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The orbits of the Hurwitz action of the braid groups on the standard generators

by

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Abstract. The Hurwitz action of the n-braid group B_n on the n-fold direct product $(B_m)^n$ of the m-braid group B_m is studied. We show that the orbit of any n- tuple of the n standard generators of B_{n+1} consists of the $(n-1)$th powers of $n+1$ elements.

1. Introduction. The n-braid group, denoted by B_n, has the following presentation [1, 3]:

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & (|i-j| = 1) \\ \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| > 1) \end{array} \right\rangle,$$

where σ_i is the ith standard generator represented by a geometric n-braid depicted in Figure 1.1.

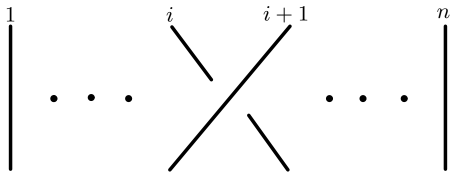

Fig. 1.1

Let G be a group. The following action of B_n on the n-fold product G^n of G is called the Hurwitz action.

Definition 1.1. The *Hurwitz action* of B_n on G^n is the right action defined by

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \cdot \sigma_i = (g_1, \dots, g_{i-1}, g_{i+1}, {g_{i+1}}^{-1} g_i g_{i+1}, g_{i+2}, \dots, g_n),$$

where $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of B_n.

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In this paper, we denote the orbit of $(g_1, \ldots, g_n) \in G^n$ under the Hurwitz action of B_n by $(g_1, \ldots, g_n) \cdot B_n$.

There is a strong relationship between the Hurwitz actions of B_n on G^n and the equivalence classes of braided surfaces when G is a braid group [7, 8, 9, 5, 10].

We study the Hurwitz action of B_n on $(B_{n+1})^n$, so $G = B_{n+1}$. Throughout this paper, we use the symbol "s_i" to denote the ith standard generator of B_{n+1}, and "σ_i" to denote that of B_n.

In [4], S. P. Humphries proved the following.

Theorem 1.2 ([4]). *The orbit $(s_1, \ldots, s_n) \cdot B_n$ consists of $(n+1)^{n-1}$ elements.*

The following is our main result.

Main Theorem 1.3. *For any permutation φ of $\{1, \ldots, n\}$, the orbit $(s_{\varphi(1)}, \ldots, s_{\varphi(n)}) \cdot B_n$ of the element $(s_{\varphi(1)}, \ldots, s_{\varphi(n)})$ consists of $(n+1)^{n-1}$ elements.*

In Section 2, we prepare some notions which are used later. Section 3 is devoted to the proof of Theorem 3.2 which is a generalization of Theorem 1.3.

Throughout this paper, n is an integer with $n \geq 2$.

2. Some notions. Throughout this section, A is a fixed subset of $\{2, \ldots, n\}$. For integers i and j with $1 \leq i < j \leq n+1$, we define $s^{\mathrm{A}}_{ij} \in B_{n+1}$ by

$$s^{\mathrm{A}}_{ij} = \Big(\prod_{k=i+1}^{j-1} (s_k)^{\epsilon_k}\Big)^{-1} s_i \prod_{k=i+1}^{j-1} (s_k)^{\epsilon_k},$$

where $\epsilon_k = 1$ if $k \in \mathrm{A}$ and $\epsilon_k = -1$ if $k \notin \mathrm{A}$ (see Example 2.1(1)).

We call s^{A}_{ij} a *band generator* of B_{n+1} associated with A. Note that a standard generator s_i of B_{n+1} is a band generator $s^{\mathrm{A}}_{i,i+1}$.

Let Σ^{A} be the set of band generators $\{s^{\mathrm{A}}_{ij} \in B_{n+1} \mid 1 \leq i < j \leq n+1\}$ associated with A.

Let $\mathrm{P}_k = (k, 0) \in \mathbb{R}^2$ for $1 \leq k \leq n+1$. Let C_1 be the circle in $\mathbb{R}^2$ passing through the points P_1 and P_{n+1} with the length of the segment $\overline{\mathrm{P}_1\mathrm{P}_{n+1}}$ in diameter. Take the points $\mathrm{Q}_k \in C_1$ for $1 \leq k \leq n+1$ such that $\mathrm{Q}_1 = \mathrm{P}_1$, $\mathrm{Q}_{n+1} = \mathrm{P}_{n+1}$ and $\mathrm{Q}_k = (k, y_k)$ for each $2 \leq k \leq n$, where $y_k < 0$ if $k \in \mathrm{A}$ and $y_k > 0$ if $k \notin \mathrm{A}$.

For $1 \leq i < j \leq n+1$, we call the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ the *segment corresponding to s^{A}_{ij}*. (See Example 2.1(2).)

Remark. The reason to call $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ the segment corresponding to s^{A}_{ij} is as follows.

Let $\mathrm{P}_0 = \mathrm{Q}_0 = (0,0) \in \mathbb{R}^2$ and $\mathrm{P}_{n+2} = \mathrm{Q}_{n+2} = (n+2,0) \in \mathbb{R}^2$. Let C_2 be the circle in $\mathbb{R}^2$ passing through the points P_0 and P_{n+2} with the length of the segment $\overline{\mathrm{P}_0\mathrm{P}_{n+2}}$ in diameter. Let D be the disk in $\mathbb{R}^2$ with $\partial D = C_2$. Take an isotopy $\{h_u\}_{u\in[0,1]}$ of D such that for each $u \in [0,1]$, $h_0 = \mathrm{id}$, $h_u|_{\partial D_{n+1}} = \mathrm{id}$, and for each $u \in [0,1]$ and each $(x,y) \in \bigcup_{i=0}^{n+1} \overline{\mathrm{Q}_i\mathrm{Q}_{i+1}}$, $h_u(x,y) = (x,(1-u)y)$.

Then $h_1(\mathrm{Q}_i) = \mathrm{P}_i$ for any i. For $1 \le i < j \le n+1$, we define α_{ij}^{A} to be the arc $h_1(\overline{\mathrm{Q}_i\mathrm{Q}_j})$ in D. Note that $\partial\alpha_{ij}^{\mathrm{A}} = \{\mathrm{P}_i, \mathrm{P}_j\}$, α_{ij}^{A} is above P_k if $k \in \mathrm{A}$ and α_{ij}^{A} is below P_k if $k \notin \mathrm{A}$ (see Example 2.1(3)).

The braid group B_{n+1} is isomorphic to the mapping class group of $(D, \{\mathrm{P}_1, \dots, \mathrm{P}_{n+1}\})$ relative to the boundary (cf. [2]).

The band generator s_{ij}^{A} corresponds to the isotopy class of a homeomorphism from $(D, \{\mathrm{P}_1, \dots, \mathrm{P}_{n+1}\})$ to itself which twists a sufficiently small disk neighborhood of the arc α_{ij}^{A} by a 180° clockwise rotation using its collar neighborhood.

By the homeomorphism $h_1 : (D, \{\mathrm{Q}_1, \dots, \mathrm{Q}_{n+1}\}) \to (D, \{\mathrm{P}_1, \dots, \mathrm{P}_{n+1}\})$, we identify the mapping class group of $(D, \{\mathrm{Q}_1, \dots, \mathrm{Q}_{n+1}\})$ and that of $(D, \{\mathrm{P}_1, \dots, \mathrm{P}_{n+1}\})$. Then the band generator s_{ij}^{A} corresponds to the isotopy class of a homeomorphism from $(D, \{\mathrm{Q}_1, \dots, \mathrm{Q}_{n+1}\})$ to itself which twists a sufficiently small disk neighborhood of the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ by a 180° clockwise rotation. Therefore, we say that the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ corresponds to the band generator $s_{ij}^{\mathrm{A}} \in \Sigma^{\mathrm{A}}$.

Example 2.1. Let $n = 4$ and $\mathrm{A} = \{2\}$.

(1) The band generator $s_{14}^{\mathrm{A}} \in \Sigma^{\mathrm{A}}$ is $s_3(s_2)^{-1}s_1s_2(s_3)^{-1}$ (see Figure 2.1).
(2) Figure 2.2 shows the segment $\overline{\mathrm{Q}_1\mathrm{Q}_4}$ corresponding to $s_{14}^{\mathrm{A}} \in \Sigma^{\mathrm{A}}$.
(3) Figure 2.3 shows the arc $\alpha_{14}^{\mathrm{A}} = h_1(\overline{\mathrm{Q}_1\mathrm{Q}_4})$.

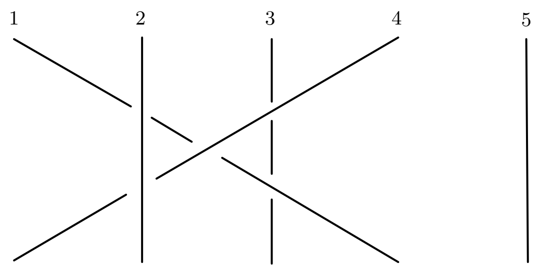

Fig. 2.1

For an element $(g_1, \dots, g_n)$ of the n-fold product $(\Sigma^{\mathrm{A}})^n$ of Σ^{A}, we call an n-tuple $(a_1, \dots, a_n)$ of the segments a_i corresponding to g_i the *segment system corresponding to* $(g_1, \dots, g_n)$.

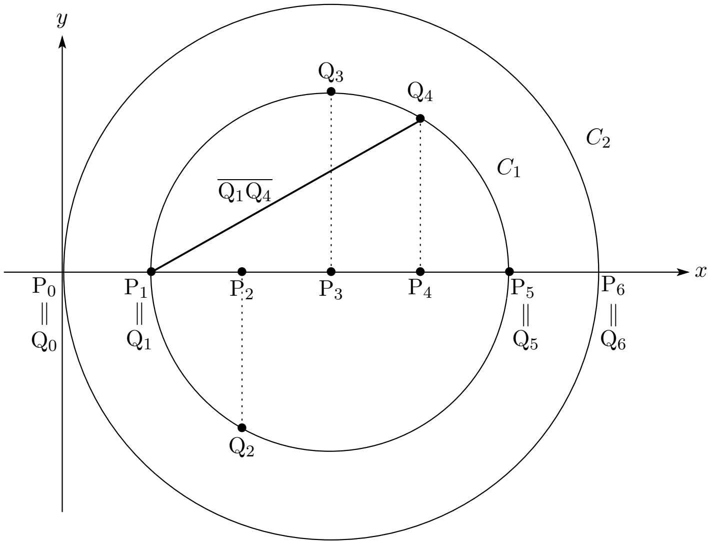

Fig. 2.2. $n = 4$ and $\mathrm{A} = \{2\}$

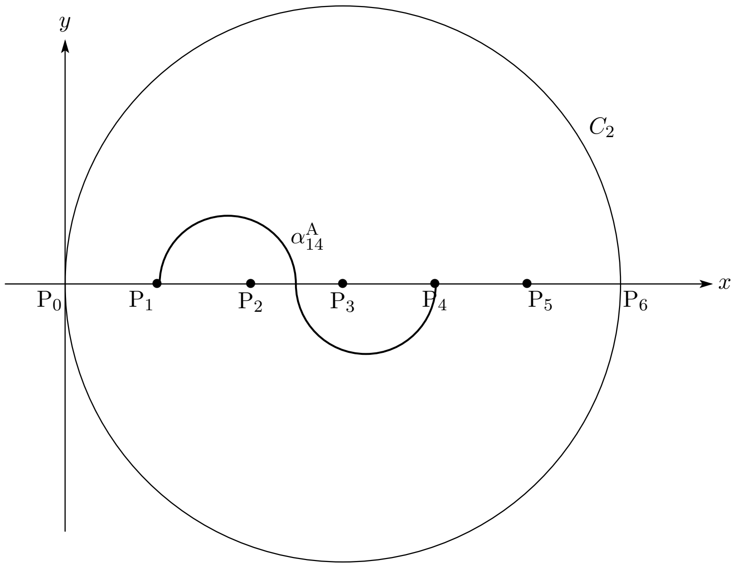

Fig. 2.3

Let a and a' be the segments corresponding to elements g and g' of Σ^{A}. If $\partial a = \{\mathrm{Q}_i, \mathrm{Q}_{i'}\}$, $\partial a' = \{\mathrm{Q}_i, \mathrm{Q}_{i''}\}$ and $\mathrm{Q}_{i'} \neq \mathrm{Q}_{i''}$, i.e., a and a' share a common end point Q_i, then we say that a and a' are *adjacent* (at Q_i). Moreover, if the end points $\mathrm{Q}_{i'}$, Q_i and $\mathrm{Q}_{i''}$ appear on C_1 counterclockwise in this order, then we say that a' is *right adjacent* to a (at Q_i), or a is *left adjacent* to a'.

DEFINITION 2.2. An element $(g_1, \dots, g_n)$ of $(\Sigma^{\mathrm{A}})^n$ *is* A-*good* if the segment system $(a_1, \dots, a_n)$ corresponding to $(g_1, \dots, g_n)$ satisfies the following conditions:

(i) If $k \neq l$, then a_k and a_l are disjoint or adjacent,
(ii) If $k < l$ and a_k and a_l intersect, then a_l is right adjacent to a'.
(iii) The union $a_1 \cup \dots \cup a_n$ is a tree as a graph.

EXAMPLE 2.3. Let $n = 4$ and $\mathrm{A} = \{2\}$. Then $(s^{\mathrm{A}}_{23}, s^{\mathrm{A}}_{24}, s^{\mathrm{A}}_{13}, s^{\mathrm{A}}_{25})$ is A-good. The segments $a_1, \dots, a_4$ corresponding to $s^{\mathrm{A}}_{23}, s^{\mathrm{A}}_{24}, s^{\mathrm{A}}_{13}, s^{\mathrm{A}}_{25}$ are depicted in Figure 2.4.

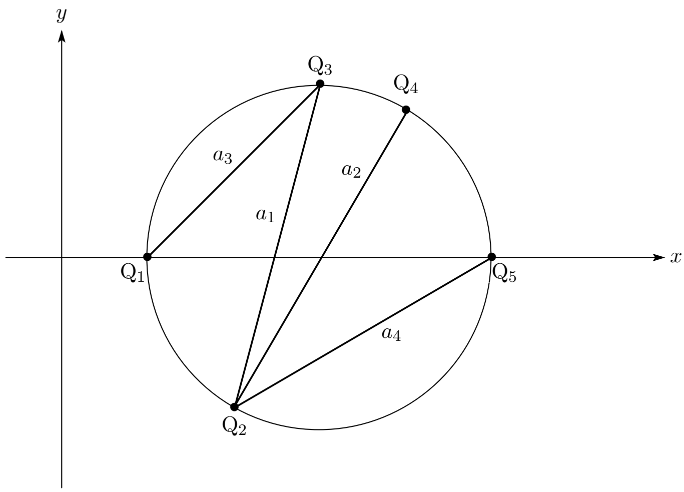

Fig. 2.4

Let $(g_1, \dots, g_n)$ be an element of $(\Sigma^{\mathrm{A}})^n$ that is A-good and let $(a_1, \dots, a_n)$ be the corresponding segment system.

Suppose that a_l is right adjacent to a_k at Q_i for some k, l $(k < l)$ and some i. Put $a_k = \overline{\mathrm{Q}_i\mathrm{Q}_{i'}}$ and $a_l = \overline{\mathrm{Q}_i\mathrm{Q}_{i''}}$. Then the points $\mathrm{Q}_{i'}$, Q_i and $\mathrm{Q}_{i''}$ appear counterclockwise in this order and the following lemma holds:

LEMMA 2.4. *If* $a_k \cap a_m \cap a_l = \{\mathrm{Q}_i\}$ *for* $m \in \{1, \dots, n\}$, $m \neq k, l$, *then* a_m *intersects* Int $\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$ *if and only if* $k < m < l$. *In particular, if* $l = k+1$, *then* a_m *and* Int $\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$ *are disjoint.*

Proof. Put $a_m = \overline{\mathrm{Q}_i\mathrm{Q}_j}$.

(Case I) Suppose that $m < k < l$. Then a_m is left adjacent to a_k and a_l by condition (ii) of Definition 2.2. Hence, the points Q_j, Q_i, $\mathrm{Q}_{i''}$ and $\mathrm{Q}_{i'}$ appear counterclockwise in this order. Then a_m and Int $\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$ are disjoint.

(Case II) Suppose that $k < m < l$. Then a_m is right adjacent to a_k and left adjacent to a_l by condition (ii) of Definition 2.2. Hence, the points $\mathrm{Q}_{i'}$,

Q_i, $\mathrm{Q}_{i''}$ and Q_j appear counterclockwise in this order. Then a_m intersects $\mathrm{Int}\,\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$.

(Case III) Suppose that $k < l < m$. Then a_m is right adjacent to a_k and a_l by condition (ii) of Definition 2.2. Hence, the points $\mathrm{Q}_{i'}$, Q_i, Q_j and $\mathrm{Q}_{i''}$ appear counterclockwise in this order. Then a_m and $\mathrm{Int}\,\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$ are disjoint.

Thus, a_m intersects $\mathrm{Int}\,\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$ if and only if $k < m < l$. ∎

3. Proof of Theorem 1.3. The following lemma is the first step towards the proof of Theorem 1.3.

LEMMA 3.1. *For an element $\varphi \in \mathrm{Sym}\{1, \ldots, n\}$, let $\mathrm{A} = \{i \in \mathbb{N} \mid \varphi^{-1}(i-1) < \varphi^{-1}(i),\ 2 \le i \le n\}$. Then $(s_{\varphi(1)}, \ldots, s_{\varphi(n)})$ is an element of $(\Sigma^{\mathrm{A}})^n$ and it is* A*-good.*

Proof. Since the standard generators of B_{n+1} belong to Σ^{A}, it follows that $(s_{\varphi(1)}, \ldots, s_{\varphi(n)}) \in (\Sigma^{\mathrm{A}})^n$. The arc a_m corresponding to $s_{\varphi(m)}$ is $\overline{\mathrm{Q}_{\varphi(m)}\mathrm{Q}_{\varphi(m)+1}}$. Suppose that $a_k \cap a_l \neq \emptyset$ for $k < l$. Then $a_k \cap a_l = \{\mathrm{Q}_{\varphi(k)}\}$ or $\{\mathrm{Q}_{\varphi(l)}\}$, and $|\varphi(k) - \varphi(l)| = 1$. Assume $\varphi(l) - \varphi(k) = 1$, so that $a_k \cap a_l = \{\mathrm{Q}_{\varphi(l)}\}$ and $\varphi(l) \in \mathrm{A}$. Then the points $\mathrm{Q}_{\varphi(k)}$, $\mathrm{Q}_{\varphi(l)}$ and $\mathrm{Q}_{\varphi(l)+1}$ appear counterclockwise in this order since the y-coordinate of $\mathrm{Q}_{\varphi(l)}$ is negative. Thus, a_l is right adjacent to a_k. Assume $\varphi(k) - \varphi(l) = 1$, so that $a_k \cap a_l = \{\mathrm{Q}_{\varphi(k)}\}$ and $\varphi(k) \notin \mathrm{A}$. Then the points $\mathrm{Q}_{\varphi(k+1)}$, $\mathrm{Q}_{\varphi(k)}$ and $\mathrm{Q}_{\varphi(l)}$ appear counterclockwise in this order since the y-coordinate of $\mathrm{Q}_{\varphi(l)}$ is positive. Thus, a_l is right adjacent to a_k. We easily see that the graph $a_1 \cup \cdots \cup a_n$ is a tree. ∎

Theorem 1.3 is obtained from the following theorem by Lemma 3.1.

THEOREM 3.2. *Let* A *be a subset of $\{2, \ldots, n\}$. For any element $(g_1, \ldots, g_n) \in (\Sigma^{\mathrm{A}})^n$ that is* A*-good, the orbit $(g_1, \ldots, g_n) \cdot B_n$ consists of $(n+1)^{n-1}$ elements.*

The rest of this paper is devoted to proving Theorem 3.2.

For $(g_1, \ldots, g_n) \in (\Sigma^{\mathrm{A}})^n$, it is not always the case that $(g_1, \ldots, g_n) \cdot B_n \subset (\Sigma^{\mathrm{A}})^n$. However, we have

LEMMA 3.3. *Let* A *be a subset of $\{2, \ldots, n\}$. If $(g_1, \ldots, g_n) \in (\Sigma^{\mathrm{A}})^n$ is* A*-good, then, for any $k \in \{1, \ldots, n-1\}$ and any $\epsilon \in \{1, -1\}$, we have:*

(1) $(g_1, \ldots, g_n) \cdot (\sigma_k)^\epsilon \in (\Sigma^{\mathrm{A}})^n$,
(2) $(g_1, \ldots, g_n) \cdot (\sigma_k)^\epsilon$ *is* A*-good.*

Proof. Let $(a_1, \ldots, a_n)$ be the segment system corresponding to $(g_1, \ldots, g_n)$, and let $(b_1, \ldots, b_n)$ be that corresponding to $(g_1, \ldots, g_n) \cdot \sigma_k$.

First we consider the case where a_k and a_{k+1} are disjoint. Then g_k and g_{k+1} are commutative, and

$$\begin{aligned}(g_1,\dots,g_{k-1},g_k,g_{k+1},g_{k+2},\dots,g_n)\cdot\sigma_k\\ &= (g_1,\dots,g_{k-1},g_k,g_{k+1},g_{k+2},\dots,g_n)\cdot(\sigma_k)^{-1}\\ &= (g_1,\dots,g_{k-1},g_{k+1},g_k,g_{k+2},\dots,g_n).\end{aligned}$$

Thus, we obtain (1).

For the proof of (2), it is enough to prove $(g_1,\dots,g_n)\cdot\sigma_k$ is A-good. Since $b_k = a_{k+1}$, $b_{k+1} = a_k$ and $b_p = a_p$ for $p \neq k, k+1$, we see that $(g_1,\dots,g_n)\cdot\sigma_k$ satisfies conditions (i) and (iii) of Definition 2.2. Suppose that b_p and b_q intersect for some p and q $(p < q)$. Let $b_p = a_{p'}$, $b_q = a_{q'}$. Note that $(p,q) \neq (k,k+1)$ and $(p',q') \neq (k+1,k)$ because $b_k \cap b_{k+1} = a_{k+1} \cap a_k = \emptyset$. Thus, we have $p' < q'$ because $p < q$. Since $a_{p'}$ and $a_{q'}$ satisfy condition (ii) of Definition 2.2, so do b_p and b_q. We have (2).

Now consider the case where a_k and a_{k+1} intersect. Let Q_x, Q_y, Q_z $(x < y < z)$ be the points such that $\{\mathrm{Q}_x,\mathrm{Q}_y,\mathrm{Q}_z\} = \partial a_k \cup \partial a_{k+1}$.

By condition (ii) of Definition 2.2, a_k and a_{k+1} satisfy one of the following conditions:

(A1) $y \in \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_x\mathrm{Q}_y}$, $a_{k+1} = \overline{\mathrm{Q}_y\mathrm{Q}_z}$,
(A2) $y \in \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_y\mathrm{Q}_z}$, $a_{k+1} = \overline{\mathrm{Q}_x\mathrm{Q}_z}$,
(A3) $y \in \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_x\mathrm{Q}_z}$, $a_{k+1} = \overline{\mathrm{Q}_x\mathrm{Q}_y}$,
(A4) $y \notin \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_y\mathrm{Q}_z}$, $a_{k+1} = \overline{\mathrm{Q}_x\mathrm{Q}_y}$,
(A5) $y \notin \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_x\mathrm{Q}_y}$, $a_{k+1} = \overline{\mathrm{Q}_x\mathrm{Q}_z}$,
(A6) $y \notin \mathrm{A}$ and $a_k = \overline{\mathrm{Q}_x\mathrm{Q}_z}$, $a_{k+1} = \overline{\mathrm{Q}_y\mathrm{Q}_z}$.

Then $(g_k, g_{k+1}) = (s^{\mathrm{A}}_{xy}, s^{\mathrm{A}}_{yz})$, $(s^{\mathrm{A}}_{yz}, s^{\mathrm{A}}_{xz})$, $(s^{\mathrm{A}}_{xz}, s^{\mathrm{A}}_{xy})$, $(s^{\mathrm{A}}_{yz}, s^{\mathrm{A}}_{xy})$, $(s^{\mathrm{A}}_{xy}, s^{\mathrm{A}}_{xz})$ or $(s^{\mathrm{A}}_{xz}, s^{\mathrm{A}}_{yz})$. By direct calculations $(g_{k+1}, (g_{k+1})^{-1}g_kg_{k+1}) = (s^{\mathrm{A}}_{yz}, s^{\mathrm{A}}_{xz})$, $(s^{\mathrm{A}}_{xz}, s^{\mathrm{A}}_{xy})$, $(s^{\mathrm{A}}_{xy}, s^{\mathrm{A}}_{yz})$, $(s^{\mathrm{A}}_{xy}, s^{\mathrm{A}}_{xz})$, $(s^{\mathrm{A}}_{xz}, s^{\mathrm{A}}_{yz})$ or $(s^{\mathrm{A}}_{yz}, s^{\mathrm{A}}_{xy})$, respectively. This implies that $(g_1,\dots,g_n)\cdot\sigma_k$ and $(g_1,\dots,g_n)\cdot(\sigma_k)^2$ are elements of $(\varSigma^{\mathrm{A}})^n$ and $(g_1,\dots,g_n)\cdot(\sigma_k)^3 = (g_1,\dots,g_n)$. Note that $(g_1,\dots,g_n)\cdot(\sigma_k)^{-1} \in (\varSigma_{\mathrm{A}})^n$ since $(g_1,\dots,g_n)\cdot(\sigma_k)^{-1} = (g_1,\dots,g_n)\cdot(\sigma_k)^2$. Thus, we obtain (1).

For (2), it is sufficient to prove $(g_1,\dots,g_n)\cdot\sigma_k$ is A-good. Note that $b_k = a_{k+1}$, b_{k+1} is the edge of the boundary of $|\mathrm{Q}_x\mathrm{Q}_y\mathrm{Q}_z|$ that is neither a_k nor a_{k+1}, and $b_p = a_p$ for $p \neq k, k+1$. Thus, we see that b_p and b_k are disjoint or $b_p \cap b_k = \{\mathrm{Q}_i\}$ for $p \neq k, k+1$ and some i, and b_p and b_q are disjoint or $b_p \cap b_q = \{\mathrm{Q}_i\}$ for $p \neq q \in \{1,\dots,n\}\setminus\{k,k+1\}$ and some i. By Lemma 2.4, for $p \neq k, k+1$, a_p and $\operatorname{Int} b_{k+1}$ are disjoint. Thus, b_p and b_{k+1} are disjoint or $b_p \cap b_{k+1} = \{\mathrm{Q}_i\}$ for $p \neq k, k+1$ and some i, and $(g_1,\dots,g_n)\cdot\sigma_k$ satisfies condition (i) of Definition 2.2.

Let X be the space defined by

$$\begin{aligned}X &= a_1 \cup \dots \cup a_{k-1} \cup |\mathrm{Q}_x\mathrm{Q}_y\mathrm{Q}_z| \cup a_{k+2} \cup \dots \cup a_n\\ &= b_1 \cup \dots \cup b_{k-1} \cup |\mathrm{Q}_x\mathrm{Q}_y\mathrm{Q}_z| \cup b_{k+2} \cup \dots \cup b_n.\end{aligned}$$

Note that X is homotopy equivalent to $a_1 \cup \cdots \cup a_n$ and $b_1 \cup \cdots \cup b_n$. Since $a_1 \cup \cdots \cup a_n$ is a tree, we see that $b_1 \cup \cdots \cup b_n$ is a tree. Thus, $(g_1, \ldots, g_n) \cdot \sigma_k$ satisfies condition (iii) of Definition 2.2.

We have already seen that b_k and b_{k+1} satisfy condition (A2), (A3), (A1), (A5), (A6) or (A4) if a_k and a_{k+1} satisfy (A1), (A2), (A3), (A4), (A5) or (A6), respectively. Let $p \neq q \in \{1, \ldots, n\} \setminus \{k, k+1\}$. If b_p $(= a_p)$ and b_q $(= a_q)$ intersect, then they satisfy condition (ii) of Definition 2.2. If b_k and b_p intersect, then $b_k = a_{k+1}$ and $b_p = a_p$ satisfy condition (ii) of Definition 2.2 since $p < k$ iff $p < k+1$.

The remainder of the proof of (2) is to check that b_{k+1} and b_p satisfy condition (ii) of Definition 2.2 if b_{k+1} and b_p intersect for $p \in \{1, \ldots, n\} \setminus \{k, k+1\}$.

Let $b_k \cap b_{k+1} = \{\mathrm{Q}_i\}$, $b_k = \overline{\mathrm{Q}_i\mathrm{Q}_{i'}}$ and $b_{k+1} = \overline{\mathrm{Q}_i\mathrm{Q}_{i''}}$. Then we have already seen that $a_k = \overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}}$, $a_{k+1} = b_k = \overline{\mathrm{Q}_i\mathrm{Q}_{i'}}$ and the points $\mathrm{Q}_{i''}$, $\mathrm{Q}_{i'}$ and Q_i appear counterclockwise in this order.

(Case 1) Suppose that b_p is adjacent to b_{k+1} at Q_i and let $b_p = \overline{\mathrm{Q}_i\mathrm{Q}_j}$.

(Case 1-1) Suppose that $p < k$. Then we have seen that b_p is left adjacent to b_k. Since b_k is left adjacent to b_{k+1}, we see that b_p is left adjacent to b_{k+1}.

(Case 1-2) Suppose that $p > k$. Then we have seen that b_p is right adjacent to b_k and the points $\mathrm{Q}_{i'}$, Q_i and Q_j appear counterclockwise in this order. By Lemma 2.4, the points $\mathrm{Q}_{i''}$, Q_i and Q_j appear counterclockwise in this order. Thus, b_p is right adjacent to b_{k+1}.

(Case 2) Suppose that b_p is adjacent to b_{k+1} at $\mathrm{Q}_{i''}$ and let $b_p = \overline{\mathrm{Q}_{i''}\mathrm{Q}_j}$.

(Case 2-1) Suppose that $p < k$. Then we have seen that b_p $(= a_p)$ is right adjacent to $\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}} = a_k$ at $\mathrm{Q}_{i''}$ by condition (ii) of Definition 2.2. Thus, Q_j, $\mathrm{Q}_{i''}$ and $\mathrm{Q}_{i'}$ appear counterclockwise in this order. By Lemma 2.4, Q_j, $\mathrm{Q}_{i''}$ and Q_i appear counterclockwise in this order. Thus, b_p is left adjacent to b_{k+1}.

(Case 2-2) Suppose that $p > k$. Then we have seen that $b_p = a_p$ is right adjacent to $\overline{\mathrm{Q}_{i'}\mathrm{Q}_{i''}} = a_k$ at $\mathrm{Q}_{i''}$ by condition (ii) of Definition 2.2. Note that a_k is right adjacent to b_{k+1}. Thus, b_p is right adjacent to b_{k+1}.

Consequently, b_{k+1} and b_p satisfy condition (ii) of Definition 2.2 in the case where $b_{k+1} = \overline{\mathrm{Q}_x\mathrm{Q}_z}$, and this completes the proof of Lemma 3.3. ∎

Let S_{n+1} be the symmetric group of degree $n+1$.

LEMMA 3.4 ([6]). *Let $\tau_1, \ldots, \tau_n$ be the transpositions in S_{n+1} satisfying $\tau_i \neq \tau_j$ $(i \neq j)$. Then the orbit of $(\tau_1, \ldots, \tau_n)$ under the Hurwitz action of B_n on $(S_{n+1})^n$ consists of $(n+1)^{n-1}$ elements.*

For groups G, H and a homomorphism $f : G \to H$, let $f^n : G^n \to H^n$ be the map defined by $(g_1, \ldots, g_n) \mapsto (f(g_1), \ldots, f(g_n))$. The following lemma is easily seen.

Lemma 3.5. *For any $\beta \in B_n$,*

$$f^n((g_1, \dots, g_n) \cdot \beta) = (f^n(g_1, \dots, g_n)) \cdot \beta.$$

Proof of Theorem 3.2. Note that the restriction $p|_{\Sigma^{\mathrm{A}}}$ of the canonical projection $p : B_{n+1} \to S_{n+1}$ to Σ^{A} is injective and the image $p(\Sigma^{\mathrm{A}})$ is the set of all transpositions of S_{n+1}. By Lemma 3.3, we see $(g_1, \dots, g_n) \cdot B_n \subset (\Sigma^{\mathrm{A}})^n$. Hence, $\#((g_1, \dots, g_n) \cdot B_n) = \#(p^n((g_1, \dots, g_n) \cdot B_n))$. By Lemma 3.5, $\#(p^n((g_1, \dots, g_n)) \cdot B_n) = \#((p^n(g_1, \dots, g_n)) \cdot B_n)$. By the definition of A-good, $g_k \neq g_l$ for $k \neq l$ (since the arcs a_k and a_l corresponding to g_k and g_l are disjoint or they meet only in their end point). Hence, $p^n(g_1, \dots, g_n)$ is an element whose components are mutually distinct transpositions of S_{n+1}. By Lemma 3.4, we obtain the result. ∎

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