FUNDAMENTA
MATHEMATICAE
210 (2010)

The absolute continuity of the invariant measure of random iterated function systems with overlaps

by

Balázs Bárány (Budapest) and **Tomas Persson** (Warszawa)

Abstract. We consider iterated function systems on the interval with random perturbation. Let Y_ε be uniformly distributed in $[1-\varepsilon, 1+\varepsilon]$ and let $f_i \in C^{1+\alpha}$ be contractions with fixpoints a_i. We consider the iterated function system $\{Y_\varepsilon f_i + a_i(1-Y_\varepsilon)\}_{i=1}^n$, where each of the maps is chosen with probability p_i. It is shown that the invariant density is in L^2 and its L^2 norm does not grow faster than $1/\sqrt{\varepsilon}$ as ε vanishes.

The proof relies on defining a piecewise hyperbolic dynamical system on the cube with an SRB-measure whose projection is the density of the iterated function system.

1. Introduction and statements of results. Let $\{f_1, \ldots, f_l\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities $(p_1, \ldots, p_l)$, with the choice of the map random and independent at each step. We assume that for each i, f_i maps $[-1, 1)$ into itself so that the image is bounded away from -1 and 1, and $f_i \in C^{1+\alpha}([-1, 1))$. Let ν be the invariant measure of our IFS, i.e.

$$\nu = \sum_{i=1}^{l} p_i \nu \circ f_i^{-1}. \tag{1.1}$$

Let $\mu = (p_1, \ldots, p_l)^{\mathbb{N}}$ be a Bernoulli measure on the space $\Sigma = \{1, \ldots, l\}^{\mathbb{N}}$. Let $h(\underline{p}) = -\sum_{i=1}^{l} p_i \log p_i$ be the entropy of the left-shift operator with respect to the Bernoulli measure μ. It was proved in [7], for non-linear, contracting on average, iterated function systems (and later extended in [3]) that

$$\dim_{\mathrm{H}}(\nu) \leq h/|\chi|,$$

where $\dim_{\mathrm{H}}(\nu)$ is the Hausdorff dimension of the measure ν, and χ is the Lyapunov exponent of the IFS associated to the Bernoulli measure μ.

2010 *Mathematics Subject Classification*: Primary 37C40; Secondary 37H15.
Key words and phrases: iterated function system, absolute continuity, random perturbation.

One can expect that, at least "typically", the measure ν is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is transversality. For example, for the linear case with uniform contraction ratios, see [8] and [10]. For the linear case and non-uniform contraction ratios, see [5] and [6]. For the non-linear case, see for example [14] and [1]. We note that there is another direction in the study of iterated function systems with overlaps, which is concerned with concrete, but non-typical systems, often of arithmetic nature, for which there is a dimension drop (see for example [4]).

Throughout this paper we are interested in studying absolute continuity with density in L^2. We will study a modification of the problem, namely we consider a random perturbation of the functions. The linear case was studied by Peres, Simon and Solomyak in [9]. They proved absolute continuity for random linear IFS, with non-uniform contraction ratios and also L^2 and continuous density in the uniform case. We extend this result by proving L^2 density with non-uniform contraction ratios and in the non-linear case.

We consider two cases. First let us suppose that for each $i \in \{1, \ldots, l\}$, f_i maps $[-1,1)$ into itself, $f_i([-1,1))$ is bounded away from -1 and 1, $f_i \in C^{1+\alpha}([-1,1))$ and

$$0 < \lambda_{i,\min} \leq |f_i'(x)| \leq \lambda_{i,\max} < 1 \tag{1.2}$$

for every $x \in [-1,1)$. Moreover suppose that for every i the fixed point of f_i is $a_i \in (-1,1)$, and

$$i \neq j \ \Rightarrow\ a_i \neq a_j. \tag{1.3}$$

Let Y_ε be uniformly distributed on $[1-\varepsilon, 1+\varepsilon]$. Denote the probability measure of Y_ε by η_ε. Let

$$f_{i,Y_\varepsilon}(x) = Y_\varepsilon f_i(x) + a_i(1 - Y_\varepsilon) \tag{1.4}$$

for every $i \in \{1, \ldots, l\}$. Then $f_{i,Y_\varepsilon}(x)$ is in $[-1,1)$ for all values of $x \in [-1,1)$ and Y_ε, provided ε is sufficiently small. The iterated maps are applied randomly according to the stationary measure μ, with the sequence of independent and identically distributed errors $y_1, y_2, \ldots$ distributed as Y_ε, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta_\varepsilon) = \mathbb{E}(\log(Y_\varepsilon f'))$$

and it is easy to see that

$$\chi(\mu, \eta_\varepsilon) < \sum_{i=1}^{l} p_i \log((1+\varepsilon)\lambda_{i,\max}) < 0$$

for sufficiently small $\varepsilon > 0$. Let Z_ε be the random variable

$$Z_\varepsilon := \lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0), \tag{1.5}$$

where the numbers i_k are i.i.d., with distribution μ on $\{1,\dots,l\}$, and $y_{k,\varepsilon}$ are pairwise independent with the distribution of Y_ε and also independent of the choice of i_k. Let ν_ε be the distribution of Z_ε.

One can easily prove the following theorem.

THEOREM 1.1. *The measure ν_ε converges weakly as $\varepsilon \to 0$ to the measure ν satisfying* (1.1).

THEOREM 1.2. *Let ν_ε be the distribution of the limit* (1.5). *Assume that* (1.2) *and* (1.3) *hold, and*

$$\sum_{i=1}^{l} p_i^2 \frac{\lambda_{i,\max}}{\lambda_{i,\min}^2} < 1. \tag{1.6}$$

Then for every sufficiently small $\varepsilon > 0$, ν_ε is absolutely continuous with respect to Lebesgue measure, with density in L^2, and there exists a constant C such that the density of ν_ε satisfies

$$\|\nu_\varepsilon\|_2 \le C/\sqrt{\varepsilon}.$$

REMARK 1. Let

$$C'_\varepsilon = \sqrt{\frac{32}{\big(1-\sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\big)C''_\varepsilon}},$$

$$C''_\varepsilon = \min_{i\ne j}\left\{\frac{|a_i-a_j|+\varepsilon(-|a_i+a_j|-2)}{1-\varepsilon^2}\right\}.$$

The proof of Theorem 1.2 will show that $\|\nu_\varepsilon\|_2 \le C'_\varepsilon/\sqrt{\varepsilon}$. Hence we can choose any $C > \lim_{\varepsilon\to 0} C'_\varepsilon$.

REMARK 2. Actually the proof of Theorem 1.2 shows that Z_ε conditioned on the perturbations $y_{1,\varepsilon}, y_{2,\varepsilon},\dots$ has density in L^2 for almost all $y_{1,\varepsilon}, y_{2,\varepsilon},\dots$.

We can state an easy corollary of the theorem.

COROLLARY 1.3. *Let $\{\lambda_i Y_\varepsilon x + a_i(1-\lambda_i Y_\varepsilon)\}_{i=1}^{l}$ be a random iterated function system. Assume that* (1.3) *holds, and*

$$\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1. \tag{1.7}$$

Then for every sufficiently small $\varepsilon > 0$, ν_ε is absolutely continuous with respect to Lebesgue measure with density in L^2, and there exists a constant C such that

$$\|\nu_\varepsilon\|_2 \le C/\sqrt{\varepsilon}.$$

We study another case of random perturbation, namely let $\widetilde{\lambda}_{i,\varepsilon}$ be uniformly distributed on $[\lambda_i-\varepsilon, \lambda_i+\varepsilon]$. Let $\{\widetilde{\lambda}_{i,\varepsilon}x + a_i(1-\widetilde{\lambda}_{i,\varepsilon})\}_{i=1}^{l}$ be our random iterated function system, where $a_i \neq a_j$ for every $i \neq j$. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_l)$, and $X_{\underline{\lambda},\varepsilon}$ be the random variable

$$X_{\underline{\lambda},\varepsilon} = \sum_{k=1}^{\infty} (a_{i_k}(1 - \widetilde{\lambda}_{i_k,\varepsilon})) \prod_{j=1}^{k-1} \widetilde{\lambda}_{i_j,\varepsilon} \tag{1.8}$$

where the numbers i_k are i.i.d. with distribution μ on $\{1, \dots, l\}$, and $\widetilde{\lambda}_{i_k,\varepsilon}$ are pairwise independent. Let $\nu_{\underline{\lambda},\varepsilon}$ denote the distribution of $X_{\underline{\lambda},\varepsilon}$. Moreover let $\nu_{\underline{\lambda}}$ be the invariant measure of the iterated function system $\{\lambda_i x + a_i(1 - \lambda_i)\}_{i=1}^{l}$ according to μ.

Theorem 1.4. *The measure $\nu_{\underline{\lambda},\varepsilon}$ converges weakly to $\nu_{\underline{\lambda}}$ as $\varepsilon \to 0$.*

To have a statement similar to Theorem 1.2 we need a technical assumption

$$\min_{i \neq j} \left| \frac{a_j \lambda_i - a_i \lambda_j}{\lambda_i - \lambda_j} \right| > 1. \tag{1.9}$$

Theorem 1.5. *Suppose that* (1.9) *and* (1.3) *hold, and moreover*

$$\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1. \tag{1.10}$$

Then for every sufficiently small $\varepsilon > 0$, the measure $\nu_{\underline{\lambda},\varepsilon}$ is absolutely continuous with respect to Lebesgue measure, with density in L^2, and there exists a constant C such that

$$\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq C/\sqrt{\varepsilon}.$$

Remark 3. Let

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1 - \sum_{i=1}^{l} p_i^2 \frac{\lambda_i + \varepsilon}{(\lambda_i - \varepsilon)^2}\right) C''_\varepsilon}},$$

$$C''_\varepsilon = \sigma \min_{i \neq j} \frac{|a_i \lambda_j - a_j \lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i \lambda_j},$$

where $0 < \sigma < 1$. As in Remark 1, the proof of Theorem 1.5 will show that $\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$ for small ε.

The main difference between Theorem 1.5 and Corollary 1.3 is the random perturbation. Namely, in Theorem 1.5 we choose the contraction ratio uniformly in the ε-neighborhood of λ_i, while in Corollary 1.3 we choose the contraction ratio uniformly in the $\lambda_i\varepsilon$-neighborhood of λ_i.

Throughout this paper we will use the method of [11].

2. Proof of Theorem 1.2. Let $Q = [-1, 1)^3$ and $m \in \mathbb{N}$. We partition the cube Q into the rectangles $\{Q_{1,k}, \ldots, Q_{l,k}\}_{k=0}^{2^m-1}$, where

$$Q_{i,k} = \Big\{ (x, y, z) \in Q : -1 + 2\sum_{j=1}^{i-1} p_j \le y < -1 + 2\sum_{j=1}^{i} p_j,$$
$$-1 + k2^{-m+1} \le z < -1 + (k+1)2^{-m+1} \Big\},$$

where we use the convention that an empty sum is 0. Hence we slice Q into 2^m slices along the z-axis and l slices along the y-axis. We thereby get $2^m l$ pieces which we call $Q_{i,k}$, according to the definition above.

Let

$$Q_i = \bigcup_{k=0}^{2^m-1} Q_{i,k}.$$

We define a map $g_{\varepsilon,m} : Q \to Q$ so that each slice $Q_{i,k}$ is expanded as much as possible in the second and third coordinates. In the first coordinate it is mapped according to a perturbation of f_i, and hence contracted. Which perturbation is chosen depends on the third coordinate. There is a picture of this in Figure 1.

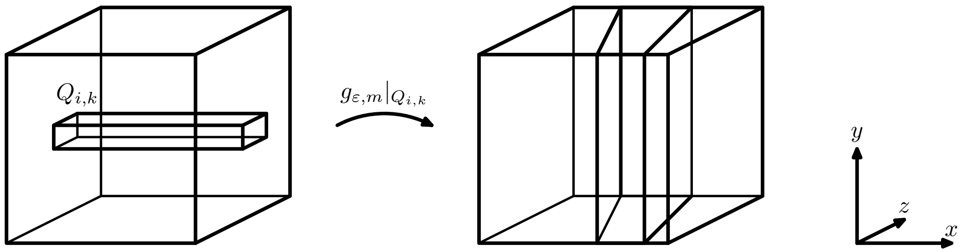

Fig. 1. The action of $g_{\varepsilon,m}$ restricted to $Q_{i,k}$

More precisely, we define $g_{\varepsilon,m}\colon Q \to Q$ by setting, for $(x, y, z) \in Q_{i,k}$,

$$g_{\varepsilon,m}\colon (x, y, z) \mapsto \Big(d(z) f_i(x) + a_i(1 - d(z)),\ \frac{1}{p_i} y + b(y),\ 2^m z + c(z) \Big),$$

where

$$d(z) = 1 + 2^m \varepsilon (z - (-1 + (k + 1/2)2^{-m+1})) \quad \text{for } (x, y, z) \in Q_{i,k},$$

$$b(y) = 1 - \frac{1}{p_i}\Big(-1 + 2\sum_{j=1}^{i} p_j\Big) \quad \text{for } (x, y, z) \in Q_{i,k},$$

$$c(z) = 2^m - 2k - 1 \quad \text{for } (x, y, z) \in Q_{i,k}.$$

Hence $g_{\varepsilon,m}$ maps each of the pieces $Q_{i,j}$ so that it is contracted in the x-direction and fully expanded in the y- and z-directions.

Let $\mathcal{L}_3$ be the normalised Lebesgue measure on Q. The measures

$$\gamma_{\varepsilon,m,n} = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}_3 \circ g_{\varepsilon,m}^{-k}$$

converge weakly to an SRB-measure $\gamma_{\varepsilon,m}$ as $n \to \infty$ (see [12] and [13]). The measure $\gamma_{\varepsilon,m}$ is ergodic by the Hopf argument, since $g_{\varepsilon,m}$ is hyperbolic and the stable and unstable manifolds are parallel to the coordinate axes and have maximal extension in the box Q. Moreover, let $\nu_{\varepsilon,m}$ be the projection of $\gamma_{\varepsilon,m}$ onto the first coordinate. More precisely, if $E \subset [-1,1)$ is a measurable set, then we define $\nu_{\varepsilon,m}(E) = \gamma_{\varepsilon,m}(E \times [-1,1) \times [-1,1))$.

The measure $\nu_{\varepsilon,m}$ is the distribution of the limit

$$\lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0),$$

where $y_{i,\varepsilon}$ are uniformly distributed on $[1-\varepsilon, 1+\varepsilon]$, but not independent. However, one can easily prove the following lemma.

LEMMA 2.1. *The measure $\nu_{\varepsilon,m}$ converges weakly to ν_ε as $m \to \infty$.*

Let

$$A_i = \{(i,0),(i,1),\ldots,(i,2^m-1)\} \quad \text{and} \quad A = \bigcup_{i=1}^{l} A_i.$$

If $a = (i,k) \in A$ we will write $\hat{Q}_a$ for $Q_{i,k}$. With this notation we have

$$Q = \bigcup_{a\in A} \hat{Q}_a \quad \text{and} \quad Q_i = \bigcup_{a\in A_i} \hat{Q}_a, \qquad i = 0,1,\ldots,l.$$

Let $\Theta_0 = A^{\mathbb{N}\cup\{0\}}$. If $p \in Q$ then there is a unique sequence $\rho_0(p) = \{\rho_0(p)_k\}_{k=0}^{\infty} \in \Theta_0$ such that

$$g_{\varepsilon,m}^k(p) \in Q_{\rho_0(p)_k}, \qquad k = 0,1,\ldots.$$

The map $\rho_0 \colon Q \to \Theta_0$ is not injective. We have $\rho_0(x,y,z) = \rho_0(x',y',z')$ if $y = y'$ and $z = z'$, but $\rho_0(x,y,z) \neq \rho_0(x',y',z')$ otherwise. Hence we can (and will) use the notation $\rho_0(y,z)$ instead of $\rho_0(x,y,z)$.

We will denote elements in Θ_0 by $\boldsymbol{a}$, $\boldsymbol{b}$ and so on. We let σ denote the left shift on Θ_0, defined in the usual way.

We can transfer the measures $\gamma_{\varepsilon,m}$ to a measure γ_{Θ_0} by $\gamma_{\Theta_0} = \gamma_{\varepsilon,m} \circ \rho_0^{-1}$.

We let Θ denote the natural extension of Θ_0. That is, Θ is the set of all two-sided infinite sequences such that any one-sided infinite subsequence of a sequence in Θ is a sequence in Θ_0. The measure γ_{Θ_0} defines an ergodic measure γ_Θ on Θ in a natural way. If $\xi \colon \Theta \to \Theta_0$ is defined by $\xi(\{i_k\}_{k\in\mathbb{Z}}) = \{i_k\}_{k\in\mathbb{N}\cup\{0\}}$, then we define $\gamma_\Theta(\xi^{-1}E) = \gamma_{\Theta_0}(E)$. We can define a map $\rho^{-1} \colon \Theta \to Q$ such that $\rho^{-1}(\sigma(\boldsymbol{a})) = g_{\varepsilon,m}(\rho^{-1}(\boldsymbol{a}))$ for any sequence $\boldsymbol{a} \in \Theta$.

We note that the L^2 norm of the density $\nu_{\varepsilon,m}$ is not larger than twice that of the density of $\gamma_{\varepsilon,m}$. If $h_{\nu_{\varepsilon,m}}(x)$ and $h_{\gamma_{\varepsilon,m}}(x,y,z)$ denote the densities of $\nu_{\varepsilon,m}$ and $\gamma_{\varepsilon,m}$ respectively, then by Cauchy–Schwarz's inequality

$$\|\nu_{\varepsilon,m}\|_2^2 \le \int_{-1}^{1} h_{\nu_{\varepsilon,m}}(x)^2\,dx = 32\int_{-1}^{1}\Big(\int_{-1}^{1}\int_{-1}^{1} h_{\gamma_{\varepsilon,m}}(x,y,z)\,\frac{dy}{2}\,\frac{dz}{2}\Big)^2\,\frac{dx}{2}$$
$$\le 32\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} h_{\gamma_{\varepsilon,m}}(x,y,z)^2\,\frac{dy}{2}\,\frac{dz}{2}\,\frac{dx}{2} = 4\|\gamma_{\varepsilon,m}\|_2^2.$$

This proves that if $\gamma_{\varepsilon,m}$ has L^2 density, then so has $\nu_{\varepsilon,m}$, and

$$\|\nu_{\varepsilon,m}\|_2 \le 2\|\gamma_{\varepsilon,m}\|_2. \tag{2.1}$$

If p is a point in Q, then we let T_pQ denote the tangent space at p. For each p in Q we define the following cone in the tangent space T_pQ:

$$C_p = \Big\{(u,v,w)\in T_pQ : \Big|\frac{u}{w}\Big|, \Big|\frac{v}{w}\Big| < \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)}\Big\},$$

where $\lambda_{\max,\max} = \max_i \lambda_{i,\max} = \max_i \sup_{x\in[-1,1)} |f_i'(x)|$. The following lemma states that the set of cones C_p defines a family of unstable cones, and that images of certain curves intersect transversally. There is an illustration of the transversality in Figure 2.

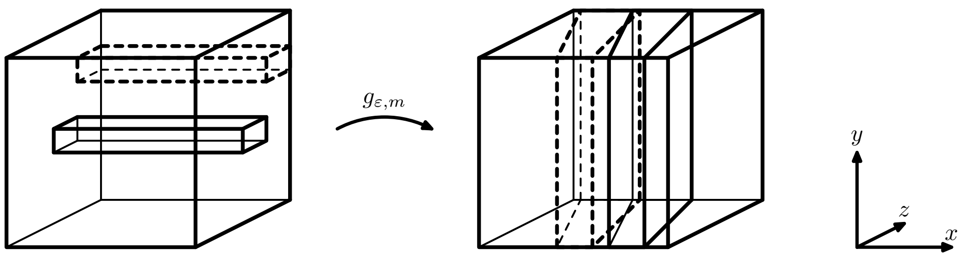

Fig. 2. Any two different $Q_{i,k}$ and $Q_{j,l}$ on the same height ($i=j$) share the same image, but in the case when $i\ne j$ their images have transversal intersection if they intersect.

Lemma 2.2. *The cones C_p make up a family of unstable cones, that is, $d_p g_{\varepsilon,m}(C_p) \subset C_{g_{\varepsilon,m}(p)}$.*

Moreover, for sufficiently large m and every $0<\varepsilon<\min_{i\ne j}\frac{|a_i-a_j|}{2+|a_i+a_j|}$, if $\zeta_1\subset Q_{\xi_1}$ and $\zeta_2\subset Q_{\xi_2}$ are two curve segments with tangents in C_p such that $\xi_1\in A_i$ and $\xi_2\in A_j$, $i\ne j$, then if $g_{\varepsilon,m}(\zeta_1)$ and $g_{\varepsilon,m}(\zeta_2)$ intersect, and if $(u_1,v_1,1)$ and $(u_2,v_2,1)$ are tangents to $g_{\varepsilon,m}(\zeta_1)$ and $g_{\varepsilon,m}(\zeta_2)$ respectively, it follows that $|u_1-u_2| > C_{\varepsilon,m}\varepsilon$, where

$$C_{\varepsilon,m} = \min_{i\ne j}\Big\{\frac{|a_i-a_j|+\varepsilon(-|a_i+a_j|-2)}{1-\varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m-\lambda_{\max,\max}(1+\varepsilon)}\Big\}.$$

Proof. The Jacobian of $g_{\varepsilon,m}$ is

$$d_p g_{\varepsilon,m} = \begin{pmatrix} d(z)f_i'(x) & 0 & 2^m\varepsilon(f_i(x)-a_i) \\ 0 & 1/p_i & 0 \\ 0 & 0 & 2^m \end{pmatrix}$$

where $p=(x,y,z)\in Q_{i,k}$. If $(u,v,w)\in C_p$, then

$$d_p g_{\varepsilon,m}(u,v,w) = \begin{pmatrix} d(z)f_i'(x)u + 2^m\varepsilon(f_i(x)-a_i)w \\ (1/p_i)v \\ 2^m w \end{pmatrix}$$

We just need to check that this vector is in C_p, provided that m is large. This is easily checked, using that $|d(z)|\le 1+\varepsilon$, $|f_i'(x)|\le\lambda_{i,\max}$ and $|f_i(x)-a_i|\le 2$. Indeed,

$$\begin{aligned}\frac{|d(z)f_i'(x)u+2^m\varepsilon(f_i(x)-a_i)w|}{|2^m w|} &\le \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{|u|}{|w|}+2\varepsilon \\ &\le \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{2^{m+1}\varepsilon}{2^m-(1+\varepsilon)\lambda_{\max,\max}}+2\varepsilon \le \frac{2^{m+1}\varepsilon}{2^m-(1+\varepsilon)\lambda_{\max,\max}}\end{aligned}$$

and

$$\frac{|(1/p_i)v|}{|2^m w|} \le \frac{1}{p_i 2^m}\frac{2^{m+1}\varepsilon}{2^m-(1+\varepsilon)\lambda_{\max,\max}} \le \frac{2^{m+1}\varepsilon}{2^m-(1+\varepsilon)\lambda_{\max,\max}}$$

proves that $d_p g_{\varepsilon,m}(C_p)\subset C_{g_{\varepsilon,m}(p)}$ if m is sufficiently large, so that $2^m-(1+\varepsilon)\lambda_{\max,\max}>0$ and $p_i 2^m>1$.

To prove the other statement of the lemma, assume that $p=(x_p,y_p,z_p)\in Q_i$ and $q=(x_q,y_q,z_q)\in Q_j$, $i\neq j$, are such that $g_{\varepsilon,m}(p)=g_{\varepsilon,m}(q)=(x,y,z)$. Then, if $p\in Q_i$,

$$d_p g_{\varepsilon,m}\colon (u,v,1)\mapsto 2^m\biggl(\frac{d(z_p)f_i'(x_p)}{2^m}u+(f_i(x_p)-a_i)\varepsilon,\ \frac{v}{p_i},\ 1\biggr).$$

Then

$$f_i(x_p)=\frac{x-a_i(1-d(z_p))}{d(z_p)} \quad\text{and}\quad f_j(x_q)=\frac{x-a_j(1-d(z_q))}{d(z_q)}.$$

Without loss of generality, assume that $a_i>a_j$. For simplicity we study the case $x\ge a_i>a_j$. The proofs for $a_i\ge x\ge a_j$ and $a_i>a_j\ge x$ are similar. Then

$$d_p g_{\varepsilon,m}(C_p)\subset\biggl\{\, w(u,v,1) : \frac{x-a_i}{1+\varepsilon}\varepsilon-\Delta_i\varepsilon\le u\le\frac{x-a_i}{1-\varepsilon}\varepsilon+\Delta_i\varepsilon\biggr\},$$

where $\varDelta_i = \frac{2(1+\varepsilon)\lambda_{i,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)}$. Therefore

$$|u_2 - u_1| \ge \frac{x - a_j}{1+\varepsilon}\varepsilon - \frac{x - a_i}{1-\varepsilon}\varepsilon - (\varDelta_i + \varDelta_j)\varepsilon$$
$$\ge \left(\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1-\varepsilon^2} - 2\max_i \varDelta_i\right)\varepsilon$$

for every $x \ge a_i > a_j$. Let $\varDelta_{\max} = \max_i \varDelta_i$. Since $0 < \varepsilon < \min_{i\ne j} \frac{|a_i - a_j|}{2+|a_i+a_j|}$, we have

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1-\varepsilon^2} > 0.$$

Therefore

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1-\varepsilon^2} - 2\varDelta_{\max} > 0,$$

for sufficiently large m. By similar methods we have for $a_i \ge x \ge a_j$,

$$|u_2 - u_1| \ge \left(\frac{a_i - a_j}{1+\varepsilon} - 2\varDelta_{\max}\right)\varepsilon,$$

and for $a_i > a_j \ge x$,

$$|u_2 - u_1| \ge \left(\frac{a_i - a_j - \varepsilon(a_i + a_j + 2)}{1-\varepsilon^2} - 2\varDelta_{\max}\right)\varepsilon.$$

Therefore we can choose

$$C_{\varepsilon,m} = \min_{i\ne j}\left\{\frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1-\varepsilon^2} - 2\varDelta_{\max}\right\}. \ \blacksquare$$

The rest of the section follows Tsujii's article [15].

Proof of Theorem 1.2. For any $r > 0$ we define the bilinear form $(\cdot,\cdot)_r$ of signed measures on $\mathbb{R}$ by

$$(\rho_1, \rho_2)_r = \int_{\mathbb{R}} \rho_1(B_r(x))\rho_2(B_r(x))\, dx$$

where $B_r(x) = [x - r, x + r]$. It is easy to see that if

$$\liminf_{r\to 0} \frac{1}{r^2}(\rho, \rho)_r < \infty$$

then the measure ρ has density in L^2 (see [15]). Moreover

$$\|\rho\|_2^2 \le \liminf_{r\to 0} \frac{1}{r^2}(\rho, \rho)_r.$$

Let γ_z denote the conditional measure of $\gamma_{\varepsilon,m}$ on the set $R_z = \{(u, v, w) \in Q : v = y,\ w = z\}$. Since the one-dimensional Lebesgue measure is invariant under the action of $g_{\varepsilon,m}$ projected to the second coordinate, we conclude that γ_z is independent of y almost everywhere. It follows that

$$\|\gamma_{\varepsilon,m}\|_2^2 = \int_{-1}^{1} \|\gamma_z\|_2^2\, dz. \tag{2.2}$$

Let

$$J(r) := \frac{1}{r^2} \int_{-1}^{1} (\gamma_z, \gamma_z)_r\, dz.$$

By the invariance of $\gamma_{\varepsilon,m}$ it follows that

$$\gamma_z = 2^{-m} \sum_{i=1}^{l} p_i \sum_{a\in A_i} \gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \tag{2.3}$$

where $g^{-a}_{\varepsilon,m}$ denotes the inverse branch of $g_{\varepsilon,m}$ with image in $\hat{Q}_a$. Recall that $a \in A_i$ means that $a = (i,k)$ for some k, so that $\hat{Q}_a = Q_{i,k}$ for some k. We denote the measure $\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}$ by $\sigma_{a,z}$. Then by (2.3) and the definition of $J(r)$,

$$J(r) = \frac{1}{2^{2m} r^2} \sum_{i=1}^{l} \sum_{j=1}^{l} p_i p_j \sum_{a\in A_i} \sum_{b\in A_j} \int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z})_r\, dz. \tag{2.4}$$

For fixed $a, b \in A_i$,

$$\begin{aligned}(\sigma_{a,z}, \sigma_{b,z})_r &\le (\sigma_{a,z}, \sigma_{a,z})_r^{1/2} (\sigma_{b,z}, \sigma_{b,z})_r^{1/2} \\ &\le (1+\varepsilon)\lambda_{i,\max} \big(\gamma_{g^{-a}_{\varepsilon,m}(z)}, \gamma_{g^{-a}_{\varepsilon,m}(z)}\big)^{1/2}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} \times \big(\gamma_{g^{-b}_{\varepsilon,m}(z)}, \gamma_{g^{-b}_{\varepsilon,m}(z)}\big)^{1/2}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} \\ &\le (1+\varepsilon)\lambda_{i,\max} \frac{\big(\gamma_{g^{-a}_{\varepsilon,m}(z)}, \gamma_{g^{-a}_{\varepsilon,m}(z)}\big)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} + \big(\gamma_{g^{-b}_{\varepsilon,m}(z)}, \gamma_{g^{-b}_{\varepsilon,m}(z)}\big)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}}{2}.\end{aligned} \tag{2.5}$$

Moreover, if $a \in A_i$ and $b \in A_j$, $i \ne j$, then

$$\begin{aligned}(\sigma_{a,z}, \sigma_{b,z})_r &= \int \sigma_{a,z}(B_r(x))\sigma_{b,z}(B_r(x))\, dx \\ &= \iiint \mathbb{I}_{\{s\,:\,|s-x|<r\}}(s)\mathbb{I}_{\{t\,:\,|t-x|<r\}}(t)\, d\sigma_{a,z}(s)\, d\sigma_{b,z}(t)\, dx \\ &\le \iint 2r\mathbb{I}_{\{(s,t)\,:\,|s-t|<2r\}}(s,t)\, d\sigma_{a,z}(s)\, d\sigma_{b,z}(t) \\ &= \iint \mathbb{I}_{\{(\boldsymbol{c},\boldsymbol{d})\,:\,|\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))|<2r\}}(\boldsymbol{c},\boldsymbol{d}) \\ &\qquad\qquad d\gamma_\Theta(\boldsymbol{c})\, d\gamma_\Theta(\boldsymbol{d}).\end{aligned}$$

Let us comment on the notation $\rho_0(z)$. Actually $\rho_0(z)$ is not defined, but rather $\rho_0(x,y,z)$. Recall that $\rho_0(x,y,z)$ is independent of x and that we therefore have introduced the notation $\rho_0(y,z)$. Moreover, as noticed above, the measures γ_z, and therefore also $\sigma_{a,z}$, are independent of y. Hence we can choose arbitrary x, y and let $\rho_0(z)$ denote $\rho_0(x,y,z) = \rho_0(y,z)$. Since all the estimates below will be independent of this choice of y, we will use the notation $\rho_0(z)$ instead of $\rho_0(x,y,z)$.

By a change of order of integration we get

$$\int_{-1}^{1} (\sigma_{a,z},\sigma_{b,z})_r\,dz \le 2r\iint \mathcal{L}_1(\{z : |\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))| < 2r\})\,d\gamma_\Theta(\boldsymbol{c})\,d\gamma_\Theta(\boldsymbol{d}). \tag{2.6}$$

We will now apply Lemma 2.2 to (2.6). Note that

$$z \mapsto \rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)), \quad \text{and} \quad z \mapsto \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))$$

are two curves with tangents in the cones C_p. Lemma 2.2 states that these curves have a transversal intersection, if they intersect, so that

$$\mathcal{L}_1(\{z : |\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))| < 2r\}) \le 4r/C_{\varepsilon,m}.$$

Hence

$$\int_{-1}^{1} (\sigma_{a,z},\sigma_{b,z})_r\,dz \le \frac{8r^2}{C_{\varepsilon,m}\varepsilon}. \tag{2.7}$$

By using (2.4) we have

$$J(r) = \frac{1}{2^{2m}r^2}\sum_{i=1}^{l} p_i^2 \sum_{a,b\in A_i}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz + \frac{1}{2^{2m}r^2}\sum_{i\ne j} p_ip_j \sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz. \tag{2.8}$$

We first give an upper bound for the first summand in (2.8), using (2.5) and an integral transformation. By (2.5) we have

$$\begin{aligned}\sum_{a,b\in A_i}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz &\le (1+\varepsilon)\lambda_{i,\max}2^m \sum_{a\in A_i}\int_{-1}^{1}(\gamma_{g_{\varepsilon,m}^{-a}(z)},\gamma_{g_{\varepsilon,m}^{-a}(z)})_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz \\ &= (1+\varepsilon)\lambda_{i,\max}2^m \sum_{k=0}^{2^m-1} 2^m \int_{-1+k2^{-m+1}}^{-1+(k+1)2^{-m+1}} (\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{2^{2m}r^2}\sum_{i=1}^{l} p_i^2 \sum_{a,b\in A_i}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz \\ &\le \frac{1}{2^{2m}r^2}\sum_{i=1}^{l} p_i^2(1+\varepsilon)\lambda_{i,\max}2^m \sum_{k=0}^{2^m-1} 2^m \int_{-1+k2^{-m+1}}^{-1+(k+1)2^{-m+1}} (\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz\end{aligned} \tag{2.9}$$

$$\leq \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2} \frac{1}{\left(\frac{r}{(1-\varepsilon)\lambda_{i,\min}}\right)^2} \int_{-1}^{1} (\gamma_z, \gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} \, dz$$
$$\leq \max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right) \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}.$$

For the second summand in (2.8), we use (2.7) to prove that it is bounded by

$$\text{(2.10)} \qquad \frac{1}{2^{2m} r^2} \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z})_r \, dz \leq \frac{1}{2^{2m} r^2} \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \frac{8r^2}{C_{\varepsilon,m}\varepsilon} \leq \frac{8}{C_{\varepsilon,m}\varepsilon}.$$

By combining (2.9) and (2.10) we have

$$\text{(2.11)} \qquad J(r) \leq \frac{8}{C_{\varepsilon,m}\varepsilon} + \beta \max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right)$$

where $\beta = \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}$ is less than 1 by (1.6).

We define recursively a strictly decreasing sequence r_k. Let $r_0 < 1/2$ be fixed. Assume that r_{k-1} has been defined. Then we define $r_k = (1-\varepsilon)\lambda_{i_k,\min} r_{k-1}$, where i_k is chosen such that

$$\max_i J\left(\frac{r_k}{(1-\varepsilon)\lambda_{i,\min}}\right) = J\left(\frac{r_k}{(1-\varepsilon)\lambda_{i_k,\min}}\right) = J(r_{k-1}).$$

Hence $r_k = r_0(1-\varepsilon)^k \prod_{n=1}^{k}(\lambda_{i_n,\min})$.

We note that r_k is a well defined sequence. By induction and (2.11), we have

$$\text{(2.12)} \qquad J(r_k) \leq \frac{8}{C_{\varepsilon,m}\varepsilon} \frac{1-\beta^k}{1-\beta} + \beta^k J(r_0)$$

for every $k \geq 1$. Hence by (2.1), (2.2) and (2.12) we get

$$\text{(2.13)} \qquad \|\nu_{\varepsilon,m}\|_2^2 \leq 4 \liminf_{r \to 0} J(r) \leq 4 \liminf_{k \to \infty} J(r_k) \leq \frac{32}{C_{\varepsilon,m}\varepsilon} \frac{1}{1 - \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}}.$$

We now use the fact that a closed ball in the Hilbert space L^2 is compact in the weak topology. (See for instance Theorem V.2.1 in Yosida's book [16].) Hence, if $h_{\nu_{\varepsilon,m}}$ is the density of $\nu_{\varepsilon,m}$, then $h_{\nu_{\varepsilon,m}}$ is in L^2, and from the above we know that there is a constant C'_ε such that $\|h_{\nu_{\varepsilon,m}}\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$.

By the compactness statement above, there is an h with $\|h\|_2 \le C'_\varepsilon/\sqrt{\varepsilon}$ such that some subsequence of $h_{\nu_{\varepsilon,m}}$ converges weakly to h. Moreover h defines a probability measure since $1 = \int 1 \cdot h_{\nu_{\varepsilon,m}}\, d\mathcal{L}_3 \to \int 1 \cdot h\, d\mathcal{L}_3$.

Since $\nu_{\varepsilon,m}$ converges weakly to ν_ε it follows that ν_ε has density in L^2 and

$$\|\nu_\varepsilon\|_2 \le \frac{1}{\sqrt{\varepsilon}}\, C'_\varepsilon, \tag{2.14}$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{\bigl(1-\sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\bigr)C''_\varepsilon}},$$

$$C''_\varepsilon = \lim_{m\to\infty} C_{\varepsilon,m} = \min_{i\neq j}\biggl\{\frac{|a_i-a_j| + \varepsilon(-|a_i+a_j|-2)}{1-\varepsilon^2}\biggr\}. \ \blacksquare$$

3. Proof of Theorem 1.5. We do not give the whole proof of Theorem 1.5, because it is similar to the proof of Theorem 1.2. We only prove a modification of Lemma 2.2, which is important as it proves transversality.

First we define a new dynamical system $\widetilde{g}_{\varepsilon,m}\colon Q \to Q$, similar to the dynamical system $g_{\varepsilon,m}\colon Q \to Q$. Let $Q_{i,k}$ and $A_{i,k}$ be as in Section 2. Let $\widetilde{g}_{\varepsilon,m}\colon Q \to Q$ be defined by

$$\widetilde{g}_{\varepsilon,m}\colon (x,y,z) \mapsto \Bigl(\widetilde{d}(z)x + a_i(1-\widetilde{d}(z)),\ \frac{1}{p_i}y + b(y),\ 2^m z + c(z)\Bigr)$$

for $(x,y,z)\in Q_i$, where

$$\begin{aligned}
\widetilde{d}(z) &= \lambda_i + 2^m\varepsilon(z-(-1+(k+1/2)2^{-m+1})) && \text{for } (x,y,z)\in Q_{i,k},\\
b(y) &= 1 - \frac{1}{p_i}\Bigl(-1+2\sum_{j=1}^{i} p_j\Bigr) && \text{for } (x,y,z)\in Q_{i,k},\\
c(z) &= 2^m - 2k - 1 && \text{for } (x,y,z)\in Q_{i,k}.
\end{aligned}$$

Hence the only difference between $\widetilde{g}_{\varepsilon,m}$ and $g_{\varepsilon,m}$ is in the first coordinate, where the perturbation of f_i is made. Figure 1 also serves to visualise the action of $\widetilde{g}_{\varepsilon,m}$.

We define the cones

$$C_p = \biggl\{(u,v,w)\in T_pQ : \biggl|\frac{u}{w}\biggr|, \biggl|\frac{v}{w}\biggr| < \frac{2^{m+1}\varepsilon}{2^m-\lambda_{\max}-\varepsilon}\biggr\},$$

where $p\in Q$ and $\lambda_{\max} = \max_i \lambda_i$. Similar to Lemma 2.2, we show that these cones define a family of unstable cones, and that a certain transversality property holds.

LEMMA 3.1. *Suppose that* (1.9) *holds. The cones C_p form a family of unstable cones, that is, $d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset C_{\widetilde{g}_{\varepsilon,m}(p)}$.*

Moreover, for sufficiently large m and every sufficiently small $\varepsilon > 0$, if $\zeta_1 \subset Q_{\xi_1}$ and $\zeta_2 \subset Q_{\xi_2}$ are two line segments with tangents in C_p such that $\xi_1 \in A_i$ and $\xi_2 \in A_j$, $i \neq j$, then if $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ intersect, and if $(u_1, v_1, 1)$ and $(u_2, v_2, 1)$ are tangents to $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ respectively, there exists a constant $C_{\varepsilon,m}$, depending on ε and m, but bounded away from 0 and infinity, such that $|u_1 - u_2| > C_{\varepsilon,m}\varepsilon$.

Proof. The Jacobian of $\widetilde{g}_{\varepsilon,m}$ is

$$d_p\widetilde{g}_{\varepsilon,m} = \begin{pmatrix} \widetilde{d}(z) & 0 & 2^m\varepsilon(x - a_i) \\ 0 & 1/p_i & 0 \\ 0 & 0 & 2^m \end{pmatrix},$$

where $p = (x, y, z) \in Q_{i,k}$. If $(u, v, w) \in C_p$, then

$$d_p\widetilde{g}_{\varepsilon,m}(u, v, w) = \begin{pmatrix} \widetilde{d}(z)u + 2^m\varepsilon(x - a_i)w \\ (1/p_i)v \\ 2^m w \end{pmatrix}.$$

The estimate

$$\begin{aligned} \frac{|\widetilde{d}(z)u + 2^m\varepsilon(x - a_i)w|}{|2^m w|} &\leq \frac{\widetilde{d}(z)|u|}{2^m|w|} + 2\varepsilon \\ &\leq \frac{\lambda_i + \varepsilon}{2^m} \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon} + 2\varepsilon \leq \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \end{aligned}$$

shows that $d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset C_{\widetilde{g}_{\varepsilon,m}(p)}$. Now we prove the other statement of the lemma. Assume that $p = (x_p, y_p, z_p) \in Q_i$ and $q = (x_q, y_q, z_q) \in Q_j$, $i \neq j$, are such that $\widetilde{g}_{\varepsilon,m}(p) = \widetilde{g}_{\varepsilon,m}(q) = (x, y, z)$. Then

$$p \in Q_i \Rightarrow d_p\widetilde{g}_{\varepsilon,m} : (u, v, 1) \mapsto 2^m\left(\frac{\widetilde{d}(z_p)}{2^m}u + (x_p - a_i)\varepsilon,\ \frac{v}{p_i},\ 1\right),$$

and

$$x_p = \frac{x - a_i(1 - \widetilde{d}(z_p))}{\widetilde{d}(z_p)}, \qquad x_q = \frac{x - a_j(1 - \widetilde{d}(z_q))}{\widetilde{d}(z_q)}.$$

Let $\widetilde{\Delta}_i = \frac{2(\lambda_i + \varepsilon)}{2^m - \lambda_{\max} - \varepsilon}$. Then

$$d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset \left\{ w(u, v, 1) : \frac{x - a_i}{\widetilde{d}(z_p)}\varepsilon - \widetilde{\Delta}_i\varepsilon \leq u \leq \frac{x - a_i}{\widetilde{d}(z_p)}\varepsilon + \widetilde{\Delta}_i\varepsilon \right\}.$$

Therefore

$$|u_2 - u_1| \geq \left(\left| \frac{x - a_i}{\widetilde{d}(z_p)} - \frac{x - a_j}{\widetilde{d}(z_q)} \right| - (\widetilde{\Delta}_i + \widetilde{\Delta}_j) \right)\varepsilon.$$

The term

$$\left|\frac{x-a_i}{\widetilde{d}(z_p)}-\frac{x-a_j}{\widetilde{d}(z_q)}\right|$$

can be estimated by

$$\left|\frac{x-a_i}{\widetilde{d}(z_p)}-\frac{x-a_j}{\widetilde{d}(z_q)}\right| \geq \left|\frac{|\widetilde{d}(z_p)-\widetilde{d}(z_q)|\,|x|-|a_j\widetilde{d}(z_p)-a_i\widetilde{d}(z_q)|}{\widetilde{d}(z_p)\widetilde{d}(z_q)}\right|.$$

Hence, this term is positive provided that

$$|a_j\widetilde{d}(z_p)-a_i\widetilde{d}(z_q)| > |\widetilde{d}(z_p)-\widetilde{d}(z_q)|.$$

Since $\lambda_i-\varepsilon \leq \widetilde{d}(z_p) \leq \lambda_i+\varepsilon$ and $\lambda_j-\varepsilon \leq \widetilde{d}(z_q) \leq \lambda_j+\varepsilon$, this is implied by (1.9) if ε is sufficiently small.

If we let

$$C_{\varepsilon,m} = \frac{1}{2}\min_{i\neq j}\frac{|a_i\lambda_j-a_j\lambda_i|-|\lambda_i-\lambda_j|}{\lambda_i\lambda_j},$$

then

$$|u_2-u_1| \geq C_{\varepsilon,m}\varepsilon,$$

provided that ε is small and m large.

In fact we can let

$$C_{\varepsilon,m} = \sigma\min_{i\neq j}\frac{|a_i\lambda_j-a_j\lambda_i|-|\lambda_i-\lambda_j|}{\lambda_i\lambda_j}$$

for some $0<\sigma<1$. ■

Acknowledgments. Research of Bárány was supported by the EU FP6 Research Training Network CODY.

References

[1] B. Bárány, M. Pollicott and K. Simon, *Stationary measures for projective transformations: the Blackwell and Furstenberg measures*, preprint, 2009.

[2] P. Diaconis and D. A. Freedman, *Iterated random functions*, SIAM Rev. 41 (1999), 45–76.

[3] A. H. Fan, K. Simon and H. Tóth, *Contracting on average random IFS with repelling fixed point*, J. Statist. Phys. 122 (2006), 169–193.

[4] K.-S. Lau, S.-M. Ngai and H. Rao, *Iterated function systems with overlaps and self-similar measures*, J. London Math. Soc. (2) 63 (2001), 99–116.

[5] J. Neunhäuserer, *Properties of some overlapping self-similar and some self-affine measures*, Acta Math. Hungar. 92 (2001), 143–161.

[6] S.-M. Ngai and Y. Wang, *Self-similar measures associated with IFS with non-uniform contraction ratios*, Asian J. Math. 9 (2005), 227–244.

[7] M. Nicol, N. Sidorov and D. Broomhead, *On the fine structure of stationary measures in systems which contract on average*, J. Theoret. Probab. 15 (2002), 715–730.

[8] Y. Peres and W. Schlag, *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. 102 (2000), 193–251.

[9] Y. Peres, K. Simon and B. Solomyak, *Absolute continuity for random iterated function systems with overlaps*, J. London Math. Soc. (2) 74 (2006), 739–756.
[10] Y. Peres and B. Solomyak, *Self-similar measures and intersections of Cantor sets*, Trans. Amer. Math. Soc. 350 (1998), 4065–4087.
[11] T. Persson, *On random Bernoulli convolutions*, Dynam. Systems, to appear.
[12] Ya. Pesin, *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergodic Theory Dynam. Systems 12 (1992), 123–151.
[13] J. Schmeling and S. Troubetzkoy, *Dimension and invertibility of hyperbolic endomorphisms with singularities*, Ergodic Theory Dynam. Systems 18 (1998), 1257–1282.
[14] K. Simon, B. Solomyak and M. Urbański, *Invariant measures for parabolic IFS with overlaps and random continued fractions*, Trans. Amer. Math. Soc. 353 (2001), 5145–5164.
[15] M. Tsujii, *Fat solenoidal attractor*, Nonlinearity 14 (2001), 1011–1027.
[16] K. Yosida, *Functional Analysis*, Springer, Berlin, 1980.

Balázs Bárány
Department of Stochastics
Institute of Mathematics
Technical University of Budapest
P.O. Box 91
1521 Budapest, Hungary
E-mail: balubsheep@gmail.com

Tomas Persson
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
P.O. Box 21
00-956 Warszawa, Poland
E-mail: tomasp@impan.pl

Received 24 February 2009;
in revised form 28 June 2010