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Torsion in Khovanov homology of semi-adequate links

by

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Abstract. The goal of this paper is to address A. Shumakovitch's conjecture about the existence of $\mathbb{Z}_2$-torsion in Khovanov link homology. We analyze torsion in Khovanov homology of semi-adequate links via chromatic cohomology for graphs, which provides a link between link homology and the well-developed theory of Hochschild homology. In particular, we obtain explicit formulae for torsion and prove that Khovanov homology of semi-adequate links contains $\mathbb{Z}_2$-torsion if the corresponding Tait-type graph has a cycle of length at least 3. Computations show that torsion of odd order exists but there is no general theory to support these observations. We conjecture that the existence of torsion is related to the braid index.

1. Introduction. In his visionary paper [Kh0] M. Khovanov revolutionized the theory of quantum knot invariants by categorifying the Jones polynomial of links. In 2003, A. Shumakovitch conjectured that any link which is not a connected or disjoint sum of Hopf links and trivial links has $\mathbb{Z}_2$-torsion in Khovanov homology [Sh1, Sh2].

In this paper we consider Khovanov bigraded homology (see Definition 2.8) of adequate and semi-adequate knots and links. Adequacy is a natural generalization of the alternating property suitable for studying Khovanov homology. Firstly, the outermost Khovanov homology group of +-adequate links is equal to $\mathbb{Z}$ [Kh0, Kh1], i.e., $H_{n,*}(D)=H_{n,n+2|D_{s_+}|}(D)=\mathbb{Z}$, where D is a +-adequate diagram of a link L with n crossings. Furthermore, M. Asaeda and J. Przytycki [AP] showed that the next nontrivial homology group $H_{n-2,n+2|D_{s_+}|-4}(D)$ has nontrivial $\mathbb{Z}_2$-torsion as long as the graph $G(D) = G_{s_+}(D)$ associated with the state s_+ is not bipartite (see Section 2.1 for definitions). An explicit formula for $H_{n-2,n+2|D_{s_+}|-4}(D)$ of a +-adequate link is derived in [PPS], showing, in particular, that for a

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nonsplit +-adequate diagram D,

$$\text{tor}\, H_{n-2,n+2|D_{s_+}|-4}(D) = \begin{cases} \mathbb{Z}_2 & \text{if } G(D) \text{ has an odd cycle,} \\ 0 & \text{if } G(D) \text{ is a bipartite graph.} \end{cases} \tag{1.1}$$

Torsion that lies in Khovanov homology one step deeper, $H_{n-4,n+2|D_{s_+}|-8}(D)$, is analyzed in [AP]. The authors show that for a strongly +-adequate diagram D with the graph $G(D)$ containing an even cycle, $H_{n-4,n+2|D_{s_+}|-8}(D)$ contains $\mathbb{Z}_2$-torsion. This statement implies Shumakovitch's result that any alternating link which is not a connected or disjoint sum of trivial links and Hopf links, has a nontrivial $\mathbb{Z}_2$-torsion in its Khovanov homology [Sh2].

In Section 4 we compute the entire $H_{n-4,n+2|D_{s_+}|-8}(D)$ for many classes of +-adequate diagrams, including strongly +-adequate diagrams. We prove that for a +-adequate diagram D,

$$\text{tor}\, H_{n-4,n+2|D_{s_+}|-8}(D) = \begin{cases} \mathbb{Z}_2^{p_1(G'(D))-1} & \text{if } G'(D) \text{ has an odd cycle,} \\ \mathbb{Z}_2^{p_1(G'(D))} & \text{if } G'(D) \text{ is a bipartite graph,} \end{cases}$$

where $G(D) = G_{s_+}(D)$ is the graph associated to the Kauffman state s_+, $G'(D)$ is a simple graph obtained from $G(D)$ by replacing multiple edges by singular edges (see Section 2), and $p_1(G)$ denotes the cyclomatic number of the graph G.

In Section 2 we provide an overview of relations between plane graphs and link diagrams, and the corresponding polynomial invariants: the Kauffman bracket polynomial and the Kauffman bracket version of the Tutte polynomial. Next, we outline the theory of Khovanov homology, categorification of the Kauffman bracket polynomial, and a related comultiplication-free version of homology of graphs (derived by L. Helme-Guizon and Y. Rong [HR] as a categorification of the chromatic polynomial).

In Section 3 we prove Main Lemma 3.1 computing homology $H_{1,v-2}(G)$, and derive important corollaries.

In Section 4 we modify the translation of Khovanov homology to graph homology by allowing one comultiplication. This allows us to compute the torsion of $H_{n-4,n+2|D_{s_+}|-8}(D)$ for any +-adequate diagram D.

In Section 5 we give examples of adequate diagrams in a braid form starting from the 3-braid $\sigma_1^3\sigma_2^3\sigma_1^2\sigma_2^2$ representing the knot 10_{152}.

Finally, in Section 6 we speculate about the existence of arbitrary torsion in Khovanov homology and its relations to the braid index.

2. Background. When developing our results for graph homology, we had in mind the application to Khovanov homology of links. This is also the reason why we modify the comultiplication-free version of Khovanov homology of graphs introduced in [HR] by allowing the "first" comultiplication (see Section 4). Thus we approximate Khovanov homology one step further but still have homology of graphs independent of a surface embedding.

In this section we provide the background material: the connection between graphs and links used in this paper. We also recall relations between graphs and link polynomials, and between Khovanov homology and its comultiplication-free version for graphs.

2.1. State graphs, state diagrams, and the Kauffman bracket polynomial. Tait was the first to notice the relation between knots and planar graphs [Ta, LT]. He colored the regions of a knot diagram alternately white and black (following Listing) and constructed the graph by placing a vertex inside each white region, and then connecting vertices by edges going through the crossing points of the diagram.

To generalize Tait's construction and associate to any Kauffman state a graph, we have to recall some preliminary definitions.

Definition 2.1. A *Kauffman state* s of D is a function from the set of crossings of D to the set $\{+1,-1\}$. Diagrammatically, we assign to each crossing of D a marker according to the convention of Figure 1:

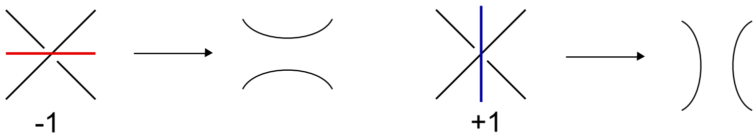


Fig. 1. Markers and associated smoothings

By D_s we denote the system of circles embedded in the plane obtained by smoothing all crossings of D according to the markers of the state s (for example see Figure 2(b)). Let $|D_s|$ denote the number of circles in the state D_s.

Definition 2.2 ([PPS]). Let D be a diagram of a link and s its Kauffman state. We form a graph $G_s(D)$, associated to D and s, as follows. Vertices of $G_s(D)$ correspond to circles of D_s. Edges of $G_s(D)$ are in bijection with crossings of D and an edge connects given vertices if the corresponding crossing connects circles of D_s corresponding to the vertices (see Figures 2, 4, 6, 7). As in the case of the Tait graph, $G_s(D)$ can be turned into a signed graph with the sign of an edge $e(p)$ associated with the crossing $p \in D$ equal to the sign of the marker of the Kauffman state s at that crossing p (notice that we will not be working with signed graphs in this paper).

The *Kauffman bracket polynomial* $\langle D\rangle_{(\mu,A,B)} \in \mathbb{Z}[\mu, A, B]$ of a diagram D is defined by:

(i) $\langle U_n\rangle = \mu^{n-1}$, where U_n is the trivial diagram of n components.
(ii) $\langle D_{\times}\rangle = A\langle D_{\asymp}\rangle + B\langle D_{)(}\rangle$.

From this we obtain the state sum formula:

$$\langle D\rangle_{(\mu,A,B)} = \sum_s A^{|s^{-1}(1)|}B^{|s^{-1}(-1)|}\mu^{|D_s|-1}.$$

In order to have invariance of the Kauffman bracket polynomial under regular isotopy (i.e. Reidemeister moves R_2 and R_3), we need $B = A^{-1}$ and $\mu = -A^2 - A^{-2}$ [Ka1, Ka2].

In this notation the Kauffman bracket polynomial of D is given by the state sum formula: $\langle D\rangle = \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{|D_s|-1}$, where $\sigma(s) = |s^{-1}(1)| - |s^{-1}(-1)| = \sum_p s(p)$ is the number of positive markers minus the number of negative markers in the state s.

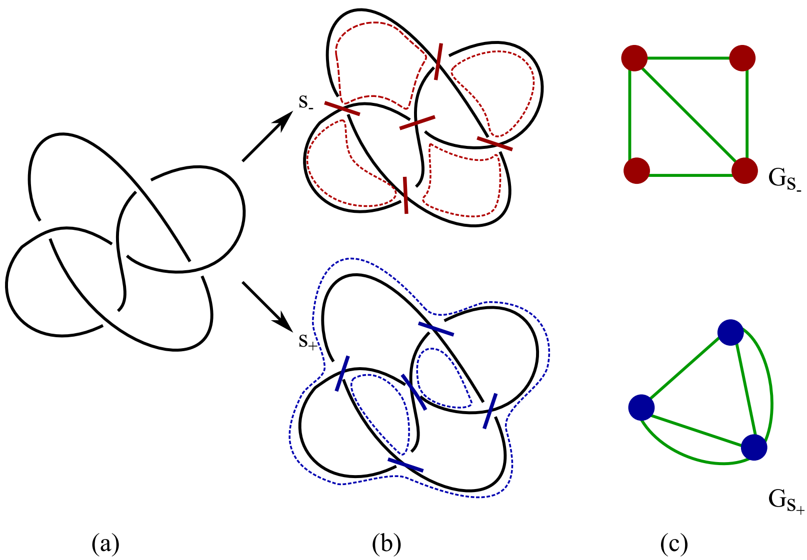


Fig. 2. (a) A minimal diagram of the Whitehead link; (b) D_{s_-} and D_{s_+}; (c) the corresponding graphs G_{s_-} and G_{s_+}.

The *unreduced Kauffman bracket* polynomial $[D]$ is defined as $[D] = (-A^2 - A^{-2})\langle D\rangle$, thus

$$[D] = \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{|D_s|}.$$

Before we move to the polynomial invariants of graphs, we describe classes of knots and links we will be analyzing in this paper, and their corresponding graphs.

Definition 2.3.

(i) In the language of graphs, a diagram D is *s-adequate* if the graph $G_s(D)$ has no loops. Similarly, D is *strongly s-adequate* if $G_s(D)$ has no loops and no multiple edges.

(ii) The *girth* $\ell(s)$ of a state s is the girth of the graph $G_s(D)$, i.e., the length of the shortest cycle in $G_s(D)$ (in case G is a forest, we define $\ell(s) = \infty$). Thus D is s-adequate iff $\ell(s) > 1$, and strongly s-adequate iff $\ell(s) > 2$.

(iii) The s_+ Kauffman state is a constant function sending all crossings to $+1$, and s_- to -1. We say that D is *+-adequate* if it is s_+-adequate, and that D is *−-adequate* if it is s_--adequate (1). Similarly, D is *strongly +-adequate* if it is strongly s_+-adequate, and D is *strongly −-adequate* if it is strongly s_--adequate.

Figure 2 shows a diagram of a Whitehead link $D = D_{\mathrm{Wh}}$, its s_- smoothing (and D_{s_-}), s_+ smoothing (and D_{s_+}), and their corresponding graphs $G(D) = G_{s_+}(D)$ and $G(\bar{D}) = G_{s_-}(D)$. Notice that D_{Wh} is strongly −-adequate. In general if $\bar{D}$ denotes the mirror image of D then $G_s(\bar{D}) = G_{-s}(D)$; in particular, $G_{s_+}(\bar{D}) = G_{s_-}(D)$.

2.2. The Kauffman bracket polynomial of graphs $[G]_{(\mu,A,B)}$. Applying the idea of Kauffman bracket polynomial of diagrams $[D]_{(\mu,A,B)}$ to graphs gives a version of the Tutte polynomial as explained below.

Definition 2.4. The *Kauffman bracket polynomial* $[G] = [G]_{(\mu,A,B)}$ of the graph G ($[G] \in \mathbb{Z}[\mu, A, B]$) is defined inductively by the following formulas (2):

(i) $[U_n] = \mu^n$, where U_n is the discrete graph on n vertices.

(ii) $[G] = A[G - e] + B[G//e]$ where $G//e = G/e$ if e is not a loop, and if e is a loop, then $G//e$ is defined to be the graph obtained from $G - e$ by adding an isolated vertex.

The Kauffman bracket satisfies the following state sum formula (see e.g. [PP], [Pr1, Chapter V]).

Lemma 2.5. *Let G be a graph with $V(G)$ the set of vertices and $E(G)$ the set of edges. Let $s \subseteq E$ denote an arbitrary set of edges of G, including the empty set, and $G - s$ the graph obtained from G by removing all edges contained in s. Let $p_0(G)$ be the number of connected components of G, and*

(1) We follow [AP, HPR, PPS] in our notation. In particular, if D is an alternating diagram then $G(D)$ is a signed Tait graph of D with all negative edges. However, we do not use signed graphs in this paper so our convention should not lead to confusion. In this paper, generally $G(D) = G_{s_+}(D)$, and a +-adequate diagram has an s_+-adequate state. Our choice of convention is dictated by the fact that we want a +-smoothing of the crossing in the diagram to correspond to the case when the edge is absent in the graph case; compare Section 4.5 in [HPR].

(2) Notice that $\langle D\rangle_{(\mu,A,B)}$ from Definition V.1.3 in [Pr1] is related to $[G]_{(\mu,A,B)}$ by $[G]_{(\mu,A,B)} = \mu\langle G\rangle_{(\mu,B,A)}$. The Tait graph $G(D)$ from [Pr1] is our $G_{s_-}(D)$ graph.

$p_1(G) = \mathrm{rank}(H_1(G,\mathbb{Z})) = |E| - v + p_0$ *the cyclomatic number of* G. *Then*

$$[G] = \sum_{s \in 2^{E(G)}} \mu^{p_0([G:s])+p_1([G:s])} A^{|E(G)\setminus s|} B^{|s|}.$$

The following formula expresses the relation between the Kauffman bracket and Tutte polynomial $\chi(G;x,y)$ of a graph G.

PROPOSITION 2.6 (see e.g. [PP], [Pr1, Chapter V]). *The following identity holds:*

$$[G]_{(\mu,A,B)} = \mu^{p_0(G)} A^{p_1(G)} B^{|E(G)|-p_1(G)} \chi(G;x,y),$$

where $x = (B+\mu A)/B$ *and* $y = (A+\mu B)/A$.

2.3. Khovanov homology via enhanced states chain complex. A convenient way of defining Khovanov homology, as noticed by O. Viro [Vi1, Vi2], is to consider enhanced Kauffman states.

DEFINITION 2.7. An *enhanced Kauffman state* S of an unoriented framed link diagram D is a Kauffman state s with an additional assignment of $+$ or $-$ sign to each circle of D_s.

Enhanced states can be used to express the Kauffman bracket polynomial as a sum of monomials:

$$[D] = (-A^2 - A^{-2})\langle D\rangle = \sum_S (-1)^{\sigma(S)} A^{\sigma(s)+2\tau(S)}, \tag{2.1}$$

where $\tau(S)$ is the number of positive circles minus the number of negative circles in the enhanced state S (notice that $\tau(S) \equiv |D_s| \bmod 2$).

DEFINITION 2.8 (Khovanov link homology). Let $\mathcal{S}(D)$ denote the set of enhanced Kauffman states of a diagram D, and let $\mathcal{S}_{i,j}(D)$ denote the set of enhanced Kauffman states S such that $\sigma(S) = i$ and $\sigma(S) + 2\tau(S) = j$. We call i a *homology grading* and j a *Kauffman bracket grading.*

(i) The *Khovanov chain group* $\mathcal{C}(D)$ (resp. $\mathcal{C}_{i,j}(D)$) is the free abelian group freely generated by $\mathcal{S}(D)$ (resp. $\mathcal{S}_{i,j}(D)$). Hence, $\mathcal{C}(D) = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{C}_{i,j}(D)$ is a bigraded free abelian group.

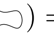
(ii) For a link diagram D with ordered crossings, we define the chain complex $(\mathcal{C}(D), d)$ with a differential $d = \{d_{i,j}\}$ determined by maps $d_{i,j} : \mathcal{C}_{i,j}(D) \to \mathcal{C}_{i-2,j}(D)$ such that $d_{i,j}(S) = \sum_{S'} (-1)^{t(S:S')}[S : S']S'$ where $S \in \mathcal{S}_{i,j}(D)$, $S' \in \mathcal{S}_{i-2,j}(D)$, and $[S : S']$ equals 0 or 1; $[S : S'] = 1$ if and only if the markers of S and S' differ exactly at one crossing, call it v, and all the circles of D_S and $D_{S'}$ not touching v have the same sign (3). Furthermore, $t(S : S')$ is the number of

(3) From our conditions it follows that at the crossing v the marker of S is positive, the marker of S' is negative, and $\tau(S') = \tau(S) + 1$.

negative markers assigned to crossings in S bigger than v in the chosen ordering.

(iii) The *Khovanov homology* of the diagram D is defined to be $H_{i,j}(D) = \ker(d_{i,j})/d_{i+2,j}(\mathcal{C}_{i+2,j}(D))$, the homology of the chain complex $(\mathcal{C}(D), d)$. The *Khovanov cohomology* of the diagram D is the homology of the dual complex.

In Khovanov's original approach every circle of a Kauffman state was decorated by a free 2-dimensional module $\mathcal{A}$ over $\mathbb{Z}$ (with basis $\mathbf{1}$ and x) with an additional structure of a Frobenius algebra $A = \mathbb{Z}[x]/(x^2)$ [Kh0, Kh1, Kh2]. According to the notation in [Vi1], we use $-$ and $+$ in place of $\mathbf{1}$ and x. On the level of algebra the differential is given by either multiplication or comultiplication, depending on whether the number of circles in the state is greater or less than that of its image.

Khovanov proved that link homology is a topological invariant [Kh0]. For the first Reidemeister move R_1 where R_{+1}() = () and R_{-1}() = () we have $H_{i+1,j+3}(R_{+1}(D)) = H_{i,j}(D) = H_{i-1,j-3}(R_{-1}(D))$. $H_{i,j}(D)$ is preserved by the second and third Reidemeister moves.

With the notation introduced before, we can write the formula for the Kauffman bracket polynomial of a link diagram in the following form:

$$[D] = \sum_j A^j \Big(\sum_i (-1)^{(j-i)/2} \sum_{S \in S_{i,j}} 1\Big) = \sum_j A^j \Big(\sum_i (-1)^{(j-i)/2} \dim C_{i,j}\Big) = \sum_j A^j \chi(C_{*,j}),$$

where

$$\chi(C_{*,j}) = \sum_{i:\, j \equiv i \,(\mathrm{mod}\, 2)} (-1)^{(j-i)/2} \dim C_{i,j}$$

is a slightly adjusted Euler characteristic of the chain complex $C_{*,j}$ for a fixed j. This explains that Khovanov homology categorifies the Kauffman bracket polynomial, as well as the Jones polynomial (4).

2.4. Khovanov-type functor on the category of graphs. The chromatic graph cohomology was introduced in [HR], as a comultiplication-free version of the Khovanov cohomology of alternating links, where alternating link diagrams are translated to plane graphs (Tait graphs). Moreover,

(4) In the narrow sense, a *categorification* of a numerical or polynomial invariant is a homology theory whose Euler characteristic or polynomial Euler characteristic (the generating function of Euler characteristics) is equal to the invariant we have started with. We quote M. Khovanov [Kh0]: "A speculative question now comes to mind: quantum invariants of knots and 3-manifolds tend to have good integrality properties. What if these invariants can be interpreted as Euler characteristics of some homology theories of 3-manifolds?".

this homology theory is a categorification of the chromatic polynomial of a graph.

The chromatic polynomial of a graph keeps track of the number of its proper vertex colorings using no more than a given number of colors, so that adjacent vertices have different colors. The analogy with the Khovanov homology construction is almost complete: instead of Kauffman states we use subgraphs $[G : s]$, containing all vertices in G and i edges from $s \subseteq E$. Analogously to labeling the circles in the enhanced Kauffman states by pluses and minuses, we define the enhanced graph states as connected components of a graph $[G : s]$ labeled by either 1 or x, the generators of the algebra $\mathcal{A} = \mathbb{Z}[x]/(x^2 = 0)$. The number $|D_s|$ of circles in the Kauffman state corresponds to the number $k(s)$ of connected components of the graph $[G : s]$ containing all vertices in G and i edges in $s \subseteq E$. Now, we consider the following state sum formula for the chromatic polynomial:

$$\begin{aligned}\chi_G(\lambda) &= \sum_{i\geq 0}(-1)^i \sum_{s\subseteq E,\,|s|=i} \lambda^{k(s)}\\ &= \sum_{i,j\geq 0}(-1)^i\lambda^j \sharp\{s \subseteq E \mid |s| = i,\, k'([G : s]) = j\},\end{aligned}$$

where $k'([G : s])$ denotes the number of components of $[G : s]$ labeled by x. Cochain groups are spanned by all subgraphs $[G : s]$, with each of $k([G : s])$ components labeled by either 1 or x, with exactly $k'([G : s]) = j$ components labeled by x.

Definition 2.9. Define the *chromatic cochain complex* and *chromatic cohomology* of a graph G over the commutative algebra $\mathcal{A} = \mathbb{Z}[x]/(x^2 = 0)$ in the following way:

(i) The cochain group is

$$C^i(G) = \bigoplus_{\substack{|s|=i\\ s\subset E(G)}} C^i_s(G),$$

with $C^i_s(G) = \mathcal{A}^{k(s)}$ where $k(s)$ denotes the number of components of the subgraph $[G : s]$. Assume that the edges of G are ordered ([5]). For a given state s and the edge $e \in E\backslash s$, let $t(s, e)$ equal the number of edges in s that are less than e in the chosen ordering. The cochain map $d^i : C^i(G) \to C^{i+1}(G)$ is a sum

$$d^i = \sum_{e\notin s}(-1)^{t(s,e)} d^i_e,$$

([5]) The chromatic graph cohomology is independent of the ordering of edges, however, the ordering is required to define the boundary map.

where the map d_e^i depends on whether e connects different components of $[G:s]$ or it connects vertices in the same component of $[G:s]$. In the latter case we assume d_e^i to be the identity [HR] (6). If e connects different components of $[G:s]$, say ith and jth, $i<j$, then $d_e^i(a_1,\dots,a_{k(s)-1})=(a_1,a_2,\dots,a_ia_j,\dots,a_{j-1},a_{j+1},\dots,a_{k(s)-1})$.

(ii) We define the chromatic cohomology, denoted by $H^*(G)$, as the cohomology of the chromatic cochain complex above.

Because $\mathcal{A}$ is graded, with $\deg(1)=0$, $\deg(x)=1$, one can consider the bigraded homology $H^{i,j}(G)$ [HR, HPR]. The chromatic graph cohomology of a graph with a loop is always zero [HR].

In this setting it is easier to work with the chain complex, similar to the classical homology theories. Therefore we perform concrete calculations in the chromatic graph homology setting and then use the universal coefficient theorem (see Proposition 2.10) to express the results in the chromatic graph cohomology setting.

Proposition 2.10. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated then*

$$H^n(C;\mathbb{Z})=H_n(C;\mathbb{Z})/\mathrm{tor}(H_n(C;\mathbb{Z}))\oplus\mathrm{tor}(H_{n-1}(C;\mathbb{Z})).$$

In particular, we have the following identities:

(i) $H^{0,v-1}(G)=H_{0,v-1}(G)/\mathrm{tor}(H_{0,v-1}(G))$,
(ii) $H^{1,v-1}(G)=H_{1,v-1}(G)\oplus\mathrm{tor}(H_{0,v-1}(G))$.

Additionally for $\ell(G)\geq 2$ we have $C_{2,v-1}(G)=0$, and $H_{1,v-1}(G)$ is the free abelian group $\ker(C_{1,v-1}(G)\to C_{0,v-1}(G))$.

In this particular bigrading the chromatic graph cohomology of a graph $G=(V(G),E(G))$ over the algebra $\mathcal{A}_2$ is equivalent to the homology with chain groups defined as in the standard graph homology and the boundary map (7) defined by $\partial(e)=\partial(\overrightarrow{V_1V_2})=V_1+V_2$ where $e=(V_1,V_2)\in E$. As a corollary we get the following lemma from [PPS]:

Proposition 2.11. *Given a connected simple graph G and the algebra $\mathcal{A}$ we have*

$$H_{0,v-1}(G)=\begin{cases}\mathbb{Z} & \text{if } G \text{ is a bipartite graph,}\\ \mathbb{Z}_2 & \text{if } G \text{ has an odd cycle.}\end{cases}\tag{2.2}$$

(6) Alternatively, d_e^i can be defined to be a zero map in this case, but this makes no difference for our purposes.

(7) An intriguing observation is that the standard graph boundary $\partial(\overrightarrow{V_1V_2})=V_2-V_1$, gives, in our range of bigradings, the odd Khovanov homology of Ozsváth, Rasmussen, and Szabó [ORS]; this is worthy of further consideration.

Consider the category of finite graphs in which they are objects, and $\mathrm{Mor}(G', G)$ are graph embeddings between G' and G which are bijections on vertices. To every graph G we associate its chain complex $\{C_{i,j}(G)\}$, and any morphism $\alpha : G' \to G$ induces a chain map $\alpha_\# : \{C_{i,j}(G')\} \to \{C_{i,j}(G)\}$. We obtain in this way a functor from the category of finite graphs to the category of graded chain complexes, and further to the category of bigraded groups $\{H_{i,j}(G)\}$. In a standard way we consider a morphism α of G' in G and related short exact sequence of chain complexes

$$0 \to C_{i,j}(G') \to C_{i,j}(G) \to C_{i,j}(G, G') \to 0,$$

where $C_{i,j}(G, G') = C_{i,j}(G)/C_{i,j}(G')$. Finally, we obtain the related long exact homology sequence:

$$\begin{aligned} (2.3) \qquad & \cdots \to H_{i,j}(G') \to H_{i,j}(G) \to H_{i,j}(G, G') \to H_{i-1,j}(G') \to \cdots \\ & \cdots \to H_{1,j}(G) \to H_{1,j}(G, G') \to H_{0,j}(G') \to H_{0,j}(G) \to H_{0,j}(G, G') \to 0. \end{aligned}$$

We write $\mathbb{Z}[i]\{j\}$ for $\mathbb{Z}$ with homological grading i and chromatic grading j.

Proposition 2.12. *Let T be a spanning tree of a connected graph G. Then*

(i) $H_{*,*}(T) = H_{0,v-1}(T) \oplus H_{0,v}(T) = \mathbb{Z}[0]\{v-1\} \oplus \mathbb{Z}[0]\{v\}$.
(ii) *$H_{i,j}(G)$ is supported on two diagonals: $H_{i,j}(G) = 0$ for $i + j \neq v, v-1$, and the torsion is trivial except possibly for $i + j = v - 1$: $\operatorname{tor} H_{i,j}(G) = 0$ for $i + j \neq v - 1$.*
(iii) *$H_{i,j}(G) = H_{i,j}(G, T)$ if $i > 1$ or $i = 1$ and $j \neq v - 1$. In particular, $H_{1,v-2}(G) = H_{1,v-2}(G, T)$.*

Proof. Let $G_1 * G_2$ denote the one-vertex product of graphs, and let K_n denote the complete graph on n vertices.

(i) Adding an edge K_1 to a graph G results in $H_{i,j}(G * K_1) = H_{i,j+1}(G)$ (see [HR]).

(ii) Part (ii) reflects the fact that Khovanov homology of alternating links lies on two adjacent diagonals [Lee]. The proof uses the long exact homology sequence with smoothings in a link case and deleting–contracting in the graph case (see [HR, AP]).

(iii) The third part follows from (i) and (ii) by applying the long exact sequence of the pair (G, T). ∎

2.5. Correspondence between Khovanov and chromatic graph homology. Based on [HPR, Pr2] we state a relation between the graph cohomology and classical Khovanov homology of alternating links (in Viro's [Vi1] notation). Proposition 2.13 is generalized in Proposition 4.7.

Proposition 2.13. *Let D be a diagram of an unoriented framed alternating link* ([8]), *and let $G = G_{s_+}(D)$. For all $i < \ell(G) - 1$, we have*

$$H^{i,j}(G) \cong H_{a,b}(D),$$

where $a = |E(G)| - 2i$, $b = |E(G)| - 2|V(G)| + 4j$ and $H_{a,b}(D)$ are the Khovanov homology groups of the unoriented framed link defined by D, as explained in Definition 2.7 *based on* [Vi1].

Furthermore, $\operatorname{tor} H^{i,j}(G) = \operatorname{tor} H_{a,b}(D)$ *for $i = \ell(G) - 1$.*

Theorem 2.14 ([PPS]). *Let G be a simple graph. Then*

(i) $H^{0,v-1}(G) = \mathbb{Z}^{p_0^{\mathrm{bi}}}$, *where p_0^{bi} is the number of bipartite components of G.*

(ii) $H^{1,v-1}(G) = \mathbb{Z}^{p_1-(p_0-p_0^{\mathrm{bi}})} \oplus \mathbb{Z}_2^{p_0-p_0^{\mathrm{bi}}}$, *where p_0 is the number of components of G and $p_1 = \operatorname{rank}(H_1(G,\mathbb{Z})) = |E| - v + p_0$ is the cyclomatic number of G.*

3. The Main Lemma and chromatic graph homology $H_{1,v-2}$. Next we compute $H_{1,v-2}(G)$ for any connected graph G, hence $H^{2,v-2}(G)$ for any graph G and, eventually, $H_{n-4,n+|D_{s_+}|-8}(D)$ for the corresponding +-adequate link diagram.

Lemma 3.1 (Main Lemma). *If G is a connected simple graph, i.e., a graph of girth $\ell(G) \geq 3$, then:*

(i) $H_{1,v-2}(G) = \mathbb{Z}_2^{p_1(G)}$ *if G is bipartite.*

(ii) $H_{1,v-2}(G) = \mathbb{Z}_2^{p_1(G)-1} \oplus \mathbb{Z}$ *if G has an odd cycle.*

Proof. Since $H_{1,v-2}(G) = H_{1,v-2}(G,T)$ for any spanning tree T of G, by Proposition 2.12, we focus on computing $H_{1,v-2}(G,T)$. We assume that both edges and vertices are ordered, although the results do not depend on this. To make the proof more comprehensible we introduce the following notation. Let $\rho(v,w)$ denote the distance between vertices $v, w \in V(T)$, equal to the length of the shortest path connecting them in T. If $(\partial_0(e_i), \partial_1(e_i))$ denotes the endpoints of the edge e_i in G, we use the short notation $\rho(e_i) = \rho(\partial_0(e_i), \partial_1(e_i))$. In particular, for $e_i \notin T$, $\rho(e_i)$ is odd if e closes an even cycle in $T \cup e_i$, and $\rho(e_i)$ is even if e closes an odd cycle in $T \cup e_i$. For $e_i, e_j \notin T$ we also use $\rho(e_i, e_j)$ to denote the distance between e_i and e_j in $T \cup e_i \cup e_j$, or equivalently, the minimal distance between endpoints of e_i and endpoints of e_j in T.

Let $e_1, \ldots, e_{p_1}$ be the edges in $E(G \backslash T)$ where $p_1(G) = |E(G)| - |E(T)| = |E(G)| - |V(G)| + 1$.

([8]) In the case of unoriented alternating links, $G_{s_-}(D)$ and $G_{s_+}(D)$ are Tait graphs, i.e. obtained from the checkerboard coloring of the projection plane.

The chain group $C_{1,v-2}(G,T)$ is freely generated by enhanced states (e_i, v_j) where the component of the graph $[G : e_i]$ containing the vertex v_j has label 1 (all other labels are x). If the vertex v_j is the endpoint of e_i, we use the short notation $(e_i, 1)$ for an enhanced state $(e_i, \partial_0(e_i)) = (e_i, \partial_1(e_i))$.

Notice that $H_{0,v-2}(G,T) = 0 = C_{0,v-2}(G,T)$ since $C_{0,v-2}(G) = C_{0,v-2}(T)$. Therefore, $\ker(d : C_{1,v-2}(G,T) \to C_{0,v-2}(G,T)) = C_{1,v-2}(G,T)$, so

$$H_{1,v-2}(G,T) = C_{1,v-2}(G,T)/d(C_{2,v-2}(G,T)).$$

Since $\ell(G) \geq 3$ the chain group $C_{2,v-2}(G,T)$ has two types of free generators (enhanced states):

(i) pairs (e_i, e), where $e \in E(T)$, generating the subgroup of $C_{2,v-2}(G,T)$ denoted by C', and

(ii) pairs (e_i, e_j) generating the subgroup C''.

Let us first compute $C_{1,v-2}(G,T)/d(C')$. For any edge $e \in T$,

$$d(e_i, e) = \pm((e_i, \partial_1(e)) + (e_i, \partial_0(e)))$$

yields the following relation in homology: $(e_i, \partial_1(e)) = -(e_i, \partial_0(e))$. Hence, we eliminate all generators of $C_{1,v-2}(G,T)$ except pairs $(e_i, \partial_0(e_i))$, satisfying the relations

$$(e_i, \partial_1(e_i)) = (-1)^{\rho(\partial_0(e_i), \partial_1(e_i))}(e_i, \partial_0(e_i)), \quad \text{that is,}$$
$$(e_i, 1) = (-1)^{\rho(e_i)}(e_i, 1).$$

Thus $C_{1,v-2}(G,T)/d(C') = \mathbb{Z}_2^{k_{\text{odd}}} \oplus \mathbb{Z}^{p_1 - k_{\text{odd}}}$, where k_{odd} is the number of edges e_i with $\rho(e_i)$ odd.

Next, we compute $(C_{1,v-2}(G,T)/d(C'))/d(C'')$. For an enhanced state (e_i, e_j) we have

$$(3.1) \quad d(e_i, e_j) = \pm\big((e_i, \partial_0(e_j)) + (e_i, \partial_1(e_j)) - (e_j, \partial_0(e_i)) - (e_j, \partial_1(e_i))\big).$$

The relation in $C_{1,v-2}(G,T)/d(C')$ corresponding to (3.1) can be written as

$$(3.2) \qquad (e_i, 1)(1 + (-1)^{\rho(e_j)}) = \varepsilon\big((e_j, 1)(1 + (-1)^{\rho(e_i)})\big),$$

where $\varepsilon = \pm 1$, or more precisely $\varepsilon = (-1)^{\rho(\partial_0(e_i), \partial_0(e_j))}$. We analyze this relation in more detail, based on the types of enhanced states generating C''. Depending on the parity of $\rho(e_i, e_j)$, we consider three different types of generators of C'':

(i) (e_i, e_j) such that both $\rho(e_i)$ and $\rho(e_j)$ are odd generate the subgroup C''_{odd},

(ii) (e_i, e_j) where exactly one of $\rho(e_i)$ and $\rho(e_j)$ is odd generate the subgroup C''_{mixed},

(iii) (e_i, e_j) such that both $\rho(e_i)$ and $\rho(e_j)$ are even generate the subgroup C''_{even}.

In the case of C''_{odd}, both sides of (3.2) are zero, so there are no new relations in $C_{1,v-2}(G,T)/d(C')$. If a graph G is bipartite, $\rho(e_i)$ is always odd and $C_{2,v-2}(G,T)$ is generated by C' and C''_{odd}, so (i) of the Main Lemma is proven.

In the second case, (3.2) reduces to $2(e_i,1)=0$, which already holds in $C_{1,v-2}(G,T)/d(C')$.

Finally, consider the third case when $2(e_i,1)=\varepsilon 2(e_j,1)$, or more precisely,

$$2((e_i,1)-(-1)^{\rho(e_i,e_j)}(e_j,1))=0. \tag{3.3}$$

To conclude the proof of (ii), let $e_1,\dots,e_k$ be edges of $G\setminus T$ with odd $\rho(e_i)$, and $e_{k+1},\dots,e_{p_1}$ the remaining edges, with $\rho(e_i)$ even. The graph H obtained by adding $e_1,\dots,e_k$ to the tree T is a bipartite graph, so

$$H_{1,v-2}(H,T)=C_{1,v-2}(H,T)/d(C')=\{\{(e_i,1)\}_{i=1}^k \mid 2(e_i,1)=0\}=\mathbb{Z}_2^k,$$

and $H_{1,v-2}(G,T)=H_{1,v-2}(H,T)/C''_{even}$. Observe now that for e_i,e_j in $E(G)\setminus E(H)$ the relation (3.3) follows from

$$2((e_i,1)-(-1)^{\rho(e_i,e_{k+1})}(e_{k+1},1))=0,$$
$$2((e_j,1)-(-1)^{\rho(e_j,e_{k+1})}(e_{k+1},1))=0,$$

since $\rho(e_i,e_j)\equiv\rho(e_i,e_{k+1})+\rho(e_j,e_{k+1}) \pmod 2$. Hence, $H_{1,v-2}(G,T)$ is generated by

$$\big(e_1,\dots,e_{k+1},e_{k+2}-(-1)^{\rho(e_{k+2},e_{k+1})}(e_{k+1},1),\dots,$$
$$e_{p_1}-(-1)^{\rho(e_{p_1},e_{k+1})}(e_{k+1},1)\big),$$

where e_{k+1} is an infinite cyclic element and all other generators have order 2. The proof of the Main Lemma is complete. ∎

As a corollary we get the following main result.

Main Theorem 3.2. *If G is a connected simple graph containing t_3 triangles and v vertices, and having cyclomatic number p_1, then:*

(i) $H_{1,v-2}(G)=\begin{cases}\mathbb{Z}_2^{p_1(G)} & \textit{if } G \textit{ is bipartite},\\ \mathbb{Z}_2^{p_1(G)-1}\oplus\mathbb{Z} & \textit{if } G \textit{ has an odd cycle}.\end{cases}$

(ii) $H_{2,v-2}(G)=\begin{cases}\mathbb{Z}^{\binom{p_1}{2}-t_3} & \textit{if } G \textit{ is bipartite},\\ \mathbb{Z}^{\binom{p_1}{2}-t_3+1} & \textit{if } G \textit{ has an odd cycle}.\end{cases}$

(iii) $H^{1,v-2}(G)=\begin{cases}0 & \textit{if } G \textit{ is bipartite},\\ \mathbb{Z} & \textit{if } G \textit{ has an odd cycle}.\end{cases}$

(iv) $H^{2,v-2}(G)=\begin{cases}\mathbb{Z}_2^{p_1}\oplus\mathbb{Z}^{\binom{p_1}{2}-t_3} & \textit{if } G \textit{ is bipartite},\\ \mathbb{Z}_2^{p_1-1}\oplus\mathbb{Z}^{\binom{p_1}{2}+1-t_3} & \textit{if } G \textit{ has an odd cycle}.\end{cases}$

Proof. (i) Follows from the Main Lemma.

(ii) Using the Euler characteristic of chromatic graph cohomology in degree $j = v-2$ we get $\operatorname{rank} H_{2,v-2}(G) - \operatorname{rank} H_{1,v-2}(G) = a_{v-2}$ where a_{v-2} denotes the coefficient of q^{v-2} in the chromatic polynomial ([9]), given by

$$a_{v-2} = \binom{|E|}{2} - t_3 - |E|(v-1) + \binom{v}{2} = \binom{p_1}{2} - t_3. \tag{3.4}$$

(iii)–(iv) Parts (iii) and (iv) follow from (i) and (ii) by applying the universal coefficient theorem: $H^{2,v-2}(G) = \mathrm{free}(H_{2,v-2}(G)) \oplus \mathrm{tor}\, H_{1,v-2}(G)$, and $H^{1,v-2}(G) = \mathrm{free}(H_{1,v-2}(G))$. ∎

The restriction to connected graphs was made only for simplicity. The Künneth formula is sufficient for recovering homology of the graph from the homology of the connected components (compare [Ha, HR]). In fact, when computing the homology of a disjoint sum of graphs, $H^{**}(G_1 \sqcup G)$, we can sometimes ignore the tor part of the formula.

COROLLARY 3.3. *Let G, G_1 and G_2 denote arbitrary graphs, G^{bi} all bipartite components of G, and $G^{\mathrm{nbi}} = G - G^{\mathrm{bi}}$ the remaining components of the graph G. Then*

(i) $H^{i,v-i}(G_1 \sqcup G_2) = \bigoplus_{\substack{p+q=i\\ s+t=j}} H^{p,s}(G_1) \otimes H^{q,t}(G_2).$

(ii) *If G^{bi} and G^{nbi} are simple graphs then:*

$$H^{2,v-2}(G^{\mathrm{bi}}) = \mathbb{Z}_2^{p_1(G^{\mathrm{bi}})} \oplus \mathbb{Z}^{\binom{p_1(G^{\mathrm{bi}})}{2}}, \tag{3.5}$$

$$H^{2,v-2}(G^{\mathrm{nbi}}) = \mathbb{Z}_2^{p_0(G^{\mathrm{nbi}})p_1(G^{\mathrm{nbi}}) - \binom{p_0(G^{\mathrm{nbi}})+1}{2}} \oplus \mathbb{Z}^{\alpha}, \tag{3.6}$$

where $\alpha = \binom{p_1(G^{\mathrm{nbi}})+1}{2} - p_0(G^{\mathrm{nbi}})p_1(G^{\mathrm{nbi}}) + \binom{p_0(G^{\mathrm{nbi}})+1}{2} - t_3(G^{\mathrm{nbi}})$.

(iii) *If G is a simple graph then*

$$\mathrm{tor}\, H^{2,v-2}(G) = \mathbb{Z}_2^{p_1(G^{\mathrm{bi}}) + p_0(G^{\mathrm{nbi}})p_1(G) - \binom{p_0(G^{\mathrm{nbi}})+1}{2}}. \tag{3.7}$$

([9]) To put our calculation in a general combinatorial context we note that we have the following identity which we use here only for $i = 2$ and in full generality in a sequel paper:

$$\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}\lambda^{v-i} = \lambda^{v-|E|}(\lambda-1)^{|E|} \overset{\lambda=q+1}{=} (q+1)^{v-|E|}q^{|E|} = q^v(1+q^{-1})^{-(|E|-v)}$$
$$= \sum_{i=0}^{\infty}(-1)^i\binom{(|E|-v+1)+i-2}{i}q^{v-i} = \sum_{i=0}^{\infty}(-1)^i\binom{p_1+i-2}{i}q^{v-i}.$$

(iv) *If G is a simple graph then*

$$\operatorname{rank} H^{2,v-2}(G) = \binom{p_1(G)+1}{2} - \dim \operatorname{tor} H^{2,v-2}(G) - t_3(G).$$

Proof. (i) The Künneth formula yields the following formula for chromatic graph cohomology over $\mathcal{A}$:

$$\begin{aligned} (3.8)\qquad & H^{i,j}(G_1 \sqcup G_2) \\ & = \Big(\bigoplus_{\substack{p+q=i\\ s+t=j}} H^{p,s}(G_1) \otimes H^{q,t}(G_2)\Big) \oplus \Big(\bigoplus_{\substack{p+q=i+1\\ s+t=j}} H^{p,s}(G_1) *_{\mathrm{Tor}} H^{q,t}(G_2)\Big), \end{aligned}$$

thus it suffices to show that

$$\bigoplus_{\substack{p+q=i+1\\ s+t=j}} H^{p,s}(G_1) *_{\mathrm{Tor}} H^{q,t}(G_2) = 0$$

in bidegrees (i,j) satisfying $i+j = v(G_1 \sqcup G_2)$.

If G is a connected graph, then

- homology is supported in bidegrees (i,j) satisfying $v(G)-1 \le i+j \le v(G)$,
- torsion is supported in bidegrees (i,j) such that $i+j = v(G)$.

By induction on the number of components and using the Künneth formula we get a well known fact (cf. [AP, HPR]) that for an arbitrary graph G:

- homology is supported in bidegrees (i,j) such that $v(G) - p_0(G) \le i+j \le v(G)$,
- torsion is supported in bidegrees (i,j) such that $v(G) - p_0(G) + 1 \le i+j \le v(G)$.

Based on the second inequality and the Künneth formula we are interested only in bidegrees satisfying $p+q+r+s = v(G_1 \sqcup G_2)+1 = v(G_1)+v(G_2)+1$. However, this implies that either $p+q \ge v(G_1)$ or $s+t \ge v(G_2)$, which contradicts the previous observation. Hence,

$$\bigoplus_{\substack{p+q=i+1\\ s+t=j}} H^{p,s}(G_1) *_{\mathrm{Tor}} H^{q,t}(G_2)$$

is trivial.

(ii) According to part (i) we have

$$\begin{aligned} H^{2,v(G_1\sqcup G_2)-2}(G_1 \sqcup G_2) &= H^{2,v(G_1)-2}(G_1) \oplus H^{2,v(G_2)-2}(G_2) \\ &\quad \oplus (H^{1,v(G_1)-1}(G_1) \otimes H^{1,v(G_2)-1}(G_2)). \end{aligned}$$

We apply this formula inductively, using Theorems 3.2(iv) and 2.14, to obtain formulas (3.5) and (3.7). An intermediate step is computation of

$H^{2,v-2}(G^{\rm bi})$ assuming that $G^{\rm bi} = G_1^{\rm bi} \sqcup \cdots \sqcup G_{p_0(G^{\rm bi})}^{\rm bi}$:

$$H^{2,v-2}(G^{\rm bi}) = \mathbb{Z}_2^{p_1(G^{\rm bi})} \oplus \mathbb{Z}^{\binom{p_1(G^{\rm bi})}{2}},$$

using the identity

$$\binom{p_1(G_1)}{2} + \cdots + \binom{p_1(G_{p_0(G)})}{2} + \sum_{i<j} p_1(G_i)p_1(G_j) = \binom{p_1(G)}{2}.$$

Similarly, assuming that $G^{\rm nbi} = G_1^{\rm nbi} \sqcup \cdots \sqcup G_{p_0(G^{\rm nbi})}^{\rm nbi}$, we have

$$H^{2,v-2}(G^{\rm nbi}) = \mathbb{Z}_2^{p_0(G^{\rm nbi})p_1(G^{\rm nbi}) + \binom{p_0(G^{\rm nbi})+1}{2}} \oplus \mathbb{Z}^{\alpha}$$

where $\alpha = \binom{p_1(G^{\rm nbi})+1}{2} - (p_0(G^{\rm nbi}) - 1)p_1(G^{\rm nbi}) - \binom{p_0(G^{\rm nbi})+1}{2} - t_3(G^{\rm nbi})$.

(iii)–(iv) Part (iii) follows from (ii), and for part (iv) notice that for any simple graph G,

$$\text{rank free}(H^{2,v(G)-2}(G)) + \dim \text{tor}(H^{2,v(G)-2}(G)) = \binom{p_1(G)+1}{2} - t_3(G). \ \blacksquare$$

Corollary 3.4. *Let K_n denote the complete graph with $n \geq 3$ vertices. Then*

$$H^{2,v-2}(K_n) = \mathbb{Z}_2^{n(n-3)/2} \oplus \mathbb{Z}^{3\binom{n}{4}+1-\binom{n}{3}}.$$

Corollary 3.5. *Let W_n denote the wheel graph with $n \geq 4$ vertices, i.e. the cone over an $(n-1)$-gon. Then*

$$H^{2,v-2}(W_n) = \mathbb{Z}_2^{n-2} \oplus \mathbb{Z}^{\binom{n-1}{2}-n+1}.$$

4. Torsion in Khovanov homology of semi-adequate links. In order to make further use of the correspondence between Khovanov and chromatic graph cohomology described in Subsection 2.5 and [AP, HPR, Pr2, PPS], we adjust the original definition by incorporating comultiplication in the differential. This modification extends the correspondence between Khovanov homology and chromatic graph cohomology to additional homological grading. In particular, this definition enables computing torsion in Khovanov homology in bidegree $(n-4, n+2|D_{s_+}|-8)$.

First, the chain complex is adjusted so that it can accommodate comultiplication. The original cochain groups contain a copy of the algebra for each connected component in the graph $[G : s]$ (see Definition 2.9). The cochain groups ${}^{\Delta}C^i(G)$ stay the same for $i < \ell(G)$, and trivial for $i > \ell(G)$. The modified cochain groups ${}^{\Delta}C^i(G)$ will contain the tensor product $\mathcal{A} \otimes \mathcal{A}$ instead of a single copy of $\mathcal{A}$ for each state containing a closed cycle. Pictorially, the component containing a closed cycle is decorated by basis elements of the tensor product $\mathcal{A} \otimes \mathcal{A}$ (see Figure 3). This description is formalized in the following definition.

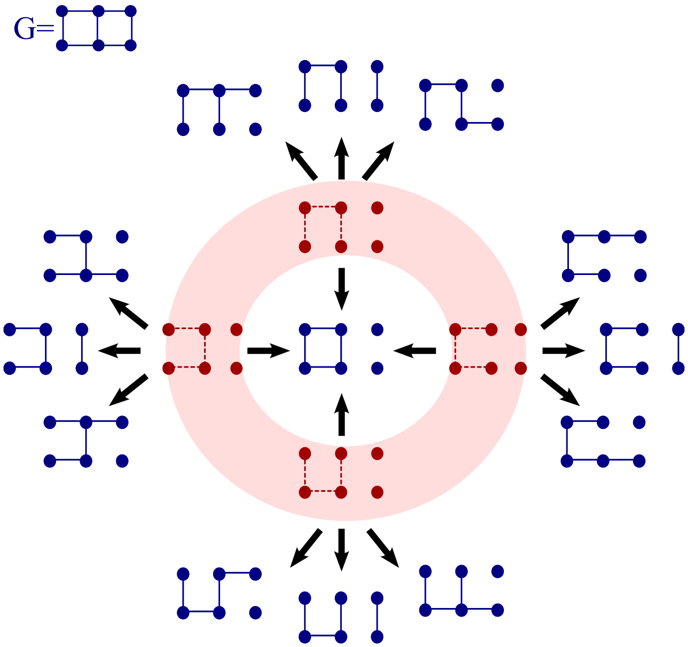


Fig. 3. The graph G and the generators of ${}^{\triangle}C_3(G)$ and ${}^{\triangle}C_4(G)$. The four generators of ${}^{\triangle}C_3(G)$ appear in the annular region; there are 13 generators of ${}^{\triangle}C_4(G)$: twelve graphs in the exterior region and one in the center of the figure.

Definition 4.1. For a given graph G of girth l, let ${}^{\triangle}C^{i,*}(G)$ denote the modified chromatic cochain groups defined in the following way:

(i) ${}^{\triangle}C^{i,*}(G) \cong C^{i,*}(G)$ for $i < l$,
(ii) ${}^{\triangle}C^{i,*}(G) \cong \bigoplus_{|s|=i} \mathcal{A}^{\otimes(p_0([G:s])+p_1([G:s]))}$ for $i = l$,
(iii) ${}^{\triangle}C^{i,*}(G) = 0$ for $i > l$.

Next, we modify the differential in the case when adding an edge for the first time does not change the number of connected components, i.e. when the added edge closes one of the shortest cycles. Let ${}^{\triangle}d_{s,e}$ denote the modified differential. If $p_1([G : s]) = p_1([G : s \cup e]) = 0$, the differential stays the same, ${}^{\triangle}d_{s,e} = d_{s,e}$.

If the edge e we are adding is an internal edge of $[G : s]$ (i.e. $1 = p_1([G : s \cup e]) = p_1([G : s]) + 1$), the differential is determined by comultiplication in $\mathcal{A}$, given by $\triangle(1) = (1 \otimes x) + (x \otimes 1)$ and $\triangle(x) = x \otimes x$.

We have all the necessary ingredients to define the new differential.

Definition 4.2. The differential map ${}^{\triangle}d^i(G) : {}^{\triangle}C^i(G) \to {}^{\triangle}C^{i+1}(G)$ is defined by

$$ {}^{\triangle}d^i[G : s] = \sum_{e \in E(G) \setminus s} (-1)^{t(s,e)} d_e([G : s]), $$

where $[G:s] \in {}^{\Delta}C^i(G)$ and $t(s,e) = |\{e' \in s \mid e' < e\}|$ for all $i < l = \ell(G)$. Let $c_1, \ldots, c_k$ denote the components of the state $[G:s]$. The definition of the map d_e varies depending on whether the edge e connects two different components of $[G:s]$, say c_m and c_n with $m < n$, or closes a shortest cycle (this can happen only in degree $i = \ell(G) - 1$):

(i) If $|s| < l - 1$, then $d_e([G:s])$ has one component less than $[G:s]$, say

$$c_1, \ldots, c_m \cup e \cup c_n, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k.$$

The label of the newly obtained component $c_m \cup e \cup c_n$ is equal to the product of the labels of the components being merged, c_m and c_n. In other words, d_e is given by multiplication in the algebra.

(ii) If $|s| = l - 1$, then

- if the number of components of s is greater than that of $s \cup e$, then d_e is the same as in case (i).
- if the number of components is preserved then $d_e([G:s]) = (c_1, \ldots, c_m \cup e, \ldots)$ and the closed component $c_m \cup e$ is decorated with $\Delta(c_m)$.

(iii) If $|s| \geq l$ (e.g. $p_1([G:s]) \geq 1$), then d_e is a zero map.

In order to have a degree-preserving differential we adjust the definition of degrees of basis elements of $\mathcal{A} \otimes \mathcal{A}$ obtained from comultiplication, according to the convention from Table 1. In general, the degree would be lowered by the cyclomatic number $p_1(G)$, but since we are closing the shortest cycle the adjustment is only by 1.

Table 1. Degrees of basis elements in $\mathcal{A} \otimes \mathcal{A}$ coming from comultiplication

Basis element	Degree
$1 \otimes 1$	-1
$1 \otimes x$, $x \otimes 1$	0
$x \otimes x$	1

The cohomology ${}^{\Delta}H^{*,*}(G)$ of the modified bigraded cochain complex ${}^{\Delta}C(G)$ is also an invariant of all graphs.

Next, we analyze the differences between the modified chromatic graph cohomology and the original one. In general, the homology of these two complexes agrees in homological degrees less than the girth of the graph $\ell(G)$.

Lemma 4.3. *For a loopless graph G, with v vertices and girth $\ell(G) = l$, $C^i(G) \cong {}^{\Delta}C^i(G)$ for $0 \leq i < l$. Moreover, there exists an injective map $\alpha : C^l(G) \to {}^{\Delta}C^l(G)$, so the homology groups are isomorphic up to homological level $l - 1$.*

We are mostly interested in the bidegree $(\ell(G), v-\ell(G))$, in particular, $(2, v-2)$. The change in the definition preserves $H^{2,v-2}(G)$ for loopless graphs even if multiple edges are allowed. The proof of this fact relies on duality between homology and cohomology, and on the following lemma.

LEMMA 4.4. *For a loopless graph G with possible multiple edges,*

$$H_{1,v-2}(G) \cong {}^{\Delta}H_{1,v-2}(G).$$

Proof. According to the original and modified definitions of chromatic graph homology, both chain groups and differentials agree on the zeroth and first level. Hence, we only need to analyze ${}^{\Delta}d_2$ and ${}^{\Delta}H_{2,v-2}(G)$ if G has double or multiple edges. Under this assumption ${}^{\Delta}C_{2,v-2}(G)$ has more generators than ${}^{\Delta}C_{2,v-2}(G')$, where G' denotes the simple graph obtained from G. Without loss of generality, denote the double edge by $e = (e_1, e_2)$. Based on the label of e we have two different cases:

(i) If e has weight $x \otimes x$ of degree 1, then all but one of the remaining vertices have labels x. Denote the special vertex by v and the state by $(e(x \otimes x), v(1))$. The image of this state ${}^{\Delta}d_2(e, v) = (e_1(x), v(1)) \pm (e_2(x), v(1))$ gives the relation $(e_1(x), v(1)) = (e_2(x), v(1))$, so there are no new generators in homology.
(ii) If e is labeled by $1 \otimes x$ or $x \otimes 1$, both of degree zero, all of the remaining vertices have to be labeled by x and ${}^{\Delta}d_2(e(1 \otimes x)) = {}^{\Delta}d_2(e(x \otimes 1)) = e_1(1) - e_2(1)$.

Therefore $\operatorname{Im} {}^{\Delta}d_2$ and $\operatorname{Im} d_2$ impose the same relations on homology, which completes the proof. ∎

COROLLARY 4.5. *For a loopless graph G with v vertices,* $\operatorname{tor} H^{2,v-2}(G) \cong \operatorname{tor} {}^{\Delta}H^{2,v-2}(G)$.

PROPOSITION 4.6. *For a connected graph G with girth $\ell(G) = 1$ we have* ${}^{\Delta}H_{0,v-1}(G) = 0$ *and* $\operatorname{tor} {}^{\Delta}H^{1,v-1}(G) = 0$.

Proof. If e_ℓ denotes a loop in G at the vertex v, notice that there is an epimorphism

$${}^{\Delta}d_1 = d_1 : {}^{\Delta}C_{1,v-1}(G) \to {}^{\Delta}C_{0,v-1}(G)$$

sending each generator of ${}^{\Delta}C_{1,v-1}(G)$ containing e_ℓ to a generator of ${}^{\Delta}C_{0,v-1}(G)$ with a label 1 or x at the vertex v, if e_ℓ had weight $1 \otimes x$, $x \otimes 1$, or $x \otimes x$. Hence ${}^{\Delta}H_{0,v-1}(G) = 0$ and ${}^{\Delta}H^{1,v-1}(G)$ is torsion free. ∎

Finally, we have a version of Proposition 2.13 for the modified chromatic homology. It holds because ${}^{\Delta}C^{*,*}(G)$ imitates the original Khovanov homology one homological degree deeper.

Proposition 4.7. *Let D be the diagram of an unoriented framed link L whose associated graph $G = G_{s_+}(D)$ has girth $l = \ell(G) > 1$, i.e., contains no loops. Then:*

(i) *For all $i < \ell$, we have ${}^{\Delta}H^{i,j}(G) \cong H_{a,b}(D)$,*
(ii) *For $i = \ell$, we have $\operatorname{tor} {}^{\Delta}H^{i,j}(G) = \operatorname{tor} H_{a,b}(D)$,*

where $a = E(G) - 2i$, $b = E(G) - 2v(G) + 4j$ and $H_{a,b}(D)$ are the Khovanov homology groups of the unoriented framed link L defined by D.

We use this result together with Corollary 3.3 to compute the torsion in Khovanov link homology.

Proposition 4.8. *Consider a $+$-adequate diagram D with n crossings. Let $G = G_{s_+}(D)$ be the graph corresponding to the diagram D and state s_+, and $G' = G'_{s_+}(D)$ be the simple graph obtained from G by replacing every multiple edge by a single one. Let $p_0(G^{\mathrm{bi}})$ denote the number of bipartite components, $p_0(G^{\mathrm{nbi}})$ the number of nonbipartite components, $p_0(G)$ the number of connected components, and $p_1(G')$ the cyclomatic number. Then*

(i) *If $G'(D)$ is connected then*

$$\operatorname{tor} H_{n-4,n+2|D_{s_+}|-8}(D) = \begin{cases} \mathbb{Z}_2^{p_1(G'(D))-1} & \textit{if } G'(D) \textit{ has an odd cycle;} \\ \mathbb{Z}_2^{p_1(G'(D))} & \textit{if } G'(D) \textit{ is a bipartite graph.} \end{cases}$$

(ii) *If we allow any $+$-adequate link diagram D (that is, $G(D)$ is not necessarily connected) then by applying Lemma 3.3 we get*

$$\operatorname{tor} H_{n-4,n+2|D_{s_+}|-8}(D) = \mathbb{Z}_2^{p_1(G'^{\mathrm{bi}}) + p_0(G^{\mathrm{nbi}})p_1(G') - \binom{p_0(G^{\mathrm{nbi}})+1}{2}}.$$

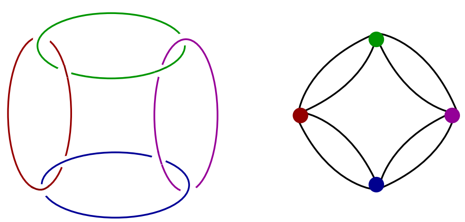


Fig. 4. The link 8^4_1 and the corresponding graph $G_{s_+}(8^4_1)$

Example 4.9. The following example illustrates the strength of Proposition 4.8 with respect to the previous results. Consider the link 8^4_1, shown in Figure 4 together with its graph $G(8^4_1) = G_{s_+}(8^4_1)$. The torsion $\operatorname{tor} H_{4,8}(8^4_1) = \mathbb{Z}_2$ in Khovanov homology of this link could not be detected by results of [AP], but can be obtained from Theorem 3.2(4) together with Proposition 4.8(1). More importantly, it answers the question raised in [AP], whether Theorem 3.2 of [AP] can be improved so that $H_{n-4,n+2|D_{s_+}|-8}(D)$ has $\mathbb{Z}_2$-torsion for any +-adequate diagram with an even n-cycle ($n \geq 4$).

Our next goal is to find an explicit formula for Khovanov homology $H_{n-2i,n+2|D_{s_+}|-4i}(D)$, $i < \ell(G(D))$, and tor $H_{n-2\ell(G(D)),n+2|D_{s_+}|-4\ell(G(D))}(D)$. We plan to use our method for computing elements of a categorification of skein modules of a product of a surface with the interval as defined in [APS].

5. Adequate positive braids. Results obtained in Section 3 can be used for finding torsion in Khovanov homology, in particular we find 2-torsion for some positive 3-braids.

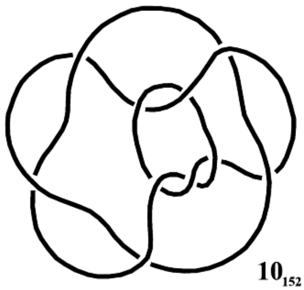


Fig. 5. The smallest nonalternating adequate knot 10_{152}

Notice that the smallest adequate nonalternating knot 10_{152} in Rolfsen's table [Ro] corresponds to the positive minimal braid is $s_1^3 s_2^2 s_1^2 s_2^3$ (see Figure 5). The graph assigned to the Kauffman state with all negative resolutions s_- has only multiple edges, so our method cannot detect torsion (see Figure 6).

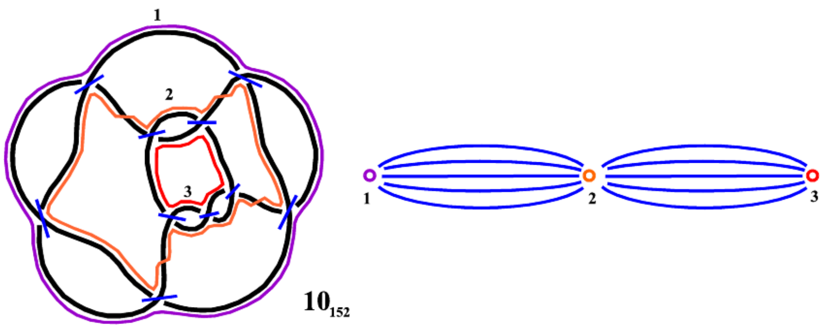


Fig. 6. s_- Kauffman state of 10_{152} and the corresponding graph

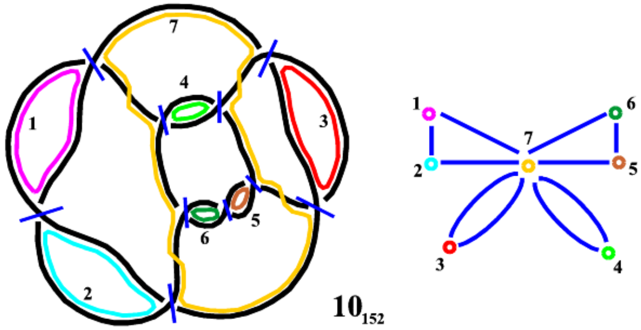


Fig. 7. s_+ Kauffman state of 10_{152} and the corresponding graph

On the other hand, the graph corresponding to the state s_+ with all positive smoothings, contains triangles [PPS], hence, $H_{8,16}(10_{152})$ contains $\mathbb{Z}_2$ (see Figure 7). More precisely,

$$\operatorname{tor} H_{8,16}(10_{152}) = \mathbb{Z}_2 = \operatorname{tor} H_{6,12}(10_{152}).$$

This example can be generalized to positive and negative 3-braids ([10]). In Proposition 5.1 we state the result for positive braids; the result for negative braids is analogous.

PROPOSITION 5.1. *Let* $\gamma = \sigma_{i_1}^{a_1} \dots \sigma_{i_k}^{a_k}$ *be a positive* 3*-braid such that* $i_j \neq i_{j+1}$, $a_i \geq 1$, *and let* $\widehat{\gamma}$ *be a closure of* γ. *Then*

(i) *The link diagram* $\widehat{\gamma}$ *is adequate if and only if* $a_j \geq 2$ *for every* $0 < j \leq k$.
(ii) *If, additionally,* $a_j \geq 3$ *for some* j, *then the link diagram* $\widehat{\gamma}$ *has* $\mathbb{Z}_2$*-torsion in Khovanov homology.*
(iii) *If* $\widehat{\gamma}$ *is an adequate knot or link of two components then its Khovanov homology contains* $\mathbb{Z}_2$*-torsion.*

Proof. Consider a standard diagram $\widehat{\gamma}$ of a positive 3-braid γ. Since the diagram is positive, the link is $+$-adequate. In this case the graph G_{s_+} has three vertices and only 2-cycles (compare with Figure 6). On the other hand, the graph G_{s_-} contains an a_i-gon for any $0 < i \leq k$. In particular, if all $a_i \geq 2$, this graph has no loops, hence $\widehat{\gamma}$ is $-$-adequate. Furthermore, if at least one $a_j \geq 3$, then the girth of the corresponding graph G_{s-} is at least 3. According to Theorems 2.14 and 3.2, Khovanov homology of such 3-braids contains $\mathbb{Z}_2$-torsion. Part (iii) follows from the fact that when all a_i are 2, then $\widehat{\gamma}$ is a link of three components. ∎

([10]) 10_{152} is a positive braid in the original (old) convention. In Proposition 5.1 we use the new convention, so 10_{152} will have all negative crossings and be a negative knot.

A weaker version of Proposition 5.1 holds for all positive n-braids.

PROPOSITION 5.2. *Let $\gamma = \sigma_{i_1}^{a_1} \dots \sigma_{i_k}^{a_k}$ be a positive n-braid such that $a_i \geq 2$ for any i. Then $\widehat{\gamma}$ is an adequate diagram. If, additionally, $a_j \geq 3$ for some j, then $\widehat{\gamma}$ has $\mathbb{Z}_2$-torsion in Khovanov homology.*

6. Conjectures. The main goal of this paper was to enhance our understanding of torsion in Khovanov homology. In order to do so, we have analyzed those gradings in chromatic graph cohomology that agree with Khovanov homology. This approach brought new insights about torsion that agree with the recent results by A. Shumakovitch [Sh3] stating that there is no other torsion except $\mathbb{Z}_2$ in Khovanov homology of alternating knots. Experimental results obtained using Shumakovitch's software *KhoHo*, *Knotscape* by M. Thistlethwaite, and *LinKnot* by S. Jablan and the second author show that there are eight positive 15-crossing knots whose 4-braid diagrams are adequate, and which have $\mathbb{Z}_4$-torsion in Khovanov homology ([11]). We suspect that the order of torsion in Khovanov homology partially depends on the minimal braid index of a given link as stated in the following conjecture.

CONJECTURE 6.1 (PS braid conjecture).

(1) *Khovanov homology of a closed 3-braid can have only $\mathbb{Z}_2$-torsion.*
(2) *Khovanov homology of a closed 4-braid cannot have an odd torsion.*
(2′) *Khovanov homology of a closed 4-braid can have only $\mathbb{Z}_2$- and $\mathbb{Z}_4$-torsion.*
(3) *Khovanov homology of a closed n-braid cannot have p-torsion for $p > n$ (p prime).*
(3′) *Khovanov homology of a closed n-braid cannot have $\mathbb{Z}_{p^r}$-torsion for $p^r > n$.*

Note that we are stating these conjectures with various degrees of confidence. The case of 3-braids was extensively tested using A. Shumakovitch's software *KhoHo*, and P. Turner proved that the Khovanov homology of $(3, q)$ torus links can only contain 2-torsion [Low, Tu]. In 2011, W. Gilliam [Gi] showed that only $\mathbb{Z}_2$-torsion is possible in their Khovanov homology. D. Bar-Natan [BN] checked that $(n, 4)$ torus knots have $\mathbb{Z}_4$-torsion for $n = 5, 7, 9, 11$.

([11]) Closures of the following braids have $\mathbb{Z}_4$-torsion in Khovanov homology:

BR[4, 1, 1, 2, 2, 1, 1, 3, 2, 2, 2, 1, 3, 2, 2, 3], BR[4, 1, 1, 2, 2, 2, 1, 1, 3, 2, 2, 1, 3, 2, 2, 3],
BR[4, 1, 2, 2, 1, 3, 2, 2, 2, 1, 3, 2, 2, 2, 3, 3], BR[4, 1, 2, 2, 1, 3, 3, 3, 2, 2, 2, 1, 3, 2, 2, 3],
BR[4, 1, 2, 2, 1, 3, 2, 2, 2, 2, 2, 1, 3, 2, 2, 3], BR[4, 1, 2, 2, 1, 3, 2, 2, 2, 1, 3, 2, 2, 3, 3, 3],
BR[4, 1, 2, 2, 1, 3, 2, 2, 2, 1, 3, 2, 2, 2, 2, 3], BR[4, 1, 2, 2, 2, 1, 3, 2, 2, 2, 1, 3, 2, 2, 2, 3],

as verified by Slavik Jablan, Cotton Seed, and Alexander Shumakovitch [Ja, Se, Sh4].

Example 6.2. As of summer of 2012, examples of knots with $\mathbb{Z}_5$-torsion in Khovanov homology were quite rare: for example 5-strand torus knots: $(6,5)$, $(7,5)$, $(8,5)$, and $(9,5)$. We predicted that the positive adequate 36-crossing knot K given by the closure of the braid

$$s_1^2 s_2^2 s_1^3 s_2^2 s_1 s_3 s_2^2 s_4^2 s_3 s_1^2 s_2^2 s_1^3 s_2^3 s_1^2 s_3 s_2^2 s_4^3 s_3^2$$

has $\mathbb{Z}_5$-torsion, which was confirmed by A. Shumakovitch using *JavaKh* [BNG]. More precisely, we show that homology mod 5 and mod 7 have different ranks while homology mod 7 and mod 11 have the same ranks. The difference between Khovanov polynomials computed mod 5 and mod 7 (see Appendix) is strictly positive:

$$KH_5(K) - KH_7(K) = (t^{12} + t^{11})q^{51} + (t^{11} + t^{10})q^{47}.$$

This means that the rank of 5-torsion is strictly greater than the one of 7-torsion, hence torsion of order 5 exists at least in degrees $(12, 51)$, and $(11, 47)$ ([12]).

Finally for the $(8,7)$ torus knot Bar-Natan computed Khovanov homology and showed that it contains $\mathbb{Z}_7$-, $\mathbb{Z}_5$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_2$-torsion but this 48-crossing 7-braid reaches the limits of current computational resources.

Appendix. Khovanov homology computations. We include the Khovanov polynomials of the positive adequate 36-crossing knot K given by the closure of the 5-braid $s_1^2 s_2^2 s_1^3 s_2^2 s_1 s_3 s_2^2 s_4^2 s_3 s_1^2 s_2^2 s_1^3 s_2^3 s_1^2 s_3 s_2^2 s_4^3 s_3^2$ computed over $\mathbb{Z}_5$ and $\mathbb{Z}_7$. Computations were done in *JavaKh* [BNG] by A. Shumakovitch using Mathematisches Forschungsinstitut Oberwolfach world-class computer facilities. Note that the Khovanov homology in the example considered is normalized to categorify the Jones polynomial, not the Kauffman bracket polynomial ([13]).

$$\begin{aligned}KH_5(K) = {}& q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2 + q^{41}t^5 + q^{43}t^5 + q^{39}t^6 \\ &+ 2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 \\ &+ 4q^{47}t^9 + 2q^{43}t^{10} + 29q^{45}t^{10} + 14q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 44q^{47}t^{11} \\ &+ 31q^{49}t^{11} + q^{51}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 42q^{51}t^{12} + 2q^{53}t^{12} \\ &+ 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} \\ &+ 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51}t^{15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 9q^{57}t^{15} \\ &+ 345q^{53}t^{16} + 5376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17}\end{aligned}$$

([12]) Theoretically, Khovanov homology can contain more 5-torsion, but then it must coincide with 7-torsion, which we predict to be trivial.

([13]) If $H_{i,j}(K)$ is the Khovanov homology of K from Definition 2.8 and $H^{c,d}(K)$ is the homology used in Example 5.2, then $c = (i - 36)/2$ corresponds to the power of t and $d = (j - 3 \cdot 36)/2$ corresponds to the power of q. Notice that $|D_{s_+}| = 5$ and $|D_{s_-}| = 21$ and the writhe $w(K)$ is 36.

$$+ 260q^{59}t^{17} + 37q^{61}t^{17} + 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18}$$
$$+ 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20}$$
$$+ 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + 2779q^{65}t^{21} + 109q^{67}t^{21}$$
$$+ 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + 3127q^{67}t^{23}$$
$$+ 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24}$$
$$+ 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26}$$
$$+ 228q^{73}t^{27} + 759q^{75}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28}$$
$$+ 219q^{77}t^{29} + 294q^{79}t^{29} + 218q^{81}t^{29} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30}$$
$$+ 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33}$$
$$+ 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36},$$

$$KH_7(K) = q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2q^{41}t^5 + q^{43}t^5 + q^{39}t^6$$
$$+ 2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9$$
$$+ 4q^{47}t^9 + 2q^{43}t^{10} + 29q^{45}t^{10} + 13q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 43q^{47}t^{11}$$
$$+ 31q^{49}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 41q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13}$$
$$+ 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} + 142q^{53}t^{14}$$
$$+ 63q^{55}t^{14} + 137q^{51}t^{15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 59q^{57}t^{15} + 345q^{53}t^{16}$$
$$+ 376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17}$$
$$+ 37q^{61}t^{17} + 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19}$$
$$+ 1501q^{61}t^{19} + 220q^{63}t^{19} + 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20}$$
$$+ 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22}$$
$$+ 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + 3127q^{67}t^{23} + 3345q^{69}t^{23}$$
$$+ 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + 137q^{69}t^{25}$$
$$+ 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27}$$
$$+ 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29}$$
$$+ 218q^{81}t^{29} + 759q^{75}t^{27} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31}$$
$$+ 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33}$$
$$+ 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}.$$

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Added in proof (February 2014). Lukas Lewark informed us that his calculations show that Conjecture 6.1 holds for the torus knot $(9,8)$, and that it has $\mathbb{Z}_8$ torsion in Khovanov homology. In particular, the torsion in one specific bigrading is equal to $\mathbb{Z}_8 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2^3$. It is interesting that the Khovanov homology of the torus knot $(9,8)$ contains no 7-torsion (email of January 27, 2014).

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