A note on singular homology groups of infinite products of compacta

by

Kazuhiro Kawamura (Tsukuba)

Abstract. Let $n$ be an integer with $n \geq 2$ and \{\(X_i\)\} be an infinite collection of \((n-1)\)-connected continua. We compare the homotopy groups of $\Sigma(\prod_i X_i)$ with those of $\prod_i \Sigma X_i$ ($\Sigma$ denotes the unreduced suspension) via the Freudenthal Suspension Theorem. An application to homology groups of the countable product of the $n(\geq 2)$-sphere is given.

1. Introduction and results. The results of the present note stem from an attempt to compute the singular homology groups of the countable product $S_n^\infty$ of the $n$-sphere ($n \geq 1$). Very little is known on these groups except for trivial facts: $\tilde{\mathbb{H}}_q(S_n^\infty) = 0$ for $q < n$ and $\mathbb{H}_n(S_n^\infty) \cong \pi_n(S_n^\infty) \cong \mathbb{Z}^\infty$, the countable product of the integers. The lack of higher local connectivity makes the computation non-trivial. A motivation for the computation is in the singular homology group of the Hawaiian earring and its $n$-dimensional analogue $\mathbb{H}_n$ (see [1]–[3]). The space $\mathbb{H}_n$ is naturally embedded in $S_n^\infty$. In [1], it is shown that for each $n \geq 2$, the singular homology group $\mathbb{H}_q(\mathbb{H}_n)$ is not zero for infinitely many $q$'s. In particular $\mathbb{H}_3(\mathbb{H}_2)$ is not zero, while Theorem 1.4 of this note shows that $\mathbb{H}_3(S_2^\infty) = 0$.

Throughout the present note, $\Sigma X$ denotes the unreduced suspension of a space $X$ obtained from $X \times [0, 1]$ by identifying $X \times \{0\}$ and $X \times \{1\}$ to points respectively. The image of $(x, t) \in X \times [0, 1]$ under the quotient map is denoted by $[x, t]$.

Let $\{X_i\}$ be an infinite collection of continua (i.e. compact connected metric spaces) and let $j : \Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ be the map defined by

$$j([((x_i)_i, t)]_i) = ([x_i, t])_i, \quad \text{where } (x_i) \in \prod_i X_i, \ t \in [0, 1].$$

It is easy to see that $j$ is a well-defined embedding. Under the above notation, our first result is stated as follows.

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**Theorem 1.1.** Let \( n \geq 2 \) be an integer and assume that each \( X_i \) is an \((n - 1)\)-connected continuum. Then:

1. The induced homomorphism \( j_q : \pi_q(\Sigma(\prod_i X_i)) \to \pi_q(\prod_i \Sigma X_i) \) is an isomorphism for each \( q < 2n \) and an epimorphism for \( q = 2n \).
2. If moreover \( H_n(X_i) \cong \mathbb{Z} \) for each \( i \), then \( j_q : \pi_{2n}(\Sigma(\prod_i X_i)) \to \pi_{2n}(\prod_i \Sigma X_i) \) is an isomorphism.

The proof is an application of the Freudenthal Suspension Theorem for unreduced suspensions of general (compactly generated) spaces (not necessarily CW-complexes). The above theorem implies the following result on homology groups of infinite products.

**Corollary 1.2.** Let \( n \geq 2 \) be an integer and assume that each \( X_i \) is \((n - 1)\)-connected. Then:

1. \( e \pi_q(\Sigma(\prod_i X_i)) \cong \pi_q(\prod_i \Sigma X_i) \) for each \( 1 \leq q < 2n \).
2. If moreover \( H_n(X_i) \cong \mathbb{Z} \) for each \( i \), then \( e \pi_q(\Sigma(\prod_i X_i)) \cong \pi_q(\prod_i \Sigma X_i) \) for each \( 1 \leq q \leq 2n \).

Applying the above corollary to the countable product \( S_1^\infty \) of the \( n \)-sphere \((n \geq 2)\), we obtain the following.

**Corollary 1.3.** Let \( n \geq 2 \) be an integer. For each integer \( k \geq 0 \), we have an isomorphism \( H_{n+k}(S_1^\infty) \cong H_{n+k+1}(S_1^\infty) \) provided \( n \geq k + 1 \).

Thus, as \( n \to \infty \), the homology group \( H_{n+k}(S_1^\infty) \) stabilizes and we make use of this fact to prove:

**Theorem 1.4.** \( H_{n+1}(S_1^\infty) = 0 \) for each \( n \geq 2 \).

The Künneth formula, applied to \( S_1^\infty \cong S_1^\infty \times S_1^\infty \), implies that \( H_2(S_1^\infty) \) contains \( \mathbb{Z}^\infty \otimes \mathbb{Z}^\infty \) as a direct summand and hence is non-zero.

Throughout, the \( n \)-sphere is denoted by \( S_n \) to keep the notation \( S_1^\infty \) for the countable product of the \( n \)-sphere.

**2. Proofs.** Let \( E : \pi_q(X) \to \pi_{q+1}(\Sigma X) \) be the unreduced suspension homomorphism described in \([6, p. 369]\). The Freudenthal Suspension Theorem for unreduced suspensions of general spaces is stated as follows.

**Theorem 2.1** ([6, Chap. VII, (7.13)] and [5, Appendix]). Let \( n \geq 2 \) be an integer and \( X \) an \((n - 1)\)-connected (compactly generated) space. Then:

1. \( E : \pi_q(X) \to \pi_{q+1}(\Sigma X) \) is an isomorphism for \( q < 2n - 1 \) and an epimorphism for \( q = 2n - 1 \).
2. The kernel of \( E : \pi_{2n-1}(X) \to \pi_{2n}(\Sigma X) \) is generated by \( \{[\alpha, \beta] \mid \alpha, \beta \in \pi_n(X)\} \), where \( [\alpha, \beta] \) denotes the Whitehead product of \( \alpha \) and \( \beta \).

**Remark.** In [5], the space \( X \) in (2) is assumed to be a CW-complex. To obtain the result for a general \( X \), take a map \( \varphi : W \to X \) which induces
isomorphisms between homotopy groups in all dimensions (see, for example, [6, Chap. V, Theorem (3.2)]). The spaces $\Sigma X$ and $\Sigma W$ are simply connected and $\Sigma \varphi$ induces isomorphisms between homology groups in all dimensions, hence also between homotopy groups. Moreover the diagram

$$
\begin{array}{ccc}
\pi_q(W) & \xrightarrow{E} & \pi_{q+1}(\Sigma W) \\
\varphi \downarrow & & \downarrow (\Sigma \varphi)_2 \\
\pi_q(X) & \xrightarrow{E} & \pi_{q+1}(\Sigma X)
\end{array}
$$

is commutative and conclusion (2) follows from the one for CW-complexes.

**Notation.** For a collection $\{\alpha_i : Y \to X_i\}_i$ of maps, $\triangle_i \alpha_i : Y \to \prod_i X_i$ denotes the diagonal product of $(\alpha_i)$, that is, the map defined by

$$
\triangle_i \alpha_i(p) = (\alpha_i(p))_i, \quad p \in Y.
$$

**Proof of Theorem 1.1.** As $\Sigma Z$ is simply connected for every path-connected space $Z$, we may restrict our attention to the case $q \geq 2$. Consider the following diagram:

$$
\begin{array}{ccc}
\pi_q(\Sigma(\prod_i X_i)) & \xrightarrow{j_2} & \pi_q(\prod_i \Sigma X_i) \\
\downarrow E & & \downarrow \Pi \\
\pi_{q-1}(\prod_i X_i) & \xrightarrow{\Pi} & \prod_i \pi_q(\Sigma X_i) \\
\downarrow & & \downarrow E^\infty \\
\prod_i \pi_{q-1}(X_i) & \xrightarrow{\Pi} & \prod_i \pi_{q-1}(X_i)
\end{array}
$$

Here $\Pi : \prod_i \pi_*(X_i) \to \pi_*(\prod_i X_i)$ and $\Pi : \prod_i \pi_*(\Sigma X_i) \to \pi_*(\prod_i \Sigma X_i)$ are the canonical isomorphisms given by $\Pi((\alpha_i)) = (\text{the homotopy class of } \triangle_i \alpha_i)$. Also $E^\infty$ is the product of the suspension homomorphisms.

It is straightforward to verify that the above diagram is commutative. Then (1) follows easily from Theorem 2.1(1). To show (2), first notice that

$$
(*) \quad \text{Ker} \left[ j_2 : \pi_{2n} \left( \Sigma \left( \prod_i X_i \right) \right) \to \pi_{2n} \left( \prod_i \Sigma X_i \right) \right] = \text{E} \Pi (\text{Ker } E^\infty)
$$

since $E : \pi_{2n-1}(\prod_i X_i) \to \pi_{2n}(\Sigma(\prod_i X_i))$ is an epimorphism. Fix a generator $e_i$ of $H_n(X_i) \cong \pi_n(X_i) \cong \mathbb{Z}$. By Theorem 2.1(2), Ker$(E : \pi_{2n-1}(X_i) \to \pi_{2n}(\Sigma X_i))$ is generated by $[e_i, e_i]$.

**Claim.** For each $\gamma = (\gamma_i)_i \in \text{Ker } E^\infty$, we have $E \circ \Pi (\gamma) = 0$.

**Proof of Claim.** Let $\gamma_i = m_i[e_i, e_i], \ m_i \in \mathbb{Z}$. Let $\varepsilon_i : S_n \to X_i$ be a map representing $e_i$. Then $m_i \cdot e_i$ is represented by $\varepsilon_i \circ \mu_i$, where $\mu_i : S_n \to S_n$ is a map of degree $m_i$. Let $p_i : \prod_j X_j \to X_i$ be the projection onto $X_i$. In what follows, for simplicity, the homotopy classes represented by $\triangle_i \varepsilon_i$ etc.
are abbreviated to $\Delta_i \varepsilon_i$ etc. The equality
\[
(p_i)_* [\Delta_i \varepsilon_i \circ \mu_i, \Delta_i \varepsilon_i] = [(p_i)_* \Delta_i \varepsilon_i \circ \mu_i, (p_i)_* \Delta_i \varepsilon_i] = [\varepsilon_i \circ \mu_i, \varepsilon_i] = m_i [\varepsilon_i, \varepsilon_i] = \gamma_i
\]
shows that $II(\gamma) = [\Delta_i \varepsilon_i \circ \mu_i, \Delta_i \varepsilon_i]$. Thus $E \circ II(\gamma) = E([\Delta_i \varepsilon_i \circ \mu_i, \Delta_i \varepsilon_i]) = 0$.

The above claim together with (*) implies that $j_*$ is a monomorphism in dimension $2n$ and hence an isomorphism. This completes the proof of Theorem 1.1.

**Proof of Corollary 1.2.** This is well known to be a direct consequence of Theorem 1.1 and a proof is provided for completeness. Statement (1) follows immediately from Theorem 1.1 via the Whitehead Theorem and the isomorphism $\tilde{H}_q(S) \cong \tilde{H}_{q-1}(Z)$ for each path-connected space $Z$.

To show (2), we identify $\Sigma(\prod_i X_i)$ with $j(\Sigma(\prod_i X_i))$. The space $\Sigma(\prod_i X_i)$ is simply connected. By Theorem 1.1, the inclusion $\Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ induces isomorphisms of homotopy groups up to dimension $2n$. Thus $\pi_q(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) = 0$ for each $q \leq 2n$ and the homomorphism $\partial : \pi_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) \to \pi_{2n}(\Sigma(\prod_i X_i))$ is trivial. Since the Hurewicz homomorphism $\pi_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) \to H_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i))$ in an isomorphism, it follows that the connecting homomorphism
\[
\partial : H_{2n+1}(\prod_i \Sigma X_i, \Sigma(\prod_i X_i)) \to H_2n(\Sigma(\prod_i X_i))
\]
is trivial. So the inclusion $\Sigma(\prod_i X_i) \to \prod_i \Sigma X_i$ induces isomorphisms of homotopy groups up to dimension $2n$.

**Proof of Theorem 1.4.** By Corollary 1.3, $H_3(S_2^\infty) \cong H_n+1(S_n^\infty)$ for each $n \geq 3$. So we may assume that $n \geq 3$. We apply Whitehead’s “certain exact sequence” [7] in the following form.

**Theorem 2.2 ([7], cf. [4, p. 36]).** *Suppose that $X$ is an $(n-1)$-connected space with $n \geq 3$. There exists a natural exact sequence*
\[
\pi_n(X) \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \pi_{n+1}(X) \xrightarrow{\theta} H_{n+1}(X) \to 0
\]
*where $\theta$ is the Hurewicz homomorphism.*

Let $p_i : S_n^\infty \to S_n$ be the projection onto the $i$th factor. We consider the following commutative diagram:
\[
\begin{array}{ccc}
\pi_n(S_n^\infty) \otimes \mathbb{Z}/2\mathbb{Z} & \xrightarrow{i} & \pi_{n+1}(S_n^\infty) \xrightarrow{\theta} H_{n+1}(S_n^\infty) \to 0 \\
\Delta_i([p_i]_* \otimes 1_{\mathbb{Z}/2\mathbb{Z}}) & & \Delta_i([p_i]_*) \\
(p_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} & \pi_{n+1}(S_n) \xrightarrow{\theta} H_{n+1}(S_n) = 0
\end{array}
\]
where the first row is the exact sequence of Theorem 2.2 for $S_n^\infty$ and the
second row is the countable product of the exact sequences of Theorem 2.2 for the $n$-sphere $S_n$.

Let $h : \pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z} \to (\pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z})^\infty$ be the homomorphism defined by $h((\alpha_i)i \otimes 1) = (\alpha_i \otimes 1)i$. It is easy to see that $h$ is an isomorphism. Now we show the following equality:

$$\Delta_i((p_i)_x \otimes 1_{\mathbb{Z}/2\mathbb{Z}}) = h \circ ((\Delta_i(p_i)_x \otimes 1_{\mathbb{Z}/2\mathbb{Z}}) : \pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z} \to (\pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z})^\infty.$$  

Indeed, for each $\alpha \otimes 1 \in \pi_n(S_n) \otimes \mathbb{Z}/2\mathbb{Z}$, we have

$$h \circ ((\Delta_i(p_i)_x \otimes 1_{\mathbb{Z}/2\mathbb{Z}})(\alpha \otimes 1) = h(((p_i)_x(\alpha))i \otimes 1) = ((p_i)_x(\alpha) \otimes 1)i,$$

which proves the desired equality.

Since $\Delta_i((p_i)_x)(= II^{-1})$ is an isomorphism, $\Delta_i((p_i)_x \otimes 1_{\mathbb{Z}/2\mathbb{Z}})$ is an isomorphism. This together with the above equality implies that $\Delta_i((p_i)_x \otimes 1_{\mathbb{Z}/2\mathbb{Z}})$ is an isomorphism. As $i^\infty$ is an epimorphism, so is $i$ and hence $H_{n+1}(S_n) = 0$. This completes the proof.

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References


Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8071, Japan
E-mail: kawamura@math.tsukuba.ac.jp

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