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The strength of the projective Martin conjecture

by

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Abstract. We show that Martin's conjecture on Π^1_1 functions uniformly $\leq_T$-order preserving on a cone implies Π^1_1 Turing Determinacy over $\mathrm{ZF}+\mathrm{DC}$. In addition, it is also proved that for $n \geq 0$, this conjecture for uniformly degree invariant $\mathbf{\Pi}^1_{2n+1}$ functions is equivalent over ZFC to $\mathbf{\Sigma}^1_{2n+2}$-Axiom of Determinacy. As a corollary, the consistency of the conjecture for uniformly degree invariant Π^1_1 functions implies the consistency of the existence of a Woodin cardinal.

1. Introduction. A *cone* C of reals with *base* z is a set of the form $\{x \mid x \geq_T z\}$ where $\leq_T$ denotes Turing reducibility. A function $F : 2^\omega \to 2^\omega$ is *degree invariant* on C if any two reals $x, y \geq_T z$ of the same Turing degree satisfy $F(x) \equiv_T F(y)$. The degree invariance is *uniform* on C if there is a function t such that if $x, y \geq_T z$, then $\Phi^x_i = y$ and $\Phi^y_j = x$ implies $\Phi^{F(x)}_m = F(y)$ and $\Phi^{F(y)}_n = F(x)$, where $t(i,j) = (m,n)$. The function F is *increasing* on C if $F(x) \geq_T x$ for all $x \geq z$, and *order preserving* on C if $z \leq_T x \leq_T y$ implies $F(z) \leq_T F(x) \leq_T F(y)$. If this order preservation is witnessed by a function $t : \omega \to \omega$, i.e., $\Phi^x_e = y \geq_T z$ implies $\Phi^{F(x)}_{t(e)} = F(y)$, then it is *uniform* (note that a uniformly order preserving function is necessarily uniformly degree invariant). Finally, given functions F and G degree invariant on a cone, write $F \geq_M G$ if $F(x) \geq_{\mathrm{T}} G(x)$ on a cone. Donald A. Martin conjectured that, under the assumption of ZF set theory plus the Axiom of Determinacy (AD) and Dependent Choice (DC):

(1) Every degree invariant function that is not increasing on a cone is a constant on a cone.
(2) $\leq_M$ prewellorders degree invariant functions which are increasing on a cone. Furthermore, if the $\leq_M$-rank of F is α, then F' has $\leq_M$-rank $\alpha+1$, where $F'(x) = (F(x))'$, the Turing jump of $F(x)$.

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Slaman and Steel [7] proved (1) for functions which are uniformly degree invariant on a cone and (2) for Borel functions which are increasing and order preserving. In [8] Steel showed (2) for uniformly degree invariant functions and conjectured that every function degree invariant on a cone is uniformly degree invariant on a cone.

While Martin [4] has shown that Borel determinacy is a theorem of $\mathrm{ZF} + \mathrm{DC}$ (hence conjectures (1) and (2) hold for $\mathbf{\Delta}^1_1$ functions that are uniformly degree invariant), it is known that AD in the analytical hierarchy beyond $\mathbf{\Delta}^1_1$ is a large cardinal axiom. An analysis of the proof in [8] shows that conjecture (2) for uniformly degree invariant $\mathbf{\Pi}^1_{2n+1}$ functions follows from $\mathbf{\Delta}^1_{2n+2}$ Determinacy. Thus a natural question for Martin's Conjectures (1) and (2) is their set-theoretic strength for uniformly degree invariant functions beyond $\mathbf{\Delta}^1_1$ in the analytical hierarchy. There is also a related question concerning the more restrictive uniformly order preserving functions, i.e. while (2) holds for such functions under AD according to Steel [8], the set-theoretic strength of (2) for these functions has not been considered.

A set of reals is degree invariant if it is closed under Turing equivalence. Martin [3] showed that under AD, every degree invariant set of reals either contains or is disjoint from a cone. By $\mathbf{\Pi}^1_{2n+1}$-Turing Determinacy ($\mathbf{\Pi}^1_{2n+1}$-TD) we mean the assertion that every $\mathbf{\Pi}^1_{2n+1}$ set of reals that is degree invariant either contains or is disjoint from a cone. We show in this paper that Conjecture (2) for uniformly order preserving Π^1_1 functions implies the existence of $0^{\#}$. Relativizing the argument to arbitrary reals x leads to the conclusion that $x^{\#}$ exists for every x, so that by Harrington [1] we have the following theorem on the strength of Conjecture (2) for uniformly order preserving $\mathbf{\Pi}^1_1$ functions.

Main Theorem 1. *If Conjecture* (2) *holds for uniformly order preserving* $\mathbf{\Pi}^1_1$ *functions then* $\mathbf{\Pi}^1_1$*-TD is true.*

We also show that in general, for $n \geq 0$, Conjecture (2) for uniformly degree invariant $\mathbf{\Pi}^1_{2n+1}$ functions implies $\mathbf{\Sigma}^1_{2n+2}$-TD, assuming $\mathbf{\Pi}^1_{2n+1}$-uniformization when $n \geq 1$. In fact, by this, Steel [8] and an unpublished work of W. H. Woodin, we have the strength of Conjecture (2) for uniformly degree invariant $\mathbf{\Pi}^1_{2n+1}$ functions measured by $\mathbf{\Sigma}^1_{2n+2}$-AD.

Main Theorem 2. *Conjecture* (2) *for uniformly degree invariant* $\mathbf{\Pi}^1_{2n+1}$ *functions is equivalent to* $\mathbf{\Sigma}^1_{2n+2}$*-AD.*

We recall some facts and notations (see Sacks [6] which is used as the standard reference in this paper). For each real x, ω^x_1 denotes the least ordinal α for which $L_\alpha[x]$ is admissible. Kleene constructed a $\Pi^1_1(x)$ complete set $\mathcal{O}^x$ with a $\Pi^1_1(x)$ well founded relation $<_{\mathcal{O}^x}$ on $\mathcal{O}^x$. The set $\mathcal{O}^x$ is the hyperjump of x. The height of the ordering $<_{\mathcal{O}^x}$ on $\mathcal{O}^x$ is exactly ω^x_1.

Furthermore, Kleene's construction of $\mathcal{O}^x$ is uniform. In other words, the relation $\{(x, \mathcal{O}^x) \mid x \in 2^\omega\}$ is Π^1_1. A fact that will be used implicitly is that given reals x and y, x is hyperarithmetic in y (written $x \leq_h y$) if and only if x is Δ^1_1 in y, and this is in turn equivalent to $x \in L_{\omega_1^y}[y]$. We work under ZF + DC. As we will only be concerned with Conjecture (2), it will be referred to as the $\leq_M$ Conjecture from here on.

2. The $\leq_M$ Conjecture for uniformly order preserving Π^1_1 functions. Let

$$\mathcal{F} = \{x \mid \forall \alpha < \omega_1^x \ \forall a \subseteq \alpha \ (a \in L_{\omega_1^x} \Rightarrow a \in L_{\alpha+3}[x])\}.$$

$\mathcal{F}$ is a degree invariant Σ^1_1 set introduced by H. Friedman [2]. We give a simpler proof of the following result given as Lemma 7.17 in [2].

LEMMA 2.1. *$\mathcal{F}$ is cofinal in the Turing degrees.*

Proof. For any real z, let

$$\mathcal{F}(z) = \{x \oplus z \mid \forall \alpha < \omega_1^z \ \forall a \subseteq \alpha \ (a \in L_{\omega_1^z} \Rightarrow a \in L_{\alpha+3}[x \oplus z])\}$$

be a degree invariant $\Sigma^1_1(z)$ set. Obviously $\mathcal{F}(z)$ is not empty. By the Gandy Basis Theorem relativized to z, there is an x such that $\omega_1^{x \oplus z} = \omega_1^z$ and $x \oplus z \in \mathcal{F}(z)$. Then $x \oplus z \in \mathcal{F}$. ∎

The following lemma follows from Lemmas 7.20–7.22 in [2].

LEMMA 2.2. *If $0^\sharp$ does not exist, then $\bar{\mathcal{F}} = 2^\omega - F$ is cofinal in the Turing degrees.*

For x a real and $n \in \omega$, let $x^{[n]}$ be the real such that $x^{[n]}(i) = x(\langle n, i\rangle)$.

THEOREM 2.3. *If the $\leq_M$ Conjecture holds for uniformly order preserving Π^1_1 functions, then $0^\sharp$ exists.*

Proof. If $0^\sharp$ does not exist, then by Lemmas 2.1 and 2.2, both $\mathcal{F}$ and $\bar{\mathcal{F}}$ are cofinal. For a contradiction, we will define a Π^1_1 function G that is uniformly order preserving such that $\{x \mid G(x) = \mathcal{O}^{\mathcal{O}^x}\}$ and $\{x \mid G(x) = \mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}\}$ are both cofinal in the Turing degrees.

Let $P(x, y)$ be an arithmetic predicate such that

$$x \in \bar{\mathcal{F}} \iff \forall y \ P(x, y).$$

CLAIM 2.4. *If $x \leq_T y$ are such that $x \in \mathcal{F}$ and $y \in \bar{\mathcal{F}}$, then $\mathcal{O}^x \leq_h y$.*

As $y \notin \mathcal{F}$, there are $\alpha < \omega_1^y$ and $a \subseteq \alpha$ with $a \in L_{\omega_1^y} \setminus L_{\alpha+3}[y]$. Clearly, $\alpha \geq \omega$. As $x \leq_T y$, $L_{\alpha+3}[x] \subseteq L_{\alpha+3}[y]$ and thus $a \notin L_{\alpha+3}[x]$. As $x \in \mathcal{F}$, $\omega_1^x \leq \alpha < \omega_1^y$. By $x \leq_T y$ again, $\mathcal{O}^x \leq_h y$.

By Claim 2.4, if $x \leq_T y$ are such that $x \in \mathcal{F}$ and $y \in \bar{\mathcal{F}}$, then $\mathcal{O}^{\mathcal{O}^x} \leq_T \mathcal{O}^y$.

Now define $G(x) = y$ as follows:

(1) $y^{[0]} = \langle 0\rangle\hat{}\mathcal{O}^{\mathcal{O}^x} \wedge x \in \bar{\mathcal{F}}$ or $y^{[0]} = \langle 1\rangle\hat{}\mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}} \wedge \exists v \leq_T y^{[0]} \neg P(x, v)$. Thus $y^{[0]}$ gives a Π^1_1 differentiation between $x \in \bar{\mathcal{F}}$ and $x \in \mathcal{F}$.
(2) $y^{[1]} = \mathcal{O}^x$.
(3) If Φ^x_e is partial then let $y^{[e+2]} = \emptyset$.
(4) If Φ^x_e is total and equal to u, the following three cases differentiate in a Π^1_1 way between $u \in \mathcal{F}$ and $x \in \bar{\mathcal{F}}$, $u, x \in \mathcal{F}$, and $u \in \bar{\mathcal{F}}$ for all $u \leq_T x$:
 (a) $y^{[0]}(0) = 0 \wedge \exists v \leq_T y^{[1]}\ \neg P(u, v) \wedge y^{[e+2]} = \langle 1\rangle\hat{}\mathcal{O}^{\Phi_i^{y^{[1]}}}$ where i is the least index so that $\mathcal{O}^{\mathcal{O}^u} = \mathcal{O}^{\Phi_i^{y^{[1]}}}$, or
 (b) $y^{[0]}(0) = 1 \wedge \exists v \leq_T y^{[1]}\ \neg P(u, v) \wedge y^{[e+2]} = \langle 1\rangle\hat{}\mathcal{O}^{\mathcal{O}^{\mathcal{O}^u}}$, or
 (c) $\forall v \leq_T y^{[1]}\ P(u, v) \wedge y^{[e+2]} = \langle 0\rangle\hat{}\mathcal{O}^{\mathcal{O}^u}$.

$G(x)$ is obviously Π^1_1.

CLAIM 2.5. *If $x \in \mathcal{F}$ then $G(x) \equiv_T \mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}$.*

Clearly $x \in \mathcal{F}$ implies $\mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}} \leq_T G(x)$ and $G(x)^{[0]} \oplus G(x)^{[1]} \leq_T \mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}$.

Given $e < \omega$, $\mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}$ can uniformly decide whether Φ^x_e is total. Suppose that Φ^x_e is total. To calculate $G(x)^{[e+2]}(n)$, one verifies clauses (4b, 4c) above. But the predicate $\forall v \leq_T y^{[1]}\ P(u, v)$ is $\Delta^1_1(\mathcal{O}^x)$, hence recursive in $\mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}$. Once this predicate is decided, $\mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}$ may use recursive functions f and g, where $u = \Phi^w_e \rightarrow \mathcal{O}^{\mathcal{O}^u} = \Phi^{\mathcal{O}^{\mathcal{O}^{\mathcal{O}^w}}}_{f(e)}$ and $u = \Phi^w_e \rightarrow \mathcal{O}^{\mathcal{O}^{\mathcal{O}^u}} = \Phi^{\mathcal{O}^{\mathcal{O}^{\mathcal{O}^w}}}_{g(e)}$, to finish the calculation.

CLAIM 2.6. *If $x \in \bar{\mathcal{F}}$ then $G(x) \equiv_T \mathcal{O}^{\mathcal{O}^x}$.*

This is similar to the above claim, except for the final step calculating $G(x)^{[e+2]}(n)$.

Now $\mathcal{O}^{\mathcal{O}^x}$ is able to decide whether (4a) or (4c) holds, as in the above claim. If (4c) holds, the calculation is the same. If (4a) holds, then $u \in \mathcal{F}$. By Claim 2.4, $\mathcal{O}^u \leq_h x$ and thus $\mathcal{O}^{\mathcal{O}^u} \leq_T \mathcal{O}^x = G(x)^{[1]}$. So i exists. Moreover, the search for i is a procedure uniformly $\Pi^1_1(\mathcal{O}^x)$. Hence $\mathcal{O}^{\mathcal{O}^x}$ uniformly computes $G(x)^{[e+2]}(n)$.

It follows from the above two claims that G is degree invariant. Moreover, G preserves $\leq_T$ by Claim 2.4.

To show that G is uniformly order preserving, let h be a recursive function such that $\forall x, y, e\ (x = \Phi^y_e \rightarrow \mathcal{O}^x = \Phi^{\mathcal{O}^y}_{h(e)})$. In addition, let s be recursive with

$$\forall x, y, e, i\ (x = \Phi^y_e \rightarrow \Phi^x_i = \Phi^y_{s(e,i)}).$$

Suppose that $x = \Phi^y_e$. Then

(1) $G(x)^{[0]} = G(y)^{[e+2]}$,

(2) $G(x)^{[1]} = \Phi_{h(e)}^{G(y)^{[1]}}$,
(3) $G(x)^{[i+2]} = G(y)^{[s(e,i)+2]}$.

Hence G is as desired. ∎

The above proof easily relativizes to any real x to guarantee the existence of $x^{\#}$. Since Harrington [1] has shown that the existence of sharps implies $\mathbf{\Pi}^1_1$-TD, we have

Main Theorem 1. *If the $\leq_M$ Conjecture holds for $\mathbf{\Pi}^1_1$ functions which are uniformly order preserving, then $\mathbf{\Pi}^1_1$-TD is true.*

3. The $\leq_M$ Conjecture for Π^1_{2n+1} functions and Σ^1_{2n+2}-TD

Lemma 3.1. *Π^1_{2n+1}-uniformization and Δ^1_{2n+2}-TD imply Σ^1_{2n+2}-TD for $n \in \omega$.*

Proof. Let $A \in \Sigma^1_{2n+2}$ be degree invariant and $\leq_T$-cofinal. Define

$$R(x,y) \ \Leftrightarrow \ x \leq_T y \wedge y \in A.$$

So $R(x,y) \in \Sigma^1_{2n+2}$. Note that Π^1_{2n+1}-uniformization implies Σ^1_{2n+2}-uniformization. Let $F \in \Sigma^1_{2n+2}$ uniformize R. Then F is actually a Δ^1_{2n+2} function.

Define

$$B = \{u \mid \exists x \leq_T u, y \equiv_T u \ (F(x) = y)\}.$$

Then B is Δ^1_{2n+2}, degree invariant and $\leq_T$-cofinal. Moreover, $B \subseteq A$. By Δ^1_{2n+2}-TD, B contains a cone of Turing degrees. Hence so does A. ∎

Corollary 3.2. *Δ^1_2-TD implies Σ^1_2-TD.*

Proof. As Π^1_1-uniformization is a theorem of ZFC, the corollary follows immediately from Lemma 3.1. ∎

We prove the next result for the lightface version. The proof for the boldface version follows with obvious changes.

Theorem 3.3. *Assume Π^1_{2n+1}-uniformization. If the $\leq_M$ Conjecture holds for uniformly degree invariant Π^1_{2n+1} functions, then Σ^1_{2n+2}-TD holds.*

Proof. Let $A \in \Delta^1_{2n+2}$, and suppose $P, Q \in \Pi^1_{2n+1}$ are such that

$$x \in A \ \Leftrightarrow \ \exists y \ P(x,y) \ \Leftrightarrow \ \forall y \ \neg Q(x,y).$$

Let $R(x,y) \Leftrightarrow P(x,y) \vee Q(x,y)$. By Π^1_{2n+1}-uniformization, let $F \in \Pi^1_{2n+1}$ uniformize R. Define $J_0(x) = z$ if and only if $z^{[0]} = F(x)$ and

$$\forall e \ ((\Phi^x_e \text{ is total} \rightarrow z^{[e+1]} = F(\Phi^x_e)) \wedge (\Phi^x_e \text{ is partial} \rightarrow z^{[e+1]} = \emptyset)).$$

Obviously $J_0 \in \Pi^1_{2n+1}$ is total. Moreover, J_0 is uniformly order preserving. To see this, let f be a recursive function such that

$$\forall e, x_0, x_1 \ (x_0 = \Phi^{x_1}_e \rightarrow \forall i \ (\Phi^{x_1}_{f(e,i)} \simeq \Phi^{x_0}_i)).$$

Suppose $x_0 = \Phi_e^{x_1}$. Then $(J_0(x_0))^{[0]} = (J_0(x_1))^{[e]}$ and $(J_0(x_0))^{[i+1]} = (J_0(x_1))^{[f(e,i)+1]}$. Thus $J_0(x_0)$ may be effectively computed from $J_0(x_1)$.

Let g be a recursive function such that $x_0 = \Phi_e^{x_1} \to J_0(x_0) = \Phi_{g(e)}^{J_0(x_1)}$.

Define $J(x) = x \oplus z_0 \oplus z_1$ if and only if $z_0 = J_0(x)$ and

$$(P(x, z_0^{[0]}) \wedge z_1 = \emptyset) \vee (Q(x, z_0^{[0]}) \wedge z_1 = \langle 1 \rangle \hat{\ } (x \oplus z_0)').$$

Note that $J \in \Pi^1_{2n+1}$. We claim that J is uniformly degree invariant. To see this, let h be a recursive function such that $x_0 = \Phi_e(x_1) \to (x_0 \oplus J_0(x_0))' = \Phi_{h(e)}^{(x_1 \oplus J_0(x_1))'}$. For each e, let $t(e)$ be the index of the procedure Ψ defined by:

1. $(\Psi^z)^{[0]} = \Phi_e^{z^{[0]}}$ and $(\Psi^z)^{[1]} = \Phi_{g(e)}^{z^{[1]}}$.
2. If $z^{[2]}(0) = 0$ then $(\Psi^z)^{[2]} = \emptyset$. Otherwise $(\Psi^z)^{[2]} = \langle 1 \rangle \hat{\ } \Phi_{h(e)}^w$, where w is such that $\langle 1 \rangle \hat{\ } w = z^{[2]}$.

If the $\leq_M$ Conjecture holds for uniformly degree invariant Π^1_{2n+1} functions, then eventually J is either $x \mapsto x \oplus J_0(x)$ or $x \mapsto x \oplus J_0(x) \oplus (x \oplus J_0(x))'$. Hence A either contains or avoids a cone of Turing degrees.

Thus we have Δ^1_{2n+2}-TD. Now Σ^1_{2n+2}-TD follows from Lemma 3.1. ∎

Main Theorem 2. *Let $n \geq 0$. The $\leq_M$ Conjecture for uniformly degree invariant $\boldsymbol{\Pi}^1_{2n+1}$ functions is equivalent to $\boldsymbol{\Sigma}^1_{2n+2}$-AD.*

Proof. An analysis of Theorem 1 in Steel [8] shows that $\boldsymbol{\Sigma}^1_{2n+2}$-AD (in fact $\boldsymbol{\Delta}^1_{2n+2}$-AD) implies the $\leq_M$ Conjecture for uniformly degree invariant $\boldsymbol{\Pi}^1_{2n+1}$ functions. We show the converse by induction on n: First note that if $n = 0$, then $\boldsymbol{\Pi}^1_1$-uniformization is the Kondo–Addison Theorem, so that by Theorem 3.3, $\boldsymbol{\Sigma}^1_2$-TD holds. Now assume by induction that $\boldsymbol{\Sigma}^1_{2n}$-TD is true. Woodin (unpublished) has shown that over ZFC, for $k \geq 1$, $\boldsymbol{\Sigma}^1_{2k}$-TD is equivalent to $\boldsymbol{\Sigma}^1_{2k}$-AD, and Moschovakis [5, Chapter 6] has shown that $\boldsymbol{\Pi}^1_{2k+1}$-uniformization is a consequence of $\boldsymbol{\Sigma}^1_{2k}$-AD. Thus $\boldsymbol{\Pi}^1_{2n+1}$-uniformization holds and so Theorem 3.3 yields $\boldsymbol{\Sigma}^1_{2n+2}$-TD, hence $\boldsymbol{\Sigma}^1_{2n+2}$-AD. ∎

The following corollary gives the consistency strength of the $\leq_M$ Conjecture.

Corollary 3.4. *If it is consistent that the $\leq_M$ Conjecture holds for uniformly degree invariant Π^1_1 functions, then it is consistent that there is a Woodin cardinal.*

Proof. The hypothesis and Theorem 3.3 imply that Π^1_2-TD is consistent. Woodin has shown that Π^1_2-TD is equiconsistent with the existence of a Woodin cardinal. ∎

Remark. We do not know if Main Theorem 1 may be strengthened to $\boldsymbol{\Delta}^1_2$-TD (hence $\boldsymbol{\Delta}^1_2$-AD). If this is true, then by Steel [8] it will give a characterization of the $\leq_M$ Conjecture for uniformly order preserving $\boldsymbol{\Pi}^1_1$

functions. In general, one would like to understand better the role of order preserving functions in the study of the $\leq_M$ Conjecture. For example, it is not clear if Corollary 3.4 applies to functions which are order preserving.

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