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A note on singular homology groups of infinite products of compacta

by

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Abstract. Let n be an integer with $n \geq 2$ and $\{X_i\}$ be an infinite collection of $(n-1)$-connected continua. We compare the homotopy groups of $\Sigma(\prod_i X_i)$ with those of $\prod_i \Sigma X_i$ (Σ denotes the unreduced suspension) via the Freudenthal Suspension Theorem. An application to homology groups of the countable product of the $n(\geq 2)$-sphere is given.

1. Introduction and results. The results of the present note stem from an attempt to compute the singular homology groups of the countable product S_n^∞ of the n-sphere ($n \geq 1$). Very little is known on these groups except for trivial facts: $\widetilde{\mathrm{H}}_q(S_n^\infty) = 0$ for $q < n$ and $\mathrm{H}_n(S_n^\infty) \cong \pi_n(S_n^\infty) \cong \mathbb{Z}^\infty$, the countable product of the integers. The lack of higher local connectivity makes the computation non-trivial. A motivation for the computation is in the singular homology group of the Hawaiian earring and its n-dimensional analogue $\mathbb{H}_n$ (see [1]–[3]). The space $\mathbb{H}_n$ is naturally embedded in S_n^∞. In [1], it is shown that for each $n \geq 2$, the singular homology group $\mathrm{H}_q(\mathbb{H}_n)$ is not zero for infinitely many q's. In particular $\mathrm{H}_3(\mathbb{H}_2)$ is not zero, while Theorem 1.4 of this note shows that $\mathrm{H}_3(S_2^\infty) = 0$.

Throughout the present note, ΣX denotes the *unreduced* suspension of a space X obtained from $X \times [0,1]$ by identifying $X \times \{0\}$ and $X \times \{1\}$ to points respectively. The image of $(x,t) \in X \times [0,1]$ under the quotient map is denoted by $[x,t]$.

Let $\{X_i\}$ be an infinite collection of continua (i.e. compact connected metric spaces) and let $j : \Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ be the map defined by

$$j([(x_i)_i, t]) = ([x_i, t])_i, \quad \text{where } (x_i) \in \prod_i X_i,\ t \in [0,1].$$

It is easy to see that j is a well-defined embedding. Under the above notation, our first result is stated as follows.

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Theorem 1.1. *Let $n \geq 2$ be an integer and assume that each X_i is an $(n-1)$-connected continuum. Then*:

(1) *The induced homomorphism $j_\sharp : \pi_q(\Sigma(\prod_i X_i)) \to \pi_q(\prod_i \Sigma X_i)$ is an isomorphism for each $q < 2n$ and an epimorphism for $q = 2n$.*

(2) *If moreover $\mathrm{H}_n(X_i) \cong \mathbb{Z}$ for each i, then $j_\sharp : \pi_{2n}(\Sigma(\prod_i X_i)) \to \pi_{2n}(\prod_i \Sigma X_i)$ is an isomorphism.*

The proof is an application of the Freudenthal Suspension Theorem for unreduced suspensions of general (compactly generated) spaces (not necessarily CW-complexes). The above theorem implies the following result on homology groups of infinite products.

Corollary 1.2. *Let $n \geq 2$ be an integer and assume that each X_i is $(n-1)$-connected. Then*:

(1) $\widetilde{\mathrm{H}}_{q-1}(\prod_i X_i) \cong \mathrm{H}_q(\prod_i \Sigma X_i)$ *for each* $1 \leq q < 2n$.

(2) *If moreover* $\mathrm{H}_n(X_i) \cong \mathbb{Z}$ *for each i, then* $\widetilde{\mathrm{H}}_{q-1}(\prod_i X_i) \cong \mathrm{H}_q(\prod_i \Sigma X_i)$ *for each* $1 \leq q \leq 2n$.

Applying the above corollary to the countable product S_n^∞ of the n-sphere ($n \geq 2$), we obtain the following.

Corollary 1.3. *Let $n \geq 2$ be an integer. For each integer $k \geq 0$, we have an isomorphism* $\mathrm{H}_{n+k}(S_n^\infty) \cong \mathrm{H}_{n+k+1}(S_{n+1}^\infty)$ *provided* $n \geq k+1$.

Thus, as $n \to \infty$, the homology group $\mathrm{H}_{n+k}(S_n^\infty)$ stabilizes and we make use of this fact to prove:

Theorem 1.4. $\mathrm{H}_{n+1}(S_n^\infty) = 0$ *for each* $n \geq 2$.

The Künneth formula, applied to $S_1^\infty \approx S_1^\infty \times S_1^\infty$, implies that $\mathrm{H}_2(S_1^\infty)$ contains $\mathbb{Z}^\infty \otimes \mathbb{Z}^\infty$ as a direct summand and hence is non-zero.

Throughout, the n-sphere is denoted by S_n to keep the notation S_n^∞ for the countable product of the n-sphere.

2. Proofs. Let $E : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ be the unreduced suspension homomorphism described in [6, p. 369]. The Freudenthal Suspension Theorem for unreduced suspensions of general spaces is stated as follows.

Theorem 2.1 ([6, Chap. VII, (7.13)] and [5, Appendix]). *Let $n \geq 2$ be an integer and X an $(n-1)$-connected (compactly generated) space. Then*:

(1) *$E : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ is an isomorphism for $q < 2n-1$ and an epimorphism for $q = 2n-1$.*

(2) *The kernel of $E : \pi_{2n-1}(X) \to \pi_{2n}(\Sigma X)$ is generated by $\{[\alpha, \beta] \mid \alpha, \beta \in \pi_n(X)\}$, where $[\alpha, \beta]$ denotes the Whitehead product of α and β.*

Remark. In [5], the space X in (2) is assumed to be a CW-complex. To obtain the result for a general X, take a map $\varphi : W \to X$ which induces isomorphisms between homotopy groups in all dimensions (see, for example, [6, Chap. V, Theorem (3.2)]). The spaces ΣX and ΣW are simply connected and $\Sigma\varphi$ induces isomorphisms between homology groups in all dimensions, hence also between homotopy groups. Moreover the diagram

$$\begin{array}{ccc} \pi_q(W) & \xrightarrow{E} & \pi_{q+1}(\Sigma W) \\ {\scriptstyle \varphi_\sharp}\downarrow & & \downarrow{\scriptstyle (\Sigma\varphi)_\sharp} \\ \pi_q(X) & \xrightarrow{E} & \pi_{q+1}(\Sigma X) \end{array}$$

is commutative and conclusion (2) follows from the one for CW-complexes.

NOTATION. For a collection $\{\alpha_i : Y \to X_i\}_i$ of maps, $\triangle_i\alpha_i : Y \to \prod_i X_i$ denotes the diagonal product of (α_i), that is, the map defined by

$$\triangle_i\alpha_i(p) = (\alpha_i(p))_i, \quad p \in Y.$$

Proof of Theorem 1.1. As ΣZ is simply connected for every path-connected space Z, we may restrict our attention to the case $q \geq 2$. Consider the following diagram:

$$\begin{array}{ccc} \pi_q(\Sigma(\prod_i X_i)) & \xrightarrow{j_\sharp} & \pi_q(\prod_i \Sigma X_i) \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle \Pi} \\ \pi_{q-1}(\prod_i X_i) & & \prod_i \pi_q(\Sigma X_i) \\ {\scriptstyle \Pi}\uparrow & & \uparrow{\scriptstyle E^\infty} \\ \prod_i \pi_{q-1}(X_i) & = & \prod_i \pi_{q-1}(X_i) \end{array}$$

Here $\Pi : \prod_i \pi_*(X_i) \to \pi_*(\prod_i X_i)$ and $\Pi : \prod_i \pi_*(\Sigma X_i) \to \pi_*(\prod_i \Sigma X_i)$ are the canonical isomorphisms given by $\Pi((\alpha_i)) =$ (the homotopy class of $\triangle_i\alpha_i$). Also E^∞ is the product of the suspension homomorphisms.

It is straightforward to verify that the above diagram is commutative. Then (1) follows easily from Theorem 2.1(1). To show (2), first notice that

$$(*) \qquad \operatorname{Ker}\Big[j_\sharp : \pi_{2n}\Big(\Sigma\Big(\prod_i X_i\Big)\Big) \to \pi_{2n}\Big(\prod_i \Sigma X_i\Big)\Big] = E\Pi(\operatorname{Ker} E^\infty)$$

since $E : \pi_{2n-1}(\prod_i X_i) \to \pi_{2n}(\Sigma(\prod_i X_i))$ is an epimorphism. Fix a generator e_i of $\mathrm{H}_n(X_i) \cong \pi_n(X_i) \cong \mathbb{Z}$. By Theorem 2.1(2), $\operatorname{Ker}(E : \pi_{2n-1}(X_i) \to \pi_{2n}(\Sigma X_i))$ is generated by $[e_i, e_i]$.

CLAIM. *For each $\gamma = (\gamma_i)_i \in \operatorname{Ker} E^\infty$, we have $E \circ \Pi(\gamma) = 0$.*

Proof of Claim. Let $\gamma_i = m_i[e_i, e_i]$, $m_i \in \mathbb{Z}$. Let $\varepsilon_i : S_n \to X_i$ be a map representing e_i. Then $m_i \cdot e_i$ is represented by $\varepsilon_i \circ \mu_i$, where $\mu_i : S_n \to S_n$ is a map of degree m_i. Let $p_i : \prod_j X_j \to X_i$ be the projection onto X_i. In what follows, for simplicity, the homotopy classes represented by $\triangle_i\varepsilon_i$ etc.

are abbreviated to $\triangle_i\varepsilon_i$ etc. The equality

$$\begin{aligned}(p_i)_\sharp[\triangle_i\varepsilon_i \circ \mu_i, \triangle_i\varepsilon_i] &= [(p_i)_\sharp\triangle_i\varepsilon_i \circ \mu_i, (p_i)_\sharp\triangle_i\varepsilon_i] \\ &= [\varepsilon_i \circ \mu_i, \varepsilon_i] = m_i[e_i, e_i] = \gamma_i\end{aligned}$$

shows that $\Pi(\gamma) = [\triangle_i\varepsilon_i\circ\mu_i, \triangle_i\varepsilon_i]$. Thus $E\circ\Pi(\gamma) = E([\triangle_i\varepsilon_i\circ\mu_i, \triangle_i\varepsilon_i]) = 0$.

The above claim together with $(*)$ implies that $j_\sharp$ is a monomorphism in dimension $2n$ and hence an isomorphism. This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. This is well known to be a direct consequence of Theorem 1.1 and a proof is provided for completeness. Statement (1) follows immediately from Theorem 1.1 via the Whitehead Theorem and the isomorphism $\widetilde{\mathrm{H}}_q(\Sigma Z) \cong \widetilde{\mathrm{H}}_{q-1}(Z)$ for each path-connected space Z.

To show (2), we identify $\Sigma(\prod_i X_i)$ with $j(\Sigma(\prod_i X_i))$. The space $\Sigma(\prod_i X_i)$ is simply connected. By Theorem 1.1, the inclusion $\Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ induces isomorphisms of homotopy groups up to dimension $2n$. Thus $\pi_q(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) = 0$ for each $q \leq 2n$ and the homomorphism $\partial : \pi_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) \to \pi_{2n}(\Sigma(\prod_i X_i))$ is trivial. Since the Hurewicz homomorphism $\pi_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) \to \mathrm{H}_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i))$ is an isomorphism, it follows that the connecting homomorphism

$$\partial : \mathrm{H}_{2n+1}\Big(\prod_i \Sigma X_i, \Sigma\Big(\prod_i X_i\Big)\Big) \to \mathrm{H}_{2n}\Big(\Sigma\Big(\prod_i X_i\Big)\Big)$$

is trivial. So the inclusion $\Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ induces isomorphisms of homology groups up to dimension $2n$.

Proof of Theorem 1.4. By Corollary 1.3, $\mathrm{H}_3(S_2^\infty) \cong \mathrm{H}_{n+1}(S_n^\infty)$ for each $n \geq 3$. So we may assume that $n \geq 3$. We apply Whitehead's "certain exact sequence" [7] in the following form.

THEOREM 2.2 ([7], cf. [4, p. 36]). *Suppose that X is an $(n-1)$-connected space with $n \geq 3$. There exists a natural exact sequence*

$$\pi_n(X) \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \pi_{n+1}(X) \xrightarrow{\theta} \mathrm{H}_{n+1}(X) \to 0$$

where θ is the Hurewicz homomorphism.

Let $p_i : S_n^\infty \to S_n$ be the projection onto the ith factor. We consider the following commutative diagram:

$$\begin{array}{ccccccc} \pi_n(S_n^\infty) \otimes \mathbb{Z}/2\mathbb{Z} & \xrightarrow{i} & \pi_{n+1}(S_n^\infty) & \xrightarrow{\theta} & H_{n+1}(S_n^\infty) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \triangle_i((p_i)_\sharp\otimes 1_{\mathbb{Z}/2\mathbb{Z}})} & & \downarrow{\scriptstyle \triangle_i(p_i)_\sharp} & & & & \\ (\pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z})^\infty & \xrightarrow{i^\infty} & \pi_{n+1}(S_n)^\infty & \xrightarrow{\theta^\infty} & H_{n+1}(S_n)^\infty = 0 & & \end{array}$$

where the first row is the exact sequence of Theorem 2.2 for S_n^∞ and the second row is the countable product of the exact sequences of Theorem 2.2 for the n-sphere S_n.

Let $h : \pi_n(S_n)^\infty \otimes \mathbb{Z}/2\mathbb{Z} \to (\pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z})^\infty$ be the homomorphism defined by $h((\alpha_i)_i \otimes 1) = ((\alpha_i \otimes 1)_i)$. It is easy to see that h is an isomorphism. Now we show the following equality:

$$\triangle_i((p_i)_\sharp \otimes 1_{\mathbb{Z}/2\mathbb{Z}}) = h \circ ((\triangle_i(p_i)_\sharp) \otimes 1_{\mathbb{Z}/2\mathbb{Z}}) : \pi_n(S_n^\infty) \otimes \mathbb{Z}/2\mathbb{Z} \to (\pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z})^\infty.$$

Indeed, for each $\alpha \otimes 1 \in \pi_n(S_n^\infty) \otimes \mathbb{Z}/2\mathbb{Z}$, we have

$$\begin{aligned} h \circ ((\triangle_i(p_i)_\sharp) \otimes 1_{\mathbb{Z}/2\mathbb{Z}})(\alpha \otimes 1) &= h(((p_i)_\sharp(\alpha))_i \otimes 1) = ((p_i)_\sharp(\alpha) \otimes 1)_i \\ &= \triangle_i((p_i)_\sharp \otimes 1_{\mathbb{Z}/2\mathbb{Z}})(\alpha \otimes 1), \end{aligned}$$

which proves the desired equality.

Since $\triangle_i((p_i)_\sharp)(= \Pi^{-1})$ is an isomorphism, $\triangle_i((p_i)_\sharp) \otimes 1_{\mathbb{Z}/2\mathbb{Z}}$ is an isomorphism. This together with the above equality implies that $\triangle_i((p_i)_\sharp \otimes 1_{\mathbb{Z}/2\mathbb{Z}})$ is an isomorphism. As i^∞ is an epimorphism, so is i and hence $H_{n+1}(S_n^\infty) = 0$. This completes the proof.

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