

## Augmented $\Gamma$ -spaces, the stable rank filtration, and a $bu$ analogue of the Whitehead conjecture

by

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**Abstract.** We explore connections between our previous paper [J. Reine Angew. Math. 604 (2007)], where we constructed spectra that interpolate between  $bu$  and  $H\mathbb{Z}$ , and earlier work of Kuhn and Priddy on the Whitehead conjecture and of Rognes on the stable rank filtration in algebraic  $K$ -theory. We construct a “chain complex of spectra” that is a  $bu$  analogue of an auxiliary complex used by Kuhn–Priddy; we conjecture that this chain complex is “exact”; and we give some supporting evidence. We tie this to work of Rognes by showing that our auxiliary complex can be constructed in terms of the stable rank filtration. As a by-product, we verify for the case of topological complex  $K$ -theory a conjecture made by Rognes about the connectivity (for certain rings) of the filtration subquotients of the stable rank filtration of algebraic  $K$ -theory.

**1. Introduction.** In [4], we introduced a sequence  $\{A_m\}$  of spectra interpolating between the connective complex  $K$ -theory spectrum  $bu$  and the integral Eilenberg–MacLane spectrum  $H\mathbb{Z}$ . These new spectra resulted from a general construction on permutative categories endowed with an “augmentation.” In the current work, we explore connections of that construction to other settings, in particular to work of Kuhn and Priddy [9] on the Whitehead Conjecture, and to work of Rognes [15] on the stable rank filtration of algebraic  $K$ -theory. Connections to Kuhn and Priddy’s work were suggested by the many properties that the spectra  $A_m$  share with the symmetric powers of the sphere spectrum,  $Sp^m(\mathbf{S})$ , which can also be given as an example of the categorical construction of [4]. This led us to call  $A_m$  the “ $bu$  analogue” of  $Sp^m(\mathbf{S})$  and to propose a  $bu$  analogue of the Whitehead Conjecture. On the other hand, our construction was also reminiscent of the stable rank filtration in algebraic  $K$ -theory, and this made it natural to ask about the exact relationship between the two filtrations for  $bu$ . Curiously, the two

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threads converged: in this paper we construct a *bu* analogue of an auxiliary complex introduced by Kuhn and Priddy in the course of their proof of the Whitehead conjecture, and it turns out to be closely related to the stable rank filtration for topological complex *K*-theory. As a by-product, we obtain a good understanding of the stable rank filtration in the case of *bu*; in particular, we are able to verify in this case the connectivity conjecture made by Rognes in the context of Euclidean domains or local rings ([15, Conjecture 12.3], though Rognes does not explicitly mention topological rings as part of his context).

To describe our results in detail, we need to recall the overall setup of [4]. Let  $\mathcal{C}$  be a permutative category. To such a category, Segal’s machine [16] associates a spectrum that we will denote  $\mathbf{k}\mathcal{C}$  in the general case. The simplest example is  $\mathbb{N}$ , the permutative “category” with nonnegative integers as objects, no nonidentity morphisms, and permutative structure given by addition. In this case the associated spectrum is the integral Eilenberg–MacLane spectrum, which we denote as usual by  $\mathbb{H}\mathbb{Z}$ , rather than  $\mathbf{k}\mathbb{N}$ . A permutative category  $\mathcal{C}$  is called augmented if there is a symmetric monoidal functor  $\epsilon : \mathcal{C} \rightarrow \mathbb{N}$  such that  $\epsilon^{-1}(0)$  is the trivial one-object category. An augmentation induces a map of spectra  $\mathbf{k}\mathcal{C} \rightarrow \mathbb{H}\mathbb{Z}$ . For example, if  $\mathcal{S}$  is the category of finite pointed sets, with augmentation given by non-basepoint cardinality, then  $\mathbf{k}\mathcal{S}$  is the sphere spectrum  $\mathbf{S}$ , and the augmentation induces the Hurewicz map  $\mathbf{S} \rightarrow \mathbb{H}\mathbb{Z}$ . Similarly, the category of complex vector spaces and unitary isomorphisms can be augmented by dimension, and this augmentation induces a map  $bu \rightarrow \mathbb{H}\mathbb{Z}$ , which is the first structure map in the tower of connective covers of complex *K*-theory.

Let  $\mathcal{C}$  be an augmented permutative category. The main construction of [4] associates to  $\mathcal{C}$  a sequence of permutative categories

$$(1.1) \quad \mathcal{C} = \mathcal{K}_0\mathcal{C} \rightarrow \mathcal{K}_1\mathcal{C} \rightarrow \cdots \rightarrow \mathcal{K}_m\mathcal{C} \rightarrow \cdots \rightarrow \mathcal{K}_\infty\mathcal{C} \simeq \mathbb{N}.$$

The associated spectra and maps refine the map  $\mathbf{k}\mathcal{C} \rightarrow \mathbb{H}\mathbb{Z}$ ,

$$(1.2) \quad \mathbf{k}\mathcal{C} \rightarrow \mathbf{k}\mathcal{K}_1\mathcal{C} \rightarrow \cdots \rightarrow \mathbf{k}\mathcal{K}_m\mathcal{C} \rightarrow \cdots \rightarrow \mathbb{H}\mathbb{Z}.$$

A key example is  $\mathcal{C} = \mathcal{S}$ , which gives  $\mathbf{k}\mathcal{K}_m\mathcal{C} \simeq \mathrm{Sp}^m(\mathbf{S})$  and recovers the classical filtration of  $\mathbb{H}\mathbb{Z} \simeq \mathrm{Sp}^\infty(\mathbf{S})$  by the finite symmetric powers of the sphere spectrum. The other primary example in [4] is the category of finite-dimensional complex vector spaces and unitary isomorphisms, augmented by dimension. In keeping with the notation in [4], we use  $A_m$  to denote the spectrum  $\mathbf{k}\mathcal{K}_m\mathcal{C}$  in the complex vector space case.

Standard manipulations of cofiber sequences allow us to recast a sequence such as (1.2) as a “chain complex” of spectra involving the successive cofibers, and one can then ask whether this complex is “exact.” (See Section 2.1.) This question at a prime  $p$  for the category of finite pointed sets

gives the classical Whitehead Conjecture, stated below, which was proved by Kuhn for  $p = 2$  and by Kuhn and Priddy for odd primes. When localized at the prime  $p$ , the sequence  $\mathrm{Sp}^m(\mathbf{S})$  only changes at powers of  $p$ , and we adopt the usual notation

$$L(k) = \Sigma^{-k} \mathrm{Sp}^{p^k}(\mathbf{S}) / \mathrm{Sp}^{p^k-1}(\mathbf{S}).$$

Note that  $L(0) = S^0$ , and that in the following theorem, the map  $L(0) \rightarrow \mathrm{HZ}$  is the Hurewicz map.

**THEOREM 1.3** (Whitehead Conjecture, [7, 9]). *The complex*

$$(1.4) \quad \cdots \rightarrow L(2) \rightarrow L(1) \rightarrow L(0) \rightarrow \mathrm{HZ}$$

*is exact at the prime  $p$ .*

Because [4] established striking similarities between the subquotients  $A_m/A_{m-1}$  and  $\mathrm{Sp}^m(\mathbf{S})/\mathrm{Sp}^{m-1}(\mathbf{S})$  (that is, between the subquotients of (1.2) for finite-dimensional complex vector spaces and finite pointed sets, respectively), we were led to conjecture a *bu* analogue of the Whitehead Conjecture. Again working at a prime  $p$ , we let

$$T(k) = \Sigma^{-(k+1)} A_{p^k} / A_{p^k-1}.$$

**CONJECTURE 1.5** (*bu* Whitehead Conjecture). *The complex*

$$(1.6) \quad \cdots \rightarrow T(2) \rightarrow T(1) \rightarrow T(0) \rightarrow \mathrm{bu} \rightarrow \mathrm{HZ}$$

*is exact at the prime  $p$ .*

In considering the potential for adapting Kuhn and Priddy's methods to prove Conjecture 1.5, we see that Kuhn and Priddy did not actually work directly with the complex  $\{L(k)\}$  in the proof of Theorem 1.3. Instead, they constructed an auxiliary complex  $\{M(k)\}$  defined by  $M(k) = \Sigma^{-k} D(k)/D(k-1)$ , where  $D(k)$  is the cofiber of the  $p$ -fold diagonal map  $\mathrm{Sp}^{p^k-1}(\mathbf{S}) \rightarrow \mathrm{Sp}^{p^k}(\mathbf{S})$ . They proved that

$$(1.7) \quad \cdots \rightarrow M(2) \rightarrow M(1) \rightarrow M(0) \rightarrow \mathrm{HZ}/p$$

is exact at the prime  $p$  (the “mod  $p$  Whitehead Conjecture”) and then used this to obtain Theorem 1.3.

In this paper, we define spectra  $\bar{M}(k)$  that may play a similar role in a *bu* version of the Whitehead Conjecture to the role played by  $M(k)$  in the classical Whitehead Conjecture. The first part of the paper constructs the spectra  $\bar{M}(k)$ , studies their properties, and gives evidence for the exactness of the complex  $\{\bar{M}(k)\}$  based on the calculus of functors. The second part of the paper sets out a fairly general categorical construction related to the rank filtration and explores some of its technical properties. The last part of the paper shows that  $\bar{M}(k)$  bears a close relationship to the categorical construction introduced in the second part, as well as to the  $k$ th filtration

quotient of the stable rank filtration of Rognes (if one were to apply the stable rank filtration to topological complex  $K$ -theory). In the remainder of this introduction, we summarize each of these segments and highlight the new results.

The spectra  $\bar{M}(k)$  are constructed in Section 2.1. We use the fact that the sequence (1.1) is natural in augmented permutative categories, and we consider the functor from finite pointed sets to finite-dimensional complex vector spaces that takes a set  $S$  with basepoint  $*$  to  $\mathbb{C}(S)/\mathbb{C}(*)$ . This functor respects the augmentation and induces maps of spectra  $\mathrm{Sp}^m(\mathbf{S}) \rightarrow A_m$  compatible with the inclusions  $\mathrm{Sp}^{m-1}(\mathbf{S}) \rightarrow \mathrm{Sp}^m(\mathbf{S})$  and  $A_{m-1} \rightarrow A_m$ . Further, the two sequences of spectra share the property that  $\mathrm{Sp}^{m-1}(\mathbf{S}) \rightarrow \mathrm{Sp}^m(\mathbf{S})$  and  $A_{m-1} \rightarrow A_m$  are equivalences at a prime  $p$  unless  $m$  is a power of  $p$ . We index logarithmically and denote the cofiber of  $\mathrm{Sp}^{p^k}(\mathbf{S}) \rightarrow A_{p^k}$  by  $C(k)$ , by analogy to Kuhn and Priddy's  $D(k)$ , and we define

$$(1.8) \quad \bar{M}(k) = \Sigma^{-(k+1)}C(k)/C(k-1),$$

by analogy to Kuhn and Priddy's  $M(k)$ . The standard manipulation fits these spectra together into a chain complex,

$$(1.9) \quad \cdots \rightarrow \bar{M}(2) \rightarrow \bar{M}(1) \rightarrow \bar{M}(0) \rightarrow bu.$$

The following conjecture is analogous to the exactness of (1.7).

**CONJECTURE 2.13** (Reduced  $bu$  Whitehead Conjecture). The complex (1.9) is exact at the prime  $p$ .

There are compelling similarities between the spectra  $M(k)$  of (1.7) and the spectra  $\bar{M}(k)$  of (1.9). As a first example, we compare the relationship of (1.4) and (1.7) in the classical case with the relationship of (1.6) and (1.9) in the  $bu$  case. In the classical case,  $M(k)$  is a cofiber of subquotients of symmetric powers of the sphere spectrum, and it sits in a cofiber sequence

$$\Sigma^{-1}L(k-1) \rightarrow L(k) \rightarrow M(k).$$

(See (2.8).) It turns out that the first map is null-homotopic, and so  $M(k)$  splits as a wedge sum,  $M(k) \simeq L(k) \vee L(k-1)$  [13]. Similarly in the  $bu$  situation,  $\bar{M}(k)$  is a cofiber of subquotients of symmetric powers of the sphere spectrum and subquotients of the spectra  $A_{p^k}$  filtering  $bu$ , and there is a cofiber sequence

$$(1.10) \quad \Sigma^{-1}L(k) \rightarrow T(k) \rightarrow \bar{M}(k).$$

(See (2.12).) We prove the following proposition.

**PROPOSITION 2.14.** In (1.10), the map  $\Sigma^{-1}L(k) \rightarrow T(k)$  is null-homotopic. Thus  $\bar{M}(k)$  splits as a wedge sum  $\bar{M}(k) \simeq T(k) \vee L(k)$ .

A second characteristic of  $M(k)$  that is important in Kuhn and Priddy's work is that the Steinberg idempotent in  $\mathbb{F}_p[\mathrm{GL}_k(\mathbb{F}_p)]$  splits  $M(k)$  off from  $\Sigma^\infty(B\Delta_k)_+$ , where  $\Delta_k \cong (\mathbb{Z}/p)^k$  is a transitive elementary abelian  $p$ -subgroup of  $\Sigma_{p^k}$ . Being a wedge summand of a suspension spectrum makes  $M(k)$  a projective spectrum in the sense of Kuhn's homological algebra of spectra. Also, because  $M(k)$  naturally splits from the suspension spectrum of a classifying space (rather than a Thom space, as for  $L(k)$ ), it is easier to apply techniques involving the transfer. We establish a similar result for our spectra  $\overline{M}(k)$ . The group that corresponds to  $\Delta_k$  in the unitary situation is the irreducible projective elementary abelian  $p$ -subgroup  $\Gamma_k$  of  $U(p^k)$ , described in [4, Section 10].

PROPOSITION 2.15.  $\overline{M}(k)$  is the stable wedge summand of  $\Sigma^\infty(B\Gamma_k)_+$  at the prime  $p$  corresponding to the symplectic Steinberg idempotent in the group ring  $\mathbb{F}_p[\mathrm{Sp}_{2k}(\mathbb{F}_p)]$ . Thus  $\overline{M}(k)$  is a projective spectrum, in the sense of Kuhn.

While the spectra  $\overline{M}(k)$  may seem at first to be an ad hoc construction, it turns out that they are an example of a general categorical construction closely related to Rognes's stable rank filtration of algebraic  $K$ -theory. We set up this construction in detail in Section 3.1, but we summarize it here. Recall that a  $\Gamma$ -space is a pointed functor from the category of finite pointed sets to the category of pointed simplicial sets. An important example of a  $\Gamma$ -space is given by the infinite symmetric product functor  $\mathrm{Sp}^\infty$ , whose stabilization is the integral Eilenberg–MacLane spectrum  $\mathrm{H}\mathbb{Z}$ . We say that a  $\Gamma$ -space  $F$  is “augmented” if it comes equipped with a natural transformation  $\epsilon : F \rightarrow \mathrm{Sp}^\infty$  such that  $\epsilon^{-1}(\mathrm{Sp}^0) = *$  (Definition 3.3). Then we give  $F$  a natural filtration  $\mathcal{R}_m F$  by pulling back the filtration of  $\mathrm{Sp}^\infty$  by finite symmetric powers. If  $F$  is obtained by applying Segal's construction to a permutative augmented category  $\mathcal{C}$  (Definition 3.7), then we denote this filtration  $\mathcal{R}_m \mathcal{C}$  and call it the “modified rank filtration” of the  $\Gamma$ -space associated to  $\mathcal{C}$ . There is a corresponding filtration of the  $K$ -theory spectrum of  $\mathcal{C}$ , and we call this the “modified stable rank filtration” of the  $K$ -theory spectrum  $k\mathcal{C}$  (Definition 3.9). It differs from the original stable rank filtration of algebraic  $K$ -theory given by Rognes in taking place within the framework of Segal's  $\Gamma$ -spaces rather than Waldhausen's  $S_\bullet$  construction, but it is otherwise similar in spirit.

In Sections 3.2 and 3.3, we do some technical work to prepare for the main goal of Section 4.1, namely to exhibit  $\overline{M}(k)$  as a particular example of the modified stable rank filtration. We say that an augmentation-preserving map  $F \rightarrow G$  of augmented  $\Gamma$ -spaces is a “strong augmented (stable) equivalence” if it induces a (stable) equivalence  $\mathcal{R}_m F \rightarrow \mathcal{R}_m G$  at each level  $m$  of the filtration associated to the augmentation. The goal of Section 3.2 is to prove

the proposition below, establishing that a bar construction of appropriate  $\Gamma$ -spaces is strongly equivalent to an objectwise homotopy pushout; this gives a tool to understand the augmentation-induced filtration of  $A_m$ , which is defined in [4] by a bar construction.

PROPOSITION 3.20. Suppose given a diagram of  $\Gamma$ -spaces

$$F_1 \leftarrow F_0 \rightarrow F_2$$

that is a diagram of augmented monoids, and suppose that  $F_{12}$  is the objectwise homotopy pushout. Then the natural map

$$F_{12} \rightarrow \text{Bar}(F_1, F_0, F_2)$$

is a strong augmented stable equivalence.

In Section 3.3, we study filtered  $\Gamma$ -spaces that are “very special,” in the sense of Segal. The primary example we have in mind is, of course, the  $\Gamma$ -space associated to a permutative augmented category, because we want to relate the modified stable rank filtration in this case to the spectra  $\bar{M}(k)$  in the reduced *bu* Whitehead Conjecture. Given a map between augmented, very special  $\Gamma$ -spaces, the main result of Section 3.3 allows us to identify the first place where the filtrations differ, and to describe the difference in terms of the value of the  $\Gamma$ -spaces on  $S^0$ .

PROPOSITION 3.24. Let  $F \rightarrow G$  be an augmentation-preserving map of augmented very special  $\Gamma$ -spaces, and suppose that  $\mathcal{R}_i[F(\underline{1})] \rightarrow \mathcal{R}_i[G(\underline{1})]$  is a homotopy equivalence for all  $i < m$ . Then  $\mathcal{R}_i[F(X)] \rightarrow \mathcal{R}_i[G(X)]$  is a homotopy equivalence for all  $i < m$  and all  $X$ , and the commuting diagram

$$\begin{array}{ccc} \mathcal{R}_m[F(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[F(X)] \\ \downarrow & & \downarrow \\ \mathcal{R}_m[G(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[G(X)] \end{array}$$

is a strong homotopy pushout diagram of augmented  $\Gamma$ -spaces, that is, it remains a homotopy pushout square after the application of  $\mathcal{R}_j$  for all  $j$ .

In the final part of the paper, we reach our goal of tying the spectra  $\bar{M}(k)$ , defined in Section 2 in terms of [4], to the stable rank filtration of Rognes. First, in Section 4.1, we study the relationship between the modified stable rank filtration and the construction of [4]. Given an augmented permutative category  $\mathcal{C}$ , the modified stable rank filtration is a filtration from  $*$  to  $\mathbf{k}\mathcal{C}$ , while the spectra  $A_m$  arising from the construction of [4] applied to  $\mathcal{C}$  give a filtration going from  $\mathbf{k}\mathcal{C}$  to  $\text{HZ}$ . The following theorem gives the relationship.

THEOREM 4.4. For every  $m$ , there is a stable homotopy pushout square of augmented  $\Gamma$ -spaces

$$\begin{array}{ccc} \mathcal{R}_m\mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m & \longrightarrow & \mathcal{K}_m\mathcal{C} \end{array}$$

and hence a homotopy pushout diagram of spectra

$$\begin{array}{ccc} \mathcal{R}_m\mathbf{k}\mathcal{C} & \longrightarrow & \mathbf{k}\mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m(\mathbf{S}) & \longrightarrow & A_m \end{array}$$

As an immediate consequence, we find that when  $\mathcal{C}$  is the category of finite-dimensional complex vector spaces, the subquotients of the stable rank filtration are actually the same as the spectra  $\overline{M}(k)$ .

COROLLARY 4.5. There is an equivalence  $\overline{M}(k) \simeq \mathcal{R}_{p^k}\mathbf{k}\mathcal{C}/\mathcal{R}_{p^{k-1}}\mathbf{k}\mathcal{C}$ .

Lastly, in Section 4.2 we address the relationship of the modified stable rank filtration and the original stable rank filtration defined by Rognes. Consider the category of finitely generated free modules over a nice commutative ring  $R$ , which is the context of [15] (where, however, the topological case is not explicitly considered). The associated spectrum is the free  $K$ -theory spectrum of  $R$ . We include here the case where  $R$  is  $\mathbb{R}$  or  $\mathbb{C}$  by considering the topologically enriched category of real or complex vector spaces, with associated spectrum  $bo$  or  $bu$ . In Section 4.2 we show that the natural map from Segal's construction of  $K$ -theory to Waldhausen's, described in Section 1.8 of [17], induces a map from the modified stable rank filtration to the original stable rank filtration of Rognes. The following proposition establishes that the map is in fact an equivalence of filtrations in the special case of topological  $K$ -theory. We note, however, that the equivalence comes about because the map on the associated graded is the inclusion of block diagonal matrices into block upper triangular matrices, which is a homotopy equivalence over the complex numbers, but not generally for discrete rings. Hence the stable rank filtration and the modified stable rank filtration will typically be different for discrete rings.

PROPOSITION 4.10. Let  $\mathcal{C}$  be the topological category of finite-dimensional complex vector spaces, let  $\mathcal{R}_m\mathbf{k}\mathcal{C}$  be the modified stable rank filtration of  $\mathbf{k}\mathcal{C} = bu$ , and let  $F_m bu$  be the  $m$ th stable rank filtration of Rognes as applied to complex topological  $K$ -theory. Then the canonical map of filtrations  $\mathcal{R}_m\mathbf{k}\mathcal{C} \rightarrow F_m bu$  is a homotopy equivalence of spectra for each  $m$ .

The proposition, together with Theorem 4.4 and the analysis in [4], yields a good understanding of the stable rank filtration of  $bu$ . Let  $F_m bu$  be the  $m$ th stage in the original stable rank filtration of  $bu$ . It follows from Proposition 4.10, Theorem 4.4, and Proposition 2.14 that there is an equivalence

$$F_m bu / F_{m-1} bu \simeq \Sigma^{-1} A_m / A_{m-1} \vee \mathrm{Sp}^m(\mathbf{S}) / \mathrm{Sp}^{m-1}(\mathbf{S}).$$

We use this to prove the following result about the connectivity of the subquotients of this filtration, which confirms for the special case of topological complex  $K$ -theory the general conjecture made by Rognes in [15] (Conjecture 12.3) for certain discrete rings.

**PROPOSITION 4.11.** The subquotient spectrum  $F_m bu / F_{m-1} bu$  of the stable rank filtration of Rognes is contractible unless  $m = p^k$  for some prime  $p$ . If  $m = p^k$ , then the bottom nontrivial homotopy group of  $F_m bu / F_{m-1} bu$  occurs in dimension  $2m - 2$ .

We conclude this introduction with an open-ended remark about the possible minimality of the various filtrations mentioned here. In [4] we constructed the complex of spectra  $T(k)$  over the fiber of  $bu \rightarrow \mathbb{H}\mathbb{Z}$ , while in this paper we construct the complex of spectra  $\bar{M}(k)$  over the spectrum  $bu$  itself. It is a consequence of Proposition 2.14 that the complex  $\{T(k)\}$  is strictly smaller than the complex  $\{\bar{M}(k)\}$ , which arises from the stable rank filtration (by Corollary 4.5). In [4] we conjectured that the complex  $\{T(k)\}$  is in fact minimal in the case of  $bu$ . Possibly it is reasonable to expect that the filtration constructed in [4] is minimal in other cases as well and is strictly smaller than the corresponding stable rank filtration. We also wonder if one can learn something interesting in the case of the algebraic  $K$ -theory of a ring  $R$  by studying the relationship between the stable rank filtration of Rognes and the modified stable rank filtration that we define in this paper.

The organization of the rest of the paper is as follows. In Section 2, we construct the complex  $\{\bar{M}(k)\}$  and prove various results about it, in particular Propositions 2.14 and 2.15 that were mentioned in the introduction. We also discuss the calculus evidence for Conjecture 2.13. In Section 3, we introduce the formalism of augmented  $\Gamma$ -spaces and define the modified stable rank filtration. We prove various technical results about constructions involving augmented  $\Gamma$ -spaces, which we need in the subsequent section. In Section 4, we prove Theorem 4.4, which gives a general result about the relationship of the modified stable rank filtration to the construction in [4]. This implies that the complex  $\{\bar{M}(k)\}$  is closely related to the modified stable rank filtration for complex  $K$ -theory,  $bu$ . Finally, we prove Proposition 4.10 to show that in the case of complex  $K$ -theory, the modified stable rank filtration agrees with Rognes's original filtration, and we use this to



prove Proposition 4.11, which confirms Rognes's connectivity conjecture for the case of  $bu$ .

**2. The reduced complex  $\overline{M}(k)$ .** As we mentioned in the introduction, Kuhn and Priddy's proof of the Whitehead Conjecture does not directly attack the complex  $\{L(k)\}$  to show that it is a projective resolution of  $H\mathbb{Z}$ . Rather, they use a related auxiliary complex  $\{M(k)\}$ , which they prove is a projective resolution of  $H\mathbb{Z}/p$ . One relevant difference between the two complexes is that the spectra  $L(k)$  are most naturally thought of as Thom spectra, while the spectra  $M(k)$  are most naturally thought of as stable summands of classifying spaces. Thus techniques from the transfer are more easily applied to the spectra  $M(k)$  than to  $L(k)$ .

In this section, we construct a complex  $\{\overline{M}(k)\}$  of spectra that appears to be the  $bu$  analogue of the complex  $\{M(k)\}$  in the classical situation. In Section 2.1, we recall the relevant definitions in Kuhn's homological algebra of spectra, we construct the spectra  $\overline{M}(k)$ , and we conjecture that the complex  $\{\overline{M}(k)\}$  is exact. In Section 2.2, we give technical material on universal spaces of collections of subgroups, which leads to two results about  $\overline{M}(k)$  (Propositions 2.14 and 2.15) that are parallel to what is known about the classical  $M(k)$  (Propositions 2.9 and 2.10). In Section 2.3, we give evidence based on the calculus of functors for the exactness of the complex  $\{\overline{M}(k)\}$ .

**2.1. Construction of the complex  $\overline{M}(k)$ .** Our first order of business is to define a complex in the context of  $bu$  that is an analogue of the projective resolution of  $H\mathbb{Z}/p$  in (1.7). We recall the relevant homological concepts from [8] and review the construction of the auxiliary complex  $\{M(k)\}$  used by Kuhn and Priddy. Then we follow the same template with different ingredients to construct the complex  $\{\overline{M}(k)\}$ , and we set out the parallels that we know and those that we conjecture between  $\{M(k)\}$  and  $\{\overline{M}(k)\}$ .

DEFINITION 2.1 ([8, Section 2]).

- (1) A spectrum is called *free* if it is the suspension spectrum of a space, and it is called *projective* if it is a wedge summand of a free spectrum.
- (2) A *chain complex (over  $E_0$ ) of spectra* is a sequence of spectra and maps between them,

$$(2.2) \quad \cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} E_0,$$

together with an extension to a diagram of the form

$$\cdots \rightarrow X_2 \xrightarrow{q_2} E_2 \xrightarrow{i_1} X_1 \xrightarrow{q_1} E_1 \xrightarrow{i_0} X_0 \xrightarrow{q_0} E_0,$$

where each sequence  $E_{n+1} \xrightarrow{i_n} X_n \xrightarrow{q_n} E_n$  is a cofiber sequence. (We will often specify explicitly only the diagram (2.2), with the extended diagram being implicitly understood.)

- (3) A cofiber sequence of spectra  $X \rightarrow Y \xrightarrow{q} Z$  is called *short exact* if the map  $\Omega^\infty q$  has a homotopy section, i.e., if there exists a map  $f : \Omega^\infty Z \rightarrow \Omega^\infty Y$  in the homotopy category such that the composed map  $(\Omega^\infty q) \circ f$  is a weak homotopy equivalence. A chain complex is called *exact* if each of the cofiber sequences  $E_{n+1} \rightarrow X_n \rightarrow E_n$  is short exact.

REMARK 2.3. Elementary diagram chasing establishes a bijective correspondence (up to a suitable notion of homotopy equivalence) between chain complexes over  $E_0$  and filtrations of the spectrum  $E_0$ , i.e., diagrams of spectra

$$F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow E_0 \simeq \text{hocolim } F_n.$$

(See [8, Remark 2.2].) To associate a chain complex to a filtration of this form, set  $X_n = \Sigma^{-n} F_n / F_{n-1}$  and  $E_n = \Sigma^{-n} E_0 / F_{n-1}$ . For the other direction, suppose we are given a chain complex as in (2.2). This structure provides maps  $E_n \rightarrow \Sigma E_{n+1}$  for each  $n$ , and hence maps  $E_0 \rightarrow \Sigma^{n+1} E_{n+1}$ . Define  $F_n$  to be the homotopy fiber of this map.

The following lemma establishes an alternative criterion for a chain complex to be exact.

LEMMA 2.4. *A chain complex of spectra*

$$\cdots \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1} \rightarrow \cdots \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} X_{-1} = E_0$$

is exact if and only if for  $n \geq -1$  there exist maps  $\epsilon_n : \Omega^\infty X_n \rightarrow \Omega^\infty X_{n+1}$  in the homotopy category such that for all  $n \geq -1$  the map

$$(2.5) \quad \epsilon_{n-1} \circ (\Omega^\infty \partial_n) + (\Omega^\infty \partial_{n+1}) \circ \epsilon_n$$

is a weak homotopy equivalence.

The maps  $\epsilon_n$  will be called a “contracting homotopy” for the chain complex. Note that if a chain complex is exact, then the homotopy spectral sequence for the associated filtration collapses at the second page.

*Proof of Lemma 2.4.* Suppose first that the chain complex is exact. Thus for each  $n$  there is a map

$$f_n : \Omega^\infty E_n \rightarrow \Omega^\infty X_n$$

that is a homotopy section of the structure map

$$\Omega^\infty q_n : \Omega^\infty X_n \rightarrow \Omega^\infty E_n.$$

For each  $n$ , we have the following homotopy commutative diagram, where

the rows are fibration sequences:

$$\begin{array}{ccccc}
 \Omega^\infty E_{n+1} & \longrightarrow & \Omega^\infty E_{n+1} \times \Omega^\infty E_n & \xrightarrow{\text{proj}} & \Omega^\infty E_n \\
 = \downarrow & & \downarrow \Omega^\infty i_n + f_n & & \downarrow = \\
 \Omega^\infty E_{n+1} & \xrightarrow{\Omega^\infty i_n} & \Omega^\infty X_n & \xrightarrow{\Omega^\infty q_n} & \Omega^\infty E_n
 \end{array}$$

The middle vertical map must be an equivalence, and we can define the map  $\epsilon_n$  to be the composite

$$\Omega^\infty X_n \xrightarrow{(\Omega^\infty i_n + f_n)^{-1}} \Omega^\infty E_{n+1} \times \Omega^\infty E_n \xrightarrow{\text{proj}} \Omega^\infty E_{n+1} \xrightarrow{f_{n+1}} \Omega^\infty X_{n+1}.$$

Elementary diagram chasing establishes that the maps  $\epsilon_n$  satisfy (2.5).

Conversely, suppose that we have a chain complex endowed with maps  $\epsilon_n$  satisfying (2.5). We define maps  $f_n : \Omega^\infty E_n \rightarrow \Omega^\infty X_n$  by  $f_n = \epsilon_{n-1} \circ (\Omega^\infty i_{n-1})$ . An inductive argument shows that  $f_n$  is a homotopy section of  $\Omega^\infty q_n$ . ■

Having established the terminology and notation for chain complexes of spectra, we now turn to the auxiliary spectra  $M(k)$  used by Kuhn and Priddy. In studying the filtration of  $\mathbb{H}\mathbb{Z}$  by  $\text{Sp}^m(\mathbf{S})$  at a particular prime  $p$ , one can focus on values of  $m$  that are powers of  $p$ , because otherwise  $\text{Sp}^{m-1}(\mathbf{S}) \rightarrow \text{Sp}^m(\mathbf{S})$  is an equivalence at the prime  $p$ . In point of fact, Kuhn and Priddy work with a version of the symmetric power filtration that is reduced mod  $p$ , as follows. Let  $D(k)$  be the cofiber of the  $p$ -fold diagonal map  $\text{Sp}^{p^{k-1}}(\mathbf{S}) \rightarrow \text{Sp}^{p^k}(\mathbf{S})$ . The filtration of  $\mathbb{H}\mathbb{Z}$  by symmetric powers of the sphere spectrum fits into a diagram with a filtration of  $\mathbb{H}\mathbb{Z}/p$  by the spectra  $D(k)$  as follows:

$$\begin{array}{ccccccc}
 * = \text{Sp}^0(\mathbf{S}) & \longrightarrow & \text{Sp}^1(\mathbf{S}) & \longrightarrow & \text{Sp}^p(\mathbf{S}) & \longrightarrow & \cdots \longrightarrow \mathbb{H}\mathbb{Z} \\
 \downarrow & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \times p \\
 (2.6) \quad \mathbf{S} = \text{Sp}^1(\mathbf{S}) & \longrightarrow & \text{Sp}^p(\mathbf{S}) & \longrightarrow & \text{Sp}^{p^2}(\mathbf{S}) & \longrightarrow & \cdots \longrightarrow \mathbb{H}\mathbb{Z} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{S} =: D(0) & \longrightarrow & D(1) & \longrightarrow & D(2) & \longrightarrow & \cdots \longrightarrow \mathbb{H}\mathbb{Z}/p
 \end{array}$$

Here the horizontal maps between symmetric powers are induced by basepoint inclusions, while the vertical maps are  $p$ -fold diagonal maps, and all spectra are implicitly localized at  $p$ . The columns are by definition homotopy cofiber sequences. To discuss the associated chain complexes, define  $L(k) := \Sigma^{-k} \text{Sp}^{p^k}(\mathbf{S})/\text{Sp}^{p^{k-1}}(\mathbf{S})$  and  $M(k) := \Sigma^{-k} D(k)/D(k-1)$ . (At the beginning of the sequence,  $L(0) = M(0) = \mathbf{S}$ .)

REMARK 2.7. Recall once again that for  $k > 0$  the maps  $\text{Sp}^{p^{k-1}}(\mathbf{S}) \rightarrow \text{Sp}^{p^k}(\mathbf{S})$  are equivalences at  $p$ . Hence the spectra  $\text{Sp}^{p^k}(\mathbf{S})/\text{Sp}^{p^{k-1}}(\mathbf{S})$  and

$\mathrm{Sp}^{p^k}(\mathbf{S})/\mathrm{Sp}^{p^k-1}(\mathbf{S})$  are equivalent after localization at  $p$ , and either one could be used to define  $L(k)$ . In fact, for  $k > 0$ , the difference between the two spectra is that the latter is  $p$ -local, while the former is not, so the latter is the  $p$ -localization of the former.

Each row in (2.6) now gives us a chain complex,

$$(2.8) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \Sigma^{-1}L(1) & \longrightarrow & \Sigma^{-1}L(0) & \longrightarrow & * & \longrightarrow & \mathrm{HZ} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \times p \\ \cdots & \longrightarrow & L(2) & \longrightarrow & L(1) & \longrightarrow & L(0) & \longrightarrow & \mathrm{HZ} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & M(2) & \longrightarrow & M(1) & \longrightarrow & M(0) & \longrightarrow & \mathrm{HZ}/p \end{array}$$

and again each column is a homotopy cofiber sequence. In fact, it turns out that this cofiber sequence splits.

PROPOSITION 2.9 ([13, Proposition 5.15]).  *$M(k)$  splits as a wedge sum:*

$$M(k) \simeq L(k) \vee L(k-1).$$

Another important feature of the chain complexes of diagram (2.8) is that each spectrum  $M(k)$  is projective, again from Mitchell and Priddy's earlier work. This is important in Kuhn and Priddy's proof because they use the standard homological technique of inducing a map from a projective complex to an acyclic complex. Let  $\Delta_k$  be a transitive elementary abelian  $p$ -subgroup of  $\Sigma_{p^k}$ . ( $\Delta_k$  is unique up to conjugacy.)

PROPOSITION 2.10 ([13, Theorem 5.1]).  *$M(k)$  splits off from  $\Sigma^\infty(B\Delta_k)_+$  as the stable wedge summand corresponding to the Steinberg idempotent in  $\mathbb{F}_p[GL_k(\mathbb{F}_p)]$ .*

To define the *bu* analogue of this setup, we follow the same template with different ingredients. Recall that the construction of [4], applied to finite pointed sets and taken at powers of  $p$ , gives exactly the filtration

$$\mathbf{S} = \mathrm{Sp}^1(\mathbf{S}) \rightarrow \mathrm{Sp}^p(\mathbf{S}) \rightarrow \mathrm{Sp}^{p^2}(\mathbf{S}) \rightarrow \cdots \rightarrow \mathrm{HZ},$$

while applying it to finite-dimensional complex vector spaces gives a filtration

$$bu = A_0 \rightarrow A_1 \rightarrow A_p \rightarrow \cdots \rightarrow \mathrm{HZ}.$$

The *bu* analogue of (2.6) retains the top row, and replaces the middle row by the spectra  $A_{p^k}$ . The map between the first row and the second row is induced by the natural map of categories from finite pointed sets to finite-dimensional complex vector spaces, and we define spectra  $C(k)$  as the ver-

tical cofibers, indexed logarithmically, by analogy <sup>(1)</sup> to the spectra  $D(k)$  of (2.6):

$$(2.11) \quad \begin{array}{ccccccc} * & \longrightarrow & \mathrm{Sp}^1(\mathbf{S}) & \longrightarrow & \mathrm{Sp}^p(\mathbf{S}) & \longrightarrow & \cdots \longrightarrow & \mathrm{HZ} \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ bu = A_0 & \longrightarrow & A_1 & \longrightarrow & A_p & \longrightarrow & \cdots \longrightarrow & \mathrm{HZ} \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ bu =: C(-\infty) & \longrightarrow & C(0) & \longrightarrow & C(1) & \longrightarrow & \cdots \longrightarrow & * \end{array}$$

We now pass to subquotients with the following notation:

$$L(k) := \Sigma^{-k} \mathrm{Sp}^{p^k}(\mathbf{S}) / \mathrm{Sp}^{p^{k-1}}(\mathbf{S}),$$

$$T(k) := \Sigma^{-(k+1)} A_{p^k} / A_{p^{k-1}},$$

$$\bar{M}(k) := \Sigma^{-(k+1)} C(k) / C(k-1).$$

(Note that  $A_{p^{k-1}} \rightarrow A_{p^{k-1}}$  is an equivalence at  $p$ , and the appropriate variation of Remark 2.7 applies to the definition of  $T(k)$ .) Thus we obtain the following diagram of chain complexes, where all spectra are implicitly localized at  $p$ , and in fact, all spectra except for the rightmost three columns were  $p$ -local from the outset.

$$(2.12) \quad \begin{array}{ccccccc} \cdots \Sigma^{-1}L(2) & \longrightarrow & \Sigma^{-1}L(1) & \longrightarrow & \Sigma^{-1}L(0) & \longrightarrow & * \longrightarrow & \mathrm{HZ} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \simeq \\ \cdots T(2) & \longrightarrow & T(1) & \longrightarrow & T(0) & \longrightarrow & bu & \longrightarrow & \mathrm{HZ} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \bar{M}(2) & \longrightarrow & \bar{M}(1) & \longrightarrow & \bar{M}(0) & \longrightarrow & bu & \longrightarrow & * \end{array}$$

(At the beginning of the sequence,  $L(0) = \mathbf{S}$ ,  $T(0) = \Sigma^\infty \mathbb{C}P^\infty$ , and  $\bar{M}(0) \simeq \Sigma^\infty \mathbb{C}P_+^\infty \simeq T(0) \vee L(0)$ .) The conjecture that corresponds to the exactness of (1.7), which was the main computation performed in [9], is the following.

**CONJECTURE 2.13 (Reduced  $bu$  Whitehead Conjecture).** *The complex*

$$\cdots \rightarrow \bar{M}(2) \rightarrow \bar{M}(1) \rightarrow \bar{M}(0) \rightarrow bu$$

*is exact at the prime  $p$ .*

In the next two subsections, we elaborate further on this conjecture in two ways. In Section 2.2, we establish analogues of Propositions 2.9 and 2.10. These establish that the spectra in Conjecture 2.13 are projective in the sense of Kuhn, and that the structure of diagram (2.12) is exactly parallel

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<sup>(1)</sup> Note that since the sequence  $\{C(k)\}$  is indexed logarithmically and the sequence  $\{A_{p^k}\}$  is not, we have an initial term  $C(-\infty)$ , corresponding to  $\log_p 0 = -\infty$ .

to that of diagram (2.6). Then in Section 2.3, we offer evidence from the calculus of functors for the correctness of Conjecture 2.13.

**2.2. Collections of subgroups.** In this subsection, we explore further the properties of the spectra  $\bar{M}(k)$  that establish them as appropriate analogues of the spectra  $M(k)$  used by Kuhn and Priddy in the proof of the mod  $p$  Whitehead Conjecture. First, comparing (2.8) to (2.12) and looking at the columns, we see that the spectra  $L(k-1)$ ,  $L(k)$ , and  $M(k)$  in the classical situation correspond, respectively, to  $L(k)$ ,  $T(k)$ , and  $\bar{M}(k)$  in the *bu* analogue. Thus the result that corresponds to Proposition 2.9 is the following.

**PROPOSITION 2.14.** *In (2.12), the map  $\Sigma^{-1}L(k) \rightarrow T(k)$  is null-homotopic. Thus  $\bar{M}(k)$  splits as a wedge sum  $\bar{M}(k) \simeq T(k) \vee L(k)$ .*

Second, we show that  $\bar{M}(k)$  is projective, corresponding to Proposition 2.10. In fact, it turns out that to describe how  $\bar{M}(k)$  splits off of a suspension spectrum, we can use the dictionary that is used in [4] between the identity functor and symmetric groups on the one hand, and the functor  $V \mapsto BU(V)$  and the unitary groups on the other. As explained in [4, Section 10], the analogue of the transitive elementary abelian  $p$ -subgroup  $\Delta_k$  of  $\Sigma_{p^k}$  is the irreducible projective elementary abelian  $p$ -subgroup  $\Gamma_k$  of  $U(p^k)$ . Just as the Weyl group of  $\Delta_k$  in  $\Sigma_{p^k}$  is  $GL_k(\mathbb{F}_p)$ , the Weyl group of  $\Gamma_k$  in  $U(p^k)$  is the symplectic group  $Sp_{2k}(\mathbb{F}_p)$ . Thus the statement that corresponds to Proposition 2.10 is the following.

**PROPOSITION 2.15.**  *$\bar{M}(k)$  is the stable wedge summand of  $\Sigma^\infty(B\Gamma_k)_+$  at the prime  $p$  corresponding to the symplectic Steinberg idempotent in the group ring  $\mathbb{F}_p[Sp_{2k}(\mathbb{F}_p)]$ . Thus  $\bar{M}(k)$  is a projective spectrum, in the sense of Kuhn.*

The objects in Propositions 2.14 and 2.15 are defined in terms of subquotients of filtrations obtained by the constructions of [4]. Since that work identifies such subquotients in terms of collections of subgroups of certain automorphism groups, we devote much of this subsection to related calculations with universal spaces. The proofs of Propositions 2.14 and 2.15 appear towards the end of the subsection.

We begin by recalling some essential background. (A good reference is [2, Section 2].) Given a group  $G$ , a *collection*  $\mathcal{C}$  of subgroups of  $G$  is a set of subgroups of  $G$  that is closed under conjugation by elements of  $G$ . There is a universal  $G$ -space  $EC$ , terminal among  $G$ -spaces whose isotropy groups are in  $\mathcal{C}$ , which can be characterized by the following two properties: (i) all isotropy groups of  $EC$  are in  $\mathcal{C}$ , and (ii) if  $H \in \mathcal{C}$ , then  $(EC)^H \simeq *$ . The space  $EC$  has a standard construction as the nerve of the category whose objects are pairs  $(O, x)$ , where  $O$  is a transitive  $G$ -set with isotropy in  $\mathcal{C}$ ,

and  $x \in O$ . If we think of  $\mathcal{C}$  as a poset under inclusions and write  $|\mathcal{C}|$  for the nerve of the corresponding category, then there is a  $G$ -equivariant map  $EC \rightarrow |\mathcal{C}|$  given by taking  $(O, x)$  to the isotropy group of  $x$ , and by [2, 2.12] that map is  $G$ -equivariant and a homotopy equivalence (but not necessarily a  $G$ -equivalence).

We are interested in particular collections of subgroups of the unitary group  $U(m)$ . These collections were studied in [4, Section 9].

DEFINITION 2.16. Let  $H$  be a subgroup of  $U(m)$ .

- (1) We say  $H$  is *standard* if  $H$  is conjugate to a proper subgroup of the form  $\prod_{i=1}^s U(m_i)$ , where  $U(m_i)$  is the group of automorphisms of a subspace  $\mathbb{C}^{m_i} \subset \mathbb{C}^m$ . The collection of standard subgroups of  $U(m)$  is denoted  $\mathcal{R}_m$ .
- (2) We say that a standard subgroup  $H$  is *complete* if it is a proper subgroup conjugate to one of the form  $\prod_{i=1}^s U(m_i)$  where  $\sum m_i = m$ . The collection of complete subgroups of  $U(m)$  is denoted  $\mathcal{L}_m$ . We observe that  $\mathcal{L}_m$  is equivalent to the poset of proper direct sum decompositions of  $\mathbb{C}^m$ .

Note that all complete subgroups are standard;  $U(m)$  itself is neither standard nor complete; the trivial subgroup is standard but is not complete.

The heart of the proofs of Propositions 2.14 and 2.15 is a commutative ladder involving both homotopy orbit spaces and strict orbit spaces of universal spaces of collections of subgroups. Let  $X^\diamond$  denote the unreduced suspension of a space  $X$  and, if  $X$  is a  $G$ -space with a based action, let  $X_{\tilde{h}G}$  be the based (reduced) homotopy orbits,  $X_{\tilde{h}G} := EG_+ \wedge_G X$ . Let  $\mathcal{R}_{m,\text{ntrv}}$  be the subcollection of nontrivial standard subgroups of  $U(m)$ . By using the map  $S^0 \rightarrow S^{2m}$  that includes  $S^0$  as the poles of  $S^{2m}$ , and the passage from homotopy orbits to actual orbits, we can construct the following commutative ladder:

$$\begin{array}{ccc}
 (E\mathcal{R}_{m,\text{ntrv}}^\diamond \wedge S^0)_{\tilde{h}U(m)} & \longrightarrow & (E\mathcal{R}_m^\diamond \wedge S^0)_{\tilde{h}U(m)} \\
 \downarrow & & \downarrow \\
 (2.17) \quad (E\mathcal{R}_{m,\text{ntrv}}^\diamond \wedge S^0)_{U(m)} & \longrightarrow & (E\mathcal{R}_m^\diamond \wedge S^0)_{U(m)} \\
 \downarrow & & \downarrow \\
 (E\mathcal{R}_{m,\text{ntrv}}^\diamond \wedge S^{2m})_{U(m)} & \longrightarrow & (E\mathcal{R}_m^\diamond \wedge S^{2m})_{U(m)}
 \end{array}$$

When  $m$  is a power of  $p$ , say  $m = p^k$ , applying  $\Sigma^\infty$  to the middle row of this ladder turns out to give the  $(k+1)$ -fold suspension of the map  $\Sigma^{-1}L(k) \rightarrow T(k)$  in (2.12), so the following lemma is an essential ingredient in Proposition 2.14.

LEMMA 2.18.  $(E\mathcal{R}_{m,\text{ntrv}}^\diamond \wedge S^0)_{U(m)} \rightarrow (E\mathcal{R}_m^\diamond \wedge S^0)_{U(m)}$  is null-homotopic.

*Proof.* Consider the lower square of (2.17). By [4, Proposition 9.13], we know that  $(E\mathcal{R}_{m,\text{ntrv}}^\diamond \wedge S^{2m})_{U(m)} \simeq *$ . On the other hand, equation (9.3) in [4] says that

$$(E\mathcal{R}_m^\diamond \wedge S^0) \rightarrow (E\mathcal{R}_m^\diamond \wedge S^{2m})$$

is a  $U(m)$ -equivalence, and so

$$(E\mathcal{R}_m^\diamond \wedge S^0)_{U(m)} \rightarrow (E\mathcal{R}_m^\diamond \wedge S^{2m})_{U(m)}$$

is an equivalence, which completes the proof. ■

The remaining ingredients for the proof of Proposition 2.14 come from the upper square of (2.17). The first step is to show that the upper square is a homotopy pushout square and that it has a contractible space in the upper right corner. The second step is to identify the suspension spectra of the remaining corners of the square with the appropriate suspensions of  $L(k)$ ,  $T(k)$ , and  $\bar{M}(k)$ .

LEMMA 2.19. *The top square of (2.17) is a homotopy pushout square with contractible upper right corner.*

*Proof.* The proof is essentially the same as that of Proposition 9.11 of [4]. The map  $E\mathcal{R}_{m,\text{ntrv}} \rightarrow E\mathcal{R}_m$  is the inclusion of the singular set, as is easily checked by looking at the isotropy groups of the chains that form the simplices of  $E\mathcal{R}_m$ . The cofiber of this map has a free action of  $U(m)$  (in the based sense), and so horizontal cofibers of the first two rows of (2.17) are equivalent. Because all of the spaces involved are simply connected (they are suspensions of connected spaces), this is sufficient to guarantee that the top square of (2.17) is a homotopy pushout square. Lastly, because  $\mathcal{R}_m$  contains the trivial subgroup, it follows that  $E\mathcal{R}_m \simeq |\mathcal{R}_m| \simeq *$ , and so  $(E\mathcal{R}_m^\diamond \wedge S^0)_{\bar{h}U(m)}$  is contractible. ■

To begin the identification of the suspension spectra of the corners of the upper square of (2.17), note that the middle row is just the inclusion map  $B\mathcal{R}_{m,\text{ntrv}}^\diamond \rightarrow B\mathcal{R}_m^\diamond$ . We recall from [4, Proposition 9.8] that the inclusion of the collection of complete subgroups into standard subgroups induces a map  $E\mathcal{L}_m \rightarrow E\mathcal{R}_{m,\text{ntrv}}$  that is a  $U(m)$ -equivalence, and therefore is still a homotopy equivalence once we take orbit spaces. This allows us to replace the middle row of (2.17) with the map  $B\mathcal{L}_m^\diamond \rightarrow B\mathcal{R}_m^\diamond$  and the top left corner with  $(E\mathcal{L}_m^\diamond)_{\bar{h}U(m)}$ . On the other hand, the  $U(m)$ -equivariant map  $E\mathcal{L}_m \rightarrow |\mathcal{L}_m|$  is a homotopy equivalence (though not a  $U(m)$ -equivalence), and so induces an equivalence on homotopy orbits. Thus Lemmas 2.18 and 2.19 tell us that the upper square of (2.17) is equivalent to a homo-



topy pushout square

$$(2.20) \quad \begin{array}{ccc} |\mathcal{L}_m|_{\hbar U(m)}^\diamond & \longrightarrow & * \\ \downarrow & & \downarrow \\ (B\mathcal{L}_m)^\diamond & \xrightarrow{*} & (B\mathcal{R}_m)^\diamond \end{array}$$

To prove Propositions 2.14 and 2.15, we need to relate (2.20) to the successive quotients of the spectra  $A_m$  and  $\mathrm{Sp}^m(\mathbf{S})$  for  $m = p^k$ . The most immediate is the lower right corner. It follows from [4, Corollary 8.3] that

$$(2.21) \quad \Sigma^\infty(B\mathcal{R}_{p^k})^\diamond \simeq A_{p^k}/A_{p^{k-1}}$$

We want to relate the lower left corner of (2.20) to the quotients of symmetric powers of spheres, which can also be identified in terms of classifying spaces of collections of subgroups, this time of symmetric groups rather than unitary groups. Let  $\mathcal{F}_m$  be the collection of proper standard subgroups of  $\Sigma_m$ , i.e., subgroups of  $\Sigma_m$  that are conjugate to subgroups of the form  $\Sigma_{m_1} \times \cdots \times \Sigma_{m_s}$ . Then by [4, Corollaries 8.3 and 8.4]

$$(2.22) \quad \Sigma^\infty(B\mathcal{F}_{p^k})^\diamond \simeq \mathrm{Sp}^{p^k}(\mathbf{S})/\mathrm{Sp}^{p^k-1}(\mathbf{S}).$$

The following proposition relates this to the lower left corner of (2.20) through the inclusion  $\Sigma_m \rightarrow U(m)$  that permutes the standard basis elements of  $\mathbb{C}^m$ .

**PROPOSITION 2.23.** *The  $\Sigma_m$ -equivariant inclusion  $\mathcal{F}_m \rightarrow \mathcal{L}_m$  induces a homotopy equivalence  $B\mathcal{F}_m \simeq B\mathcal{L}_m$ .*

To prove the proposition, one shows that the orbit categories whose nerves give  $B\mathcal{F}_m$  and  $B\mathcal{L}_m$  have isomorphic subcategories containing at least one object from each isomorphism class. The essential point is that standard subgroups of  $\Sigma_m$  are complete, and complete subgroups of  $U(m)$  and of  $\Sigma_m$  have isomorphic Weyl groups.

*Proof of Proposition 2.14.* We have a commuting diagram

$$\begin{array}{ccc} \Sigma^\infty(B\mathcal{F}_{p^k})^\diamond & \xrightarrow{\simeq} & \Sigma^\infty(B\mathcal{L}_{p^k})^\diamond & \xrightarrow{\simeq} & \Sigma^\infty(B\mathcal{R}_{p^k, \mathrm{ntrv}})^\diamond \\ \downarrow \simeq & & & & \downarrow * \\ & & & & \Sigma^\infty(B\mathcal{R}_{p^k})^\diamond \\ & & & & \downarrow \simeq \\ \mathrm{Sp}^{p^k}(\mathbf{S})/\mathrm{Sp}^{p^k-1}(\mathbf{S}) & \longrightarrow & & \longrightarrow & A_{p^k}/A_{p^{k-1}} \end{array}$$

that results from combining equations (2.21) and (2.22) with the results of Proposition 2.23, Lemma 2.18, and the naturality of the identification

of filtration quotients in terms of classifying spaces of collections of subgroups. The  $(k+1)$ -fold desuspension of  $\mathrm{Sp}^{p^k}(\mathbf{S})/\mathrm{Sp}^{p^k-1}(\mathbf{S}) \rightarrow A_{p^k}/A_{p^k-1}$  is  $\Sigma^{-1}L(k) \rightarrow T(k)$ , so this finishes the proof of the proposition. ■

*Proof of Proposition 2.15.* From (2.20)–(2.22), we know that there is a homotopy cofiber sequence

$$(|\mathcal{L}_m|^\diamond)_{\tilde{h}U(m)} \rightarrow \mathrm{Sp}^m(\mathbf{S})/\mathrm{Sp}^{m-1}(\mathbf{S}) \rightarrow A_m/A_{m-1}.$$

Since  $\bar{M}(k)$  is defined in Section 2.1 by

$$\bar{M}(k) \simeq \Sigma^{-(k+1)} \mathrm{cofiber}(A_{p^{k-1}}/\mathrm{Sp}^{p^{k-1}}(\mathbf{S}) \rightarrow A_{p^k}/\mathrm{Sp}^{p^k}(\mathbf{S})),$$

this establishes that

$$(2.24) \quad \bar{M}(k) \simeq S^{-k} \wedge (|\mathcal{L}_{p^k}|^\diamond)_{\tilde{h}U(p^k)}.$$

On the other hand, by [4, Proposition 10.3] and its proof, we know that the spectrum  $S^{-k} \wedge (|\mathcal{L}_{p^k}|^\diamond)_{\tilde{h}U(p^k)}$  is a wedge summand of  $\Sigma^\infty(B\Gamma_k^\diamond)_+$  by the symplectic Steinberg idempotent, which finishes the proof. ■

### 2.3. Calculus evidence for variants of the Whitehead conjecture.

In this subsection, we present evidence from the calculus of functors for Conjecture 2.13, the “reduced  $bu$  Whitehead conjecture,” as well as for the classical “mod  $p$  Whitehead Conjecture” when  $p = 2$ . This follows on from the discussion in [4, Sections 1 and 11], where we discussed a tantalizing link between, on the one hand, the chain complexes of spectra

$$(2.25) \quad \cdots \rightarrow L(1) \rightarrow L(0) \rightarrow \mathrm{HZ},$$

$$(2.26) \quad \cdots \rightarrow T(1) \rightarrow T(0) \rightarrow bu \rightarrow \mathrm{HZ},$$

and, on the other hand, certain “Taylor towers” arising from the calculus of functors of Goodwillie and Weiss. For (2.25), the link is that  $\Omega^{k-1}\Omega^\infty L(k)$  is equivalent to the  $k$ th nontrivial layer of the Goodwillie tower at the prime  $p$  of the identity functor evaluated at  $S^1$ . Similarly for (2.26),  $\Omega^{k-1}\Omega^\infty T(k)$  is equivalent to the  $k$ th nontrivial layer of the Weiss tower of the functor  $V \mapsto BU(V)$  at the prime  $p$ , evaluated at  $\mathbb{C}$ . This suggests that there may be a deeper connection between the complexes (2.25) and (2.26) and the Taylor towers. In particular, the number of loops involved to relate the infinite loop spaces of the spectra in (2.25) and (2.26) to the layers of the Taylor towers is just right; it allows the possibility that deloopings of the structure maps in the Taylor towers may serve as a contracting homotopy for the complexes, which would provide conceptual proofs of exactness. (See [3] for more on such deloopings.) We have not been able to prove these speculations.

There exists a similar link to a Taylor tower for the reduced  $bu$  Whitehead Conjecture. In order to obtain a Taylor tower that potentially provides

a contracting homotopy for the complex

$$\cdots \rightarrow \bar{M}(2) \rightarrow \bar{M}(1) \rightarrow \bar{M}(0) \rightarrow bu$$

of Conjecture 2.13, we evaluate the Weiss tower for the functor  $V \mapsto BU(V)$  at the vector space  $\mathbb{C}^0$ . We find (by [1]) that the fibers in the tower have the form

$$\Omega^\infty \text{map}(|\mathcal{L}_{p^k}|^\diamond, \Sigma^\infty S^{\text{ad}_{p^k}})_{\tilde{h}U(p^k)}.$$

However, by Theorem 10.1 of [4], there is a mod  $p$  equivalence of spectra

$$\text{map}(|\mathcal{L}_{p^k}|^\diamond, \Sigma^\infty S^{\text{ad}_{p^k}}) \simeq_p S^{-2k+1} \wedge |\mathcal{L}_{p^k}|^\diamond.$$

By (2.24), we know that  $\bar{M}(k) \simeq S^{-k} \wedge (|\mathcal{L}_{p^k}|^\diamond)_{\tilde{h}U(p^k)}$ . Thus we conclude that when the Weiss tower for  $V \mapsto BU(V)$  is evaluated at  $\mathbb{C}^0$ , there is a mod  $p$  equivalence between the  $k$ th layer and the appropriate loop space of  $\bar{M}(k)$ :

$$\Omega^\infty \text{map}(|\mathcal{L}_{p^k}|^\diamond, \Sigma^\infty S^{\text{ad}_{p^k}})_{\tilde{h}U(p^k)} \simeq \Omega^{k-1} \Omega^\infty \bar{M}(k).$$

The connecting maps in the Taylor tower give us maps

$$(2.27) \quad \Omega^{k-1} \Omega^\infty \bar{M}(k) \rightarrow B\Omega^k \Omega^\infty \bar{M}(k+1) \simeq \Omega^{k-1} \Omega^\infty \bar{M}(k+1).$$

Application of [3, Corollary 7.2] shows that the connecting map (2.27) deloops  $k-1$  times, and so a delooping could provide a contracting homotopy, as suggested in the following conjecture.

**CONJECTURE 2.28.**  *$(k-1)$ -fold deloopings of (2.27) give a contracting homotopy for the complex of Conjecture 2.13.*

Finally, we record as a curiosity Bill Dwyer's observation that the "mod 2 Whitehead conjecture," which asserts the exactness of the complex (1.7) for the prime 2, also possesses a Taylor tower that might give a contracting homotopy. (A contracting homotopy was constructed in [9] by other methods.) However, we do not have a calculus analogue for the odd primary analogue of this statement. The point for  $p=2$  is that  $S^0$  can be identified with  $\mathbb{Z}/2 \simeq \Omega^\infty H\mathbb{Z}/2$ . Thus, the Taylor tower of the identity evaluated at  $S^0$  gives rise to a tower of fibrations converging to  $\Omega^\infty H\mathbb{Z}/2$ , or at least trying to converge <sup>(2)</sup>. Proposition 2.29 below, together with Theorem 1.17 and Corollary 9.6 of [2], tells us that the layers in this Taylor tower coincide, up to the right number of deloopings, with the terms  $M(k)$  in (1.7). That is, Proposition 2.29 tells us that  $\Omega^k \Omega^\infty M(k)$  is the  $k$ th layer in the 2-localized Taylor tower for the identity evaluated at  $S^0$ . Hence the connecting maps for this Taylor tower give maps

$$\Omega^k \Omega^\infty M(k) \rightarrow \Omega^k \Omega^\infty M(k+1),$$

---

<sup>(2)</sup> It is not known if the Taylor tower for the identity converges at  $S^0$ . We speculate that it does.

and deloopings of these would provide potential contracting maps for (1.7), as in the other situations we have examined.

PROPOSITION 2.29. *At the prime 2,*

$$M(k) \simeq_2 S^{-(k-1)} \wedge (|\mathcal{P}_{2^k}|^\diamond)_{\tilde{h}\Sigma_{2^k}}$$

*Proof.* We know from Theorem 1.17 and Corollary 9.6 of [2] (applied with  $p = 2$  and  $X = S^0$ ) that  $S^{-(k-1)} \wedge (|\mathcal{P}_{2^k}|^\diamond)_{\tilde{h}\Sigma_{2^k}}$  is the Steinberg summand of  $\Sigma^\infty(B\Delta_k)_+$ . On the other hand, Mitchell and Priddy [13] proved that (at all primes)  $M(k)$  is the Steinberg wedge summand of  $\Sigma^\infty(B\Delta_k)_+$ , and this concludes the proof. ■

**3. General constructions.** In this section, we make technical preparations for Section 4, where our goal is to compare the filtration constructed in [4] to Rognes’s stable rank filtration of algebraic  $K$ -theory [15]. Rognes defined his stable rank filtration only for the case of the algebraic  $K$ -theory of a discrete ring, but his construction can be applied to topological  $K$ -theory as well, and the topological case is covered by our discussion. The comparison in Section 4 goes through an intermediary, the “modified stable rank filtration” of a  $K$ -theory spectrum, which we introduce and study in this section. It is based on Segal’s  $\Gamma$ -space and  $\Gamma$ -category constructions, as opposed to Waldhausen’s  $S_\bullet$  construction, which formed the setting of Rognes’s original work. In other respects, however, it follows the spirit of [15].

In Section 3.1, we define the notion of a filtered  $\Gamma$ -space, we show how it arises from an augmented permutative category, and we define the modified stable rank filtration of a  $K$ -theory spectrum. In Section 3.2, we compare bar constructions and homotopy pushouts of monoidal augmented  $\Gamma$ -spaces. Finally, in Section 3.3 we study the particular situation of “very special”  $\Gamma$ -spaces, and we give a homotopy pushout diagram that allows an inductive understanding of the modified stable rank filtration for a permutative category.

**3.1. Filtered and augmented  $\Gamma$ -spaces.** We begin by reviewing the relationship between  $\Gamma$ -spaces and spectra and discussing how filtered  $\Gamma$ -spaces can serve as convenient models for filtered spectra. We then consider how an augmented permutative category gives rise to a filtered  $\Gamma$ -space. We end this subsection with the definition of the modified stable rank filtration of the spectrum associated to an augmented permutative category, such as the algebraic  $K$ -theory spectrum of a ring, and we give examples.

Let  $\Gamma^{op}$  be the skeletal category of finite pointed sets, having one object for each cardinality. We will denote a generic object of  $\Gamma^{op}$  by  $\underline{n} = \{0, 1, \dots, n\}$ , with zero acting as the basepoint. As usual, a  $\Gamma$ -space  $F$  is a pointed functor from  $\Gamma^{op}$  to the category of pointed simplicial sets. A

$\Gamma$ -space may be prolonged to a functor from the category of pointed simplicial sets to itself via the diagonal of the levelwise evaluation of the functor on a simplicial set, once we replace it with its left Kan extension from the category of finite pointed sets to the category of all pointed sets.

A  $\Gamma$ -space comes equipped with a natural “assembly map”

$$X \wedge F(Y) \rightarrow F(X \wedge Y),$$

where  $X$  and  $Y$  are pointed simplicial sets, and so there are suspension maps  $S^1 \wedge F(X) \rightarrow F(S^1 \wedge X)$ . It follows that the sequence

$$F(S^0), F(S^1), \dots, F(S^j), \dots$$

forms a prespectrum, which we call the *stabilization* of  $F$ . A map  $\alpha : F \rightarrow G$  between  $\Gamma$ -spaces is, by definition, a natural transformation between the underlying functors. The map  $\alpha$  is called an *equivalence* if  $\alpha(X)$  is a weak equivalence for all  $X$ , and it is called a *stable equivalence* if the associated map of stabilizations

$$\{\alpha(S^j)\}: \{F(S^j)\} \rightarrow \{G(S^j)\}$$

is a weak homotopy equivalence of spectra. The category of  $\Gamma$ -spaces provides a good model for the category of  $(-1)$ -connected spectra, via the functor  $F \mapsto \{F(S^j)\}$  [5], [12].

We are interested in filtrations of certain spectra, so we also set up a notion of *filtered  $\Gamma$ -space* as a model for a filtered spectrum.

DEFINITION 3.1. A *filtered  $\Gamma$ -space* is a sequence of  $\Gamma$ -spaces of the form

$$F_0 \rightarrow F_1 \rightarrow \dots \rightarrow F_m \rightarrow \dots \rightarrow F = \operatorname{colim}_m F_m$$

such that each of the maps  $F_{m-1} \rightarrow F_m$  is an (objectwise) injection.

There is a self-evident notion of a map between filtered  $\Gamma$ -spaces. We will denote a generic filtered  $\Gamma$ -space sometimes by  $F$ , and sometimes by  $\{F_m\}$ .

DEFINITION 3.2. Let  $F = \{F_m\}$  and  $G = \{G_m\}$  be filtered  $\Gamma$ -spaces. Let  $\alpha = \{\alpha_m\}$  be a filtered morphism from  $F$  to  $G$ . We say that  $\alpha$  is a *filtered stable equivalence* if  $\alpha_m$  is a stable equivalence for all  $m$ .

The prototypical example of a filtered  $\Gamma$ -space is the infinite symmetric product functor  $\operatorname{Sp}^\infty$ , filtered by the functors  $\operatorname{Sp}^m$  defined by  $\operatorname{Sp}^m(X) = X^m/\Sigma_m$ . In this paper, we are mostly concerned with filtrations that are pulled back from this filtration of  $\operatorname{Sp}^\infty$ .

DEFINITION 3.3. An *augmentation* of a  $\Gamma$ -space  $F$  is a map of  $\Gamma$ -spaces  $\epsilon : F \rightarrow \operatorname{Sp}^\infty$  such that for all  $X$ , the inverse image of the basepoint of  $\operatorname{Sp}^\infty(X)$  under the augmentation  $\epsilon(X)$  consists of just the basepoint of  $F(X)$ . In this case  $F$  will be called an *augmented  $\Gamma$ -space*.

We can use an augmentation to endow a  $\Gamma$ -space  $F$  with a filtration by  $\Gamma$ -spaces as follows.

DEFINITION 3.4. Let  $F$  be an augmented  $\Gamma$ -space. For  $m = 0, 1, 2, \dots$ , define  $\mathcal{R}_m F$  to be the strict limit (not the homotopy limit) of the following diagram:

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ \mathrm{Sp}^m & \longrightarrow & \mathrm{Sp}^\infty \end{array}$$

The  $\Gamma$ -spaces  $\mathcal{R}_m F$  endow  $F$  with a filtration

$$(3.5) \quad * = \mathcal{R}_0 F \rightarrow \mathcal{R}_1 F \rightarrow \dots \rightarrow F = \operatorname{colim}_m \mathcal{R}_m F$$

and we call it the *filtration associated with the augmentation* of  $F$ . Because the composites  $\mathcal{R}_m F \rightarrow F \rightarrow \mathrm{Sp}^\infty$  provide compatible augmentations for the  $\Gamma$ -spaces  $\mathcal{R}_m F$ , the filtration  $\{\mathcal{R}_m F\}$  is actually a filtration of  $F$  in the category of augmented  $\Gamma$ -spaces.

From here on we will work mostly in the category of augmented  $\Gamma$ -spaces.

DEFINITION 3.6.

- (1) Morphisms between augmented  $\Gamma$ -spaces that respect the augmentations are called *augmented morphisms*.
- (2) An augmented morphism  $\alpha : F \rightarrow G$  between augmented  $\Gamma$ -spaces is called a *strong augmented equivalence* (resp., *strong augmented stable equivalence*) if the induced map  $\mathcal{R}_m \alpha : \mathcal{R}_m F \rightarrow \mathcal{R}_m G$  is an equivalence (resp. stable equivalence) for each  $m$ .
- (3) Two augmented  $\Gamma$ -spaces are said to be *strongly (stably) equivalent* if they are related by a possibly zigzagging chain of strong augmented (stable) equivalences.

In order to define the modified stable rank filtration, we actually want to filter augmented  $\Gamma$ -spaces that come from Segal's construction of the  $\Gamma$ -space associated to a symmetric monoidal category. For technical convenience, we use a *permutative category*, that is, a symmetric monoidal category in which the unit and associativity isomorphisms are actually identity morphisms. A basic example is the "category" of nonnegative integers  $\mathbb{N}$ , where the only morphisms are the identity morphisms, and the permutative structure is given by addition.

Let  $\mathcal{C}$  be a category with sums such that the associated symmetric monoidal category is in fact permutative. (For example, take  $\mathcal{C}$  to have just one object in each isomorphism class.) Segal gave a construction that associates to  $\mathcal{C}$  a  $\Gamma$ -category, i.e., a functor  $S \mapsto \mathcal{C}(S)$  from  $\Gamma^{op}$  to categories, satisfying certain conditions. The category  $\mathcal{C}(S)$  has objects  $(f, \alpha_*)$  consist-

ing of (i) a function  $f$  taking pointed subsets of  $S$  to objects of  $\mathcal{C}$ , and (ii) a collection of compatible isomorphisms  $\alpha_{S_1, S_2} : f(S_1) \oplus f(S_2) \xrightarrow{\cong} f(S_1 \vee S_2)$ , one isomorphism for each pair of subsets  $S_1$  and  $S_2$  of  $S$  whose intersection is the singleton set containing the basepoint. Morphisms of such objects are required to be isomorphisms on the objects of  $\mathcal{C}$  (or weak equivalences, in the context in which this is meaningful). There is a  $\Gamma$ -space associated to a  $\Gamma$ -category by taking nerves, and for simplicity, we do not distinguish between the two. That is, given a set  $S$ , we will write  $\mathcal{C}(S)$  for either the category  $\mathcal{C}(S)$  or its nerve, trusting to context to clarify which is meant. For example, the  $\Gamma$ -space associated with permutative category  $\mathbb{N}$  is  $\mathbb{N}(X) = \mathrm{Sp}^\infty(X)$ .

**DEFINITION 3.7.** An *augmentation* of a permutative category  $\mathcal{C}$  is a functor  $\epsilon : \mathcal{C} \rightarrow \mathbb{N}$  that respects the monoidal structure and has the property that  $\epsilon^{-1}(0)$  consists of just the zero object and its identity morphism. We write  $\mathcal{C}_m$  for the full subcategory of  $\mathcal{C}$  given by  $\epsilon^{-1}(m)$ .

If  $\mathcal{C}$  is an augmented permutative category, then the associated  $\Gamma$ -space  $\mathcal{C}(X)$  is augmented (Definition 3.3) because the augmentation  $\epsilon : \mathcal{C} \rightarrow \mathbb{N}$  induces a natural transformation  $\mathcal{C}(X) \rightarrow \mathbb{N}(X) = \mathrm{Sp}^\infty(X)$ . Therefore,  $\mathcal{C}(X)$  is equipped with a natural filtration as in (3.5), pulled back from the filtration of  $\mathrm{Sp}^\infty(X)$ .

**DEFINITION 3.8.** For an augmented permutative category  $\mathcal{C}$ , we denote the filtration of  $\mathcal{C}(X)$  associated to the augmentation in the following way:

$$\mathcal{R}_0[\mathcal{C}(X)] \rightarrow \mathcal{R}_1[\mathcal{C}(X)] \rightarrow \cdots \rightarrow \mathcal{R}_m[\mathcal{C}(X)] \rightarrow \cdots \rightarrow \mathcal{R}_\infty[\mathcal{C}(X)].$$

Here, by definition we have  $\mathcal{R}_0[\mathcal{C}(X)] = *$  and  $\mathcal{R}_\infty[\mathcal{C}(X)] = \mathcal{C}(X)$ . Observe that  $\mathcal{R}_m/\mathcal{R}_{m-1}[\mathcal{C}(\underline{1})] \cong (B\mathcal{C}_m)_+$ .

Recall that given a permutative category  $\mathcal{C}$ , the spectrum

$$\mathcal{C}(S^0), \mathcal{C}(S^1), \dots, \mathcal{C}(S^j), \dots$$

is called the (Segal) *K-theory spectrum* of the category  $\mathcal{C}$  and is denoted  $\mathbf{k}\mathcal{C}$ . In fact,  $\mathcal{C}(-)$  is a *very special*  $\Gamma$ -space, i.e., the  $n$  collapse maps  $\underline{n} \rightarrow \underline{1}$  induce a homotopy equivalence

$$\mathcal{C}(\underline{n}) \rightarrow \mathcal{C}(\underline{1})^n.$$

As a consequence the spectrum  $\{\mathcal{C}(S^j)\}$  is actually an  $\Omega$ -spectrum for  $j \geq 1$  [16].

As soon as we filter  $\mathcal{C}(X)$  as in Definition 3.8, however, the  $\Gamma$ -spaces  $\mathcal{R}_m[\mathcal{C}(-)]$  are no longer very special. Nonetheless, the mere fact that  $\mathcal{R}_m[\mathcal{C}(-)]$  is a  $\Gamma$ -space gives us a suspension map

$$S^1 \wedge \mathcal{R}_m[\mathcal{C}(S^j)] \rightarrow \mathcal{R}_m[\mathcal{C}(S^{j+1})],$$

and thus  $\{\mathcal{R}_m[\mathcal{C}(S^j)]\}_j$  is a spectrum, even if not an  $\Omega$ -spectrum.

DEFINITION 3.9. If  $\mathcal{C}$  is a permutative category, we define the spectrum  $\mathcal{R}_m \mathbf{k}\mathcal{C} := \{\mathcal{R}_m[\mathcal{C}(S^j)]\}_j$  to be the  $m$ th *modified stable rank filtration* of the  $K$ -theory spectrum  $\mathbf{k}\mathcal{C}$ .

The terminology “modified stable rank filtration” is justified in Section 4.2, where we compare this filtration to that of Rognes and show that they are closely related in result as well as in construction.

EXAMPLE 3.10.

- (1) If  $\mathcal{C} = \mathbb{N}$ , then by definition we have  $\mathcal{R}_m[\mathcal{C}(X)] = \mathrm{Sp}^m(X)$ , and so  $\mathcal{R}_m \mathbf{H}\mathbb{Z} = \mathcal{R}_m \mathbf{k}\mathbb{N} = \mathrm{Sp}^m(\mathbf{S})$ .
- (2) Let  $\mathcal{C}$  be the skeletal category of finite pointed sets with objects  $\underline{n}$ , morphisms given by isomorphisms, and permutative structure given by wedge sum with, for each  $n$  and  $k$ , a fixed choice of isomorphism of  $\underline{n} \vee \underline{k}$  with  $\underline{n+k}$ . It is well known that the associated spectrum  $\mathbf{k}\mathcal{C}$  is the sphere spectrum  $\mathbf{S}$ .

The usual model for  $\mathcal{C}(X)$  is

$$\left( \prod_{i \in \mathbb{N}} E\Sigma_i \times_{\Sigma_i} X^i \right) / \sim$$

where  $\sim$  identifies points in the  $i$ th summand to the  $(i-1)$ st summand if they have coordinates at the basepoint of  $X$ . The augmentation to  $\mathrm{Sp}^\infty(X)$  collapses each  $E\Sigma_i$  to a point. We can see from this model that if  $\mathcal{C}$  is finite pointed sets, then

$$\mathcal{R}_m[\mathcal{C}(X)] / \mathcal{R}_{m-1}[\mathcal{C}(X)] \simeq (E\Sigma_m)_+ \wedge_{\Sigma_m} X^{\wedge m} \simeq (X^{\wedge m})_{\tilde{h}\Sigma_m}.$$

If  $m > 1$ , then the quotient is out of the stable range, and the inclusion  $\mathcal{R}_{m-1}\mathcal{C}(-) \rightarrow \mathcal{R}_m\mathcal{C}(-)$  is actually an augmented stable equivalence of  $\Gamma$ -spaces. The modified stable rank filtration is trivial in this case:  $\mathcal{R}_0\mathbf{k}\mathcal{C} = *$ ,  $\mathcal{R}_1\mathbf{k}\mathcal{C} = \mathbf{k}\mathcal{C} = \mathbf{S}$ , and  $\mathcal{R}_m\mathbf{k}\mathcal{C} / \mathcal{R}_{m-1}\mathbf{k}\mathcal{C} \simeq *$  for  $m > 1$ .

**3.2. Sums, products, and pushouts of augmented  $\Gamma$ -spaces.** In this subsection, we set up machinery to use in Section 4.1 for the comparison of the modified rank filtration constructed in Section 3.1 with the filtration constructed in [4]. The filtration of [4] is based on an inductive bar construction at the level of permutative categories and infinite loop spaces. However, we would like to make our comparison in Section 4.1 using an inductive homotopy pushout construction in the category of augmented  $\Gamma$ -spaces. Thus our main result of this section, Proposition 3.20, translates between the two, giving a strong augmented stable equivalence.

Homotopy pushouts are based on sums, while bar constructions are based on products, so our first order of business is to compare sums and products. Let  $F$  and  $G$  be  $\Gamma$ -spaces (not necessarily filtered or augmented). It is a



standard fact that the natural inclusion  $F \vee G \rightarrow F \times G$  is a stable equivalence, and we need to extend this to filtered  $\Gamma$ -spaces. So let  $F = \{F_m\}$  and  $G = \{G_m\}$  be filtered  $\Gamma$ -spaces. We define filtrations of the  $\Gamma$ -spaces  $F \vee G$  and  $F \times G$  as follows:

$$(3.11) \quad (F \vee G)_m = F_m \vee G_m,$$

$$(3.12) \quad (F \times G)_m = \operatorname{colim}_{i+k \leq m} (F_i \times G_k).$$

The inclusion  $F \vee G \rightarrow F \times G$  respects these filtrations.

**LEMMA 3.13.** *If  $F$  and  $G$  are filtered  $\Gamma$ -spaces, then the inclusion map  $F \vee G \rightarrow F \times G$  is a filtered stable equivalence. More generally, for any positive integer  $l$  and filtered  $\Gamma$ -spaces  $F^1, \dots, F^l$ , the inclusion map  $F^1 \vee \dots \vee F^l \rightarrow F^1 \times \dots \times F^l$  is a filtered stable equivalence.*

*Proof.* We need to show that, for each  $m$ , the inclusion map

$$F_m \vee G_m \rightarrow \operatorname{colim}_{i+k \leq m} F_i \times G_k$$

is a stable equivalence. We consider the factorization

$$(3.14) \quad F_m \vee G_m \rightarrow \operatorname{colim}_{\{i+k \leq m | i, k=0\}} F_i \vee G_k \rightarrow \operatorname{colim}_{i+k \leq m} F_i \vee G_k \rightarrow \operatorname{colim}_{i+k \leq m} F_i \times G_k,$$

and we assert that each of these maps is a stable equivalence. In fact, it is easy to check that the first map is an isomorphism, and the second map is likewise an isomorphism because the target diagram is the left Kan extension of the source diagram. Finally, consider the diagram

$$\begin{array}{ccc} \operatorname{hocolim}_{i+k \leq m} F_i \vee G_k & \longrightarrow & \operatorname{hocolim}_{i+k \leq m} F_i \times G_k \\ \downarrow & & \downarrow \\ \operatorname{colim}_{i+k \leq m} F_i \vee G_k & \longrightarrow & \operatorname{colim}_{i+k \leq m} F_i \times G_k \end{array}$$

The vertical maps are equivalences, because both diagrams are cofibrant [6], and thus colimits are equivalent to homotopy colimits. The top row is a stable equivalence because each  $F_i \vee G_k \rightarrow F_i \times G_k$  is a stable equivalence; thus the bottom row is a stable equivalence, completing the proof that (3.14) is a stable equivalence.

The more general statement follows by induction. ■

If  $F$  and  $G$  are augmented  $\Gamma$ -spaces, then there are natural augmentations on the  $\Gamma$ -spaces  $F \vee G$  and  $F \times G$  defined by the compositions

$$(3.15) \quad F \vee G \rightarrow \operatorname{Sp}^\infty \vee \operatorname{Sp}^\infty \xrightarrow{\text{fold}} \operatorname{Sp}^\infty,$$

$$(3.16) \quad F \times G \rightarrow \operatorname{Sp}^\infty \times \operatorname{Sp}^\infty \xrightarrow{+} \operatorname{Sp}^\infty.$$

The filtrations of  $F \vee G$  and  $F \times G$  that are induced by these augmentations turn out to coincide with those defined by (3.11) and (3.12), as established in the following lemma.

LEMMA 3.17. *If  $F$  and  $G$  are augmented  $\Gamma$ -spaces and  $F \vee G$  and  $F \times G$  are augmented by (3.15) and (3.16), then*

$$\mathcal{R}_m(F \vee G) \cong (\mathcal{R}_m F) \vee (\mathcal{R}_m G), \quad \mathcal{R}_m(F \times G) \cong \operatorname{colim}_{i+k \leq m} (\mathcal{R}_i F \times \mathcal{R}_k G).$$

*Proof.* The lemma follows for the product by inspection of the two stacked strict pullback diagrams

$$\begin{array}{ccc} \operatorname{colim}_{i+k \leq m} (\mathcal{R}_i F \times \mathcal{R}_k G) & \longrightarrow & F \times G \\ \downarrow & & \downarrow \\ \operatorname{colim}_{i+k \leq m} (\operatorname{Sp}^i \times \operatorname{Sp}^k) & \longrightarrow & \operatorname{Sp}^\infty \times \operatorname{Sp}^\infty \\ \downarrow \operatorname{colim} (+) & & \downarrow + \\ \operatorname{Sp}^m & \longrightarrow & \operatorname{Sp}^\infty \end{array}$$

because the outer square is the pullback that identifies  $\mathcal{R}_m(F \times G)$ . The proof for the wedge is similar, but easier. ■

COROLLARY 3.18. *If  $F$  and  $G$  are augmented  $\Gamma$ -spaces and  $F \vee G$  and  $F \times G$  are augmented by (3.15) and (3.16), then the natural inclusion  $F \vee G \rightarrow F \times G$  is a strong augmented stable equivalence.*

Finally, we come to our goal for this subsection, which is to compare bar constructions and homotopy pushouts of  $\Gamma$ -spaces. We call an augmented  $\Gamma$ -space  $F$  an “augmented monoid” if there is an associative and unital map of augmented  $\Gamma$ -spaces  $F \times F \rightarrow F$ . Suppose given a diagram

$$(3.19) \quad F_1 \leftarrow F_0 \rightarrow F_2$$

of augmented monoids. We can define the augmented  $\Gamma$ -space  $\operatorname{Bar}(F_1, F_0, F_2)$  as the diagonal of the standard simplicial object

$$q \mapsto F_1 \times (F_0)^q \times F_2,$$

where the augmentation is given levelwise by (3.16).

On the other hand, we can define  $F_{12}$  as the objectwise homotopy pushout of (3.19), and  $F_{12}$  has an augmentation because there is a (strictly) commuting diagram

$$\begin{array}{ccc} F_0(X) & \longrightarrow & F_1(X) \\ \downarrow & & \downarrow \\ F_2(X) & \longrightarrow & \operatorname{Sp}^\infty(X) \end{array}$$

A common canonical model for  $F_{12}$  is the geometric realization (i.e., diagonal) of the simplicial object

$$q \mapsto F_1 \vee \underbrace{F_0 \vee \cdots \vee F_0}_q \vee F_2.$$

With this model, we see that, levelwise, the augmentation of  $F_{12}$  is just the wedge of the augmentations of  $F_0$ ,  $F_1$ , and  $F_2$ . There is a natural augmentation-preserving map  $F_{12} \rightarrow \text{Bar}(F_1, F_0, F_2)$  induced by the simplicial map that is given in degree  $q$  by the inclusion map

$$F_1 \vee (F_0)^{\vee q} \vee F_2 \rightarrow F_1 \times (F_0)^q \times F_2.$$

We now have all of the ingredients for the main goal of this subsection.

**PROPOSITION 3.20.** *Suppose given a diagram of  $\Gamma$ -spaces*

$$F_1 \leftarrow F_0 \rightarrow F_2$$

*that is a diagram of augmented monoids, and suppose that  $F_{12}$  is the object-wise homotopy pushout. Then the natural map*

$$F_{12} \rightarrow \text{Bar}(F_1, F_0, F_2)$$

*is a strong augmented stable equivalence.*

*Proof.* By Corollary 3.18,  $F_{12} \rightarrow \text{Bar}(F_1, F_0, F_2)$  is a strong augmented stable equivalence in each simplicial degree. It follows that the induced map of geometric realizations is a strong augmented stable equivalence. ■

**3.3. Very special  $\Gamma$ -spaces.** In this subsection, we study the structure of an augmented  $\Gamma$ -space a little more closely. We consider a map  $F \rightarrow G$  of augmented  $\Gamma$ -spaces that are “very special” in the sense of Segal. We show that if  $\mathcal{R}_i F \rightarrow \mathcal{R}_i G$  is an equivalence for  $i < m$ , then we can compare  $\mathcal{R}_m F$  and  $\mathcal{R}_m G$  using the values of  $F$  and  $G$  on the space  $S^0$  (Proposition 3.24). Further, unlike the previous subsection, this result does not require stabilization, but is true on the level of spaces rather than spectra. It generalizes the result of Theorem 1.1 of [11] (Remark 3.30).

The main tool for the comparison is a particular instance of the assembly map discussed in Section 3.1. Let  $F$  be an augmented  $\Gamma$ -space and let  $\underline{k}$  denote the set  $\{0, 1, \dots, k\}$ , with 0 as the basepoint. We can define a new augmented  $\Gamma$ -space  $X \mapsto F(\underline{1}) \wedge X$  by using the assembly map to provide an augmentation via the composition

$$F(\underline{1}) \wedge X \rightarrow F(\underline{1} \wedge X) \xrightarrow{\cong} F(X) \rightarrow \text{Sp}^\infty(X).$$

By construction, this gives a map of augmented  $\Gamma$ -spaces  $F(\underline{1}) \wedge X \rightarrow F(X)$ .

**EXAMPLE 3.21.**

- (1) If  $F = \text{Sp}^\infty$ , then  $F(\underline{1}) = \mathbb{N}$ , and the map  $\text{Sp}^\infty(\underline{1}) \wedge X \rightarrow \text{Sp}^\infty(X)$  is given by  $m \wedge x \mapsto mx$ .

- (2) If  $F$  is the  $\Gamma$ -space associated to the Segal construction for the category of finite pointed sets (so that for connected  $X$  we have  $F(X) \simeq \Omega^\infty \Sigma^\infty X$ ), then  $F(\underline{1}) \simeq \bigvee_i (B\Sigma_i)_+$ . On each component of  $F(\underline{1}) \wedge X$ , the map  $F(\underline{1}) \wedge X \rightarrow F(X)$  becomes the map

$$B\Sigma_{i+} \wedge X \rightarrow \left( \prod E\Sigma_i \times_{\Sigma_i} X^i \right) / \sim$$

induced by the  $i$ -fold diagonal map  $X \rightarrow X^i$ .

We need more detail on the filtration associated to the augmentation of the  $\Gamma$ -space  $X \mapsto F(\underline{1}) \wedge X$ . Example 3.21(1) shows that  $\mathcal{R}_m[\mathrm{Sp}^\infty(\underline{1}) \wedge X] = \mathrm{Sp}^m(\underline{1}) \wedge X$ , and this turns out to be the key to the general case, as shown in the proof of the following lemma.

LEMMA 3.22.  $\mathcal{R}_m[F(\underline{1}) \wedge X] = \mathcal{R}_m[F(\underline{1})] \wedge X$ .

*Proof.* We consider the following diagram of augmented  $\Gamma$ -spaces:

$$(3.23) \quad \begin{array}{ccc} F(\underline{1}) \wedge X & \longrightarrow & F(\underline{1} \wedge X) \cong F(X) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^\infty(\underline{1}) \wedge X & \longrightarrow & \mathrm{Sp}^\infty(\underline{1} \wedge X) \cong \mathrm{Sp}^\infty(X) \end{array}$$

The horizontal maps are assembly maps for the  $\Gamma$ -spaces  $F$  and  $\mathrm{Sp}^\infty$ , and the vertical maps are given by the augmentation of  $F$ . The diagram strictly commutes, because the assembly map is natural in maps of  $\Gamma$ -spaces.

By definition, the augmentation of  $F(\underline{1}) \wedge X$  is given by the clockwise path from top left to bottom right, and thus  $\mathcal{R}_m[F(\underline{1}) \wedge X]$  is defined as the inverse image of  $\mathrm{Sp}^m(X) \subseteq \mathrm{Sp}^\infty(X)$  by this path. However, since (3.23) commutes, we can go the other way around the square. As noted above, the inverse image of  $\mathrm{Sp}^m(X)$  in  $\mathrm{Sp}^\infty(\underline{1}) \wedge X$  is  $\mathrm{Sp}^m(\underline{1}) \wedge X$ . Because the inverse image of  $\mathrm{Sp}^m(\underline{1}) \wedge X$  in  $F(\underline{1}) \wedge X$  is, by definition,  $\mathcal{R}_m[F(\underline{1})] \wedge X$ , the lemma follows. ■

Now we want to use the assembly map to compare two  $\Gamma$ -spaces. Let  $F \rightarrow G$  be an augmentation-preserving map of augmented, very special  $\Gamma$ -spaces. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_m[F(\underline{1}) \wedge X] & \longrightarrow & \mathcal{R}_m[F(X)] \\ \downarrow & & \downarrow \\ \mathcal{R}_m[G(\underline{1}) \wedge X] & \longrightarrow & \mathcal{R}_m[G(X)] \end{array}$$

By using Lemma 3.22 to replace  $\mathcal{R}_m[F(\underline{1}) \wedge X]$  and  $\mathcal{R}_m[G(\underline{1}) \wedge X]$  with  $\mathcal{R}_m[F(\underline{1})] \wedge X$  and  $\mathcal{R}_m[G(\underline{1})] \wedge X$ , respectively, we obtain the diagram of the following theorem, which is our goal for this subsection.

PROPOSITION 3.24. *Let  $F \rightarrow G$  be an augmentation-preserving map of augmented very special  $\Gamma$ -spaces, and suppose that  $\mathcal{R}_i[F(\underline{1})] \rightarrow \mathcal{R}_i[G(\underline{1})]$  is*

a homotopy equivalence for all  $i < m$ . Then  $\mathcal{R}_i[F(X)] \rightarrow \mathcal{R}_i[G(X)]$  is a homotopy equivalence for all  $i < m$  and all  $X$ , and the commuting diagram

$$\begin{array}{ccc} \mathcal{R}_m[F(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[F(X)] \\ \downarrow & & \downarrow \\ \mathcal{R}_m[G(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[G(X)] \end{array}$$

is a strong homotopy pushout diagram of augmented  $\Gamma$ -spaces, that is, it remains a homotopy pushout square after the application of  $\mathcal{R}_j$  for all  $j$ .

Before tackling the proof of Proposition 3.24, we establish a lemma about the righthand side of its diagram in the case that  $X$  is a finite pointed set. If  $k \in \mathbb{N}$ , then because  $F$  is very special, we have an equivalence  $F(\underline{k}) \rightarrow F(\underline{1})^k$  induced by the  $k$  collapse maps. We would like to understand  $\mathcal{R}_m[F(\underline{k})]$  in terms of  $F(\underline{1})^k$ .

LEMMA 3.25. *Suppose that  $F$  is an augmented very special  $\Gamma$ -space. Then*

$$\mathcal{R}_m[F(\underline{k})] \simeq \operatorname{colim}_{i_1 + \dots + i_k \leq m} \mathcal{R}_{i_1}[F(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[F(\underline{1})].$$

*Proof.* First consider the special case of symmetric powers. Let

$$C := \operatorname{colim}_{i_1 + \dots + i_k \leq m} \operatorname{Sp}^{i_1}(\underline{1}) \times \dots \times \operatorname{Sp}^{i_k}(\underline{1}) \subset \operatorname{Sp}^\infty(\underline{1})^k,$$

where the colimit is taken over the poset of  $k$ -tuples with  $(i_1, \dots, i_k) \leq (i'_1, \dots, i'_k)$  if  $i_j \leq i'_j$  for all  $j = 1, \dots, k$ . The isomorphism of discrete sets  $\operatorname{Sp}^\infty(\underline{k}) \xrightarrow{\cong} \operatorname{Sp}^\infty(\underline{1})^k$  gives us an isomorphism between  $\operatorname{Sp}^m(\underline{k})$  and  $C$ .

Then we consider the map of diagrams

$$\left( \begin{array}{ccc} & F(\underline{k}) & \\ & \downarrow & \\ \operatorname{Sp}^m(\underline{k}) & \longrightarrow & \operatorname{Sp}^\infty(\underline{k}) \end{array} \right) \longrightarrow \left( \begin{array}{ccc} & F(\underline{1})^k & \\ & \downarrow & \\ C & \longrightarrow & \operatorname{Sp}^\infty(\underline{1})^k \end{array} \right)$$

which is an isomorphism on the lower row. Indeed, each bottom row is an inclusion of discrete sets, and the map of upper right corners is a homotopy equivalence, which is sufficient to guarantee that the map between the (strict) pullbacks is a homotopy equivalence. However, the pullback of the left diagram is  $\mathcal{R}_m[F(\underline{k})]$  by definition, and the pullback of the right diagram is  $\operatorname{colim}_{i_1 + \dots + i_k \leq m} \mathcal{R}_{i_1}[F(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[F(\underline{1})]$ . The lemma follows. ■

*Proof of Proposition 3.24.* It suffices to establish the proposition for  $X = \underline{k}$  for arbitrary  $k \in \mathbb{N}$ . Furthermore, since  $\mathcal{R}_i \mathcal{R}_m = \mathcal{R}_i$  if  $i \leq m$  and  $\mathcal{R}_j \mathcal{R}_m = \mathcal{R}_m$  if  $j > m$ , it actually suffices to show that for  $i \leq m$  we have a

homotopy pushout diagram

$$\begin{array}{ccc} \mathcal{R}_i[F(\underline{1})] \wedge \underline{k} & \longrightarrow & \mathcal{R}_i[F(\underline{k})] \\ \downarrow & & \downarrow \\ \mathcal{R}_i[G(\underline{1})] \wedge \underline{k} & \longrightarrow & \mathcal{R}_i[G(\underline{k})] \end{array}$$

We will deal with the case  $i = m$  in detail. The case  $i < m$  is proved in the same way, except it is easier.

By Lemma 3.25, we need to show that the following is a homotopy pushout:

$$(3.26) \quad \begin{array}{ccc} \mathcal{R}_m[F(\underline{1})] \wedge \underline{k} & \longrightarrow & \operatorname{colim}_{i_1+\dots+i_k \leq m} \mathcal{R}_{i_1}[F(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[F(\underline{1})] \\ \downarrow & & \downarrow \\ \mathcal{R}_m[G(\underline{1})] \wedge \underline{k} & \longrightarrow & \operatorname{colim}_{i_1+\dots+i_k \leq m} \mathcal{R}_{i_1}[G(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[G(\underline{1})] \end{array}$$

As in the proof of Lemma 3.13, we note that the diagram over which the colimits are taken is cofibrant, so the colimits can be replaced by homotopy colimits [6]. Further, the left column consists of  $k$ -fold wedge sums, and these can be written as homotopy colimits over the same category by taking  $(i_1, \dots, i_k)$  to a point if  $i_j < m$  for all  $i_j$ , and taking  $(0, \dots, 0, m, 0, \dots, 0)$  to  $\mathcal{R}_m[F(\underline{1})]$  (for the upper row) or  $\mathcal{R}_m[G(\underline{1})]$  (for the lower row).

With this setup, we conclude that the proposition follows because (3.26) is a homotopy colimit over the category of  $k$ -tuples of the following two types of squares, all of which are themselves homotopy pushout squares:

- (1) when all  $i_j < m$ ,

$$\begin{array}{ccc} * & \longrightarrow & \mathcal{R}_{i_1}[F(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[F(\underline{1})] \\ \downarrow = & & \downarrow \simeq \\ * & \longrightarrow & \mathcal{R}_{i_1}[G(\underline{1})] \times \dots \times \mathcal{R}_{i_k}[G(\underline{1})] \end{array}$$

- (2) if some  $i_j = m$ ,

$$\begin{array}{ccc} \mathcal{R}_m[F(\underline{1})] & \xrightarrow{\cong} & \mathcal{R}_0[F(\underline{1})] \times \dots \times \mathcal{R}_m[F(\underline{1})] \times \dots \times \mathcal{R}_0[F(\underline{1})] \\ \downarrow & & \downarrow \\ \mathcal{R}_m[G(\underline{1})] & \xrightarrow{\cong} & \mathcal{R}_0[G(\underline{1})] \times \dots \times \mathcal{R}_m[G(\underline{1})] \times \dots \times \mathcal{R}_0[G(\underline{1})] \blacksquare \end{array}$$

In Section 4, we apply a special case of Proposition 3.24, which we single out as a corollary. First we need to introduce one more definition.

**DEFINITION 3.27.** Let  $F$  be an augmented very special  $\Gamma$ -space. We say that  $F$  is  $m$ -reduced if the following natural map is a weak homotopy equivalence:

$$\mathcal{R}_m[F(\underline{1})] \xrightarrow{\cong} \mathcal{R}_m[\operatorname{Sp}^\infty(\underline{1})].$$

Since  $\mathrm{Sp}^\infty(\underline{1}) = \mathbb{N}$ , this is saying that the first  $m$  pieces (usually components) of  $F(\underline{1})$  are contractible. For example, if  $F$  is the Segal construction for the category of finite pointed sets, then  $F(\underline{1}) \simeq \coprod_i B\Sigma_i$ , and  $F$  is 1-reduced because  $B\Sigma_1 \simeq *$ . An augmented  $\Gamma$ -space is always 0-reduced by definition.

Since  $\mathcal{R}_m[F(X)]$  is defined as the inverse image of  $\mathrm{Sp}^m(X)$  under the augmentation  $F(X) \rightarrow \mathrm{Sp}^\infty(X)$ , and  $\mathrm{Sp}^\infty(\underline{1})$  is discrete, we see that

$$\mathcal{R}_m[F(\underline{1})] \cong \bigvee_{1 \leq i \leq m} \mathcal{R}_i/\mathcal{R}_{i-1}[F(\underline{1})].$$

We can use this splitting to define the maps in the left square of the following diagram:

$$(3.28) \quad \begin{array}{ccccc} \mathcal{R}_m/\mathcal{R}_{m-1}[F(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[F(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[F(X)] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_m/\mathcal{R}_{m-1}[\mathrm{Sp}^\infty(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m[\mathrm{Sp}^\infty(\underline{1})] \wedge X & \longrightarrow & \mathrm{Sp}^m(X) \end{array}$$

However,  $\mathcal{R}_m/\mathcal{R}_{m-1}[\mathrm{Sp}^\infty(\underline{1})]$  is just  $S^0$ , and so we arrive at the following corollary of Proposition 3.24.

**COROLLARY 3.29.** *Suppose that  $F$  is an augmented very special  $\Gamma$ -space that is  $(m-1)$ -reduced. Then there is a homotopy pushout diagram of augmented  $\Gamma$ -spaces*

$$\begin{array}{ccc} \mathcal{R}_m/\mathcal{R}_{m-1}[F(\underline{1})] \wedge X & \longrightarrow & \mathcal{R}_m F(X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Sp}^m(X) \end{array}$$

where the identity  $\Gamma$ -space in the lower left corner is given the augmentation  $X \rightarrow \mathrm{Sp}^\infty X$  by  $x \mapsto mx$ .

*Proof.* The left square of (3.28) is a homotopy pushout because  $F$  is  $(m-1)$ -reduced, and the right square is a homotopy pushout by direct application of Proposition 3.24. Thus the outer square is a homotopy pushout, and the corollary follows. ■

**REMARK 3.30.** Proposition 3.24 is closely related to Theorem 1.1 of [11], the proof of which comes in two halves, [11, Lemmas 6.1 and 6.2]. Proposition 3.24 is a slightly strengthened version of [11, Lemma 6.1]. We also obtain a version of the second half: Lemma 4.7 of this paper contains Lemma 6.3 of [11] in the case of the family of all subgroups, but it is not as general as the original lemma.

**4. Comparisons.** Our goal in this section is to establish the precise relationship of the modified stable rank filtration of a  $K$ -theory spectrum  $\mathbf{k}\mathcal{C}$ ,

defined in Section 3.1, to two other filtrations of  $\mathbf{k}\mathcal{C}$ : the filtration defined by the authors in [4], and the stable rank filtration of a  $K$ -theory spectrum that was defined by Rognes in [15]. In Section 4.1, we do the first of these comparisons, and we show that the filtration constructed by the authors in [4] is related to the modified stable rank filtration through a pushout diagram involving the symmetric powers of the sphere spectrum and the spectrum  $\mathbf{k}\mathcal{C}$  itself. This allows us to verify the claim made in the introduction that the spectra  $\bar{M}(k)$  in the reduced *bu* Whitehead Conjecture are in fact the subquotients of the modified stable rank filtration.

Then in Section 4.2, we compare the modified stable rank filtration to the original construction of the stable rank filtration by Rognes. There is a map from the first to the second, which need not be, in general, an equivalence of filtrations in the case of the algebraic  $K$ -theory of a discrete ring. However, we show that it is an equivalence in the case of topological  $K$ -theory. Thus, in this case the modified stable rank filtration provides another model for Rognes's filtration. In combination with the results of Section 4.1, we find that the spectra  $\bar{M}(k)$  introduced in Section 2 are actually filtration quotients in the stable rank filtration of Rognes.

**4.1. Comparing the modified stable rank filtration to [4].** Let  $\mathcal{C}$  be an augmented permutative category. Our previous work in [4], and the modified rank filtration constructed in Section 3.1, give two filtered  $\Gamma$ -spaces that we can associate with  $\mathcal{C}$ , and our goal in this subsection is to compare them. The sequence  $(\mathcal{K}_m\mathcal{C})(-)$  constructed in [4] is a sequence of augmented, very special  $\Gamma$ -spaces interpolating between the  $\Gamma$ -spaces  $\mathcal{C}(-)$  and  $\mathbb{N}(-) = \mathrm{Sp}^\infty(-)$ . The stabilization of this sequence is a sequence of spectra beginning with  $\mathbf{k}\mathcal{C}$  and ending with  $\mathrm{H}\mathbb{Z}$ . In the case that  $\mathcal{C}$  is the category of finite-dimensional complex vector spaces, these are the spectra whose subquotients appear in the *bu* Whitehead conjecture (Conjecture 1.5). On the other hand, we have the modified rank filtration, i.e., the augmented  $\Gamma$ -spaces  $\mathcal{R}_m[\mathcal{C}(-)]$  constructed in Section 3.1. This is a sequence of augmented, but not very special  $\Gamma$ -spaces interpolating between  $*$  and  $\mathcal{C}(-)$ .

Certainly the sequences  $(\mathcal{K}_m\mathcal{C})(-)$  and  $\mathcal{R}_m[\mathcal{C}(-)]$  cannot be the same. The spectra  $A_m$  that are the stabilizations of  $(\mathcal{K}_m\mathcal{C})(-)$  go from  $\mathbf{k}\mathcal{C}$  to  $\mathrm{H}\mathbb{Z}$ , while the stabilizations  $\mathcal{R}_m\mathbf{k}\mathcal{C}$  of  $\mathcal{R}_m[\mathcal{C}(-)]$  go from  $*$  to  $\mathbf{k}\mathcal{C}$ . However, the main technical result of this subsection (Theorem 4.3) has an immediate consequence (Theorem 4.4) giving a homotopy pushout diagram of spectra

$$(4.1) \quad \begin{array}{ccc} \mathcal{R}_m\mathbf{k}\mathcal{C} & \longrightarrow & \mathbf{k}\mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m(\mathbf{S}) & \longrightarrow & A_m \end{array}$$

a result that is heuristically plausible because the vertical fibers on the



left go from  $*$  to the fiber of the augmentation  $\mathbf{k}\mathcal{C} \rightarrow \mathbb{H}\mathbb{Z}$ , and so do the vertical fibers on the right. This gives a precise stable relationship between the modified stable rank filtration in the upper left corner of (4.1), and the filtration defined in [4] in the lower right corner, though the result is only true stably and not on the level of the corresponding  $\Gamma$ -spaces.

To set up the argument to obtain (4.1), for each  $m$  we define an augmented  $\Gamma$ -space  $\mathcal{E}_m[\mathcal{C}(X)]$  by an objectwise pushout diagram

$$(4.2) \quad \begin{array}{ccc} \mathcal{R}_m[\mathcal{C}(X)] & \longrightarrow & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m(X) & \longrightarrow & \mathcal{E}_m[\mathcal{C}(X)] \end{array}$$

where the maps  $\mathcal{R}_m[\mathcal{C}(X)] \rightarrow \mathcal{C}(X)$  and  $\mathcal{R}_m[\mathcal{C}(X)] \rightarrow \mathrm{Sp}^m(X)$  are the maps that define  $\mathcal{R}_m[\mathcal{C}(X)]$  as a pullback. It is easy to check that (4.2) is a strong pushout diagram of augmented  $\Gamma$ -spaces, that is, that the diagram remains a pushout when any  $\mathcal{R}_i$  is applied to it.

**THEOREM 4.3.** *For each  $m$ , there is a chain of augmented stable equivalences*

$$\mathcal{E}_m[\mathcal{C}(-)] \simeq \mathcal{K}_m[\mathcal{C}(-)].$$

*The stable equivalences commute with the maps from the  $(m-1)$ st stage to the  $m$ th stage on both sides.*

To obtain diagram (4.1) from the theorem, note that diagram (4.2) is a homotopy pushout as well as a pushout diagram, because the top map is an objectwise cofibration. The following theorem is then an immediate corollary, and is the main result of this section.

**THEOREM 4.4.** *For every  $m$ , there is a stable homotopy pushout square of augmented  $\Gamma$ -spaces*

$$\begin{array}{ccc} \mathcal{R}_m\mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m & \longrightarrow & \mathcal{K}_m\mathcal{C} \end{array}$$

*and hence a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} \mathcal{R}_m\mathbf{k}\mathcal{C} & \longrightarrow & \mathbf{k}\mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m(\mathbf{S}) & \longrightarrow & A_m \end{array}$$

Theorem 4.4 implies that

$$\mathcal{R}_m\mathbf{k}\mathcal{C}/\mathcal{R}_{m-1}\mathbf{k}\mathcal{C} \simeq \Sigma^{-1}(A_m/\mathrm{Sp}^m(\mathbf{S})) / (A_{m-1}/\mathrm{Sp}^{m-1}(\mathbf{S})).$$

When  $m = p^k$ , the spectrum on the right is the spectrum  $\bar{M}(k)$ , by definition from (1.8). Thus the spectra  $\bar{M}(k)$  are, in fact, subquotients of the modified rank filtration.

COROLLARY 4.5. *There is an equivalence  $\bar{M}(k) \simeq \mathcal{R}_{p^k} \mathbf{k}\mathcal{C} / \mathcal{R}_{p^{k-1}} \mathbf{k}\mathcal{C}$ .*

The proof of Theorem 4.3 is by induction on  $m$ , the case  $m = 0$  being obvious because in that case both sides are  $\mathcal{C}(-)$ . The main idea of the proof is that the augmented  $\Gamma$ -spaces  $\mathcal{E}_m \mathcal{C}(-)$  and  $\mathcal{K}_m \mathcal{C}(-)$  are obtained from  $\mathcal{E}_{m-1} \mathcal{C}(-)$  and  $\mathcal{K}_{m-1} \mathcal{C}(-)$ , respectively, by means of the same inductive formula.

We first analyze the construction  $\mathcal{K}_m \mathcal{C}$ . Recall that  $\mathcal{C}$  has an augmentation  $\epsilon: \mathcal{C} \rightarrow \mathbb{N}$ . We write  $\mathcal{C}_m$  for  $\epsilon^{-1}(m)$ , and we call  $\mathcal{C}$  *m-reduced* if  $|\mathcal{C}_i| \simeq *$  for  $i \leq m$ . (An augmented permutative category  $\mathcal{C}$  is always 0-reduced by Definition 3.7.) The inductive construction of [4] begins with  $\mathcal{K}_0 \mathcal{C} := \mathcal{C}$ , and the inductive step is equivalent to taking the augmented permutative category  $\mathcal{K}_{m-1} \mathcal{C}$ , which is  $(m-1)$ -reduced, and using a bar construction to “kill” its  $m$ th component,  $(\mathcal{K}_{m-1} \mathcal{C})_m$ , thereby obtaining the  $m$ -reduced augmented permutative category  $\mathcal{K}_m \mathcal{C}$  (Construction 3.8 and Proposition 2.5 of [4]). Theorem 3.9 of [4] establishes a stable homotopy pushout diagram involving the associated spectra, and the following lemma strengthens this result by using Section 3.2 of the current work to show that the homotopy pushout diagram in question is still a stable homotopy pushout at each level of the modified stable rank filtration, i.e., we have a *strong* augmented stable homotopy pushout diagram.

LEMMA 4.6. *There is a strong stable homotopy pushout square of augmented  $\Gamma$ -spaces*

$$\begin{array}{ccc} B(\mathcal{K}_{m-1} \mathcal{C})_{m+} \wedge X & \longrightarrow & (\mathcal{K}_{m-1} \mathcal{C})(X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & (\mathcal{K}_m \mathcal{C})(X) \end{array}$$

where the augmentation  $X \rightarrow \mathrm{Sp}^\infty(X)$  of the lower left corner is given by  $x \mapsto mx$ .

*Proof.* Recall from Definition 3.1 of [4] that if  $\mathcal{D}$  is a small category, then  $\mathrm{Free}(\mathcal{D})$  is the free permutative category generated by  $\mathcal{D}$ . By [4, Proposition 2.5 and Construction 3.8],  $\mathcal{K}_m \mathcal{C}$  is equivalent to the (augmented) bar construction

$$\mathcal{K}_m \mathcal{C} \simeq \mathrm{Bar}(\mathrm{Free}\{m\}, \mathrm{Free}(\mathcal{K}_{m-1} \mathcal{C})_m, \mathcal{K}_{m-1} \mathcal{C}).$$

It now follows by Proposition 3.20 that there is a strong augmented stable

homotopy pushout square

$$\begin{array}{ccc} \text{Free}(\mathcal{K}_{m-1}\mathcal{C})_m(X) & \longrightarrow & \mathcal{K}_{m-1}\mathcal{C}(X) \\ \downarrow & & \downarrow \\ (\text{Free}\{m\})(X) & \longrightarrow & \mathcal{K}_m\mathcal{C}(X) \end{array}$$

Now consider the diagram

$$\begin{array}{ccccc} (\mathcal{K}_{m-1}\mathcal{C})_{m+} \wedge X & \longrightarrow & \text{Free}(\mathcal{K}_{m-1}\mathcal{C})_m(X) & \longrightarrow & \mathcal{K}_{m-1}\mathcal{C}(X) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & (\text{Free}\{m\})(X) & \longrightarrow & \mathcal{K}_m\mathcal{C}(X) \end{array}$$

where the horizontal maps are given by the natural inclusions. We just saw that the right square is a strong augmented stable homotopy pushout. The horizontal maps on the left are stable equivalences because they have the form  $Y \mapsto QY$ , from [4, Proposition 3.3]; this map is a strong augmented stable equivalence with the standard augmentation, and the augmentation here in the bottom row is simply the standard one multiplied by  $m$ .

It follows that the outer square is a strong augmented stable homotopy pushout, which is what we wanted to prove. ■

LEMMA 4.7. *There is a strong stable homotopy pushout square of augmented  $\Gamma$ -spaces*

$$\begin{array}{ccc} \mathcal{R}_m[\mathcal{K}_{m-1}\mathcal{C}(X)] & \longrightarrow & \mathcal{K}_{m-1}\mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathcal{R}_m[\mathcal{K}_m\mathcal{C}(X)] & \longrightarrow & \mathcal{K}_m\mathcal{C}(X) \end{array}$$

*Proof.* We tack the desired square onto the square obtained by applying Corollary 3.29 to the  $(m-1)$ -reduced augmented category  $\mathcal{K}_{m-1}\mathcal{C}$ :

$$(4.8) \quad \begin{array}{ccccc} ((\mathcal{K}_{m-1}\mathcal{C})_m)_+ \wedge X & \longrightarrow & \mathcal{R}_m[\mathcal{K}_{m-1}\mathcal{C}(X)] & \longrightarrow & \mathcal{K}_{m-1}\mathcal{C}(X) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{R}_m[\mathcal{K}_m\mathcal{C}(X)] & \longrightarrow & \mathcal{K}_m\mathcal{C}(X) \end{array}$$

The left square is a strong augmented stable homotopy pushout square by Corollary 3.29, since  $\mathcal{K}_m\mathcal{C}(X)$  being  $m$ -reduced implies that  $\mathcal{R}_i[\mathcal{K}_m\mathcal{C}(X)] \simeq \mathcal{R}_i[\mathbb{N}(X)] = \text{Sp}^i(X)$  for  $i \leq m$ . The outer square of (4.8) is a strong augmented stable homotopy pushout square by Lemma 4.6. Hence the right square of (4.8) is a strong augmented stable homotopy pushout square. ■

*Proof of Theorem 4.3.* Our plan is to establish an inductive formula for  $\mathcal{E}_m\mathcal{C}$  that corresponds to the formula provided by Lemma 4.7 for  $\mathcal{K}_m\mathcal{C}$ .

Consider the following diagram of augmented  $\Gamma$ -spaces:

$$(4.9) \quad \begin{array}{ccc} \mathcal{R}_m[\mathcal{C}(X)] & \longrightarrow & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathcal{R}_m[\mathcal{E}_{m-1}\mathcal{C}(X)] & \longrightarrow & \mathcal{E}_{m-1}\mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^m(X) & \longrightarrow & \mathcal{E}_m\mathcal{C}(X) \end{array}$$

By definition from diagram (4.2), the outer square is an augmented pushout square, and it remains so after application of  $\mathcal{R}_i$ , so it is a strong augmented pushout square. The top horizontal map is a cofibration, so the outer square is also a strong augmented homotopy pushout square. Therefore  $\mathcal{R}_m[\mathcal{E}_m\mathcal{C}(X)] \simeq \mathrm{Sp}^m(X)$ .

If we prove that the bottom square is an augmented stable homotopy pushout square, then we will have an inductive formula for  $\mathcal{E}_m\mathcal{C}(X)$  that matches the one for  $\mathcal{K}_m\mathcal{C}(X)$ . To accomplish this, it is enough to prove that the upper square is a strong augmented stable homotopy pushout square.

Diagram (4.2) for  $m - 1$  gives us a strong augmented pushout square

$$\begin{array}{ccc} \mathcal{R}_{m-1}[\mathcal{C}(X)] & \longrightarrow & \mathcal{C}(X) \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{m-1}(X) & \longrightarrow & \mathcal{E}_{m-1}\mathcal{C}(X) \end{array}$$

and applying  $\mathcal{R}_m$  to it and using the fact that  $\mathcal{R}_m\mathcal{R}_{m-1} = \mathcal{R}_{m-1}$  gives the strong augmented pushout square

$$\begin{array}{ccc} \mathcal{R}_{m-1}[\mathcal{C}(X)] & \longrightarrow & \mathcal{R}_m[\mathcal{C}(X)] \\ \downarrow & & \downarrow \\ \mathrm{Sp}^{m-1}(X) & \longrightarrow & \mathcal{R}_m[\mathcal{E}_{m-1}\mathcal{C}(X)] \end{array}$$

Consider the diagram

$$\begin{array}{ccccc} \mathcal{R}_{m-1}[\mathcal{C}(X)] & \longrightarrow & \mathcal{R}_m[\mathcal{C}(X)] & \longrightarrow & \mathcal{C}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp}^{m-1}(X) & \longrightarrow & \mathcal{R}_m[\mathcal{E}_{m-1}\mathcal{C}(X)] & \longrightarrow & \mathcal{E}_{m-1}\mathcal{C}(X) \end{array}$$

The outer square and left square are augmented pushouts, respectively by definition and by the above discussion. It follows that the right square is an augmented pushout, and since the top map is a cofibration, it is also a homotopy pushout. Thus the upper square of (4.9) is a homotopy pushout, and therefore the lower square is likewise.

To prove the theorem, suppose given (by induction) a strong augmented stable equivalence  $\mathcal{K}_{m-1}\mathcal{C} \rightarrow \mathcal{E}_{m-1}\mathcal{C}$ . The induced map  $\mathcal{R}_m[\mathcal{K}_{m-1}\mathcal{C}] \rightarrow \mathcal{R}_m[\mathcal{E}_{m-1}\mathcal{C}]$  is also a strong augmented stable equivalence, and we have the following diagram in which all the vertical maps are strong augmented stable equivalences, in the case of the leftmost arrow by Corollary 3.29 because  $\mathcal{K}_m\mathcal{C}$  is  $m$ -reduced:

$$\begin{array}{ccccc} \mathcal{R}_m[\mathcal{K}_m\mathcal{C}] & \longleftarrow & \mathcal{R}_m[\mathcal{K}_{m-1}\mathcal{C}] & \longrightarrow & \mathcal{K}_{m-1}\mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp}^m & \longleftarrow & \mathcal{R}_m[\mathcal{E}_{m-1}\mathcal{C}] & \longrightarrow & \mathcal{E}_{m-1}\mathcal{C} \end{array}$$

As a consequence, there is an augmented stable equivalence from the homotopy pushout of the top row to the homotopy pushout of the bottom row. But that is a stable augmented equivalence from  $\mathcal{K}_m\mathcal{C}$  to  $\mathcal{E}_m\mathcal{C}$ , as required. ■

**4.2. Comparing stable rank filtrations.** In the previous section, we related two filtrations of a  $K$ -theory spectrum  $\mathbf{k}\mathcal{C}$ : the modified stable rank filtration constructed in Section 3.1, and the filtration of  $\mathbf{k}\mathcal{C}$  constructed in [4]. In this section, we relate the construction of Section 3.1 to the original stable rank filtration of Rognes constructed in [15]. We also justify our use of the terminology “modified stable rank filtration” by explaining how the two filtrations come from the same idea applied to two different infinite loop space machines.

Our plan is to use a comparison of Segal’s and Waldhausen’s  $K$ -theory constructions to describe a canonical map from the modified stable rank filtration to the original stable rank filtration. In general, this map need not be an equivalence of filtrations. In particular, if the category being considered is the category of finite-dimensional free modules over a ring  $R$  satisfying dimension invariance, then the comparison of filtration quotients amounts to including the set of diagonal matrices into the set of upper triangular matrices. However, for matrices over a contractible topological ring, such as  $\mathbb{R}$  or  $\mathbb{C}$ , this inclusion is in fact a homotopy equivalence. Thus in the special case of complex topological  $K$ -theory, we establish the following equivalence. (The proposition also applies to real topological  $K$ -theory.)

**PROPOSITION 4.10.** *Let  $\mathcal{C}$  be the topological category of finite-dimensional complex vector spaces, let  $\mathcal{R}_m\mathbf{k}\mathcal{C}$  be the modified stable rank filtration of  $\mathbf{k}\mathcal{C} = bu$ , and let  $F_mbu$  be the  $m$ th stable rank filtration of Rognes as applied to complex topological  $K$ -theory. Then the canonical map of filtrations  $\mathcal{R}_m\mathbf{k}\mathcal{C} \rightarrow F_mbu$  is a homotopy equivalence of spectra for each  $m$ .*

As a consequence, the homotopy pushout square of spectra in Theorem 4.4, when applied to topological complex  $K$ -theory, gives us a good understanding of Rognes’s stable rank filtration for this case. In particu-

lar, Rognes conjectures in [15] that the subquotient  $F_m \mathbf{k}R / F_{m-1} \mathbf{k}R$  is  $(2m - 3)$ -connected for a large class of rings  $R$ , and we establish that this connectivity conjecture actually holds sharply for infinitely many filtration quotients of topological complex  $K$ -theory.

**PROPOSITION 4.11.** *The subquotient spectrum  $F_m bu / F_{m-1} bu$  of the stable rank filtration of Rognes is contractible unless  $m = p^k$  for some prime  $p$ . If  $m = p^k$ , then the bottom nontrivial homotopy group of  $F_m bu / F_{m-1} bu$  occurs in dimension  $2m - 2$ .*

To begin the work of comparing the modified stable rank filtration of Definition 3.9 and the original stable rank filtration defined by Rognes, we recapitulate from [17, Section 1.8] some elements of the comparison between Segal's  $\Gamma$ -space construction and Waldhausen's  $S_\bullet$  construction. This is because the modified and original stable rank filtrations are based, respectively, on filtrations of these constructions.

We reformulate slightly, in two stages. First, an alternative (isomorphic) construction of the Segal  $K$ -theory of  $\mathcal{C}$  comes from thinking of  $S^k$  as the  $k$ -simplicial set  $S^1 \wedge \cdots \wedge S^1$ , evaluating the nerve of  $\mathcal{C}(-)$  levelwise to obtain a  $k$ -simplicial space, and then taking the geometric realization to obtain the  $k$ th space in the  $K$ -theory spectrum of  $\mathcal{C}$ . (Note that for  $k = 1$ , a  $k$ -simplicial set is just a classical simplicial set, and for  $k = 2$ , it is a bisimplicial set, or simplicial space.) Second, for a pointed set  $S$ , the category  $\mathcal{C}(S)$  is again a category with sums. That is, it is a category to which Segal's construction may be applied. Thus we can iterate by applying Segal's construction with the simplicial category  $\mathcal{C}(S^1)$  to the simplicial set  $S^1$  to obtain a bisimplicial category. The following lemma formalizes the equivalence of these constructions.

**LEMMA 4.12.** *The natural map  $\mathcal{C}(S \wedge T) \rightarrow \mathcal{C}(S)(T)$  is an equivalence of categories.*

The lemma follows from the fact that both categories are equivalent to the product category  $\prod_{S \wedge T} \mathcal{C}$ .

**COROLLARY 4.13.** *The natural map  $\mathcal{C}(S^1 \wedge S^1) \rightarrow [\mathcal{C}(S^1)](S^1)$  of bisimplicial sets is an equivalence on geometric realizations.*

By induction, we conclude that we can construct the  $k$ th space in the Segal  $K$ -theory spectrum of  $\mathcal{C}$  by iterating Segal's construction for the category  $\mathcal{C}$ : we take the  $k$ -simplicial category with sums that we have at the  $k$ th stage and use it to evaluate Segal's construction on the simplicial set  $S^1$  to obtain a  $(k + 1)$ -simplicial category. This is a useful phrasing because Waldhausen describes his  $K$ -theory of  $\mathcal{C}$  in terms of an iterated construction.

We summarize Waldhausen's construction next. Recall that if  $\mathcal{C}$  is a category with cofibrations and weak equivalences, then  $wS_\bullet \mathcal{C}$  is a simplicial

category whose objects are determined by data of the following form, where  $\twoheadrightarrow$  denotes a cofibration in  $\mathcal{C}$ :

$$(B_1 \twoheadrightarrow \cdots \twoheadrightarrow B_q, \text{ choices of subquotients}).$$

Morphisms are pointwise weak equivalences between such objects. See [17] for more details. If  $\mathcal{C}$  is a category with cofibrations and weak equivalences, then  $S_\bullet \mathcal{C}$  is again a category with cofibrations and weak equivalences, and thus one may iterate the  $S_\bullet$  construction  $k$  times to obtain the  $k$ -simplicial category  $S_\bullet^k \mathcal{C}$ . Considering only the morphisms that are weak equivalences gives us the  $k$ -simplicial category  $wS_\bullet^k \mathcal{C}$ : the objects of the category  $wS_{q_1} \dots S_{q_k} \mathcal{C}$  are cofibrant  $q_1 \times \dots \times q_k$ -arrays of objects in  $\mathcal{C}$ , together with choices of certain subquotients, and morphisms are objectwise weak equivalences of such arrays. As with the Segal construction, we take the sequence  $w\mathcal{C}, wS_\bullet \mathcal{C}, \dots, wS_\bullet^k \mathcal{C}, \dots$  and obtain the prespectrum that is Waldhausen's  $K$ -theory of  $\mathcal{C}$  by applying nerves levelwise and then taking the geometric realization of the resulting  $k$ -simplicial spaces. This prespectrum is in fact an  $\Omega$ -spectrum after the first map.

To compare the modified stable rank filtration to the original stable rank filtration, we need a map between Segal's construction and Waldhausen's construction. Recall from Section 3.1 that for Segal's  $\Gamma$ -category  $S \mapsto \mathcal{C}(S)$ , the objects of  $\mathcal{C}(S)$  are pairs  $(f, \alpha_*)$ , where  $f$  is a function taking pointed subsets of  $S$  to objects of  $\mathcal{C}$ , and  $\alpha_*$  is a collection of compatible isomorphisms  $\alpha_{S_1, S_2} : f(S_1) \oplus f(S_2) \xrightarrow{\cong} f(S_1 \vee S_2)$ . If we take the simplicial model for  $S^1$  that has the set  $q$  in simplicial dimension  $q$ , then there is a functor sending the  $q$ -simplices  $(f, \alpha_*)$  of  $\mathcal{C}(S^1)$  to  $wS_q \mathcal{C}$  as follows. Let  $A_i$  denote  $f(\{i\})$  and let  $B_i = f(\{1, \dots, i\})$ , where  $\{i\}$  and  $\{1, \dots, i\}$  implicitly contain the basepoint. Then  $(f, \alpha_*)$  provides structure maps

$$\alpha_{\{1, \dots, i\}, \{i+1\}} : B_i \oplus A_{i+1} \rightarrow B_{i+1}$$

and the object  $(f, \alpha_*)$  in the Segal category is sent to the object in the  $S_\bullet$  construction

$$B_1 \twoheadrightarrow \cdots \twoheadrightarrow B_q,$$

together with the choices of quotient maps given by the inverse of the isomorphisms from  $\alpha_*$  followed by projection:  $B_{i+1} \xrightarrow{\cong} B_i \oplus A_{i+1} \twoheadrightarrow A_{i+1}$ .

This simplicial functor generalizes to a  $k$ -simplicial functor that for each  $k$  takes the  $k$ -fold iterate of Segal's construction for  $\mathcal{C}$  on  $S^1$  to the  $k$ -fold iterate  $wS_\bullet^k \mathcal{C}$ . Hence it induces a  $k$ -simplicial functor

$$\mathcal{C}(S^k) \rightarrow wS_\bullet^k \mathcal{C}.$$

Together these simplicial functors induce a map from Segal's  $K$ -theory to Waldhausen's  $K$ -theory, which is a homotopy equivalence when cofibrations in  $\mathcal{C}^n$  are "splittable up to weak equivalence" ([17, Theorem 1.8.1]).

The example that is most relevant to us is the category of free modules over a ring  $R$  satisfying the dimension invariance property. In this case, both constructions give a model for the free  $K$ -theory of  $R$  that coincides with Quillen's  $K$ -theory above dimension zero ([17, Sections 8 and 9]). Here we include the case when  $R$  is  $\mathbb{R}$  or  $\mathbb{C}$ , in which case  $\mathcal{C}$  is the topologically enriched category of (real or complex) vector spaces, and both constructions give a model for connective topological  $K$ -theory.

We need to understand the two stable rank filtrations in these terms. Recall from Section 3.1 that the modified rank filtration of an augmented  $\Gamma$ -space  $F$  is defined as the pullback of the filtration of  $\mathrm{Sp}^\infty$  by  $\{\mathrm{Sp}^m\}_{m=0}^\infty$ . On  $\mathcal{C}(S^k)$ , this amounts to filtering by the maximal dimension of the modules appearing in an object  $(f, \alpha_*)$ . Similarly, the original stable rank filtration is obtained by filtering  $wS_\bullet^k\mathcal{C}$  by the maximal dimension of the modules in the  $k$ -dimensional array of cofibrations (see [15] for more details). It is easy to see that the map from Segal's spectrum to Waldhausen's spectrum respects the two filtrations.

Now we are ready to prove Propositions 4.10 and 4.11.

*Proof of Proposition 4.10.* Let  $R$  be  $\mathbb{R}$  or  $\mathbb{C}$ . The map of  $k$ -simplicial categories

$$(4.14) \quad \mathcal{C}(S^k) \rightarrow wS_\bullet^k\mathcal{C}$$

respects the filtrations (modified rank filtration in the domain, rank filtration in the codomain). Therefore it is sufficient to check that it induces a homotopy equivalence on the filtration quotients.

Both the domain and the codomain are  $k$ -simplicial groupoids, so they are equivalent, in each dimension, to a disjoint union of groups, and the map (4.14) is a map of  $k$ -simplicial groupoids. The category  $\mathcal{C}(S^k)$  is equivalent, in total dimension exactly  $n$ , to the disjoint union of automorphism groups of certain ordered direct-sum decompositions of  $R^n$ . Such a group is equivalent to a group of invertible block diagonal matrices. The category  $wS_\bullet^k\mathcal{C}$  is equivalent, in each dimension, to a disjoint union of groups of automorphisms of a certain lattice of subspaces of  $R^n$  for various  $n$ . Such a group is equivalent to a certain group of invertible block upper triangular matrices. In these terms, the map  $\mathcal{C}(S^k) \rightarrow wS_\bullet^k\mathcal{C}$  amounts, on each connected component of the filtration quotients, to the inclusion of a group of block diagonal matrices into a corresponding group of block upper triangular matrices. In the topological case, such an inclusion is always a homotopy equivalence. Thus the map  $\mathcal{C}(S^k) \rightarrow wS_\bullet^k\mathcal{C}$  induces a homotopy equivalence on each connected component of each filtration quotient. Therefore, the whole map between  $K$ -theory spectra is a homotopy equivalence. ■



As a consequence, we may verify Rognes's connectivity conjecture for complex  $K$ -theory. Let  $F_m bu$  be the  $m$ th stage in the original stable rank filtration of  $bu$ , as defined by Rognes.

*Proof of Proposition 4.11.* By Proposition 4.10, we may substitute the modified stable rank filtration for the original one, that is,

$$F_m bu / F_{m-1} bu \simeq \mathcal{R}_m / \mathcal{R}_{m-1} bu.$$

We use the homotopy pushout square of spectra in Theorem 4.4 to find the connectivity of  $\mathcal{R}_m bu / \mathcal{R}_{m-1} bu$  by considering  $\mathrm{Sp}^m(\mathbf{S}) / \mathrm{Sp}^{m-1}(\mathbf{S})$  and  $\Sigma^{-1}A_m / A_{m-1}$ . It follows from Theorems 9.5 and 9.7 of [4] that  $\Sigma^{-1}A_m / A_{m-1}$  is  $(2m - 2)$ -connected when  $m = p^k$  and is contractible if  $m \neq p^k$ . On the other hand, we know that  $\mathrm{Sp}^m(\mathbf{S}) / \mathrm{Sp}^{m-1}(\mathbf{S})$  is also contractible unless  $m = p^k$  for some prime  $p$ , and that a basis for the mod  $p$  cohomology of  $\mathrm{Sp}^{p^k}(\mathbf{S}) / \mathrm{Sp}^{p^k-1}(\mathbf{S})$  is given by admissible words of length  $k$  in  $\mathcal{A} / \mathcal{A}\beta$ , where  $\mathcal{A}$  is the Steenrod algebra [14]. The lowest dimension of an admissible sequence of length  $k$  is  $2p^k - 2$ , which completes the proof of the proposition. ■

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