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Finite-to-one maps and dimension

by

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Abstract. It is shown that *for every at most k-to-one closed continuous map f from a non-empty n-dimensional metric space X, there exists a closed continuous map g from a zero-dimensional metric space onto X such that the composition $f \circ g$ is an at most $(n+k)$-to-one map.* This implies that f is a composition of $n+k-1$ *simple* (= at most two-to-one) closed continuous maps. Stronger conclusions are obtained for maps from Anderson–Choquet spaces and ones that satisfy W. Hurewicz's condition (α). The main tool is a certain extension of the Lebesgue–Čech dimension to finite-to-one closed continuous maps.

This paper deals with a composition problem, which originates in papers of K. Borsuk, R. Molski, and K. Sieklucki [6, 25]. In [15] the present author proved that *every at most k-to-one closed map from an n-dimensional space is a composition of* $(n+1)k-1$ *simple* (= *at most two-to-one*) *closed maps.* Herein, we improve the bound $(n+1)k-1$ to $n+k-1$ without other assumptions added, and to $n+k-2$ if either the domain X of the map to be factored contains no pair of disjoint homeomorphic n-dimensional closed subspaces, or the map is irreducible and X satisfies the following condition of W. Hurewicz [12]:

(α) *every nowhere dense subspace of X has dimension less than that of X.*

Our main tool is the use of the concepts of covering and partition dimensions of a map, which were defined in [16] and extend the notion of the Lebesgue–Čech dimension of a space to closed maps. We prove two formulas for calculating the covering dimension cdim of a map and we use them to show that *every at most k-to-one closed map f from an n-dimensional space has* $\operatorname{cdim} f \le n+k-1$. (The statement emphasized in the abstract is, in fact, equivalent to this inequality.) Our formulas for cdim imply well-known theorems on dimension-raising maps by Hurewicz [11, 12], K. Morita [21],

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J. E. Keesling [13], and A. V. Zarelua [29, 30]. The paper ends with some remarks about Anderson–Choquet and Cook type continua.

Throughout the paper, all spaces are assumed to be metrizable and all maps to be continuous.

1. The covering dimension of closed maps: basic facts. Let X be a topological space and $\mathcal{A} = (A_s)_{s\in S}$ be an indexed collection of subsets of X. It is important that the sets A_s may be equal for distinct indices s. We denote the boundary of A_s by ∂A_s. Given a function f on X, we denote the collections $(\overline{A}_s)_{s\in S}$, $(\partial A_s)_{s\in S}$ and $(f(A_s))_{s\in S}$ by $\overline{\mathcal{A}}$, $\partial\mathcal{A}$ and $f(\mathcal{A})$, respectively. All coverings and refinements will be considered as indexed collections of sets, and from now on we omit the word "indexed".

By $|S| \in \mathbb{N}\cup\{\infty\}$ we denote the number of elements in S. For any $x\in X$ we define $\operatorname{ord}_x \mathcal{A} = |\{s \in S : x \in A_s\}|$ and $\operatorname{ord}\mathcal{A} = \sup_{x\in X} \operatorname{ord}_x \mathcal{A}$. If X is empty, then $\operatorname{ord}\mathcal{A} = 0$.

1.1. Definition ([16]). Let f be a closed map from a space X. The *covering dimension* $\operatorname{cdim} f$ is the least integer n such that every finite open cover of X has a finite closed refinement $\mathcal{F}$ with $\operatorname{ord} f(\mathcal{F}) \le n + 1$. If such n's do not exist, we write $\operatorname{cdim} f = \infty$.

Images of distinct elements of $\mathcal{F}$ may be equal, and then such an image is counted two or more times among the elements of the collection $f(\mathcal{F})$.

It is easily seen that $\dim X \le \operatorname{cdim} f$ and $\dim f(X) \le \operatorname{cdim} f$. Since $\dim X = \operatorname{cdim} \mathrm{id}_X$ for every space X, cdim may be viewed as an extension of the Lebesgue–Čech dimension. Moreover, certain properties of dimension extend to maps: the countable and locally finite sum theorems, the completion theorem, and the enlargement theorem—see [16]. The importance of cdim to the study of finite-to-one maps comes from the following

1.2. Theorem. *If f is a closed map from a non-empty space X, then statements* (a) *and* (b) *below are equivalent, and they imply* (c).

(a) $\operatorname{cdim} f \le n$.
(b) *There exists a closed map g from a zero-dimensional space onto X such that the composition $f \circ g$ is at most $(n + 1)$-to-one.*
(c) *f is a composition of n simple closed maps.* ∎

The equivalence (a)⇔(b) is proved in [16, Theorem 6.1]. It extends the well-known Hurewicz–Morita theorem (Morita [21], see also R. Engelking [9, Theorem 4.3.15]), which corresponds to the case $f = \mathrm{id}_X$. The implication (b)⇒(c) is Theorem 2.2 of [15].

Finally, note that *for every finite-to-one map f from a space X, we have* $\dim X \le \dim f(X)$—by the theorem on dimension-lowering maps, see [9, Theorem 4.3.4].

2. The first formula for $\operatorname{cdim} f$**.** Let f be a function into a set Y. We define the *order* of f by $\operatorname{ord} f = \sup_{y\in Y} |f^{-1}(y)|$, and for $k \in \mathbb{N}$ we let

$$C_k(f) = \{y \in Y : |f^{-1}(y)| \geq k\}, \quad E_k(f) = f^{-1}(C_k(f)).$$

If f is a closed map, then $C_k(f)$ and $E_k(f)$ are F_σ-sets; cf. [9, Lemma 4.3.5], see also our Lemma 2.4(a).

Here we prove the following

2.1. Theorem. *If f is a closed map from a non-empty space and* $\operatorname{ord} f < \infty$, *then*

$$\operatorname{cdim} f = \max\{\dim C_j(f) + j - 1 : j = 1, \ldots, \operatorname{ord} f\}.$$

The formula above enables easy calculation of the covering dimension cdim. For example, the original Peano map from $[0,1]$ onto $[0,1]^2$ has order 4, and its covering dimension turns out to be 3. Another example: if f is an exactly k-to-one closed map from an n-dimensional space, then $\operatorname{cdim} f$ turns out to be $n+k-1$ (to see this use also Lemma 3.2 below). More essential applications will be given in Section 3.

Before the proof of Theorem 2.1, we need some preparation.

Suppose that $\mathcal{V} = (V_i)_{i=1}^m$ is a finite open cover of a space X. By the *reduction* of $\mathcal{V}$ we understand the closed cover, denoted by $[\mathcal{V}]$, that consists of sets $[V_1], \ldots, [V_m]$ defined as follows:

$$\begin{aligned} [V_1] &= \overline{V}_1, \\ [V_i] &= \overline{V}_i \setminus (V_1 \cup \cdots \cup V_{i-1}) \quad \text{for } i = 2, \ldots, m; \end{aligned}$$

cf. J. M. Aarts, R. J. Fokkink and H. Vermeer [1, Definition 12 and Lemma 13] and S. A. Bogatyĭ [5, Lemma 3].

Observe that for each $s \geq 2$ and each sequence $1 \leq i_1 < \cdots < i_s \leq m$, we have

$$[V_{i_1}] \cap \cdots \cap [V_{i_s}] \subset \partial V_{i_1} \cap \cdots \cap \partial V_{i_{s-1}}. \tag{1}$$

Indeed, if $x \in [V_{i_1}] \cap \cdots \cap [V_{i_s}]$, then $x \in \overline{V}_{i_t}$ for $t \leq s$. However, $x \notin V_{i_t}$ for $t < s$ by the definition of $[V_{i_s}]$. Hence $x \in \partial V_{i_1} \cap \cdots \cap \partial V_{i_{s-1}}$.

2.2. Lemma. *Suppose that $f : X \to Y$ is a function, and $\mathcal{V}$ is a finite open cover of X. Then for every point $y \in Y$ we have*

$$\operatorname{ord}_y f([\mathcal{V}]) \leq \operatorname{ord}_y f(\partial\mathcal{V}) + |f^{-1}(y)|.$$

Proof. Let $\mathcal{V} = (V_i)_{i=1}^m$ and $T = \{i : y \in f([V_i])\}$. To each index $i \in T$ we assign a point $\varphi(i) \in f^{-1}(y) \cap [V_i]$. We have

$$\operatorname{ord}_y f([\mathcal{V}]) = |T| = \sum_{x\in f^{-1}(y)} |\varphi^{-1}(x)|. \tag{2}$$

For every $x \in f^{-1}(y)$, we can write $\varphi^{-1}(x) = \{i_1^x < \cdots < i^x_{|\varphi^{-1}(x)|}\}$ whenever this point-inverse is non-empty.

According to the inclusion (1), we have

$$y \in \bigcap_{\substack{x\in f^{-1}(y)\\ |\varphi^{-1}(x)|>1}} f(\partial V_{i_1^x}) \cap \cdots \cap f(\partial V_{i^x_{|\varphi^{-1}(x)|-1}}).$$

Then, since $\{i_1^x, \ldots, i^x_{|\varphi^{-1}(x)|}\} \cap \{i_1^{\widehat{x}}, \ldots, i^{\widehat{x}}_{|\varphi^{-1}(\widehat{x})|}\} = \emptyset$ for $x \neq \widehat{x}$, we obtain

$$\sum_{x\in f^{-1}(y)} (|\varphi^{-1}(x)| - 1) \le \sum_{\substack{x\in f^{-1}(y)\\ |\varphi^{-1}(x)|>1}} (|\varphi^{-1}(x)| - 1) \le \operatorname{ord}_y f(\partial\mathcal{V}).$$

This inequality together with (2) completes the proof. ∎

The next statement is a direct consequence of Corollary 2.5 and Theorem 4.3 in [16], where the notion of the *partition dimension* pdim of a map is used in formulations and proofs.

2.3. Lemma (cf. Aarts, Fokkink and Vermeer [1, Lemma 11]; Bogatyĭ [5, Lemma 2]). *Suppose that $f : X \to Y$ is a closed map all of whose point-inverses are zero-dimensional. Let $Y_j \subset Y$, where $j \in \mathbb{N}$, be F_σ-sets such that* $\dim Y_j = n_j$. *Moreover, take subsets $K_1 \subset U_1, \ldots, K_m \subset U_m$ of X, where all the K_i are closed, and U_i are open. Then there exist open subsets $V_1, \ldots, V_m \subset X$ such that $K_i \subset V_i \subset \overline{V}_i \subset U_i$ for each i, and* $\operatorname{ord}(f(\partial V_i) \cap Y_j)_{i=1}^m \le n_j$ *for every j.* ∎

2.4. Lemma. *Suppose that $f : X \to Y$ is a closed map, $k \ge 2$, and define*

$$C_k(f,\varepsilon) = \Big\{y \in Y : \textit{if } 1 \le m < k \textit{ and } \widehat{x}_1, \ldots, \widehat{x}_m \in X \textit{ are distinct}, \textit{ then } f^{-1}(y) \not\subset \bigcup_{i=1}^m \mathrm{B}(\widehat{x}_i, \varepsilon)\Big\},$$

where B *indicates a ball. Then*:

(a) *Each $C_k(f,\varepsilon)$ is a closed subset of Y, and $C_k(f) = \bigcup_{\varepsilon>0} C_k(f,\varepsilon)$.*
(b) *If $y \in C_k(f,\varepsilon)$, then there are $x_1, \ldots, x_k \in f^{-1}(y)$ such that $\varrho(x_i, x_j) \ge \varepsilon$ for $i \neq j$ (where ϱ is the metric on X).*

Proof. $C_k(f,\epsilon)$ is the intersection of the sets

$$\Big\{y \in Y : f^{-1}(y) \not\subset \bigcup_{i=1}^m \mathrm{B}(\widehat{x}_i, \varepsilon)\Big\} = f\Big(X \setminus \bigcup_{i=1}^m \mathrm{B}(\widehat{x}_i, \varepsilon)\Big),$$

which are closed by the closedness of f. ∎

Proof of Theorem 2.1. The inequality "$\le$". Let $f : X \to Y$ and $k = \operatorname{ord} f$. Take a finite open cover $(U_i)_{i=1}^m$ of X. Choose a closed shrinking $(K_i)_{i=1}^m$ of $(U_i)_{i=1}^m$. By Lemma 2.3, there are open sets V_i, $i = 1, \ldots, m$, such that $K_i \subset V_i \subset \overline{V}_i \subset U_i$ for each i, and $\operatorname{ord}(f(\partial V_i) \cap C_j(f))_{i=1}^m \le \dim C_j(f)$ for $j = 1, \ldots, k$. Consider the reduction $[\mathcal{V}]$ of the cover $\mathcal{V} = (V_i)_{i=1}^m$. If $y \in Y$

and $|f^{-1}(y)| = j$, then (by Lemma 2.2) we have $\operatorname{ord}_y f([\mathcal{V}]) \le \operatorname{ord}_y f(\partial\mathcal{V}) + j \le \dim C_j(f) + j$. Thus, $\operatorname{ord} f([\mathcal{V}]) \le \max\{\dim C_j(f) + j : j = 1, \ldots, k\}$.

The inequality "$\ge$". Since $\dim f(X) \le \operatorname{cdim} f$, it remains to check that $\dim C_j(f) + j - 1 \le \operatorname{cdim} f$ for $j = 2, \ldots, k$. Fix j, and write $n = \dim C_j(f)$. The countable sum theorem (see Engelking [8, Theorem 7.2.1]) implies that $\dim C_j(f, \varepsilon) = n$ for some $\varepsilon > 0$.

Now, we confine our attention to the sets $Y' = C_j(f, \varepsilon)$, $X' = f^{-1}(C_j(f, \varepsilon))$, and the restriction $f|X'$. As $\operatorname{ord} f = k$, for every $y \in Y'$ there are pairwise disjoint open (in X') sets V_i^y, where $i = 1, \ldots, k$, such that $\operatorname{diam} V_i^y < \varepsilon$ for each i, and $f^{-1}(y) \subset \bigcup_{i=1}^k V_i^y$. (Some of these V_i^y may be empty.) Choose a closed neighbourhood F^y of y such that $f^{-1}(F^y) \subset \bigcup_{i=1}^k V_i^y$. By the paracompactness of Y', we can assume that the family $\{F^y : y \in Y'\}$ is locally finite. Define $K_i^y = V_i^y \cap f^{-1}(F^y)$, which are closed sets. According to the locally finite sum theorem (see [8, Theorem 7.2.3]), we have $\dim f(K_{i_0}^{y_0}) \ge n$ for some y_0 and i_0. Hence $\operatorname{cdim} f|K_{i_0}^{y_0} \ge n$, and there exists a finite open cover $\mathcal{U}$ of $K_{i_0}^{y_0}$ that has no finite closed refinement $\mathcal{F}$ with $\operatorname{ord} f(\mathcal{F}) \le n$.

The sets of the following three kinds:

- $U \cup (V_{i_0}^{y_0} \setminus K_{i_0}^{y_0})$, where $U \in \mathcal{U}$,
- $V_i^{y_0}$, where $i \ne i_0$, and
- $X' \setminus f^{-1}(F^{y_0})$

form an open cover, say $\mathcal{V}$, of X'. We claim that $\mathcal{V}$ *has no finite closed refinement* $\mathcal{G}$ *such that* $\operatorname{ord} f(\mathcal{G}) \le n + j - 1$. Indeed, suppose that a finite closed cover $\mathcal{G}$ refines $\mathcal{V}$. Since the sets $G \cap K_{i_0}^{y_0}$, where $G \in \mathcal{G}$, form a refinement of $\mathcal{U}$, there is a point $y \in \bigcap_{s=1}^{n+1} f(G_s \cap K_{i_0}^{y_0})$ for distinct $G_s \in \mathcal{G}$. Since $\mathcal{G}$ refines $\mathcal{V}$, we have $G_s \subset V_{i_0}^{y_0}$ for each s. By Lemma 2.4, there are points $x_t \in f^{-1}(y)$, where $t = 1, \ldots, j$, such that $\varrho(x_t, x_{t'}) \ge \varepsilon$ for $t \ne t'$. At most one of these points may belong to each of the sets $V_i^{y_0}$. There is one in $V_{i_0}^{y_0}$, maybe. Hence, there are further distinct sets $G_{t+n+1} \in \mathcal{G}$, where $t = 1, \ldots, j-1$, such that $x_t \in G_{t+n+1}$. Thus, $\operatorname{ord} f(\mathcal{G}) \ge n+1+j-1 = n+j$. The foregoing claim means that $n + j - 1 \le \operatorname{cdim} f|X' \le \operatorname{cdim} f$. ∎

3. The second formula for $\operatorname{cdim} f$ **and main corollaries.** The next formula is more suitable for applications than that of Theorem 2.1.

3.1. THEOREM. *If f is a closed map from a non-empty space and* $\operatorname{ord} f < \infty$, *then*

$$\operatorname{cdim} f = \max\{\dim E_j(f) + j - 1 : j = 1, \ldots, \operatorname{ord} f\}.$$

To prove this we need the following two lemmata.

3.2. Lemma (J. Suzuki [26]). *If f is an exactly k-to-one closed map from a space X, then* $\dim X = \dim f(X)$. ∎

3.3. Lemma (Keesling [13, Corollary III.6], K. Nagami [23, Theorem 24-4]; the separable case: Hurewicz [12]). *If f is a finite-to-one closed map from a space X, then*

$$\dim C_k(f) \le \max\{\dim X, \dim C_{k+1}(f) + 1\} \quad \textit{for each } k. \ \blacksquare$$

Proof of Theorem 3.1. Write $k = \operatorname{ord} f$. According to Theorem 2.1, it suffices to prove that

$$\begin{aligned}\max\{\dim E_j(f) + j - 1 : j = 1, \dots, k\}\\ = \max\{\dim C_j(f) + j - 1 : j = 1, \dots, k\}.\end{aligned}$$

The inequality "$\le$". By the theorem on dimension-lowering maps (see [9, Theorem 4.3.4]), we have $\dim E_j(f) \le \dim C_j(f)$.

The inequality "$\ge$". Write $M = \max\{\dim E_j(f) + j - 1 : j = 1, \dots, k\}$. By downward induction we shall show that $\dim C_j(f) + j - 1 \le M$ for each j. Indeed, it follows from Lemma 3.2 that $\dim C_k(f) + k - 1 = \dim E_k(f) + k - 1 \le M$ (for $j = k$). Assume that $\dim C_{j+1}(f) + j \le M$. By Lemma 3.3 applied to the restriction $f|E_j(f)$ we obtain

$$\begin{aligned}\dim C_j(f) + j - 1 &\le \max\{\dim E_j(f), \dim C_{j+1}(f) + 1\} + j - 1\\ &= \max\{\dim E_j(f) + j - 1, \dim C_{j+1}(f) + j\} \le M. \ \blacksquare\end{aligned}$$

Since always $\dim f(X) \le \operatorname{cdim} f$, Theorem 3.1 implies that *if f is a closed map from a non-empty space X and* $\operatorname{ord} f < \infty$, *then* $\dim f(X) \le \max\{\dim E_j(f) + j - 1 : j = 1, \dots, \operatorname{ord} f\}$. This theorem on dimension-raising maps was proved by Zarelua [29, 30] in a more general setting.

The next statement is a common generalization of several theorems on dimension-raising maps: Hurewicz [12], Zarelua [29, Corollary 1], [30, Proposition 4.5], Keesling [13, Theorem III.2], Nagami [23, Theorem 24-5] (see also A. Lelek [18] for a survey concerning maps and dimension inequalities).

3.4. Theorem. *Suppose that f is a finite-to-one closed map from a non-empty space X. If the function $y \mapsto |f^{-1}(y)|$ takes on finitely many values $m_1 < \dots < m_k$ on $f(X)$, then*

$$\dim f(X) \le \max\{\dim E_{m_j}(f) + j - 1 : j = 1, \dots, k\}.$$

Proof. Note that $f(X) = C_{m_1}(f)$, and repeat the reasoning in the proof of Theorem 3.1 (the inequality "$\ge$"). Apply Lemma 3.2 to the restriction $f|E_{m_k}(f)$ and Lemma 3.3 to $f|E_{m_j}(f)$. ∎

Theorems 3.1 and 1.2 immediately yield

3.5. Corollary. *If f is an at most k-to-one closed map from a non-empty n-dimensional space X, then*:

(a) $\operatorname{cdim} f \le n+k-1$.
(b) *There exists a closed map g from a zero-dimensional space onto X such that the composition $f \circ g$ is at most $(n+k)$-to-one.*
(c) *f is a composition of $n+k-1$ simple closed maps.* ∎

3.6. Remark. (i) In view of Remark 6.2 in [16] we infer that for any countably many at most k_i-to-one closed maps f_i from an n-dimensional space X there exists a closed map g from a zero-dimensional space onto X such that each composition $f_i \circ g$ is at most $(n+k_i)$-to-one.

(ii) If either $n=0$, or $k=3$ and n is odd, then the number $n+k-1$ in (c) cannot in general be replaced by a smaller one; see [15, p. 151] and [25, Theorem 2].

4. Maps from spaces with Hurewicz's property (α)**.** Motivated by the work of Hurewicz [11, 12], we impose additional assumptions on the map under which the number $n+k-1$ in Corollary 3.5 can be diminished.

A map $f : X \xrightarrow{\text{onto}} Y$ is called *irreducible* if for every proper closed subset $F \subset X$, we have $f(F) \neq Y$.

4.1. Lemma (cf. G. T. Whyburn [28, Theorem 2], I. A. Vaĭnšteĭn [27, p. 30, Corollary 2]). *If f is an irreducible closed map from a space X, then $E_2(f)$ is a first category F_σ-set in X.*

Proof. In view of Lemma 2.4(a), it suffices to show that $f^{-1}(C_2(f,\varepsilon))$ is nowhere dense for every $\varepsilon > 0$. Take any $\widehat{x} \in X$ and any number $\delta < \varepsilon$. Since f is irreducible, there is a point x such that $f^{-1}f(x) \subset \mathrm{B}(\widehat{x},\delta)$. Hence $x \in \mathrm{B}(\widehat{x},\varepsilon) \setminus f^{-1}(C_2(f,\varepsilon))$. ∎

4.2. Corollary. *Suppose that f is an irreducible, at most k-to-one $(k \ge 2)$, and closed map from an n-dimensional space X that satisfies Hurewicz's condition (α). Then* $\operatorname{cdim} f \le n+k-2$, *and so the conclusions of Corollary* 3.5 *remain true with $n+k$ replaced by $n+k-1$.*

Proof. Since X has property (α), we obtain $\dim E_2(f) \le n-1$ by Lemma 4.1 and the countable sum theorem for dim (see [8, Theorem 7.2.1]). Use Theorems 3.1 and 1.2 to complete the proof. ∎

We cannot drop the assumption of irreducibility in Corollary 4.2, for any exactly k-to-one map f from an n-dimensional space has $\operatorname{cdim} f = n+k-1$.

5. Maps from n-ACh spaces. In this section we find n-dimensional spaces that are not domains of at most k-to-one $(k \ge 2)$ closed maps f with $\operatorname{cdim} f = n+k-1$.

R. D. Anderson and G. Choquet [2] constructed a continuum that contains no pair of homeomorphic distinct non-degenerate subcontinua; see also J. J. Andrews [3]. (All known examples of such continua are one-dimensional.) These continua are a prototype for the following classes of spaces:

We say that a space of dimension $\geq n$ is *n-ACh* if it contains no pair of disjoint homeomorphic closed subspaces of dimension $\geq n$. Spaces that are 1-ACh are called *Anderson–Choquet* (abbrev. *ACh*).

It follows from the countable sum theorem (see [8, Theorem 7.2.1]) that *in an n-ACh space, every F_σ-subspace of dimension $\geq n$ is also n-ACh.*

This example shows that for every $n \geq 1$, there exists an n-dimensional n-ACh compactum:

5.1. Example. Consider the n-dimensional cube I^n, where $n \geq 2$, and choose a sequence of pairwise disjoint sets $D_m \subset I^n$ such that $|D_m| = m$ for each m, $\lim_{m\to\infty} \operatorname{diam} D_m = 0$, and $\bigcup_{m=1}^\infty D_m$ is dense in I^n. Consider the decomposition $\mathcal{D}$ of I^n into all the sets D_m and all remaining singletons in I^n. It is easily checked that this decomposition is upper semicontinuous. We prove that *the quotient space $A_n = I^n/\mathcal{D}$ is an n-dimensional n-ACh space.*

Write $q : I^n \to A_n$ for the natural quotient map. By Lemma 3.3 with $k = 1$, we obtain $\dim A_n \leq n$. We claim that $\dim q^{-1}(G) = n$ *iff* $\dim G = n$, *for every subset $G \subset A_n$*. Indeed, if $\dim q^{-1}(G) < n$, then $\dim G < n$—apply Lemma 3.3 to the restriction $q|q^{-1}(G)$. If $\dim q^{-1}(G) = n$, the theorem on dimension-lowering maps yields $\dim G = n$. In consequence, $\dim A_n = n$.

In order to show that A_n *has property* (α), take an n-dimensional set $G \subset A_n$. Then $\dim q^{-1}(G) = n$, $\operatorname{int} q^{-1}(G) \neq \emptyset$, and hence $\overline{I^n \setminus q^{-1}(G)}$ is a proper subset of I^n. Since q is an irreducible map, we have

$$\begin{aligned}\emptyset \neq A_n \setminus q[\overline{I^n \setminus q^{-1}(G)}] &= A_n \setminus \overline{q[\overline{I^n \setminus q^{-1}(G)}]} \\ &\subset A_n \setminus \overline{q[I^n \setminus q^{-1}(G)]} = A_n \setminus \overline{A_n \setminus G} = \operatorname{int} G.\end{aligned}$$

Therefore, *every n-dimensional subset of A_n has non-empty interior.*

Now, suppose that $h : M \xrightarrow{\text{onto}} N$ is a homeomorphism between disjoint n-dimensional sets $M, N \subset A_n$. Then h maps $U = M \setminus [\partial M \cup h^{-1}(\partial N)]$ onto $V = N \setminus [h(\partial M) \cup \partial N]$. As M and N have non-empty interiors, U and V are non-empty open subsets of A_n. Any point $D_m \in U$ disconnects every small enough connected neighbourhood W of D_m into m components of $W \setminus \{D_m\}$. A contradiction: V does not contain such a point with the same m. ∎

It is not difficult to construct an n-dimensional n-ACh continuum without property (α). This can be done by starting with the product $S \times I^{n-1}$ instead of I^n, where S denotes the $\sin 1/x$ curve.

5.2. THEOREM. *If $k \geq 2$, then there exists no exactly k-to-one closed map from an n-ACh space.*

Proof. Suppose that $f : X \xrightarrow{\text{onto}} Y$ is such a map. It follows from Lemma 2.4(a) and the countable sum theorem that $\dim f^{-1}(C_k(f,\varepsilon)) \geq n$ for some $\varepsilon > 0$. Lemma 2.4(b) implies that for every $x \in f^{-1}(C_k(f,\varepsilon))$, the map f is one-to-one on $\mathrm{B}(x,\varepsilon/2) \cap f^{-1}(C_k(f,\varepsilon))$. The restriction $f|f^{-1}(C_k(f,\varepsilon))$ is a covering map, being an exactly k-to-one, locally one-to-one and closed map; cf. Krzempek [17, Lemma 1].

There is an open cover $(U_i)_{i\in I}$ of $C_k(f,\varepsilon)$ such that each $f^{-1}(\overline{U}_i)$ is the union of pairwise disjoint closed subsets $G_i^j \subset X$, where $j = 1,\ldots,k$, and each $f|G_i^j$ is a homeomorphism onto $\overline{U}_i$. By the paracompactness of $C_k(f,\varepsilon)$, we can assume that the cover $(U_i)_{i\in I}$ is locally finite. From the locally finite sum theorem (see [8, Theorem 7.2.3]), we infer that some G_i^j has dimension $\geq n$. This is impossible since X is n-ACh. ∎

5.3. COROLLARY. *Suppose that f is an at most k-to-one* ($k \geq 2$) *closed map from an n-dimensional n-ACh space X. Then* $\operatorname{cdim} f \leq n+k-2$, *and so the conclusions of Corollary* 3.5 *remain true with $n+k$ replaced by $n+k-1$.*

Proof. If $E_k(f)$ were n-dimensional, it would be n-ACh. Hence, Theorem 5.2 leads to $\dim E_k(f) \leq n-1$. Use Theorems 3.1 and 1.2. ∎

Applying Theorems 5.2 and 3.4, we obtain the following new result of the Hurewicz type; cf. Hurewicz [11, 12], Keesling [13, Theorem III.7].

5.4. COROLLARY. *Suppose that f is a closed map from an n-dimensional n-ACh space X. If the function $y \mapsto |f^{-1}(y)|$ takes on k distinct values on $f(X)$, where $k \geq 2$, then* $\dim f(X) \leq n+k-2$. ∎

Observe that *an n-dimensional space X is n-ACh iff every simple closed map from X is n-dimensional.* Indeed, the sufficiency follows from Corollary 5.3. On the other hand, if X contains two disjoint closed copies of the same n-dimensional space, then the simple map that glues these copies together has covering dimension $n+1$.

6. Open problems and remarks. Comparing the bounds of $\dim f(X)$ or $\operatorname{cdim} f$ (for a map f from a space X) in Corollaries 3.5, 4.2, 5.3, 5.4 and in theorems on dimension-raising maps from [11–13, 21], we can ask about stronger inequalities:

6.1. QUESTION. *Does there exist an n-dimensional compact space X, $n \geq 2$, such that* $\dim f(X) \leq n+k-3$ [*or even* $\operatorname{cdim} f \leq n+k-3$] *for every at most k-to-one map f from X, $k \geq 3$?*

The next proposition shows that if we knew Anderson–Choquet spaces of dimensions greater than one, they would be appropriate examples to the foregoing question.

6.2. Proposition. *Suppose f is an at most k-to-one closed map from an n-dimensional Anderson–Choquet space X. Then* $\operatorname{cdim} f \le \max\{n, k-1\}$, *and so the conclusions of Corollary* 3.5 *remain true with $n+k$ replaced by* $\max\{n+1, k\}$.

Proof. We can assume that $k = \operatorname{ord} f$. We claim that $\dim E_j(f) \le \dim E_{j+1}(f) + 1$ *for each* $j \ge 2$. Indeed, it follows from the enlargement theorem (see [9, Theorems 4.1.19 and 4.1.3]) that there exists a G_δ-subset $H \subset X$ such that $E_{j+1}(f) \subset H$, $\dim H = \dim E_{j+1}(f)$, and $H = f^{-1}f(H)$. The restriction $f|E_j(f) \setminus H$ is exactly j-to-one, and $E_j(f) \setminus H$ is an F_σ-set. If it had positive dimension, it would be an ACh space, and this would contradict Theorem 5.2. Thus $\dim[E_j(f) \setminus H] \le 0$. The claim follows from the decomposition theorem (see [9, Theorems 4.1.16 and 4.1.3]).

We have $\dim E_{k+1}(f) = -1$, and our claim implies that $\dim E_j \le k - j$ for $j \ge 2$. The proposition is a consequence of Theorems 3.1 and 1.2. ∎

6.3. Question. *Do there exist Anderson–Choquet spaces of dimensions greater than one?*

We shall state a negative result in this direction.

The next definition is much more restrictive than that of an Anderson–Choquet space: By a *Cook continuum* we understand a continuum K such that every map $f : L \to K$ from a subcontinuum $L \subset K$ is either a constant map or the identity id_L. The first example of such a continuum K was constructed by H. Cook [7] (for a detailed description see A. Pultr and V. Trnková [24, Appendix A]); it was one-dimensional and hereditarily indecomposable.

T. Maćkowiak [19] showed that *there exists no Cook continuum of dimension greater than two,* and asked whether all Cook continua are curves. Since every continuum of dimension greater than n contains a hereditarily indecomposable continuum of dimension at least n (R. H. Bing [4]), the following statement implies Maćkowiak's:

6.4. Proposition. *There exists no hereditarily indecomposable Cook continuum of dimension greater than one.*

Proof. We modify the argument given in [19, (4.1)]. Recall that a map is *monotone* if each of its point-inverses is a continuum. A map $f : X \to Y$ is *weakly confluent* if for every continuum $B \subset Y$ there is a continuum $A \subset X$ such that $f(A) = B$.

Let K be any hereditarily indecomposable continuum K with $\dim K \ge 2$. Take a proper subcontinuum $X \subset K$ with $\dim X \ge 2$. Since X is hereditarily indecomposable, by a theorem of J. L. Kelley [14] (see also K. P. Hart, J. van Mill and R. Pol [10]) there is a monotone open map $f : X \to Y$ onto an infinite-dimensional continuum Y. By a theorem of S. Mazurkiewicz [20] (see also S. B. Nadler [22, Theorem 13.56]) there is a weakly confluent map $g : Y \to I^3$ onto the cube I^3. Take any one-dimensional continuum $Z \subset K \setminus X$ and an embedding $i : Z \to I^3$.

As g is weakly confluent, there is a continuum $A \subset Y$ with $g(A) = i(Z)$. As f is monotone, $L = f^{-1}(A) \subset X$ is a continuum; cf. [22, Exercise 8.46]. The map $i^{-1} \circ g \circ f|L : L \xrightarrow{\text{onto}} Z$ shows that K is not a Cook continuum. ■

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