}.$$

Thus a_{kj} can be obtained by a backward double recursion. Normalizing by setting $a_{r-2n-2,r-2n-2} = 1$, we obtain

PROPOSITION 4.1. *For* $0 \leq k,j \leq r-2n-2$,

$$a_{r-2n-2-j,r-2n-2-k} = P_j(\lambda_{r-n},(x_l)_{l\geq 1}) \cdot P_k(\lambda_{r-n-j},(x_l)_{l\geq 1}),$$

where

$$x_l = \frac{[r-n-1-l][n+r-l]}{[r-l-1][r-l]}, \qquad l \geq 1,$$
$$\lambda_m = -t^{2m-2} - t^{-2m+2}, \qquad m \geq 0.$$

REMARK 4.2. This formula should be contrasted with those in [21] and [8].

4.2. The T-matrix and the twist on the boundary. Because $T(0,1) = (0,1)T$ and $(0,1)$ has 1-dimensional eigenspaces, it follows that T is diagonal, say $T = (b_{j,j})_j$, $0 \leq j \leq r-2n-2$. The equality

$$(1,0)Tv_{2n,n+j} = T(1,1)v_{2n,n+j}$$

yields the recursive relation $b_{j,j} = -t^{2n+2j+1} b_{j-1,j-1}$. As T is only projectively defined, we are free to choose $b_{1,1}$, and we set it equal to $(-1)^n t^{n^2-1}$. Then $b_{j,j} = (-1)^{n+j} t^{(n+j)^2-1}$, the well known formula.

The twist T_1 on the boundary commutes with all operators in $K_t(\Sigma_{1,1} \times [0,1])$. Because the representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$ is irreducible, T_1 acts as multiplication by a scalar, which we may choose to be $t^{(2n)^2-1}$.

5. The quantum group quantization of the moduli space of flat su(2)**-connections on the punctured torus.** As explained in [10] and [11], we are actually concerned with the quantum group setting, since it is our paradigm that the quantum group quantization of the moduli space of flat connections on a surface is an analogue of Weyl quantization.

We look at quantization in Heisenberg's formalism. In this formalism, to a symplectic manifold, which is the phase space of a classical system, one associates a Hilbert space, which is the space of states of a quantum system. Then, to C^∞ functions on the manifold, which are the classical observables, one associates linear operators on the Hilbert space, which are the quantum observable. This construction is done for a specified value of Planck's constant.

In our situation, the space that is quantized is the moduli space of flat su(2)-connections on the punctured torus, with fixed connection on the boundary. It is a well known fact of Chern–Simons theory that this is a compact symplectic manifold, which topologically is a ball. There is a standard procedure for producing the Hilbert space of the quantization, called *geometric quantization.* In this procedure the Hilbert space is obtained as the space of holomorphic sections of a holomorphic line bundle obtained in turn as a tensor product between a line bundle of curvature $(1/h)\omega$ and the metaplectic correction. Here h is the reduced Planck's constant, and ω is the symplectic form. Since $(1/h)\omega$ needs to be an integral class in cohomology, this puts a constraint on Planck's constant.

In Chern–Simons theory with gauge group SU(2), h is already constrained to be the reciprocal of an even integer $2r$. This then imposes a constraint on the "size" of the moduli space, which becomes a constraint on the holonomy around the boundary. It was explained in [5] that in our situation the holonomy is constrained to be, modulo a gauge transformation,

$$\begin{pmatrix} e^{2\pi i n/r} & 0 \\ 0 & e^{-2\pi i n/r} \end{pmatrix}, \qquad n = 0, 1, \ldots, r-2.$$

This is equivalent to requiring that the trace of the holonomy is one of the numbers $2\cos(2\pi i n/r)$, $n = 0, 1, \ldots, r-2$.

The operators are quantizations of Wilson lines. Recall that for a simple closed curve γ on the punctured torus, the *Wilson line* W_γ is the function on the space of su(2)-connections on the punctured torus obtained by taking the trace of the holonomy of the connections along γ.

There is a quantization scheme performed using quantum groups, as explained in [11] and [1]. This quantization is of combinatorial nature, and both the Hilbert space and the operators are described by graphs/knots decorated by irreducible representations. The fact that this construction is indeed a quantization, namely that it satisfies Dirac's requirements, is a consequence of the properties of the R-matrix, and was proved in [1].

The vector space has an orthogonal basis specified by the same diagrams as those for $v_{2n,m}$, this time with strands colored by the $m+1$- respectively $2n+1$-dimensional irreducible representations of the quantum group of SU(2).

The quantum group quantization can be described using skein modules (this is done in [11] using the skein modules with skein relations derived in [15]); the skein associated to a Wilson line W_γ is γ itself. The formulas for this action are the same as the ones derived above, except that the minus sign in front of t is deleted in each of the formulas for $(0,1)$ and $(1,1)$.

For each γ, W_γ is real valued, so the associated operator $\mathrm{Op}(W_\gamma)$ must be self-adjoint. This shows that, while the vectors $v_{2n,m}$ are orthogonal, they are not unit vectors. We normalize them to

$$w_{2n+1,m+1} = \left(\prod_{j=n+1}^{m} \frac{[j-n][j+n+1]}{[j][j+1]} \right)^{-1/2} v_{2n,m},$$

which are now unit vectors. The shift in the indices is such that they agree with the dimensions of the corresponding irreducible representation of the quantum group. We thus have

Proposition 5.1. *The quantum group quantization at $h = 1/(2r)$ of the moduli space of flat* su(2)*-connections on the punctured torus with the trace of the holonomy on the boundary equal to $2\cos(2\pi i n/r)$, for some $n \in \{0, 1, \ldots, r-2\}$, has the Hilbert space $\mathcal{H}_{r,n}$ with orthonormal basis $w_{2n+1,m}$, $n+1 \le m \le r-1-n$, and with the algebra of quantum observables acting by*

$$\mathrm{Op}(W_{(1,0)}) w_{2n+1,m} = \sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}\, w_{2n+1,m+1} + \sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}\, w_{2n+1,m-1},$$

$$\mathrm{Op}(W_{(0,1)})w_{2n+1,m} = (t^{2m} + t^{-2m})w_{2n+1,m},$$

$$\mathrm{Op}(W_{(1,1)})w_{2n+1,m} = t^{-2m-1}\sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}\, w_{2n+1,m+1},$$

$$+\, t^{2m-1}\sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}\, w_{2n+1,m-1}.$$

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Jea-Pil Cho, Răzvan Gelca
Department of Mathematics and Statistics
Texas Tech University
Lubbock, TX 79409, U.S.A.
E-mail: rgelca@gmail.com

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