A twisted dimer model for knots

by

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Abstract. We develop a dimer model for the Alexander polynomial of a knot. This recovers Kauffman's state sum model for the Alexander polynomial using the language of dimers. By providing some additional structure we are able to extend this model to give a state sum formula for the twisted Alexander polynomial of a knot depending on a representation of the knot group.

1. Introduction. A dimer is an edge in a bipartite graph, and a dimer covering is a perfect matching for that graph. The study of dimer coverings started in the 1960's with the work of Kasteleyn [Kas63] and Temperley–Fisher [TF61] who used it as a tool for studying statistical physics. Kasteleyn showed that the partition function on weighted bipartite planar graphs can be expressed as the determinant of a suitable matrix. The last ten years have seen a resurgence of the study of dimers and the application of this theory to many other areas of mathematics.

Our interest is in exploring the opposite direction. We have a given matrix, and we want to find the corresponding dimer model that expresses the determinant of the matrix as the partition function on the graph. Two well-known polynomial knot invariants, the classical Alexander polynomial and Xiao-Song Lin's twisted Alexander polynomial, are defined as determinants. The goal of the current work is to use the language of dimers to find a combinatorial model for the Alexander polynomial and the twisted Alexander polynomial.

Given a knot K in S^3 and some generic diagram for the knot we construct an associated bipartite planar graph with one set of vertices corresponding to crossings and the other set corresponding to faces. Edges signify incidence between crossings and faces. Using this graph along with a certain weighting

of the edges we provide a state sum model for the Alexander polynomial in terms of dimer coverings. This model recovers Kauffman's state sum model for the Alexander polynomial [Kau87].

Consider a representation ρ of the fundamental group of the knot complement. Associated to this representation one defines the twisted Alexander polynomial $\Delta_{\rho,K}$, which is an invariant of the knot together with the representation ρ. We extend our dimer model for the Alexander polynomial to provide a state sum model for the twisted Alexander polynomial.

In Section 2 we review some basic definitions and theorems dealing with dimer coverings. In Section 3 we recall the definition of the Alexander polynomial and show how to translate it into a dimer model. We will see that it is equivalent to Kauffman's state sum model. Section 4 begins with the definition of the twisted Alexander polynomial and shows how the dimer model can be augmented to provide a twisted dimer model. We provide examples throughout.

2. Dimer background. In this section we review some facts and results about dimer coverings of graphs. For the interested reader, Kenyon provides an excellent introductory set of lectures on this subject [Ken09]. We will also need to recall some results of Kasteleyn [Kas63], a good explanation of which can be found in Kuperberg's work [Kup94, Kup02, Kup98].

Let $\Gamma = (V_1, V_2, E)$ be a bipartite graph with V_1 and V_2 the two vertex sets and E the collection of edges in Γ each of which has one endpoint in V_1 and one endpoint in V_2.

DEFINITION 2.1. A *dimer* is an edge in E. A *dimer covering* is a subset m of E such that each vertex in Γ is an endpoint of exactly one edge in m. In other words a dimer covering is a perfect matching on Γ. Let $\mathcal{M}$ be the set of all dimer coverings of Γ; note that $\mathcal{M} = \emptyset$ whenever $|V_1| \neq |V_2|$.

Let $\mu : E \to \mathbb{C}[t]$ be a weighting of the graph Γ, and denote the weighted graph by Γ_μ. Then we consider the following partition function $Z(\Gamma_\mu)$ which is of particular interest in statistical physics:

$$Z(\Gamma_\mu) = \sum_{m \in \mathcal{M}} \prod_{e \in m} \mu(e).$$

DEFINITION 2.2. Given a weighting μ of Γ and an ordering of the vertex sets

$$V_1 = \{v_{1,1}, \ldots, v_{|V_1|,1}\} \quad \text{and} \quad V_2 = \{v_{1,2}, \ldots, v_{|V_2|,2}\},$$

construct the matrix $M(\Gamma_\mu)$ of dimension $|V_1| \times |V_2|$ with entries specified by the weight function μ as follows: The ijth entry of $M(\Gamma_\mu)$ is given by the sum of all weights μ assigned to edges between $v_{i,1}$ and $v_{j,2}$. We call this the *weight matrix* for Γ_μ. Figure 1 gives an example.
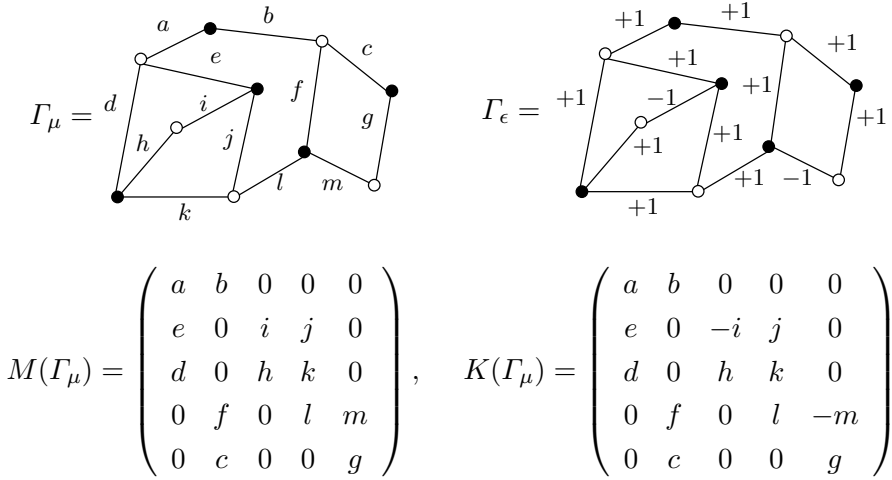
$$M(\Gamma_\mu) = \begin{pmatrix} a & b & 0 & 0 & 0 \\ e & 0 & i & j & 0 \\ d & 0 & h & k & 0 \\ 0 & f & 0 & l & m \\ 0 & c & 0 & 0 & g \end{pmatrix}, \quad K(\Gamma_\mu) = \begin{pmatrix} a & b & 0 & 0 & 0 \\ e & 0 & -i & j & 0 \\ d & 0 & h & k & 0 \\ 0 & f & 0 & l & -m \\ 0 & c & 0 & 0 & g \end{pmatrix}$$

Fig. 1. A weighted bipartite graph, a Kasteleyn weighting, the weight matrix $M(\Gamma_\mu)$ and the Kasteleyn matrix $K(\Gamma_\mu)$

Let $\mathrm{Perm}(M(\Gamma_\mu))$ denote the permanent (or unsigned determinant) of $M(\Gamma_\mu)$. Thus, we see that $Z(\Gamma_\mu) = \mathrm{Perm}(M(\Gamma_\mu))$. In the case that $|V_1| \neq |V_2|$ both values are 0. A natural question to ask is: Under what conditions can the weighting μ be modified to get a new weighting μ' with the property that the partition function for $\Gamma_{\mu'}$ is the determinant of the weight matrix for Γ_μ. In other words does there exist a weighting $\mu' : E \to \mathbb{C}[t]$ such that $Z(\Gamma_{\mu'}) = \mathrm{Det}(M(\Gamma_\mu))$? In the case that Γ is planar, Kasteleyn proves that such a modification is always possible. He accomplishes this by using what is now called a Kasteleyn weighting.

DEFINITION 2.3. Let Γ be a bipartite planar graph, that is, a bipartite graph together with a fixed embedding of that graph in the plane. A *Kasteleyn weighting* $\epsilon : E \to \{\pm 1\}$ is a choice of ± 1 for each edge with the property that each bounded face with 0 mod 4 edges has an odd number of -1 assignments and each bounded face with 2 mod 4 edges has an even number of -1 assignments.

PROPOSITION 2.4 (Kasteleyn). *Every bipartite planar graph Γ has a Kasteleyn weighting.*

Proof. We can prove this fact by providing an algorithm for finding a Kasteleyn weighting. Begin by choosing a spanning tree $T = (V_1, V_2, E_T) \subset \Gamma$. If $\bar{\Gamma}$ is the dual planar graph of Γ (with vertices given by faces of Γ, edges transverse to edges of Γ, and faces given by vertices of Γ) then there is an associated spanning tree $\bar{T} \subset \bar{\Gamma}$ which is disjoint from T. Consider $\bar{T}$ to be rooted at the vertex corresponding to the unbounded face.

Let $\epsilon : E_T \to \{\pm 1\}$ be arbitrarily given. Choose a valence 1 vertex v_1 of $\bar{T}$ that is not the root vertex. This vertex represents a face of Γ that has all but one bounding edge present in the tree T. Let e_1 be the edge that is missing in that face of Γ. Since all other edges bounding the face have been assigned weights, the choice of value for $\epsilon(e_1)$ that will satisfy the properties of a Kasteleyn weighting is forced. Remove the vertex v and the edge incident on v from $\bar{T}$, and define $\epsilon(e_1)$ as necessary.

Repeat this process, pruning the non-root valence 1 vertices and their edges as you go. Eventually a single edge connecting a vertex v to the root is all that remains in $\bar{T}$. This represents a single edge e in Γ shared by a bounded face and the unbounded face. Assign the necessary value to $\epsilon(e)$ in order to complete to a Kasteleyn weighting. $\blacksquare$

DEFINITION 2.5. Let Γ_μ be a weighted bipartite planar graph. We call the weight matrix $M(\Gamma_{\epsilon \cdot \mu})$ the *Kasteleyn matrix*, and we give it the special notation $K(\Gamma_\mu)$. See Figure 1 for a calculation of a Kasteleyn matrix.

A proof of the following result, which again is due to Kasteleyn, can be found in [Kup98].

THEOREM 2.6 (Kasteleyn). *Let Γ_μ be a weighted bipartite planar graph. Then*

$$Z(\Gamma_{\epsilon \cdot \mu}) = \mathrm{Perm}(K(\Gamma_\mu)) = \pm \mathrm{Det}(M(\Gamma_\mu)),$$

or equivalently

$$Z(\Gamma_\mu) = \mathrm{Perm}(M(\Gamma_\mu)) = \pm \mathrm{Det}(K(\Gamma_\mu)).$$

REMARK 2.7. It is known that Kasteleyn's theorem does not hold in general for non-planar graphs. In fact, it holds if and only if the graph does not have $K_{3,3}$ as a minor [LP09].

3. The Alexander polynomial. We begin by giving a determinant definition of the Alexander polynomial due to Fox, which can be found in [CF77]. By using Kasteleyn's theorem we will construct a bipartite planar graph such that the partition function of this graph is the Alexander polynomial. Finally, we will show that this approach yields Kauffman's state sum model [Kau87].

While it might initially seem strange to express a determinant by a partition function, this combinatorial model proved to be useful, for example in the study of Ozsváth–Szabó knot Floer homology theory; see e.g. [OS03, Low08] and compare with [DL10].

Moreover, some properties of the Alexander polynomial follow directly from this approach; for instance, it is an easy exercise to show that the Alexander polynomial of an alternating knot has coefficients of alternating signs.

3.1. The Alexander polynomial as a determinant. Consider a knot $K \subset S^3$ along with some fixed generic diagram D_K. Label the faces of D_K with $a_0, \ldots, a_m$, where a_0 is the unbounded face. Choose some base point above the plane of projection. Let $A_0, \ldots, A_m$ be loops in $\pi := \pi_1(S^3 - K)$ given by passing through face a_i and returning through a_0 to the base point. Thus, the loop A_0 is trivial in π. Using this notation we have the *Dehn presentation* for the knot group:

$$\pi = \langle A_0, \ldots, A_m | r_1, \ldots, r_{m-1}, A_0 \rangle,$$

where the r_i are relations coming from the crossings in D_K. In particular the Dehn relation for a crossing shown in Figure 2 is $r : A_4 A_2^{-1} = A_3 A_1^{-1}$.
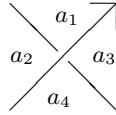
Fig. 2. A crossing with labeled faces

Setting $A_0 = 1$ and incorporating that change in the relations r_i we get a presentation of π with m generators and $m - 1$ relations:

$$\pi = \langle A_1, \ldots, A_m : r_1, \ldots, r_{m-1} \rangle.$$

Let $\mathcal{F}_m$ be the free group on m generators $A_1, \ldots, A_m$. Then the free derivative $\frac{\partial}{\partial A_i}$ is a map from $\mathcal{F}_m$ to $\mathbb{Z}[\mathcal{F}_m]$ recursively defined by

- $\frac{\partial}{\partial A_i}(1) = 0$,
- $\frac{\partial}{\partial A_i}(A_j) = \delta_{ij}$,
- $\frac{\partial}{\partial A_i}(-A_j) = -\delta_{ij} A_j^{-1}$, and
- $\frac{\partial}{\partial A_i}(w A_j) = \frac{\partial}{\partial A_i}(w) + w \frac{\partial}{\partial A_i}(A_j)$ for any word $w \in \mathcal{F}_m$.

Consider the map $\phi : \mathcal{F}_m \to \pi$ defined by $\phi(A_i) = A_i$. We can extend this to a map $\phi : \mathbb{Z}[\mathcal{F}_m] \to \mathbb{Z}[\pi]$, and the kernel of this map will be generated by the relations r_i in $\mathcal{F}_m$. Let $\psi : \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}]$ be the abelianization mapping which will take meridians positively linking the knot to the variable t.

Let M_K be the $(m-1) \times m$ matrix with ijth entry given by $\psi \circ \phi\left(\frac{\partial r_i}{\partial A_j}\right)$. Remove any one column corresponding to a face of the diagram D_K that is adjacent to the unbounded face, obtaining a square matrix M_K'. Up to sign and multiplication by a power of $t^{\pm 1}$, the determinant of M_K' is independent of the choice of the adjacent face and it is invariant under Reidemeister moves. Thus we define the *Alexander polynomial K*, denoted $\Delta_K(t)$, to be

$$\Delta_K(t) \doteq \det(M_K'),$$

where $\doteq$ means equality up to multiplication with $\pm t^k$ for some power k.

It will turn out that the entries of the matrix M_K are all either 0 or ± 1 or $\pm t$. More specifically, the free derivatives of all the relations r_i will end up being 0 or 1 or meridians in π. In other words, while we form the matrix M_K using the Dehn presentation for the knot group, the free derivatives of the relations are up to a sign the so-called Wirtinger generators. Since it will become important in Section 4.1 we recall the Wirtinger presentation for π and then show the calculation of the free derivatives.

Given the diagram D_K we can label the arcs of the knot $c_1, \dots, c_\ell$. Again choose some base point above the plane of projection. For $1 \le i \le \ell$ let the loop x_i be the meridian that leaves the base point, positively links the arc c_i, and returns to the base point. We again get a relation r'_j at each crossing. Using these generators and relations we get another presentation for π known as the *Wirtinger presentation*:

$$\pi = \langle x_1, \dots, x_\ell \mid r'_1, \dots, r'_{\ell-1} \rangle.$$

Under abelianization we see that $\psi(x_i) = t$ for all i.

LEMMA 3.1. *The free derivatives of Dehn relations are either* 0, ± 1 *or, up to a sign, Wirtinger generators. Thus the matrix M'_K has entries either* 0, ± 1 *or* $\pm t$. *By multiplying suitable rows and colums with* -1 *we can assume that M'_K has only non-negative entries* 0, 1 *or* t.

Proof. As we noted above, the Dehn relation coming from the crossing in Figure 2 has the form $A_4 A_2^{-1} = A_3 A_1^{-1}$. Furthermore if x is the Wirtinger generator assigned to the overcrossing strand in Figure 2 then we have $A_4 A_2^{-1} = x$.

Now we take the free derivatives of the relation r with respect to each variable and see that the results can be written completely in terms of x and 1. Indeed, we get

$$\frac{\partial}{\partial A_1}(r) = A_4 A_2^{-1} = x, \quad \frac{\partial}{\partial A_2}(r) = -A_4 A_2^{-1} = -x,$$
$$\frac{\partial}{\partial A_3}(r) = -1, \quad\quad\quad \frac{\partial}{\partial A_4}(r) = 1.$$

It remains to show that the matrix can be transformed into a matrix with only non-negative entries by multiplying suitable rows or columns by -1. For that, color the faces of the diagram black/white so that no two adjacent faces have the same color. We see that locally the partial derivatives are negative at the two generators corresponding to either the black faces or the white faces. By multiplying all entries locally by -1, i.e. multiplying a row by -1, we can assume that the partial derivatives are negative at, say, the black faces. By multiplying all columns corresponding to black faces we obtain the result. ■

3.2. The dimer state sum. Using Kasteleyn's Theorem 2.6 we get the following construction which expresses the Alexander polynomial, given as a determinant, as a partition function of a certain bipartite graph.

For this we take a knot diagram and choose two adjacent faces that we disregard. As in Lemma 3.1 the Alexander polynomial is the determinant of a matrix M_K' that is indexed by the remaining faces and the relations which correspond to the crossings of the diagram. For each relation (i.e. crossing) locally the entries in the matrix are described by the picture in Figure 3 as proven in Lemma 3.1.
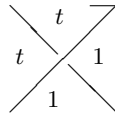
Fig. 3. Local weights at a crossing

Thus we obtain:

ALGORITHM 3.2 (The dimer state sum model).

(D1) Construct a bipartite planar graph $\Gamma = (V_1, V_2, E)$ as follows:
- The vertex set V_1 is the set of crossings of the diagram.
- The vertex set V_2 is the set of faces of the diagram.
- Given vertices $x \in V_1$ and $y \in V_2$, the edge (x, y) is in the set E if and only if the crossing x is incident on the face y.

We will call it the *Alexander graph*.
(D2) Use the weighting system of Figure 3 to define a weighting $\alpha : E \to \mathbb{C}[t]$ on Γ.
(D3) Choose a Kasteleyn weighting. The next paragraph will describe a particular weighting due to Kauffman.
(D4) Calculate the partition function $Z(\Gamma_\alpha) = \sum_{m \in \mathcal{M}} \prod_{e \in m} \alpha(e)$.

3.2.1. *Kauffman's Kasteleyn weighting.* The following proposition describes a way to choose a Kasteleyn weighting which is due to Kauffman. The proof that it gives a Kasteleyn weighting is an easy exercise.

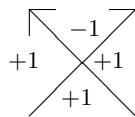PROPOSITION 3.3. *The assignments of weights given by Figure 4 form a Kasteleyn weighting.*

Fig. 4. Kauffman's Kasteleyn weighting

The next result immediately follows by construction and Kasteleyn's Theorem 2.6.

PROPOSITION 3.4. *The dimer state sum model described in Algorithm* 3.2 *calculates the Alexander polynomial. In other words for the weighted graph* Γ_α *we have* $\Delta_L(t) \doteq Z(\Gamma_\alpha)$.

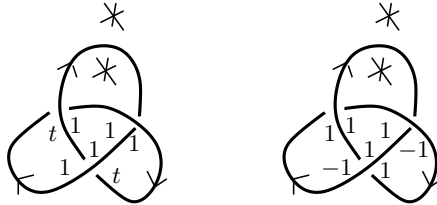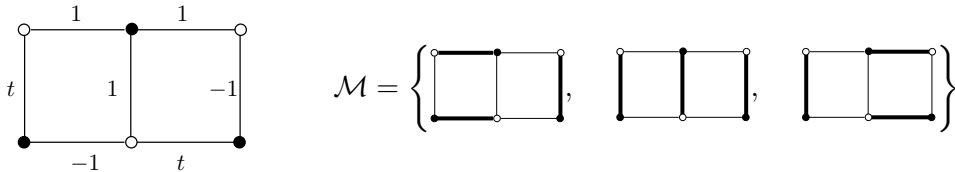EXAMPLE 3.5. Consider the trefoil given in Figure 5.

Fig. 5. Local weights and Kauffman's Kasteleyn weights for the trefoil

Associated to that diagram we have the following bipartite planar graph where the black vertices correspond to the faces and the white vertices to the crossings of the diagram. This graph has three dimer coverings.

Then we calculate the Alexander polynomial using Algorithm 3.2 as follows:

$$Z(\Gamma_\alpha) = \alpha \left(\text{□□} \right) + \alpha \left(\text{□□} \right) + \alpha \left(\text{□□} \right)$$
$$= t^2 - t + 1.$$

3.3. Kauffman's state sum. It turns out that with the dimer state sum model described in the previous section we recover the state sum model of Kauffman for the Alexander polynomial. We will briefly describe this model here.

Again let D_K be a generic diagram of a knot K. Choose the unbounded face of D_K and one other face that is adjacent to the unbounded face. We will disregard these faces in our calculation. A simple Euler characteristic argument tells us that the number of crossings of the diagram is equal to the number of faces remaining. Say this number is m.

ALGORITHM 3.6 (Kauffman state sum model).

(K1) Decorate the diagram D_K with the product of the two weights around each crossing that are depicted in Figures 3 and 4.

(K2) Find all possible ways to distribute m markers on the diagram so that each remaining face and each crossing has exactly one marker. Each of these configurations is called a *state*. Denote the set of all states by $\mathcal{S}$.

(K3) For each $s \in \mathcal{S}$ let $w(s)$ be the product of the weights associated to the state. Then

$$\Delta_K(t) \doteq \sum_{s \in \mathcal{S}} w(s).$$

The weights given in the lefthand diagram in step (K1) are encoding the free derivatives described in the previous section. The values shown are not exactly $\psi \circ \phi\left(\frac{\partial r_i}{\partial A_j}\right)$, but simple matrix operations discussed in [Kau87] give us these unsigned weights, which are more convenient for calculation.

EXAMPLE 3.7. Consider the following weighted diagram of the trefoil and its three states. (The two starred regions are the disregarded faces.)
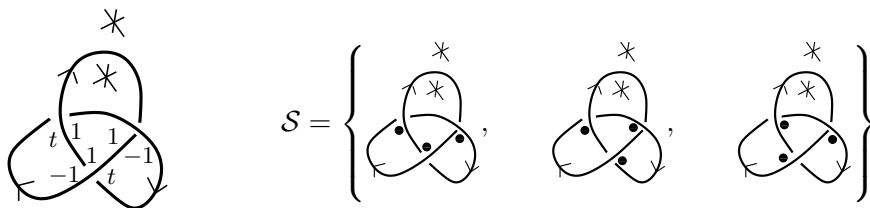
Then using Algorithm 3.6 we calculate the Alexander polynomial as follows:

$$\Delta_K(t) \doteq w\left(\vcenter{\hbox{\includegraphics{s1}}}\right) + w\left(\vcenter{\hbox{\includegraphics{s2}}}\right) + w\left(\vcenter{\hbox{\includegraphics{s3}}}\right) = t^2 - t + 1.$$

4. The twisted dimer model. We begin with a brief description of the twisted Alexander polynomial. This polynomial was originally described by Lin [Lin01] using the Seifert matrix, and has also been described by Kirk and Livingston [KL99] using the language of Reidemeister torsion. Our description is adapted from Wada's exposition in [Wad94] which defines the twisted Alexander polynomial for all finitely presented groups. We choose this description because it most clearly shows the generalization of the matrix described in Section 3.1. The polynomial that Wada defines to be the twisted Alexander polynomial is a certain quotient. At this point we depart from Wada's terminology and follow the work of Lin who refers to the numerator of Wada's quotient as the twisted Alexander polynomial.

4.1. The twisted Alexander polynomial for knots. Recall the maps $\phi : \mathbb{Z}[\mathcal{F}_m] \to \mathbb{Z}[\pi]$ and $\psi : \mathbb{Z}[\pi] \to \mathbb{Z}[t^{\pm 1}]$ from Section 3.1. Let R be an integral domain, and let ρ be a finite-dimensional representation

of π, where $\rho : \pi \to \mathrm{GL}_n(R)$. We can extend ρ to a ring homomorphism $\rho : \mathbb{Z}[\pi] \to M_n(R)$.

We put these maps together to define

$$\Phi = (\rho \otimes \psi) \circ \phi : \mathbb{Z}[\mathcal{F}_m] \to M_n(R[t^{\pm 1}]).$$

Now using this map we construct a block matrix $M_{K,\rho}$ with ijth block entry $\Phi\left(\frac{\partial r_i}{\partial A_j}\right)$. Wada calls it the *Alexander matrix* associated to the representation ρ. The matrix has dimensions $n(m-1) \times nm$. Consider the submatrix $M'_{K,\rho}$ that comes from deleting any block column corresponding to a face of D_K that is adjacent to the unbounded face.

Up to sign and multiplication by a power of $t^{\pm 1}$ the determinant of $M'_{K,\rho}$ is well-defined and invariant under Reidemeister moves. Thus we define the *twisted Alexander polynomial* of the pair K, ρ, denoted $\Delta_{K,\rho}(t)$, to be

$$\Delta_{K,\rho}(t) \doteq \det(M'_{K,\rho}).$$

Given the trivial representation $\rho : \pi \to \mathbb{C}$ we see that $\Delta_{K,\rho}(t) \doteq \Delta_K(t)$. In general the rows in the twisted Alexander matrix replace occurrences of 1 in Algorithms 3.6 and 3.2 with $\mathrm{Id} \in \mathrm{GL}_n(R)$, and occurrences of t with tX where X is the element of $\mathrm{GL}_n(R)$ assigned to the Wirtinger generator linking the overstrand at the associated crossing.

We conclude this subsection with an example calculation of the twisted Alexander polynomial.

EXAMPLE 4.1. Consider the following diagram of the trefoil. We have labeled the regions of the diagram that give the Dehn generators as well as the arcs of the knot that give the Wirtinger generators.
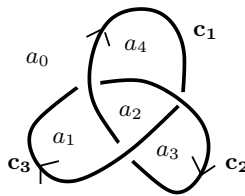
As noted by Lemma 3.1 we only need to know what our representation ρ does to Wirtinger generators. We consider the coloring representation $\rho : \pi \to \mathrm{GL}_3(\mathbb{Z})$ given by

$$\rho(C_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(C_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(C_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We set $A_0 = 1$ in the knot group, and we have three relations remaining:

(r$_1$) $A_2 A_1^{-1} = A_4 = C_1$,

(r$_2$) $A_2 A_4^{-1} = A_3 = C_2$,

(r$_3$) $A_2 A_3^{-1} = A_1 = C_3$.

Now we build a block matrix with entries $\Phi\left(\frac{\partial r_i}{\partial A_j}\right)$. The map Φ will assign identity matrices to occurrences of 1, and various representation matrices scaled by t when Wirtinger generators occur. We get the matrix

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
0 & -t & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
-t & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -t & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -t & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -t \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -t & 0 \\
\hline
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & -t & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & -t & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Finally we remove the last block column corresponding to the face a_4, and we take the determinant. We get $\Delta_{\text{Tref},\rho}(t) = -t^6 + t^5 + t^4 - 2t^3 + t^2 + t - 1 = -(-1+t)^2(1+t)^2(1-t+t^2)$.

4.2. The twisted Alexander graph. We want to build a graph that encodes the Alexander matrix for a pair K, ρ. This graph, which we will call the twisted Alexander graph, is a generalization of the Alexander graph defined in the dimer state sum algorithm. The twisted Alexander graph replaces single edges in the Alexander graph by "twisted edges". For an n-dimensional representation these "twisted edges" are each a copy of $K_{n,n}$ (the complete bipartite graph on $2n$ vertices) that will eventually encode the associated block entry in the Alexander matrix.

Begin by fixing a knot K with diagram D_K and representation $\rho : \pi \to \mathrm{GL}_n(R)$. As with the original Alexander graph, choose two adjacent faces of the diagram to disregard, one of which is the unbounded face.

DEFINITION 4.2. Construct a bipartite graph $\Gamma' = (V_1', V_2', E')$ as follows.

- The vertex set V_1' has n vertices for each crossing of the diagram.
- The vertex set V_2' has n vertices for each face of the diagram.
- If a face and a crossing are incident, insert a copy of $K_{n,n}$ going between the vertices corresponding to that face and crossing. If the face and crossing are not incident, no edges should connect their corresponding vertex sets.

We will call this the *twisted Alexander graph*.

DEFINITION 4.3. Let $M \in M_n(R([t^{\pm 1}]))$. The complete bipartite graph $K = K_{n,n}$ has all possible edges between two sets of V_r and V_c, each consisting of n vertices. Let K_M be the complete bipartite graph weighted according to M. More precisely, begin by enumerating the vertices in V_r with $v_{1,r}, \ldots, v_{n,r}$ and the vertices in V_c with $v_{1,c}, \ldots, v_{n,c}$. Now the edge between $v_{i,r}$ and $v_{j,c}$ gets the entry in the ijth position of M. If the entry in M is zero, we do not include the edge. We will call the weighted graph K_M the *graph encoding M*.

EXAMPLE 4.4. Let $M = \left(\begin{smallmatrix} a & b & 0 \\ 0 & 0 & c \\ d & 0 & e \end{smallmatrix} \right)$. Then the graph encoding M is
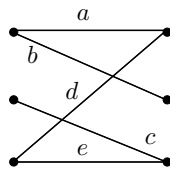
Enumerate the vertices at every crossing and every face of the twisted Alexander graph Γ'. We endow Γ' with the weighting $\alpha_\rho : E' \to R[t^{\pm 1}]$ so that the copy of $K_{n,n}$ (or the "twisted edge") connecting the collection of vertices for a crossing and a face is the graph encoding the matrix shown in Figure 6. Here X is the element of $\mathrm{GL}_n(R)$ assigned by ρ to the Wirtinger generator corresponding to the overstrand in the figure.
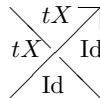

Fig. 6. Weights for the twisted Alexander graph

The following is an immediate consequence of our definitions of Γ' and α_ρ.

PROPOSITION 4.5. *The permanent of the matrix $M'_{K,\rho}$ is equal to the partition function of the graph Γ' weighted by α_ρ. In other words $Z(\Gamma'_{\alpha_\rho}) = \mathrm{Perm}(M'_{K,\rho})$.*

4.3. Kuperberg's tricks. As we discussed at length in Section 2 we would like to find a modification of the weighting α_ρ that would allow us to encode the twisted Alexander polynomial directly. The problem is that by replacing single edges in the Alexander graph by twisted edges, we no longer necessarily have a planar graph.

We will use two techniques due to Kuperberg [Kup98] to solve this problem: edge tripling and butterflies. This will enable us to modify an embedding of the graph Γ' and the weighting α_ρ to get a weighted planar graph

that will encode the twisted Alexander polynomial as desired. For the remainder of this section, fix an embedding of Γ'.

In order to force planarity we need to require that each pair of edges in Γ' intersects at most once. We can accomplish this by repeatedly tripling edges as shown in Figure 7. If our weight function α_ρ assigns $a \in R[t^{\pm 1}]$ to the edge, we modify the weights as shown in the figure.
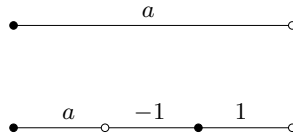

Fig. 7. Replace a single edge with three edges

Call the new graph obtained after tripling all necessary edges Γ'', and call the updated weight function $\alpha_{\rho,t}$. We denote the associated matrix by $M''_{K,\rho}$. Kuperberg argues that this operation changes the determinant at most up to a sign. In other words

$$\det(M''_{K,\rho}) = \pm \det(M'_{K,\rho}).$$

Indeed, this is easy to see as the Alexander matrix for the pair K, ρ changes as follows for each edge tripling:

$$\left(\begin{array}{c|c} a & * \\ \hline * & * \end{array} \right) \xrightarrow{\text{tripling}} \left(\begin{array}{cc|ccc} -1 & a & 0 & \cdots & 0 \\ 1 & 0 & & * & \\ \hline 0 & & & & \\ \vdots & * & & * & \\ 0 & & & & \end{array} \right).$$

Now we assume that our graph Γ'' has edges which pairwise intersect at most once. The final step in forcing planarity is to replace each pair of crossing edges with what Kuperberg calls a butterfly. This object and the necessary weight modifications are shown in Figure 8.
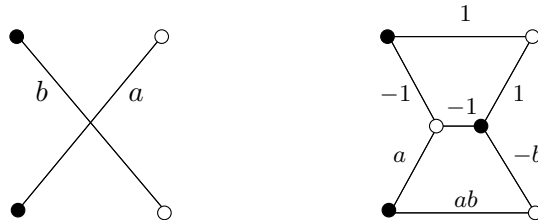

Fig. 8. Replace crossing edges with the butterfly

Call the new graph obtained after inserting all necessary butterflies Γ'''_{α_ρ}, and call the updated weight function $\alpha_{\rho,b}$. The associated matrix is $M'''_{K,\rho}$. Again Kuperberg argues that $\det(M'''_{K,\rho}) = \pm \det(M'_{K,\rho})$. We can also see this directly by examining the changes to the Alexander matrix for the pair K, ρ:

$$
\begin{pmatrix}
\begin{matrix} 0 & a \\ b & 0 \end{matrix} & * \\
\hline
* & *
\end{pmatrix}
\xrightarrow{\text{butterfly}}
\left(
\begin{array}{ccc|ccc}
-1 & -1 & a & 0 & \cdots & 0 \\
1 & 1 & 0 & & & \\
-b & 0 & ab & & * & \\
\hline
0 & & & & & \\
\vdots & & * & & * & \\
0 & & & & &
\end{array}
\right).
$$

4.4. The twisted dimer model. We can now put together the last two subsections to state the twisted dimer model. Fix a knot K, a generic diagram D_K, and a representation $\rho : \pi \to \mathrm{GL}_n(R)$.

ALGORITHM 4.6 (Twisted dimer model).

(T1) Build the twisted Alexander graph as described in Definition 4.2 with associated weight function α_ρ.

(T2) Choose an embedding of Γ' that minimizes the number of edge crossings.

(T3) Triple any edges necessary so that each pair of edges intersects at most once, and update the weight function as shown in Figure 7. Call the new graph Γ'' and the new weight function $\alpha_{\rho,t}$.

(T4) Replace any crossing pairs of edges with butterflies, and update the weight function as shown in Figure 8. Call the resulting graph Γ''' and the new weight function $\alpha_{\rho,b}$.

(T5) Use the algorithm described in the proof of Proposition 2.4 to get a Kasteleyn weighting ϵ for Γ'''.

(T6) Calculate the partition function

$$
Z(\Gamma'''_{\epsilon \cdot \alpha_{\rho,b}}) = \sum_{m \in \mathcal{M}} \left(\prod_{e \in m} \epsilon(e) \cdot \alpha_{\rho,b}(e) \right).
$$

Our main theorem follows by construction.

THEOREM 4.7. *The twisted dimer model described above calculates the twisted Alexander polynomial for the pair K, ρ. In other words*

$$
Z(\Gamma'''_{\epsilon \cdot \alpha_{\rho,b}}) \doteq \Delta_{K,\rho}(t).
$$

We conclude by applying the twisted dimer model to the pair Tref$, \rho$ from Example 4.1.

EXAMPLE 4.8. Recall that we are considering the following diagram for the trefoil and the representation $\rho : \pi \to \mathrm{GL}_3(\mathbb{Z})$ that comes from a nontrivial 3-coloring. We have labeled the diagram to indicate the weight function α_ρ that will be assigned to the twisted Alexander graph.
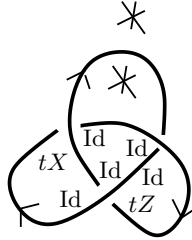

Recall that the representation matrices are

$$\rho(c_1) = X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(c_2) = Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(c_3) = Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The twisted Alexander graph will have three times the number of vertices in the original Alexander graph. Each edge in the original graph will be replaced with a copy of $K_{3,3}$. The following labeling of the Alexander graph helps us to see what the twisted graph will look like:
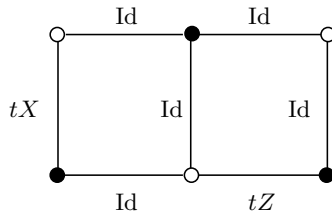
Finally, then, we see that the twisted Alexander graph for the pair Tref, ρ has the following form. To simplify the pictures, all thickened edges have weight t, and all other edges have weight 1. As we mentioned in Definition 4.3, we do not draw weight 0 edges since any perfect matching that includes a weight 0 edge will not contribute to the value of the partition function.

We can see through the sequence of manipulations in Figure 9 that, in this case, the twisted Alexander graph is a planar graph. While the embedding has changed, on the level of abstract graphs we have $\Gamma' = \Gamma'' = \Gamma'''$ and $\alpha_\rho = \alpha_{\rho,t} = \alpha_{\rho,b}$.
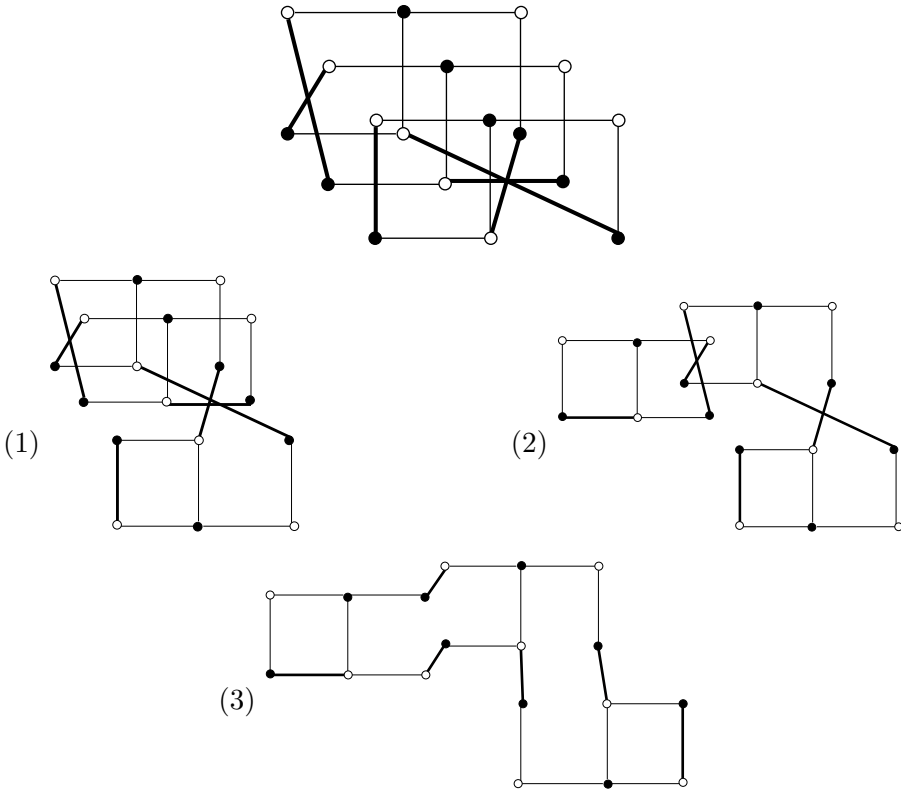
(1)

(2)

(3)

Fig. 9. Unraveling the twisted dimer graph

We then find a spanning tree in the graph Γ''' by the solid line segments transverse to the edges. We assign an ϵ-weighting of $+1$ to all of these edges. This is shown in Figure 10.
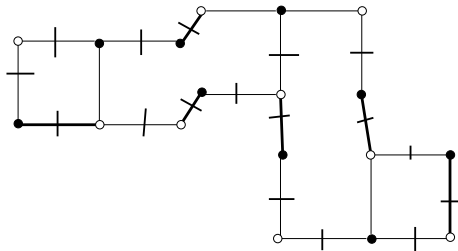
Fig. 10. Finding a spanning tree

Finally, as is shown in Figure 11, we complete ϵ to a Kasteleyn weighting. In the picture, -1 weights are indicated by double line segments. Calculating
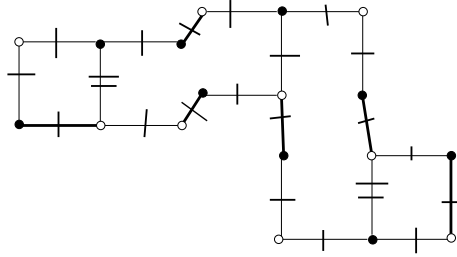
Fig. 11. A Kasteleyn weighting

the partition function one can see that the polynomial obtained is

$$\Delta_{\mathrm{Tref},\rho}(t) = t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1,$$

which agrees up to multiplication by -1 with Example 4.1.

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