

## Finite type invariants for cyclic equivalence classes of nanophrases

by

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**Abstract.** We define finite type invariants for cyclic equivalence classes of nanophrases and construct universal invariants. Also, we identify the universal finite type invariant of degree 1 essentially with the linking matrix. It is known that extended Arnold basic invariants to signed words are finite type invariants of degree 2, by Fujiwara's work. We give another proof of this result and show that those invariants do not provide the universal one of degree 2.

**1. Introduction.** Turaev [T1, T2] developed the theory of words based on the analogy with curves in the plane, knots on the 3-sphere, virtual knots, etc. A word is a sequence of letters, belonging to a given set, called an alphabet. Let  $\alpha$  be a set. A nanoword over  $\alpha$  (defined in [T2]) is a pair of a word in which each letter appears exactly twice and a map from the set of letters appearing in the word to  $\alpha$ . A nanophrase is a generalization of a nanoword, defined in [T1].

Vassiliev [V] developed the theory of finite type invariants of knots, which is conjectured to classify knots. Ito [I1] defined a notion of finite type invariants for curves on surfaces, constructed a large family of finite type invariants, called  $SCI_m$ , and showed that they become a complete invariant for stable homeomorphism classes. On the other hand, Fujiwara [Fuj] provided a simple idea to define finite type invariants for cyclic equivalence classes of signed words by introducing a new type of crossing, called singular crossing, which plays an intermediate role between an actual and virtual crossing. Here the signed word is a nanoword over  $\alpha = \{+, -\}$  and it is known from [T1] that the set of cyclic equivalence classes of signed words bijectively corresponds to the set of stable homeomorphism classes of curves

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on surfaces. We extend Fujiwara's finite type invariants to those for cyclic equivalence classes of nanophrases over a general  $\alpha$  not necessarily  $\{+, -\}$ .

The first aim of this paper is to construct universal intervals in Theorem 5.4, by following the approach in [GPV]. In addition, we identify the universal finite type invariant of degree 1 essentially with the linking matrix. As a related work, we should mention that Gibson and Ito [GI] extended the universal finite type invariant for nanophrases under different equivalence relations, called homotopy and closed homotopy.

To see the second aim, recall that Polyak [P] reconstructed Arnold's basic invariants [A1, A2] for isotopy classes of generic curves on the plane or sphere by using Gauss diagrams and showed that those invariants are finite type invariants of degree 1 in his sense. Ito [I1] connected Arnold's basic invariants of planar curves with his finite type invariants. On the other hand, Fujiwara [Fuj] extended Arnold's basic invariants of spherical curves to signed words and showed that those are finite type invariants of degree 2 in his sense. We give another proof of Fujiwara's result and show that Fujiwara's invariants do not provide the universal invariant of degree 2 in Fujiwara's sense.

This paper is organized as follows. In Section 2, following Turaev, we give formal definitions of words, phrases and so on. In Section 3, following Fujiwara, we give definitions of singular crossings and singular letters. In Section 4, we define finite type invariants for cyclic equivalence classes of nanophrases. In Section 5, we construct the universal finite type invariants for cyclic equivalence classes of nanophrases. In Section 6, we identify the universal finite type invariant of degree 1 essentially with the linking matrix. In Section 7, we restrict our discussion to nanowords corresponding to spherical curves, and clarify the relation between Arnold's basic invariants and our finite type invariants.

**2. Nanowords and nanophrases.** In this section, following Turaev [T1, T2], we review the formal definitions of words, phrases and so on.

**2.1. Words and phrases.** An *alphabet* is a finite set and its elements are called *letters*. A *word* of length  $m$  is a finite sequence of  $m$  letters. The unique word of length 0 is called the *trivial word* and is written  $\emptyset$ . An  $n$ -component *phrase* is a sequence of  $n$  words, called *components*, separated by  $\cdot$ . The unique  $n$ -component phrase each of whose components is the trivial word is called the *trivial  $n$ -component phrase* and is denoted by  $\emptyset_n$ . In this paper, we will regard words as 1-component phrases.

**2.2. Nanowords and nanophrases.** Let  $\alpha$  be a finite set. An  $\alpha$ -*alphabet* is an alphabet  $\mathcal{A}$  together with an associated map from  $\mathcal{A}$  to  $\alpha$ . This map is called a *projection*. The image of any  $A \in \mathcal{A}$  in  $\alpha$  will be denoted

by  $|A|$ . An *isomorphism*  $f$  of an  $\alpha$ -alphabet  $\mathcal{A}_1$  to  $\mathcal{A}_2$  is a bijection such that  $|f(A)|$  is equal to  $|A|$  for any letter  $A$  in  $\mathcal{A}_1$ . A *Gauss word* on  $\mathcal{A}$  is a word on  $\mathcal{A}$  such that every letter in  $\mathcal{A}$  appears exactly twice. Similarly, a *Gauss phrase* on  $\mathcal{A}$  is a phrase which satisfies the same condition. By definition, a 1-component Gauss phrase is a Gauss word.

Let  $p$  be a Gauss word or a Gauss phrase. The *rank* of  $p$  is the number of distinct letters appearing in  $p$ . We denote it by  $\text{rank}(p)$ . Note that the rank of a Gauss word must be half of its length. For example, the rank of  $ABCBAC$  is 3 and the rank of  $A : B : \emptyset : BA$  is 2.

An  $n$ -component *nanophrase* over  $\alpha$  is a pair  $(\mathcal{A}, p)$  where  $\mathcal{A}$  is an  $\alpha$ -alphabet and  $p$  is an  $n$ -component Gauss phrase on  $\mathcal{A}$ . When  $n$  is equal to 1, we call  $(\mathcal{A}, p)$  a *nanoword*.

Let  $(\mathcal{A}_1, p_1)$  and  $(\mathcal{A}_2, p_2)$  be nanophrases over  $\alpha$ . An *isomorphism*  $f$  of a nanophrase  $(\mathcal{A}_1, p_1)$  to  $(\mathcal{A}_2, p_2)$  is an isomorphism  $f$  of  $\alpha$ -alphabets such that  $f$  applied letterwise to the  $i$ th component of  $p_1$  gives the  $i$ th component of  $p_2$  for all  $i$ . If such an isomorphism exists, we say that  $(\mathcal{A}_1, p_1)$  and  $(\mathcal{A}_2, p_2)$  are isomorphic.

We can define the rank of a nanoword and of a nanophrase similarly to that of a Gauss word and of a Gauss phrase.

**2.3. Shift move on nanowords and nanophrases.** Turaev [T1] defined a *shift move* on nanophrases. Let  $\nu$  be an involution on  $\alpha$ . Suppose  $p$  is an  $n$ -component nanophrase over  $\alpha$ . A  $\nu$ -*shift move* on the  $i$ th component of  $p$  is a transformation which produces a new nanophrase  $p'$  as follows. If the  $i$ th component of  $p$  is empty or only a single letter, then  $p'$  is  $p$ . If not, we can write the  $i$ th component of  $p$  as  $Ax$ , where  $x$  is a word. Then the  $i$ th component of  $p'$  is  $xA$  and the other components of  $p'$  are the same as the corresponding components of  $p$ . Furthermore, if we write  $|A|_p$  for  $|A|$  in  $p$ , and  $|A|_{p'}$  for  $|A|$  in  $p'$ , then  $|A|_{p'}$  equals  $\nu(|A|_p)$  when  $x$  contains the letter  $A$ , and otherwise  $|A|_{p'}$  equals  $|A|_p$ .

**3. Singular crossings and singular letters on signed words.** Fujiwara [Fuj] introduced singular letters on signed words, which are nanowords over  $\alpha = \{+, -\}$ . In this section, following Fujiwara, we review the definitions of singular crossings and singular letters on signed words.

**3.1. Curves and signed words.** In this paper, a *curve* means a generic immersion from an oriented circle to a closed oriented surface. Here *generic* means that the curve has only a finite set of self-crossings which are all transversally double points, does not have triple points and self-tangencies, and has a regular neighborhood. A *pointed curve* is a curve endowed with a base point away from the double points.

Let  $\alpha$  be the set  $\{+, -\}$  and  $\mathcal{A}$  an  $\alpha$ -alphabet. We then call a nanoword  $(\mathcal{A}, w)$  over  $\alpha$  a *signed word*.

We consider a pointed curve on a surface. We label the crossings of the curve in an arbitrary way by different letters  $A_1, \dots, A_m$ , where  $m$  is the number of crossings. The Gauss word of a curve is obtained by the following. We start from the base point, move along the curve following the orientation and finish when we get back to the base point. Then we write down all the letters of the crossings in the order we meet them. The resulting word  $w$  on the alphabet  $\mathcal{A} = \{A_1, \dots, A_m\}$  contains each letter twice. To make  $w$  a signed word, we consider the crossing of the curve labeled by  $A$ . Then we define the sign of a letter as illustrated in Figure 3.1. When moving along the curve as above, if we first traverse the crossing labeled by  $A$  from the bottom-left to the top-right, then  $|A| = -$ , otherwise  $|A| = +$ . Here the orientation of the ambient surface is counterclockwise. The trivial curve corresponds to the signed word  $\emptyset$ . If we choose a different choice of the labeling of the crossing, we get an isomorphic signed word. We assign to each curve on a surface the isomorphism class of this signed word.

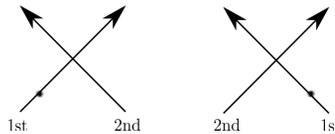


Fig. 3.1. On the left the sign is  $-$ , and on the right  $+$ .

Two curves are *stably homeomorphic* if there is an orientation preserving homeomorphism between their regular neighborhoods which maps one curve onto the other. Similarly, two pointed curves are *pointed stably homeomorphic* if they are stably homeomorphic via a map that sends one base point to the other.

If we change the curve by a stable homeomorphism, then the associated signed word does not change, since it is defined by the behavior of the curve in its regular neighborhood. It is proved in [T1] that the set of isomorphism classes of signed words bijectively corresponds to the set of stable homeomorphism classes of pointed curves on surfaces.

Let  $\nu$  be an involution on  $\alpha$  which maps  $+$  to  $-$ . The *cyclic equivalence relation* on signed words is defined by the relation generated by the  $\nu$ -shift move. It is proved in [T1] that the set of cyclic equivalence classes of signed words bijectively corresponds to the set of stable homeomorphism classes of curves on surfaces.

**3.2. Singular curves and singular signed words.** Consider a surface standardly embedded in  $\mathbb{R}^3$ . When we project a curve on a surface to the

plane, some crossings which are not crossings on a surface may appear in the planar curve. We call such a crossing a *virtual crossing* and denote it by a crossing with a small circle surrounding it (see Figure 3.2). Such crossings do not contribute at all to the associated signed word. A virtual crossing is not just an ordinary graphical vertex, but a non-actual crossing.

Let  $c$  be a curve on a surface, and  $w_c$  a signed word corresponding to  $c$ . If we replace some crossings of a planar curve which come from actual crossings of  $c$  by virtual crossings, we get a new curve on some surface. The signed word corresponding to the new curve is obtained from  $c$  by deleting the letters assigned to the actual crossing in question.

Fujiwara [Fuj] introduced a new type of crossing called *singular crossing*, which is an intermediate notion between an actual crossing and a virtual crossing, and denoted it by a small box (see Figure 3.2). A *singular curve* is a planar curve which may contain singular crossings.

We call a signed word corresponding to a singular curve a *singular signed word*. To get signed words corresponding to singular curves, we label actual and singular crossings by letters such as  $A$  and letters with an asterisk such as  $A^*$ , respectively. The  $A^*$  are called *singular letters*. For a word  $w$  without singular letters, denote by  $w^*$  the word obtained by changing all letters of  $w$  to singular letters keeping signs.

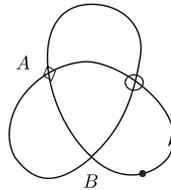


Fig. 3.2. The singular curve corresponding to the signed word  $A^*BA^*B$ , where  $|A^*| = +$  and  $|B| = -$

**4. Definitions of finite type invariants.** Fujiwara [Fuj] defined finite type invariants for cyclic equivalence classes of signed words. We now extend this definition to general  $\alpha$ .

Let  $\alpha$  be any finite set. Let  $P(\alpha, n)$  denote the set of isomorphism classes of  $n$ -component nanophrases over  $\alpha$ . Let  $\nu$  be any involution on  $\alpha$ . We define the *cyclic equivalence relation* over  $n$ -component nanophrases as the equivalence relation generated by isomorphisms and  $\nu$ -shift moves. Let  $P(\alpha, \nu, n)$  denote the set of cyclic equivalence classes of  $n$ -component nanophrases over  $\alpha$ .

We define  $\alpha^* = \{a^* \mid a \in \alpha\}$  and let  $\mathcal{A}$  be an  $\alpha \cup \alpha^*$ -alphabet. Then the letter whose projection is contained in  $\alpha^*$  is called a *singular letter* and is denoted with an asterisk, say by  $A^*$ , for easy distinction. Let  $P_m(\alpha, n)$  denote

the set of isomorphism classes of  $n$ -component nanophrases over  $\alpha \cup \alpha^*$  with  $m$  singular letters. We call the phrase with singular letters a *singular phrase*. By definition,  $P_0(\alpha, n)$  is the set of non-singular  $n$ -component nanophrases and  $P_0(\alpha, n) = P(\alpha, n)$ .

Given an involution  $\nu$  over  $\alpha$ , we extend  $\nu$  to  $\alpha \cup \alpha^*$  as follows. For any  $a \in \alpha$ , we define

$$\nu(a^*) = \nu(a)^*.$$

Then let  $P_m(\alpha, \nu, n)$  denote the set of cyclic equivalence classes of  $n$ -component nanophrases over  $\alpha \cup \alpha^*$  with  $m$  singular letters, where  $m$  is at least 0. By definition,  $P(\alpha, \nu, n) = P_0(\alpha, \nu, n)$ . Put  $\mathcal{P}(\alpha, \nu, n) = \bigcup_{m \geq 0} P_m(\alpha, \nu, n)$ .

An *invariant for cyclic equivalence classes of nanophrases* is a map  $u$  from the set of  $n$ -component nanophrases to an abelian group  $G$ , which takes equal values on nanophrases related by isomorphisms and  $\nu$ -shift moves. In other words, it is a map  $u : P(\alpha, \nu, n) \rightarrow G$ .

Given an invariant  $u : P(\alpha, \nu, n) \rightarrow G$ , we define its extension  $\hat{u} : \mathcal{P}(\alpha, \nu, n) \rightarrow G$  by the following rule

$$\begin{aligned} \hat{u}(p) &= u(p) && \text{if } p \in P_0(\alpha, \nu, n), \\ \hat{u}(xA^*yA^*z) &= \hat{u}(xAyAz) - \hat{u}(xyz) && \text{if } xA^*yA^*z \in P_m(\alpha, \nu, n) \ (m \geq 1), \end{aligned}$$

where  $A$  is a non-singular letter such that  $|A|^*$  is equal to  $|A^*|$ , and  $x, y$  and  $z$  are arbitrary sequences of letters, possibly including ‘:’ or ‘ $\emptyset$ ’. This map is well defined because the result does not depend on the order of the singular letters which we exclude.

Let  $p$  and  $q$  be  $n$ -component nanophrases. A nanophrase  $q$  is a *subphrase* of  $p$ , denoted by  $q \triangleleft p$ , if it is a nanophrase obtained from  $p$  by deleting pairs of letters. Here each letter preserves the value of the projection. We use this word even if we eliminate no-letters. If the rank of  $p$  is  $k$ , then  $p$  has exactly  $2^k$  subphrases.

EXAMPLE 4.1. Let  $p = ABA:B$ . Then the subphrases of  $p$  are  $ABA:B$ ,  $AA:\emptyset$ ,  $B:B$  and  $\emptyset:\emptyset$ .

For any nanophrases  $p$  and  $q$ , we define  $\delta(p, q)$  by

$$\delta(p, q) = \text{rank}(p) - \text{rank}(q).$$

PROPOSITION 4.2. For any  $n$ -component nanophrase  $p$  in  $\mathcal{P}(\alpha, \nu, n)$  and any invariant  $u : P(\alpha, \nu, n) \rightarrow G$ ,

$$(4.1) \quad \hat{u}(p) = \sum_{p' \triangleleft q \triangleleft p'} (-1)^{\delta(p', q)} u(q),$$

where  $p'$  is the non-singular phrase obtained by replacing all singular letters of  $p$  with the corresponding non-singular letters, and  $p''$  is the subphrase of  $p$  obtained from  $p$  by deleting all singular letters.

*Proof.* We use induction on the number  $k$  of singular letters. If  $p$  does not have singular letters, then the assertion is trivial. We assume that phrases with  $k$  singular letters satisfy (4.1). If  $p$  has  $k + 1$  singular letters, then we can write  $p = xA^*yA^*z$ , where  $x$ ,  $y$  and  $z$  are some singular phrases. By the definition of extension and the induction assumption, we have

$$\begin{aligned} \hat{u}(p) &= \hat{u}(xAyAz) - \hat{u}(xyz) \\ &= \sum_{x''Ay''Az'' \triangleleft q \triangleleft x' Ay' Az'} (-1)^{\delta(x' Ay' Az', q)} u(q) - \sum_{x''y''z'' \triangleleft q \triangleleft x' y' z'} (-1)^{\delta(x' y' z', q)} u(q) \\ &= \sum_{x''y''z'' \triangleleft q \triangleleft x' Ay' Az'} (-1)^{\delta(x' Ay' Az', q)} u(q), \end{aligned}$$

where  $x''$ ,  $y''$  and  $z''$  are the phrases obtained from  $x$ ,  $y$  and  $z$  by deleting all singular letters, and  $x'$ ,  $y'$  and  $z'$  are the non-singular phrases obtained by replacing all singular letters of  $x$ ,  $y$  and  $z$  with the corresponding non-singular letters, respectively. ■

A map  $u : P(\alpha, \nu, n) \rightarrow G$  is a *finite type invariant* if there exists a non-negative integer  $m$  such that for any  $n$ -component nanophrase  $p$  with more than  $m$  singular letters,  $\hat{u}(p)$  is zero. The minimal such  $m$  is called the *degree* of  $u$ .

The map  $u : P(\alpha, \nu, n) \rightarrow G$  is a *universal finite type invariant* of degree  $m$  if for any finite type invariant  $v$  of degree less than or equal to  $m$  taking values in some abelian group  $H$ , there exists a homomorphism  $f$  from  $G$  to  $H$  such that

$$\begin{array}{ccc} P(\alpha, \nu, n) & \xrightarrow{u} & G \\ & \searrow v & \downarrow f \\ & & H \end{array}$$

○

In particular, if  $p$  and  $q$  are two  $n$ -component nanophrases over  $\alpha$  which can be distinguished by a finite type invariant of degree less than or equal to  $m$  and  $u$  is a universal invariant of degree  $m$ , then  $u(p)$  is not equal to  $u(q)$ .

Goussarov, Polyak and Viro [GPV] constructed universal finite type invariants of virtual knots and links. Gibson and Ito [GI] constructed universal finite type invariants for homotopy classes of nanophrases. In a similar way, we construct universal finite type invariants for cyclic equivalence classes of nanophrases in the next section.

**5. Universal finite type invariants.** In this section, following the approach by Goussarov, Polyak and Viro [GPV] we construct universal finite type invariants for cyclic equivalence classes of nanophrases over general  $\alpha$ .

Let  $\mathbb{Z}P(\alpha, n)$  be the additive abelian group freely generated by isomorphism classes of  $n$ -component nanophrases. Let  $\mathbb{Z}P(\alpha, \nu, n)$  be the additive abelian group freely generated by  $P(\alpha, \nu, n)$ .

Let  $G(\alpha, \nu, n)$  be the group obtained from  $\mathbb{Z}P(\alpha, n)$  by taking the quotient by the following relations:

$$\begin{cases} w: AxAy:t - w:xB yB:t = 0, \\ w:Az:t - w:zA:t = 0, \end{cases}$$

where  $|B| = \nu(|A|)$ ,  $x, y$  and  $z$  do not contain ‘:’ and  $z$  does not contain  $A$ .

The projection from  $\mathbb{Z}P(\alpha, n)$  to  $\mathbb{Z}P(\alpha, \nu, n)$  is a homomorphism, denoted by  $I$ .

PROPOSITION 5.1. *The homomorphism  $I$  induces an isomorphism  $\hat{I} : G(\alpha, \nu, n) \rightarrow \mathbb{Z}P(\alpha, \nu, n)$ .*

*Proof.* It is clear that  $I$  is onto and the subgroup generated by the element  $w: AxAy:t - w:xB yB:t$  and  $w:Az:t - w:zA:t$  in  $\mathbb{Z}P(\alpha, n)$  is a subset of the kernel of  $I$ . Conversely, we show that the kernel of  $I$  is a subset of the subgroup in question. For any  $p$  in  $\mathbb{Z}P(\alpha, n)$ , there exists a finite subset  $X$  of  $P(\alpha, n)$  such that  $p = \sum_{x \in X} e_x x$ , where  $e_x$  is an integer. If  $p$  is contained in the kernel of  $I$ , we have

$$0 = I\left(\sum_{x \in X} e_x x\right) = \sum_{x \in X} e_x [x] = \sum_{x \in X'} \left(\sum_{x \sim y, y \in X} e_y\right) [x],$$

where  $X'$  is a set of representatives. The above equality is true if and only if for any  $x \in X'$ ,  $\sum_{x \sim y, y \in X} e_y = 0$ . Thus we have

$$\sum_{x \sim y, y \in X} e_y y = \sum_{x \sim y, y \in X} e_y y - \sum_{x \sim y, y \in X} e_y x = \sum_{x \sim y, y \in X} e_y (y - x).$$

Therefore  $\sum_{x \sim y, y \in X} e_y y$  is contained in the above subgroup and so is  $p$ . ■

This proposition means that the abelian group  $G$  is isomorphic to the free abelian group. We then define a homomorphism  $\theta_n : \mathbb{Z}P(\alpha, n) \rightarrow \mathbb{Z}P(\alpha, n)$  as follows. For any  $n$ -component nanophrase  $p$ ,  $\theta_n(p)$  is the sum of all the subphrases of  $p$  considered as an element of  $\mathbb{Z}P(\alpha, n)$ . We then extend  $\theta_n$  linearly to all of  $\mathbb{Z}P(\alpha, n)$ . Note that for any nanophrase  $p$ ,  $\theta_n(p)$  can be written as

$$\theta_n(p) = \sum_{q \triangleleft p} q.$$

We then define another homomorphism  $\varphi_n : \mathbb{Z}P(\alpha, n) \rightarrow \mathbb{Z}P(\alpha, n)$  as follows. For any  $n$ -component nanophrase  $p$ ,

$$\varphi_n(p) = \sum_{q \triangleleft p} (-1)^{\delta(p,q)} q.$$

We then extend  $\varphi_n$  linearly to all of  $\mathbb{Z}P(\alpha, n)$ .

PROPOSITION 5.2. *The homomorphism  $\theta_n$  is an isomorphism and its inverse is given by  $\varphi_n$ .*

*Proof.* For any  $n$ -component nanophrase  $p$ ,

$$(5.1) \quad \begin{aligned} \varphi_n(\theta_n(p)) &= \varphi_n\left(\sum_{q \triangleleft p} q\right) = \sum_{q \triangleleft p} \sum_{r \triangleleft q} (-1)^{\delta(q,r)} r = \sum_{r \triangleleft p} \left(\sum_{r \triangleleft q \triangleleft p} (-1)^{\delta(q,r)} r\right) \\ &= \sum_{r \triangleleft p} \left(\sum_{i=0}^{\delta(p,r)} \binom{\delta(p,r)}{i} (-1)^{\delta(p,r)-i} r\right) = p. \end{aligned}$$

In the first line we take a sum over  $q$  satisfying  $r \triangleleft q \triangleleft p$ . Here  $i$  is the difference between the rank of  $p$  and  $q$ .

Thus  $\varphi_n(\theta_n(p)) = p$  and so  $\varphi_n \circ \theta_n$  is the identity map. Similarly,

$$\theta_n(\varphi_n(p)) = \theta_n\left(\sum_{q \triangleleft p} (-1)^{\delta(p,q)} q\right) = \sum_{q \triangleleft p} (-1)^{\delta(p,q)} \left(\sum_{r \triangleleft q} r\right) = p.$$

Thus  $\theta_n \circ \varphi_n$  is also the identity map. Therefore  $\theta_n$  is an isomorphism and its inverse is given by  $\varphi_n$ . ■

PROPOSITION 5.3. *The map  $\theta_n$  induces an isomorphism  $\hat{\theta}_n : G(\alpha, \nu, n) \rightarrow G(\alpha, \nu, n)$ .*

*Proof.* For any  $w : Ax Ay : t - w : x By B : t$  in  $\mathbb{Z}P(\alpha, n)$ , we have

$$(5.2) \quad \begin{aligned} \theta_n(w : Ax Ay : t - w : x By B : t) &= \sum_{q \triangleleft Ax Ay} w : q : t - \sum_{q \triangleleft x By B} w : q : t \\ &= \left( \sum_{AA \triangleleft q \triangleleft Ax Ay} w : q : t + \sum_{q \triangleleft xy} w : q : t \right) \\ &\quad - \left( \sum_{BB \triangleleft q \triangleleft x By B} w : q : t + \sum_{q \triangleleft xy} w : q : t \right) \\ &= \sum_{AA \triangleleft q \triangleleft Ax Ay} w : q : t - \sum_{BB \triangleleft q \triangleleft x By B} w : q : t. \end{aligned}$$

The last line of (5.2) can be written as a finite sum of elements of the form  $w : Ax Ay : t - w : x By B : t$ . Similarly  $\theta_n(w : Az : t - w : z A : t)$  can be written as a finite sum of elements of the form  $w : Az : t - w : z A : t$ . Therefore  $\hat{\theta}_n$  is a homomorphism and so is  $\hat{\varphi}_n$ . Thus  $\hat{\theta}_n$  is an isomorphism. ■

For each non-negative integer  $m$ , we define a map  $O_m : P(\alpha, n) \rightarrow P(\alpha, n)$  by

$$(5.3) \quad O_m(p) = \begin{cases} p & \text{if } \text{rank}(p) \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

We extend  $O_m$  linearly to all of  $\mathbb{Z}P(\alpha, n)$ .

We introduce the following relation on  $G(\alpha, \nu, n)$ :

$$p = 0 \quad \text{if } p \text{ is an } n\text{-component nanophrase with } \text{rank}(p) > m.$$

Let  $G_m(\alpha, \nu, n)$  be the group obtained from  $G(\alpha, \nu, n)$  by taking the quotient with the above relation which depends on  $m$ . Then  $G_m(\alpha, \nu, n)$  is generated by the set of  $n$ -component nanophrases with rank  $m$  or less. As this set is finite,  $G_m(\alpha, \nu, n)$  is a finitely generated abelian group. Clearly  $O_m$  induces a homomorphism  $\hat{O}_m : G(\alpha, \nu, n) \rightarrow G_m(\alpha, \nu, n)$ .

Let  $\Gamma_{m,n}$  be the composition  $\hat{O}_m \circ \hat{\theta}_n \circ \hat{I}^{-1}$ . It is a homomorphism from  $\mathbb{Z}P(\alpha, \nu, n)$  to  $G_m(\alpha, \nu, n)$ .

The main theorem of this section is as follows.

**MAIN THEOREM 5.4.** *The map  $\Gamma_{m,n} : \mathbb{Z}P(\alpha, \nu, n) \rightarrow G_m(\alpha, \nu, n)$  is a universal finite type invariant of degree  $m$  for cyclic equivalence classes of nanophrases.*

**COROLLARY 5.5.** *For any finite type invariant for  $P(\alpha, \nu, n)$  taking values in an abelian group  $G$ , there is a finite subset  $H$  of  $G$  such that for any  $p$  in  $P(\alpha, \nu, n)$ ,  $v(p)$  is represented by a linear combination of elements of  $H$ .*

We prepare some lemmas for the proof of the theorem.

For any subphrases  $p_1$  and  $p_2$  of  $p$ , let  $p_1 \setminus_p p_2$  be the subphrase of  $p$  obtained from  $p_1$  by deleting the letters contained in  $p_2$ .

**EXAMPLE 5.6.** Let  $p = AADBCBCD$ ,  $p_1 = AABB$  and  $p_2 = BCBC$ . Then  $p_1 \setminus_p p_2 = AA$ .

**LEMMA 5.7.** *Let  $p$  be an  $n$ -component nanophrase. The sum of all the subphrases of  $p$  regarded as an element of  $\mathbb{Z}P(\alpha, n)$  can be written for any subphrase  $t$  of  $p$  as follows:*

$$(5.4) \quad \sum_{q \triangleleft p} q = \sum_{t \triangleleft r \triangleleft p} \left( \sum_{(r \setminus_p t) \triangleleft s \triangleleft r} s \right).$$

*Proof.* For any subphrase  $q$  of  $p$ , the right hand side of (5.4) contains  $q$ . In fact, letting  $r$  be  $t \cup_p q$ , we have  $t \triangleleft r \triangleleft p$  and  $(r \setminus_p t) \triangleleft q \triangleleft r$ . For the left hand side of (5.4), the number of terms is  $2^{\text{rank}(p)}$ . For the right hand side, the number of subphrases  $r$  such that  $t \triangleleft r \triangleleft p$  is  $2^{\delta(p,t)}$ . On the other hand, the number of subphrases  $s$  such that  $(r \setminus t) \triangleleft s \triangleleft r$  does not depend on  $r$  and is  $2^{\text{rank}(t)}$ . Therefore the number of terms on the right hand side of (5.4) is  $2^{\text{rank}(p)}$ . This yields lemma. ■

**LEMMA 5.8.** *Let  $p$  be an  $n$ -component nanophrase. For any subphrase  $t$  of  $p$ ,*

$$\sum_{t \triangleleft q \triangleleft p} (-1)^{\delta(p,q)} \left( \sum_{t \triangleleft r \triangleleft q} r \right) = p.$$

*Proof.* The proof is similar to that of Proposition 5.2, and we omit it. ■

*Proof of Theorem 5.4.* First of all, we prove that  $\Gamma_{m,n}$  is a finite type invariant of degree less than or equal to  $m$ . Let  $\hat{\Gamma}_{m,n} : \mathcal{P}(\alpha, \nu, n) \rightarrow G_m(\alpha, \nu, n)$  be an extension of  $\Gamma_{m,n}$ . Let  $p$  be an element in  $P_k(\alpha, \nu, n)$  where  $k > m$ . Let  $p'$  be the non-singular phrase obtained from  $p$  by replacing all singular letters of  $p$  with the corresponding non-singular ones, and  $p''$  the subphrase of  $p$  obtained from  $p$  by deleting all singular letters. Then by Proposition 4.2, the definition of  $\Gamma_{m,n}$  and Lemma 5.7, we have

$$\begin{aligned} \hat{\Gamma}_{m,n}(p) &= \Gamma_{m,n}\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} q\right) = \hat{O}_m \circ \hat{\theta} \circ \hat{I}^{-1}\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} q\right) \\ &= \hat{O}_m \circ \hat{\theta}\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} q\right) = \hat{O}_m\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} \left(\sum_{r \triangleleft q} r\right)\right) \\ &= \hat{O}_m\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} \left(\sum_{p'' \triangleleft s \triangleleft q} \sum_{(s \setminus_s p'') \triangleleft u \triangleleft s} u\right)\right). \end{aligned}$$

Now we define a map  $g : P(\alpha, \nu, n) \rightarrow \mathbb{Z}P(\alpha, \nu, n)$  as follows. We fix a nanophrase  $p''$  and then for any  $s$  in  $P(\alpha, \nu, n)$  such that  $p'' \triangleleft s$ , we set

$$g_{p''}(s) = \sum_{(s \setminus_s p'') \triangleleft u \triangleleft s} u.$$

We extend  $g$  linearly to all of  $\mathbb{Z}P(\alpha, \nu, n)$  and denote it by  $g$  again. Thus

$$(5.5) \quad \sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} \left(\sum_{p'' \triangleleft s \triangleleft q} \left(\sum_{(s \setminus_s p'') \triangleleft u \triangleleft s} u\right)\right) = \left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} \left(\sum_{p'' \triangleleft s \triangleleft q} g_{p''}(s)\right)\right).$$

Since  $g$  is linear, Lemma 5.8 implies that the right hand side of (5.5) is equal to

$$g_{p''}\left(\sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p',q)} \left(\sum_{p'' \triangleleft s \triangleleft q} s\right)\right) = g_{p''}(p') = \sum_{(p' \setminus_{p'} p'') \triangleleft u \triangleleft p'} u.$$

Since  $p' \setminus_{p'} p''$  is a subphrase of  $p'$  which has rank  $k (> m)$ , each phrase of the above term has rank more than  $m$ . Thus the image of this term under  $\hat{O}_m$  is zero.

Secondly, we prove that  $\Gamma_{m,n}$  is a finite type invariant of degree  $m$ . For any  $n$ -component nanophrase  $p$  without singular letters, let  $p^*$  be the phrase obtained from  $p$  by replacing all letters with the corresponding singular ones. We then have

$$\hat{\Gamma}_{m,n}(p^*) = \hat{O}_m\left(\sum_{p \triangleleft s \triangleleft p} s\right) = \hat{O}_m(p) = p.$$

Therefore  $\hat{\Gamma}_{m,n}(p^*)$  is not equal to zero, and so the degree of  $\Gamma_{m,n}$  is  $m$ .

Finally, we prove the universality as follows. For any finite type invariant  $v$  of degree less than or equal to  $m$  taking values in  $H$ , we will show that there exists a homomorphism  $f$  from  $G_m(\alpha, \nu, n)$  to  $H$  such that  $v = f \circ \Gamma_{m,n}$ . We extend  $v$  linearly to all of  $\mathbb{Z}P(\alpha, \nu, n)$  and denote it by  $v$  again. We define a map  $f$  as follows. For any  $n$ -component nanophrase  $p$ ,

$$f(p) = v \circ \hat{I} \circ \hat{\theta}_n^{-1}(q),$$

where  $q$  is in  $G(\alpha, \nu, n)$  and  $\hat{O}_m(q) = p$ . Note that  $q$  is a sum of phrases. Then  $f$  is well defined. In fact, for any  $q$  and  $q'$  such that

$$\hat{O}_m(q) = \hat{O}_m(q') = p,$$

$\hat{O}_m(q - q')$  is equal to zero in  $G_m(\alpha, \nu, n)$  and so  $q - q'$  consists of phrases with rank more than  $m$ . We then have

$$v(q^*) = v\left(\sum_{r \triangleleft q} (-1)^{\delta(q,r)} r\right) = v \circ I \circ \hat{\theta}_n^{-1}(q),$$

where  $q^*$  is the sum of phrases obtained from  $q$  by replacing all letters of each phrase of  $q$  with the corresponding singular ones. By the above and the fact that  $v$  is a finite type invariant of degree less than or equal to  $m$ ,

$$v \circ I \circ \hat{\theta}_n^{-1}(q - q') = v(q^* - q'^*) = 0,$$

where  $q'^*$  is constructed in the same way as  $q^*$ . Therefore  $f$  is well defined. We then obtain

$$v = f \circ \hat{O}_m \circ \hat{\theta}_n \circ \hat{I}^{-1} = f \circ \Gamma_{m,n}.$$

This yields the universality. ■

For any nanophrase  $w$ , we denote by  $[w]$  the sum of all isomorphism classes of the set of nanophrases which are obtained from  $w$  by shift moves. For example, consider the case when  $\alpha = \{\pm\}$  and  $\nu$  sends  $+$  to  $-$ . If  $w = XX\bar{Y}\bar{Y}$ , where  $X$  and  $\bar{Y}$  mean a letter  $X$  with  $|X| = +$  and  $Y$  with  $|Y| = -$ , respectively, then  $[w] = XX\bar{Y}\bar{Y} + YXXY + \bar{Y}\bar{Y}XX + \bar{X}\bar{Y}\bar{Y}\bar{X}$ . If  $w = XXY\bar{Y}$ , then  $[w] = XXY\bar{Y} + \bar{Y}XX\bar{Y}$ .

For two arbitrary nanophrases  $w$  and  $v$ , we define  $\langle \cdot, \cdot \rangle$  by

$$\langle w, v \rangle = \#\{\text{subwords of } v \text{ isomorphic to } w\}.$$

For example,  $\langle AA, BCBC \rangle = 2$ . We extend  $\langle \cdot, \cdot \rangle$  bilinearly, so that it is a map  $\mathbb{Z}P(\alpha, \nu, n) \times \mathbb{Z}P(\alpha, \nu, n) \rightarrow \mathbb{Z}$ .

LEMMA 5.9. *For any  $w$  in  $P(\alpha, n)$ ,  $\langle [w], \cdot \rangle : P(\alpha, \nu, n) \rightarrow \mathbb{Z}$  is a finite type invariant of degree  $\text{rank}(w)$ .*

*Proof.* It is easy to check that it is a homomorphism. Suppose that  $w$  is a nanophrase whose rank is  $m$ . Let  $p$  be an  $n$ -component nanophrase with

more than  $m$  singular letters. By Proposition 4.2,

$$(5.6) \quad \langle [w], p \rangle = \sum_{p'' \triangleleft q \triangleleft p'} (-1)^{\delta(p', q)} \langle [w], q \rangle,$$

where  $p'$  is the non-singular phrase obtained by replacing all singular letters of  $p$  with the corresponding non-singular ones, and  $p''$  is the subphrase of  $p'$  obtained from  $p$  by deleting all singular letters. For any subphrase  $r$  of  $p'$  with rank  $m$ , we consider the number of  $q$ 's such that  $r \triangleleft q$  and  $p'' \triangleleft q \triangleleft p'$ . Let  $k$  be the rank of the phrase  $p' \setminus_{p'} (p'' \cup r)$ . Then  $k \geq 1$ . The number of  $q$ 's satisfying  $r \triangleleft q$  and  $p'' \triangleleft q \triangleleft p'$  is

$$\sum_{s=0}^k \binom{k}{s} (-1)^s = 0.$$

So  $\langle [w], p \rangle = 0$ . Since  $\langle [w], w^* \rangle = \langle [w], w \rangle = 1$ , it is a finite type invariant of degree  $m$ . ■

Let  $\varphi_{m,n}$  denote a map

$$P(\alpha, \nu, n) \rightarrow \bigoplus_{v \in P(\alpha, \nu, n), \text{rank}(v) \leq m} \mathbb{Z}\langle v \rangle,$$

defined as follows:  $\varphi_{m,n}(w)$  is the direct sum of  $\langle [v], w \rangle$  for  $v$  in  $P(\alpha, \nu, n)$  whose rank is less than or equal to  $m$ , where  $\mathbb{Z}\langle v \rangle$  is an infinite cyclic group generated by  $v$ . We extend  $\varphi_{m,n}$  linearly to all of  $\mathbb{Z}P(\alpha, \nu, n)$ .

PROPOSITION 5.10. *There exists an isomorphism*

$$f : \bigoplus_{v \in P(\alpha, \nu, n), \text{rank}(v) \leq m} \mathbb{Z}\langle v \rangle \rightarrow G_m(\alpha, \nu, n)$$

such that  $f \circ \varphi_{m,n} = \Gamma_{m,n}$ .

*Proof.* We define  $f$  as follows. For each  $v$ ,  $f$  sends  $a$  in  $\mathbb{Z}\langle v \rangle$  to  $av$  in  $G_m(\alpha, \nu, n)$ . The proof is an easy check. ■

REMARK. Fujiwara [Fuj] showed that when  $\alpha = \{+, -\}$ , this finite type invariant is a complete invariant for cyclic equivalence classes of signed words, so a complete invariant for stable homeomorphism classes of curves on closed oriented surfaces.

REMARK. We compare  $\Gamma_{m,1}$  with Ito's  $\text{SCI}_m$ . Let  $c$  be a generic immersed curve with  $n$  crossings, and  $w_c$  a signed word corresponding to  $c$ . Ito [I1] defined a map  $\text{SCI}_m(c) : W_m \rightarrow k$ , where  $k$  is a field and  $W_m$  is a  $k$ -linear space generated by the isomorphism classes of signed words with rank  $m$ .  $\text{SCI}_m(c)$  has information about all subwords of  $w_c$  with rank  $m$ . Therefore, we suppose that  $\alpha$  is  $\{\pm\}$ ,  $\nu$  a map from  $+$  to  $-$ , that is, signed words. Then  $\Gamma_{m,1}(w_1) = \Gamma_{m,1}(w_2)$  if and only if  $\text{SCI}_i(c_1) = \text{SCI}_i(c_2)$  for any  $i$  between 0 and  $m$ , for any signed words  $w_1$  and  $w_2$  corresponding to generic immersed curves  $c_1$  and  $c_2$  respectively.

**6. Example.** Here is an example of finite type invariants.

EXAMPLE 6.1. The rank of a nanophrase over  $\alpha$  is a finite type invariant of degree 1.

EXAMPLE 6.2. Let  $\pi_{ij}$  ( $1 \leq i < j \leq n$ ) be a free abelian group generated by elements in  $\alpha$ . Let  $\pi_{ii}$  ( $1 \leq i \leq n$ ) be a free abelian group generated by elements in  $\alpha$  with the relations  $a = \nu(a)$  for all  $a$  in  $\alpha$ .

For an  $n$ -component nanophrase  $p$ , the *linking matrix* of  $p$  is defined as follows. Let  $\mathcal{A}_{ij}(p)$  ( $1 \leq i < j \leq n$ ) be the set of letters which have one occurrence in the  $i$ th component of  $p$  and another occurrence in the  $j$ th component of  $p$ . Let  $\mathcal{A}_{ii}(p)$  ( $1 \leq i \leq n$ ) be the set of letters which have two occurrences in the  $i$ th component of  $p$ . For any  $i$  and  $j$ , define

$$l_{ij}(p) = \sum_{A \in \mathcal{A}_{ij}(p)} |A| \in \pi_{ij}.$$

Let the *linking matrix*  $L(p)$  be the symmetric  $n \times n$  matrix with entries  $l_{ij}(p)$ . It is easy to see that the linking matrix of a nanophrase is an invariant for cyclic equivalence classes of nanophrases.

We define a map  $\iota : P(\alpha, \nu, n) \rightarrow \mathbb{Z}$  by setting  $\iota(p) = 1$  for any  $p$  in  $P(\alpha, \nu, n)$  and extend the map linearly.

THEOREM 6.3. *The linking matrix is a finite type invariant of degree 1, and a direct sum of the linking matrix and  $\iota$  is a universal finite type invariant of degree 1.*

*Proof.* We first prove that the linking matrix  $L(p)$  is a finite type invariant of degree 1. Let  $p_{A^*B^*}$  be a nanophrase with two singular letters  $A^*$  and  $B^*$ . Let  $p_{AB}$  be the nanophrase obtained from  $p$  by replacing  $A^*$  and  $B^*$  with  $A$  and  $B$ , respectively. Let  $p_A$  be the nanophrase obtained from  $p$  by replacing  $A^*$  with  $A$  and deleting  $B^*$ . Let  $p_B$  be the nanophrase obtained from  $p$  by replacing  $B^*$  with  $B$  and deleting  $A^*$ . Let  $p$  be the nanophrase obtained from  $p$  by deleting  $A^*$  and  $B^*$ . Then

$$L(p_{A^*B^*}) = L(p_{AB}) - L(p_A) - L(p_B) + L(p).$$

We show that  $L(p_{A^*B^*})$  is zero. It is sufficient to prove that  $l_{ij}(p_{A^*B^*})$  is zero. We first consider the case  $i = j$ . Suppose the letters  $A$  and  $B$  appear twice in the  $i$ th component. Then

$$\begin{aligned} l_{ii}(p_{AB}) &= l_{ii}(p) + |A| + |B|, \\ l_{ii}(p_A) &= l_{ii}(p) + |A|, \\ l_{ii}(p_B) &= l_{ii}(p) + |B|. \end{aligned}$$

Therefore  $l_{ii}(p_{A^*B^*})$  is zero.

Suppose the letter  $A$  appears twice in the  $i$ th component and the letter  $B$  does not appear twice in the  $i$ th component. Then

$$l_{ii}(p_{AB}) = l_{ii}(p_A) = l_{ii}(p) + |A|, \quad l_{ii}(p_B) = l_{ii}(p).$$

Hence  $l_{ii}(p_{A^*B^*})$  is zero.

Suppose neither  $A$  nor  $B$  appears twice in the  $i$ th component. Then

$$l_{ii}(p_{AB}) = l_{ii}(p_A) = l_{ii}(p_B) = l_{ii}(p).$$

Therefore  $l_{ij}(p_{A^*B^*})$  is zero.

The same conclusion can be obtained for  $i \neq j$ . Let  $q_*$  be a nanophrase with only one singular letter. If the  $i$ th and  $j$ th components contain a singular letter, then  $l_{ij}(q_*) \neq 0$ . Thus  $L$  is a finite type invariant of degree 1.

The proof is completed by showing universality. We show that

$$G_1(\alpha, \nu, n) \cong \bigoplus_{1 \leq i < j \leq n} \pi_{ij} \oplus \bigoplus_{1 \leq i \leq n} \pi_{ii} \oplus \mathbb{Z}.$$

We note that a nanophrase with rank 1 is of the form  $\emptyset : \dots : \emptyset : A : \emptyset : \dots : \emptyset : A : \emptyset : \dots : \emptyset$  or  $\emptyset : \dots : \emptyset : AA : \emptyset : \dots : \emptyset$ . Let  $g_{ija}$  ( $1 \leq i < j \leq n$ ) be the nanophrase of the form  $\emptyset : \dots : \emptyset : A : \emptyset : \dots : \emptyset : A : \emptyset : \dots : \emptyset$  where  $A$  appears in both the  $i$ th and  $j$ th components and  $|A| = a$  in  $\alpha$ , and  $g_{iia}$  ( $1 \leq i \leq n$ ) be the nanophrase of the form  $\emptyset : \dots : \emptyset : AA : \emptyset : \dots : \emptyset$  where  $A$  appears in the  $i$ th component and  $|A| = a$  in  $\alpha$ . Then the generators of  $G_1(\alpha, \nu, n)$  are  $g_{ija}$  ( $1 \leq i \leq j \leq n$ ) and  $\emptyset_n$ . Since the  $\nu$ -shift move sends  $g_{iia}$  to  $g_{iiv(a)}$  and  $g_{ija}$  to itself, the relations of  $G_1(\alpha, \nu, n)$  are  $g_{iia} = g_{iiv(a)}$  for any  $i$  and  $a$ . Thus  $G_1(\alpha, \nu, n)$  is isomorphic to the abelian group generated by  $g_{ija}$  ( $1 \leq i \leq j \leq n$ ) and  $\emptyset_n$  with the relations  $g_{iia} = g_{iiv(a)}$  for all  $a$  in  $\alpha$  and  $1 \leq i \leq n$ . This abelian group is isomorphic to

$$\bigoplus_{1 \leq i < j \leq n} \pi_{ij} \oplus \bigoplus_{1 \leq i \leq n} \pi_{ii} \oplus \mathbb{Z},$$

by a direct sum  $f$  of maps sending  $a$  in  $\pi_{ij}$  to  $g_{ija}$  for any  $i$  and  $j$ , and sending 1 in  $\mathbb{Z}$  to 1 in  $\mathbb{Z}\langle \emptyset_n \rangle$ , and so  $f \circ (L \oplus \iota) = \Gamma_{1,n}$ . ■

REMARK. This linking matrix differs from Fukunaga's [Fuk] on diagonal components. Fukunaga's linking matrix is a universal finite type invariant for homotopy classes of nanophrases (see [GI]). But his linking matrix is not a universal finite type invariant for cyclic equivalence classes of nanophrases.

**7. Finite type invariants for spherical curves.** From now on, we work only with the set  $P(\alpha, \nu, 1)$ , where  $\alpha = \{\pm\}$  and  $\nu$  is the map which sends  $+$  to  $-$ , that is, with signed words.

Let  $P(\alpha, 1)_s$  be a subset of  $P(\alpha, 1)$ , each of which has the corresponding generic curve on a sphere. Let  $P(\alpha, \nu, 1)_s$  be the union of the set of cyclic equivalence classes of  $P(\alpha, 1)_s$  and  $P(\alpha, 1) \setminus P(\alpha, 1)_s$ . Note that according

to our definition of finite type invariants, when we extend an invariant  $u$  to  $\hat{u}$ , we must extend the domain to  $P(\alpha, \nu, 1)_s$ . Therefore we consider not the set of cyclic equivalence classes of  $P(\alpha, 1)_s$  but  $P(\alpha, \nu, 1)_s$ .

Ito [I2] introduced regular homotopy moves on signed words. We review this definition.

DEFINITION 7.1 (Regular homotopy moves on signed words). Three *regular homotopy moves* on signed words are defined as follows.

A *first regular homotopy* is  $(\mathcal{A}, xyz) \rightarrow (\mathcal{A} \cup \{A, B\}, xA^\pm B^\mp yA^\pm B^\mp z)$ , where  $x, y$  and  $z$  are words.

A *second regular homotopy* is  $(\mathcal{A}, xyz) \rightarrow (\mathcal{A} \cup \{A, B\}, xA^\pm B^\mp yB^\mp A^\pm z)$ , where  $x$  and  $y$  are words.

A *third regular homotopy* is

$$(\mathcal{A}, xA^\pm B^\pm yA^\pm C^\pm zB^\pm C^\pm t) \rightarrow (\mathcal{A}, xB^\pm A^\pm yC^\pm A^\pm zC^\pm B^\pm t),$$

where  $x, y, z$  and  $t$  are words.

These moves correspond to moves of curves illustrated in Figure 7.1.

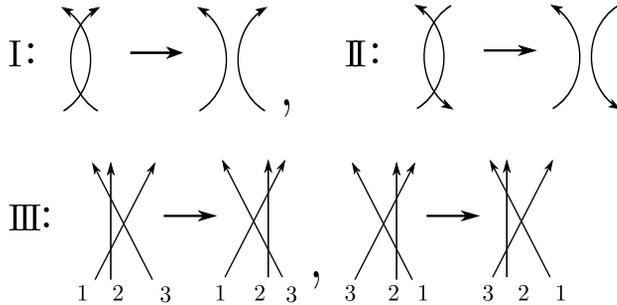


Fig. 7.1. Regular homotopy moves

Arnold's basic invariants  $J_s^+, J_s^-$  and  $St_s$  are defined by their behavior under the moves and the inverse moves in Figure 7.1. Fujiwara [Fuj] extended Arnold's basic invariants of spherical curves to invariants on signed words (see [P]). We review these definitions.

DEFINITION 7.2 (Arnold's basic invariants on signed words).  $J_s^+$  decreases (respectively increases) by 2 under the first regular homotopy move (respectively inverse move), and does not change under the other moves.

$J_s^-$  increases (respectively decreases) by 2 under the second regular homotopy move (respectively inverse move), and does not change under the other moves.

$St_s$  increases (respectively decreases) by 1 under the third regular homotopy move (respectively inverse move), and does not change under the other moves.

Normalized conditions for Arnold’s basic invariants are as follows. Let  $w_i$  be a signed word  $A_1A_1A_2A_2\dots A_iA_i$  (or  $\bar{A}_1\bar{A}_1\bar{A}_2\bar{A}_2\dots\bar{A}_i\bar{A}_i$ ). Then

$$J_s^+(w_i) = \frac{(i-1)^2}{2}, \quad J_s^-(w_i) = \frac{(i-2)^2}{2} - \frac{3}{2}, \quad St_s(w_i) = -\frac{(i-1)^2}{4}.$$

By using Polyak’s formulation of Arnold’s basic invariants [P], we extend Arnold’s basic invariants to maps  $P(\alpha, \nu, 1)_s \rightarrow \mathbb{Z}$  defined by

$$\begin{aligned} J_s^+(w) &= \langle\langle AABB - ABBA - 3ABAB, w \rangle\rangle - \frac{1}{2}([AA], w) + \frac{1}{2}, \\ J_s^-(w) &= \langle\langle AABB - ABBA - 3ABAB, w \rangle\rangle - \frac{3}{2}([AA], w) + \frac{1}{2}, \\ St(w) &= \frac{1}{2}\langle\langle -AABB + ABBA + ABAB, w \rangle\rangle + \frac{1}{4}\langle[AA], w\rangle - \frac{1}{4}, \end{aligned}$$

where  $\langle\langle \cdot, \cdot \rangle\rangle$  is defined as follows. For any Gauss word  $v$  and signed word  $w$ , we define

$$\langle\langle v, w \rangle\rangle = \sum_{w' \triangleleft w, w' \cong v} \prod_{A \text{ in } w', |A|=-1} (-1),$$

where  $\cong$  means isomorphic as Gauss words. We extend the map bilinearly.

**THEOREM 7.3.** *Arnold’s basic invariants are finite type invariants of degree 2, but they do not provide a universal finite type invariant of degree 2.*

*Proof.* It is easy to see that  $\langle\langle AABB - ABBA, \cdot \rangle\rangle$  is an invariant for  $P(\alpha, \nu, 1)_s$ . From [P] we see that  $\langle\langle ABAB, \cdot \rangle\rangle$  is also an invariant for  $P(\alpha, \nu, 1)_s$ . Thus  $J_s^+$ ,  $J_s^-$  and  $St$  are invariants for  $P(\alpha, \nu, 1)_s$ . Similar to Lemma 5.9, we have finite type invariants of degree 2.

We compare a signed word  $AA\bar{B}\bar{B}CC$  with  $\bar{A}\bar{A}\bar{B}\bar{B}CC$ .

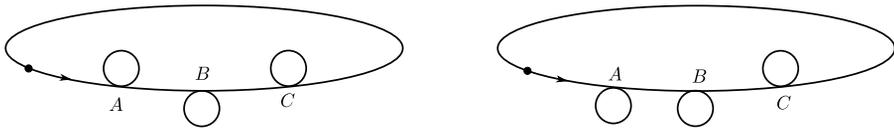


Fig. 4

$$\begin{aligned} J_s^+(AA\bar{B}\bar{B}CC) &= -2 = J_s^+(\bar{A}\bar{A}\bar{B}\bar{B}CC), \\ J_s^-(AA\bar{B}\bar{B}CC) &= -5 = J_s^-(\bar{A}\bar{A}\bar{B}\bar{B}CC), \\ St_s(AA\bar{B}\bar{B}CC) &= 1 = St_s(\bar{A}\bar{A}\bar{B}\bar{B}CC). \end{aligned}$$

On the other hand, because  $\varphi_{2,1}$  is an invariant for  $P(\alpha, \nu, 1)$ ,  $\varphi_{2,1}$  is also an invariant for  $P(\alpha, \nu, 1)_s$ , and we have

$$\varphi_{2,1}(AA\bar{B}\bar{B}CC) \neq \varphi_{2,1}(\bar{A}\bar{A}\bar{B}\bar{B}CC).$$

Therefore the triple  $J_s^+$ ,  $J_s^-$  and  $St$  does not provide the universal finite type invariant of degree 2. ■

**REMARK.** Since  $\langle[AA], \cdot\rangle$ ,  $\langle[AA\bar{B}\bar{B}], \cdot\rangle$ ,  $\langle[\bar{A}\bar{A}\bar{B}\bar{B}], \cdot\rangle$ ,  $\langle[ABAB], \cdot\rangle$ ,  $\langle[\emptyset], \cdot\rangle$  and  $\langle\langle ABAB, \cdot \rangle\rangle$  are independent, the linear space generated

by finite type invariants of degree at most 2 for spherical curves has dimension 7 or more.

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