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Some examples of hyperarchimedean lattice-ordered groups

by

Anthony W. Hager (Middletown, CT) and
Chawne M. Kimber (Easton, PA)

Abstract. All ℓ-groups shall be abelian. An a-extension of an ℓ-group is an extension preserving the lattice of ideals; an ℓ-group with no proper a-extension is called a-closed. A hyperarchimedean ℓ-group is one for which each quotient is archimedean. This paper examines hyperarchimedean ℓ-groups with unit and their a-extensions by means of the Yosida representation, focussing on several previously open problems. Paul Conrad asked in 1965: If G is a-closed and M is an ideal, is G/M a-closed? And in 1972: If G is a hyperarchimedean sub-ℓ-group of a product of reals, is the f-ring which G generates also hyperarchimedean? Marlow Anderson and Conrad asked in 1978 (refining the first question above): If G is a-closed and M is a minimal prime, is G/M a-closed? If G is a-closed and hyperarchimedean and M is a prime, is G/M isomorphic to the reals? Here, we introduce some techniques of a-extension and construct a several parameter family of examples. Adjusting the parameters provides answers "No" to the questions above.

1. Preliminaries. In the following, a *lattice-ordered group*, or an *ℓ-group*, is an abelian group $(G,+)$ with a lattice order $\leq$ for which $a \leq b$ implies $a+c \leq b+c$ for all c. Moreover, $G^+ = \{g \in G \mid g \geq 0\}$ is the positive cone of G. We shall use the references [AF], [BKW], [D], and [LZ] for various aspects of ℓ-group theory. We sketch some particular ideas which we need here.

An *ideal* in an ℓ-group is a convex sub-ℓ-group; these are the kernels of ℓ-homomorphisms. The collection of all ideals in the ℓ-group G is denoted $\mathrm{Idl}(G)$. Let G be a sub-ℓ-group of H; we write $G \leq H$. When the contraction map $\mathrm{Idl}(H) \to \mathrm{Idl}(G)$ is one-to-one and onto, H is called an *a-extension* of G, and we write $G \leq_a H$.

Proposition 1.1 ([C1, 2.1]). *$G \leq_a H$ if and only if $G \leq H$ and for each $h \in H^+$ there is $g \in G^+$ such that $g \sim_a h$, i.e., $h \leq mg$ and $g \leq nh$ for some positive integers m, n.*

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If $G \leq_a H$ implies $G = H$, then G is called *a-closed*. When $G \leq_a H$ and H is a-closed, H is called an *a-closure* of G. The first systematic study of a-extensions is Conrad's [C1]. A large literature has developed subsequently on this complicated subject; see especially [AF] and [D]. Two main themes, which this paper continues, are: non-uniqueness of a-closure, even in quite restricted contexts; what does "a-closed" mean?

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the natural numbers, the integers, the rational numbers, and the reals, respectively, as one sort of mathematical structure or another, which being clear from context.

Let G be an ℓ-group, and $u \in G^+$. An element $g \in G$ is *infinitesimal* with respect to h, written $g \ll h$, if $0 \leq ng \leq h$ for all $n \in \mathbb{N}$. G is *archimedean* if $g \ll h$ implies $g = 0$. A positive element u is a *weak unit* in G if $|g| \wedge u = 0$ implies $g = 0$; and u is a *strong unit* if $\langle u \rangle = G$, where $\langle u \rangle = \{g \in G \mid \exists n \in \mathbb{N}\ (|g| \leq nu)\}$ denotes the ideal in G generated by u.

To illustrate: $\mathbb{R}$ is archimedean, and the class of archimedean ℓ-groups is closed under product and sub-ℓ-group formation. So, whenever X is a topological space, which we always assume to be completely regular and Hausdorff (see [GJ]), the ℓ-group $C(X)$ of continuous $\mathbb{R}$-valued functions is archimedean. A function $u \in C(X)^+$ is a weak unit if and only if $\operatorname{coz}(u) = \{x \in X \mid u(x) \neq 0\}$ is dense in X; so the constant function 1 is a weak unit. For compact X, 1 is a strong unit since each $f \in C(X)$ is bounded.

The following is the classical first representation theorem of Yosida [Y], elaborated and augmented somewhat. Many useful variations and generalizations are discussed in [AF], [LZ], [HR] and [BH].

Theorem 1.2 (Representation of objects). *Let G be an archimedean ℓ-group with strong unit u.*

(a) *The set of maximal ideals of G, with the hull-kernel topology, is a compact Hausdorff space, denoted $Y(G,u)$ or just YG.*
(b) *There is an ℓ-isomorphism $G \approx \widehat{G} \leq C(YG)$ for which $\widehat{u} = 1$ and $\widehat{G}$ separates the points of YG.*
(c) *If $G \approx \widetilde{G} \leq C(X)$ is an ℓ-isomorphism with X compact Hausdorff and $\widetilde{u} = 1$, then there is a continuous surjection $\tau : X \to YG$ for which $\widetilde{g} = \widehat{g} \circ \tau$ for each $g \in G$. The group $\widetilde{G}$ separates points of X if and only if τ is a homeomorphism.*
(d) *For each $p \in YG$, $M_p = \{g \mid \widehat{g}(p) = 0\}$ is a maximal ideal, and each maximal ideal is of this form for unique p. For each p, we have $\widehat{G}/M_p \approx \{\widehat{g}(p) \mid g \in G\} \leq \mathbb{R}$.*

Theorem 1.3 (Representation of morphisms). *Let (G,u) and (H,v) be archimedean ℓ-groups with strong units u,v, and let $\varphi : G \to H$ be an ℓ-homomorphism with $\varphi(u) = v$. Define a function $Y\varphi : YG \leftarrow YH$ by: $(Y\varphi)(p) = q$ means $\varphi^{-1}(M_p) = M_q$. Then $Y\varphi$ is continuous and is the* *unique map for which* $\widehat{\varphi}(g) = \widehat{g} \circ (Y\varphi)$ *for each* $g \in G$. *Further*, φ *is one-to-one if and only if* $Y\varphi$ *is onto*; *if* φ *is onto*, *then* $Y\varphi$ *is one-to-one.*

COROLLARY 1.4. *Suppose* G *is archimedean with strong unit* u, *and* $G \leq_a H$. *Then* H *is archimedean*, u *is a strong unit in* H, *and* $YH = YG$.

For the rest of the paper, we shall identify each archimedean ℓ-group with strong unit with its Yosida representation: $1 \in G \leq C(YG)$, and that notation means "G is an archimedean ℓ-group with strong unit in its Yosida representation".

For X a topological space, $\mathrm{clop}(X)$ denotes the Boolean algebra of clopen subsets of X. For $U \subseteq X$, $\chi(U)$ denotes the characteristic function of U and $g\chi(U)$ stands for the function on YG which vanishes off U and agrees with g on U. The expression "$g\chi(U) \in G$" means there is $h \in G$ which in the Yosida representation is $g\chi(U)$. The following is simple but crucial.

PROPOSITION 1.5. *Suppose that* $1 \in G \leq C(YG)$. *Then* $g\chi(U) \in G$ *for any* $U \in \mathrm{clop}(YG)$ *and* $g \in G$.

Proof. We first show that $\chi(U) \in G$: Let $\{h < a\}$ stand for $\{x \in YG \mid h(x) < a\}$, and let $U' = YG - U$. For each $p \in U$ and each $q \in U'$, choose $g_{pq} \in G$ which is 2 at p and -1 at q. (From Theorem 1.2, there is $g \in G$ with $g(p) = 1$ and $g(q) = 0$. Let $g_{pq} = 3g - 1$.) Fixing p, the set $\{\{g_{pq} < 0\} \mid q \in U'\}$ covers U', so for some finite $E \subseteq U'$, the set $\{\{g_{pq} < 0\} \mid q \in E\}$ covers U'. Then $g_p = (\bigwedge_{q\in E} g_{pq}) \vee 0$ is 2 at p and 0 on U'. Now $\{\{g_p > 1\} \mid p \in U\}$ covers U, so for some finite $F \subseteq U$, the set $\{\{g_p > 1\} \mid p \in F\}$ covers U. Then $\chi(U) = (\bigvee_{p\in F} g_p) \wedge 1 \in G$.

If $g \geq 0$, there is n with $g \leq n$, and then $g\chi(U) = g \wedge n\chi(U) \in G$. Finally, $g\chi(U) = (g \vee 0)\chi(U) - ((-g) \vee 0)\chi(U) \in G$. ∎

2. Hyperarchimedean ℓ-groups with unit. A *hyperarchimedean ℓ-group* is one for which each quotient is archimedean. We abbreviate "hyperarchimedean" to "HA". In this section we indicate some basic features of HA ℓ-groups.

For G an abelian ℓ-group, dG denotes the *divisible hull*. This is the unique divisible ℓ-group containing G for which $b \in dG$ implies there exists $a \in G$ for which $nb = a$ for some $n \in \mathbb{N}$. For G archimedean with strong unit, viewed *per* Theorem 1.2 as $1 \in G \leq C(YG)$, it is easily seen that $\{rg \mid g \in G,\ r \in \mathbb{Q}\} \leq C(YG)$ is an explicit presentation of dG.

PROPOSITION 2.1 ([C2]). (a) *If* H *is* HA *and* $G \leq H$, *then* G *is* HA.
(b) *If* G *is* HA *and* $G \leq_a H$, *then* H *is* HA.
(c) *If* G *is* HA, *then* dG *is* HA.
(d) *If* G *is* HA, *then any weak unit in* G *is strong.*

We turn to a discussion of the central role of "zero-sets".

For $f \in \mathbb{R}^X$, the *zero-set* of f is $Z(f) = f^{-1}(\{0\})$ and the *cozero-set* of f is $\operatorname{coz}(f) = X - Z(f)$. If $f \in C(X)$, then $Z(f)$ is closed and $\operatorname{coz}(f)$ is open. For $1 \in G \leq C(YG)$, let $\mathcal{Z}G = \{Z(g) \mid g \in G\}$; note that Theorem 1.3 says $\operatorname{clop}(YG) \subseteq \mathcal{Z}(G)$. For X a space, $S(X) = \{f \in C(X) \mid \operatorname{range} f \text{ is finite}\} \leq C(X)$, and $S(X, \mathbb{Z}) = \{f \in S(X) \mid \operatorname{range} f \subseteq \mathbb{Z}\} \leq S(X)$. Note that $\mathcal{Z}S(X) = \mathcal{Z}S(X, \mathbb{Z}) = \operatorname{clop}(X)$.

Let G be an ℓ-group and $g \in G^+$. A *component* of g is an $h \in G$ for which $h \wedge (g - h) = 0$. The following is immediate.

PROPOSITION 2.2. *Let* $1 \in G \leq C(YG)$, *and let* $h \in G$. *Then* h *is a component of* 1 *if and only if there is* $U \in \operatorname{clop}(YG)$ *for which* $h = \chi(U)$.

In view of [BKW], Theorem 1.2 and Proposition 2.1 we obtain the following characterization of HA groups.

THEOREM 2.3. *Let* G *be an abelian* ℓ*-group with weak unit* u. *The following are equivalent*:

(a) G *is HA*.
(b) G *is archimedean*, u *is a strong unit*, *and in the Yosida representation* $\mathcal{Z}G = \operatorname{clop}(YG)$.
(c) G *is archimedean*, u *is a strong unit*, *and for some representation* $G \approx \widetilde{G} \leq C(X)$ *with* X *compact Hausdorff*, $\widetilde{u} = 1$, *and* $\mathcal{Z}\widetilde{G} \subseteq \operatorname{clop}(X)$.
(d) u *is a strong unit*, *and for each* $g \in G^+$ *there is a pair* (χ, n), *where* χ *is a component of* u *and* $n \in \mathbb{N}$, *with* $ng \geq \chi$ *and* $g \wedge (u - \chi) = 0$.
(e) G *is archimedean*, *and in the Yosida representation* $S(YG, \mathbb{Z}) \leq_a G$.

Proposition 2.1, Theorem 2.3 and [C2] give the following.

COROLLARY 2.4. (a) *If* G *is HA with strong unit*, *then* YG *is zero-dimensional* (*i.e.*, $\operatorname{clop}(YG)$ *is a base for the open sets*).
(b) *For* X *a space*, $S(X)$ *is HA*.
(c) *Let* G *be an* ℓ*-group. Then* G *is HA with strong unit if and only if there is a space* Y *with* $S(Y) \leq_a G$.
(d) *For* X *a space*, $C(X)$ *is HA if and only if the sub-*ℓ*-group of bounded functions* $C^*(X)$ *is HA if and only if* X *is finite*.
(e) *If* G *is HA with strong unit*, *then*: G *is a-closed if and only if* ($G \leq_a H \leq C(YG)$ *implies* $H = G$).

Some of Theorem 2.3 generalizes as follows. First, if G is archimedean with *weak* unit, in its Yosida representation (see [HR]), and $g \in G^+$ then $Z(g)$ is open if and only if g has a (χ, n) *per* Theorem 2.3(c). Second, for G abelian with weak unit u, and $G^* = \langle u \rangle$, these are equivalent: G is archimedean and G^* is *HA*; G is archimedean and $\mathcal{Z}G = \operatorname{clop}(YG)$; each

$g \in G^+$ has a (χ, n). These ℓ-groups are called *bounded away*, and were introduced in the preprint [KM], and are discussed in [HKM].

This section concludes with the following useful results which show that if an ℓ-group G has ℓ-group generators which seem hyperarchimedean, then G actually is hyperarchimedean.

Let H be an ℓ-group, let B be a *subgroup* of H, and let $\langle B\rangle$ denote the sub-ℓ-group of H generated by B. The elements of $\langle B\rangle$ are the elements of H of the form $\bigwedge_k \bigvee_j b_{jk}$, for $b_{jk} \in B$ and finite index sets. (See [BKW].) The usual problem with analyzing $\langle B\rangle$ is the inscrutability of the expressions $\bigwedge \bigvee b_{jk}$.

Proposition 2.5. *Suppose X is compact and zero-dimensional, and that B is a point-separating subgroup of $C(X)$ with $1 \in B$ and $Z(b)$ open for all $b \in B$. Then $\langle B\rangle$ is HA and $\langle B\rangle = \{\sum_{i\in I} b_i\chi(U_i) \mid b_i \in B,\, U_i \in \mathrm{clop}(X);$ I finite$\}$.*

Corollary 2.6. *Suppose G is archimedean with strong unit: $1 \in G \le C(YG)$, and suppose $b \in C(YG)$ has $Z(g+zb)$ open for each $g \in G$, $z \in \mathbb{Z}$. Then*

$$\langle G + \mathbb{Z}\cdot b\rangle = G + \Big\{ \sum_{i\in I} z_i b\chi(U_i) \,\Big|\, z_i \in \mathbb{Z},\, U_i \in \mathrm{clop}(YG);\, I \text{ finite}\Big\},$$

$\langle G + \mathbb{Z}\cdot b\rangle$ is HA and hence $G \le_a \langle G + \mathbb{Z}\cdot b\rangle$. Moreover, if for each $U \in \mathrm{clop}(YG)$, one of $b\chi(U), b\chi(YG - U)$ is in G, then $\langle G + \mathbb{Z}\cdot b\rangle = G + \mathbb{Z}\cdot b$; when G is divisible, the converse holds.

The following lemma is necessary.

Lemma 2.7. *If I is finite and $h = \sum\{b_i\chi(U_i) \mid i \in I\}$, then for a finite set $\mathcal{E}$, there is a rewriting $h = \sum\{b_E\chi(V_E) \mid E \in \mathcal{E}\}$ with the sets V_E disjoint and non-empty and $X = \bigcup\{V_E \mid E \in \mathcal{E}\}$.*

Proof. For $E \subseteq I$, let $V_E = \bigcap\{U_i \mid i \in E\} \cap \bigcap\{X - U_i \mid i \notin E\}$, and let $\mathcal{E} = \{E \mid E \subseteq I,\, V_E \ne \emptyset\}$. Note that $V_\emptyset = X - \bigcup\{U_i \mid i \in E\}$. For $\emptyset \ne E \in \mathcal{E}$, let $b_E = \sum\{b_i \mid i \in E\}$, and set $b_\emptyset = 0$. If $E \ne F$, say $i \in E$ but $i \notin F$, then $V_E \subseteq U_i$ while $V_F \subseteq X - U_i$, so $V_E \cap V_F = \emptyset$. For $p \in X$, let $E(p) = \{i \in I \mid p \in U_i\}$. Then $V_{E(p)} \ne \emptyset$, and so $E(p) \in \mathcal{E}$, and

$$h(p) = \sum\{b_i(p) \mid i \in E(p)\} = b_{E(p)}\chi(V_{E(p)})(p) = \sum\{b_E\chi(V_E) \mid E \in \mathcal{E}\}(p)$$

since the V_E's are disjoint. ∎

Proof of Proposition 2.5. Of course, $\langle B\rangle$ is archimedean with strong unit 1; by Theorem 1.2, $Y\langle B\rangle = X$ and the presentation of $\langle B\rangle$ is the Yosida representation. Let B_0 be the set of expressions $\sum b_i\chi(U_i)$. By Proposition 1.5, each $b\chi(u) \in \langle B\rangle$, so that $B_0 \subseteq \langle B\rangle$, and B_0 clearly is a group.

We shall show that $Z(h)$ is open for $h \in B_0$ and then that B_0 is an ℓ-group. So $B_0 = \langle B \rangle$, and by Theorem 2.3, $\langle B \rangle$ is HA.

Now let $B_0 \ni h = \sum b_E \chi(V_E)$ be written as in Lemma 2.7. Then $Z(h) = \bigcup\{Z(h) \cap V_E\} = \bigcup\{Z(b_E) \cap V_E\}$, by disjointness. Since each $Z(b_E)$ is supposed open, and each V_E is open, $Z(h)$ is open.

For B_0 to be an ℓ-group, it suffices that $h \in B_0$ imply $h \vee 0 \in B_0$ (see [BKW]). Again write $h = \sum b_E \chi(V_E)$ as in Lemma 2.7. Disjointness implies $h \vee 0 = \sum (b_E \chi(V_E) \vee 0)$ and $b_E \chi(V_E) \vee 0 = (b_E \vee 0)\chi(E)$. So $h \vee 0 \in B_0$. ∎

Proof of Corollary 2.6. For the first part, just apply Proposition 2.5 to $B = G + \mathbb{Z} \cdot b$. For the second part: Let $U' = YG - U$. Suppose one of $b\chi(U), b\chi(U')$ is in G for all $U \in \operatorname{clop}(YG)$. For all U, $b\chi(U) = g + z \cdot b$ for some $g \in G$, $z \in \mathbb{Z}$. If $b\chi(U) \in G$, use $z = 0$. If $b\chi(U') \in G$, then $B = b\chi(U) + b\chi(U')$, and we can use $z = -1$. This shows that any expression $g + \sum z_i b\chi(U_i)$ actually lies in $G + \mathbb{Z} \cdot b$. Conversely, suppose $\langle G + \mathbb{Z} \cdot b \rangle = G + \mathbb{Z} \cdot b$, and $U \in \operatorname{clop}(YG)$. Suppose $b\chi(U) \in G$, so $b\chi(U) = g + zb$ with $z \neq 0$. Then $b\chi(U) = g + zb = g\chi(U) + g\chi(U') + zb\chi(U) + zb\chi(U')$. For $x \in U'$, this equation becomes $0 = g(x) + zb(x)$. This shows $g\chi(U') + zb\chi(U') = 0$. Since G is divisible, $z \neq 0$, and $g\chi(U') \in G$ by Proposition 1.5, we have $b\chi(U') \in G$. ∎

It is not difficult to construct an HA group such that $\langle G + \mathbb{Z} \cdot b \rangle \neq G + \mathbb{Z} \cdot b$; we omit this.

3. The P-groups. $\alpha\mathbb{N} = \mathbb{N} \cup \{\alpha\}$ denotes the one-point compactification of the discrete space $\mathbb{N}$. In this section we describe all the groups G with $S(\alpha\mathbb{N}, \mathbb{Z}) \leq_a G$. This is a generalization of

EXAMPLE 3.1. Let $C_{00} = \{f \in C(\alpha\mathbb{N}) \mid f$ vanishes on a neighborhood of $\alpha\}$. This is the weak product of countably many copies of $\mathbb{R}$, and $C_{00} \leq S(\alpha\mathbb{N})$. Let $b(n) = \pi + 1/n$, so $b \in C(\alpha\mathbb{N}) - S(\alpha\mathbb{N})$ and $b(\alpha) = \pi$. Write $\mathbb{R} = A \oplus \mathbb{Q}\pi$, as a direct sum of $\mathbb{Q}$-vector spaces.

(a) (P. Conrad [C2, 7.1]) Let $J = (C_{00} + \mathbb{Q}) + \mathbb{Q} \cdot b$. Then $S(\alpha\mathbb{N}, \mathbb{Z}) \leq_a J \not\subseteq S(\alpha\mathbb{N})$. Then J is HA, cannot be represented as an ℓ-group of step functions, and the vector lattice hull, vJ, is not HA.

(b) (M. Anderson and P. Conrad [AC, 4.1]) Let $K = (C_{00} + A) + \mathbb{Q} \cdot b$. Then $J \leq_a K$ and $K/M_p = \mathbb{R}$ for all $p \in \alpha\mathbb{N}$. So K is a-closed, thus an a-closure of J. But K is not a vector lattice. Note that, actually, [AC, p. 239] says "K is the a-closure of J". It can be seen from Proposition 3.5 below that J has $2^{\mathfrak{c}}$ a-closures.

Some of these assertions about J and K are not particularly obvious. Some, not all, will be shown below. But we emphasize what is going on here:

The signature of the hyperarchimedean situation, which is the basic theme of this paper, is the interplay between the locally constant functions and the non-locally constant functions. This takes a rather simple form for the groups in Example 3.1 above, and indeed for general G with $YG = \alpha\mathbb{N}$, as described in the rest of this section. For more general G, the interplay is highly visible, but more complicated and more difficult to describe. We quantify some of this at a rudimentary level:

For archimedean G with strong unit and $YG = \alpha\mathbb{N}$, let

$$L(G,\alpha) = \{\delta \in \mathbb{R} \mid \exists g \in G \ (g = \delta \text{ on a nbhd of } \alpha)\},$$

$$nL(G,\alpha) = \{\delta \in \mathbb{R} \mid \exists g \in G \ (g(\alpha) = \delta,\ g \text{ is not constant on any nbhd of } \alpha)\}.$$

Regarding Example 3.1, we have

$$J/M_\alpha = \mathbb{Q} \oplus \mathbb{Q}\cdot\pi, \qquad L(J,\alpha) = \mathbb{Q}, \qquad nL(J,\alpha) = J/M_\alpha - \mathbb{Q};$$
$$K/M_\alpha = A \oplus \mathbb{Q}\cdot\pi = \mathbb{R}, \qquad L(K,\alpha) = A, \qquad nL(J,\alpha) = \mathbb{R} - A.$$

Here we have $L(J,\alpha) \cap nL(J,\alpha) = \emptyset$, likewise for K, and this is why the groups are hyperarchimedean; and $K/M_\alpha = \mathbb{R}$, and this is why K is a-closed. These ideas will be articulated fully for $YG = \alpha\mathbb{N}$.

It will be helpful to keep these guidelines in mind as we proceed.

Suppose $S(\alpha\mathbb{N},\mathbb{Z}) \le S \le S(\alpha\mathbb{N})$, let $F = S \cap C_{00}$ and $A = S/M_\alpha$. For each $s \in S$ and $n \in \mathbb{N}$, we have $s\chi(\{n\}) \in S$. Thus, $S/M_n = F/M_n$ for each $n \in \mathbb{N}$, for each $\delta \in A$ there is $n(\delta)$ such that $\delta \in F/M_n$ for $n \ge n(\delta)$, and $S = F + A$,

We prefer to start with F and A as initial data.

Proposition 3.2. *Suppose $F \le C_{00}, 1 \in A \le \mathbb{R}$, and suppose that for each $\delta \in A$ there is $n(\delta)$ such that $\delta \in F/M_n$ for $n \ge n(\delta)$. Let $b \in C(\alpha\mathbb{N}) - S(\alpha\mathbb{N})$ with $b(n) \in F/M_n$ for each $n \in \mathbb{N}$. The following are equivalent*:

(a) $b(\alpha) \notin dA = \mathbb{Q}\cdot A$.
(b) *The sub-ℓ-group of $C(\alpha\mathbb{N})$ generated by $A + \mathbb{Z}\cdot b$ is HA.*
(c) $(F + A) + \mathbb{Z}\cdot b$ *(or $(\mathbb{Q}\cdot F + \mathbb{Q}A) + \mathbb{Q}\cdot b$) is a sub-$\ell$-group of $C(\alpha\mathbb{N})$ which is HA.*
(d) $(C_{00} + A) + \mathbb{Z}\cdot b$ *(or $(C_{00} + \mathbb{Q}\cdot A) + \mathbb{Q}\cdot b$) is a sub-$\ell$-group of $C(\alpha\mathbb{N})$ which is HA.*

Proof. (a)$\Rightarrow$(d). We use Corollary 2.6 with $G = C_{00} + A$. We first want to see that $Z(g + zb)$ is open for $g \in G$. This is obvious if $z = 0$. So suppose $z \neq 0$. Note that $g(\alpha) \in A$. Thus $g(\alpha) + zb(\alpha) \neq 0$ by (a), so $Z(g + zb)$ is a finite subset of $\mathbb{N}$, thus open in $\alpha\mathbb{N}$. Now using the second part of Corollary 2.6, consider $b\chi(U)$ for $U \in \text{clop}(\alpha\mathbb{N})$. Then $\alpha \notin U$ implies U is finite, so $b\chi(U) \in C_{00} \subseteq G$, and $\alpha \in U$ implies $\alpha\mathbb{N} - U$ is finite, so $b\chi(\alpha\mathbb{N} - U) \in C_{00} \subseteq G$. The parenthetical part of (d) follows from Proposition 2.1(c).

(d)⇒(c)⇒(b) follow by Proposition 2.1(a), using Proposition 2.1(c) for the parenthetical part of (c).

(b)⇒(a). Let $G = \langle A+\mathbb{Z}\cdot b\rangle \le C(\alpha\mathbb{N})$. By (b) and Theorem 2.3, $Z(\delta+zb)$ is open for each $z \in \mathbb{Z}$ and $\delta \in A$. As in (a)⇒(d), this means that if $z \neq 0$, then $\delta + zb(\alpha) \neq 0$, which says (a). ∎

Proposition 3.2 is a straightforward generalization of Example 3.1(a). We now use it to describe explicitly *all* the divisible HA groups, G, with strong unit for which $YG = \alpha\mathbb{N}$. The "divisible" restriction is simplifying, and from most points of view, without loss of generality.

So consider Proposition 3.2 supposing F and A are divisible, and let

$$P = (F + A) + \mathbb{Q}\cdot b$$

be the HA ℓ-group in Proposition 3.2(c). Then $P \cap C_{00} = F$, $P \cap S(\alpha\mathbb{N}) = F + A$, $P/M_n = F/M_n$ for each $n \in \mathbb{N}$, $P \cap S(\alpha\mathbb{N})/M_\alpha = A$ and $P/M_\alpha = A \oplus \mathbb{Q}\cdot b(\alpha)$ as a direct sum of $\mathbb{Q}$-vector spaces.

Construction 3.3. (a) Let F and A be as in Proposition 3.2 and divisible. Let $D \le \mathbb{R}$ be divisible and $\mathbb{Q}$-linearly independent of A, i.e., $D \cap A = \{0\}$. Let $\varphi : D \to C(\alpha\mathbb{N})$ be a $\mathbb{Q}$-module homomorphism for which

(i) $\varphi(D) \cap S(\alpha\mathbb{N}) = \{0\}$,
(ii) $\forall\delta \in D\ \forall n \in \mathbb{N}\ (\varphi(\delta)(n) \in F/M_n)$,
(iii) $\forall\delta \in D\ (\varphi(\delta)(\alpha) = \delta)$, which implies φ is one-to-one.

Let $P(F, A, (D, \varphi)) = (F + A) + \varphi(D)$.

(b) Before proceeding, we take note that plenty of these exist: Given F, A, D, let D_1 be a $\mathbb{Q}$-basis for D. Then let $\varphi_1 : D_1 \to C(\alpha\mathbb{N}) - S(\alpha\mathbb{N})$ be "constructed" like this: A and each F/M_n are topologically dense in $\mathbb{R}$, so given $\delta \in D_1$, there is a sequence $a_n \in A$ with $|\delta - a_n| < 1/n$. Let n_1 be the first integer such that $(n > n_1 \Rightarrow a_1 \in F/M_n)$, let n_2 be the first integer $> n_1$ such that $(n > n_2 \Rightarrow a_2 \in F/M_n)$, and so on. Now let $f_n = 1$ for $1 \le n \le n_1$, so these $f_n \in F/M_n$; let $f_n = a_n$ for $n_1 < n \le n_2$, so these $f_n \in F/M_n$, and so on.

Thus, $f_n \in F/M_n$ for all n, and $f_n \to \delta$. Define $\varphi_1(\delta)(n) = f_n$ and $\varphi_1(\delta)(\alpha) = \delta$, and now extend φ_1 to $\varphi : D \to C(\alpha\mathbb{N})$ by $\mathbb{Q}$-linearity. Clearly φ satisfies (ii) and (iii) above. Easy examples show that the preceding precautions are necessary to ensure (i), which we now show: Since φ is 1-1, (i) just means that for all $\delta \in D - \{0\}$, $\varphi(\delta)$ is constant on no neighborhood of α. But in fact there exists n with $\varphi(\delta)(\alpha) = \delta$ if and only if $\delta = 0$, since $\varphi(\delta)(n) \in A$ and A and D are $\mathbb{Q}$-linearly independent.

Theorem 3.4. (I) $P = P(F, A, (D, \varphi))$ *is a divisible hyperarchimedean* ℓ*-group with strong unit* 1 *and* $YP = \alpha\mathbb{N}$, *with the features*:

$P \cap C_{00} = F$ and $P \cap S(\alpha\mathbb{N}) = F + A$;
$P/M_n = F/M_n$ for all $n \in \mathbb{N}$ and $P \cap S(\alpha\mathbb{N})/M_\alpha = L(P,\alpha) = A$;
$P/M_\alpha = A \oplus D$ and $nL(P,\alpha) = P/M_\alpha - A = (D - \{0\}) + A$.

(II) *If G is any divisible hyperarchimedean ℓ-group with strong unit, and $YG = \alpha\mathbb{N}$, then there are F, A, D, φ for which, in the Yosida representation of G, we have $G = P(F, A, (D, \varphi))$.*

Note that in (II), given G, the sets F and A are determined by (I). But, by (I), D only needs to have $A \oplus D = G/M_\alpha$, and situations ($D_1 \neq D_2$ with $A \oplus D_1 = A \oplus D_2$) are common. Then of course, having fixed D, various φ are possible. See below.

Proof. Note that $P(F, A, (D, \varphi)) = \bigcup\{(F + A) + b \mid b \in \varphi(D)\}$.

(I) Clearly, $1 \in P \subseteq C(\alpha\mathbb{N})$ and P separates the points of $\alpha\mathbb{N}$. If $0 \neq b = \varphi(\delta)$ for some $\delta \in D$, then $\delta \neq 0$ and $\varphi(\delta)(\alpha) \notin A$, so Proposition 3.2(a) holds. Thus Proposition 3.2(c) holds, and $(F + A) + \mathbb{Q} \cdot b$ is an HA ℓ-group with Yosida space $\alpha\mathbb{N}$. Therefore, if $g_i = (f_i + a_i) + b_i \in P$ ($i = 1, 2$, $b_i \in \varphi(D)$, etc.) then $Z(g_1)$ is open by Theorem 2.3, and $g_1 \vee 0 \in (F + A) + \mathbb{Q} \cdot b_1 \subseteq P$. Also, if $b_i = \varphi(\delta_i)$, then since φ is $\mathbb{Q}$-linear, $b_0 = b_1 - b_2 = \varphi(\delta_1 - \delta_2)$, so that $g_1 - g_2 \in (F + A) + b_0 \subseteq P$. So P is an ℓ-group with open zero-sets, thus HA by Theorem 2.3. By Theorem 1.2, $YP = \alpha\mathbb{N}$, and the other claims for P are clear.

(II) Let G be given, let $F = G \cap C_{00}$ and $A = G \cap S(\alpha\mathbb{N})/M_\alpha = L(G, \alpha)$. Thus $nL(G, \alpha) = G/M_\alpha - A$.

Let D_1 be a subset of G/M_α maximal $\mathbb{Q}$-linearly independent of A, and let $\varphi_1 : D_1 \to G - S(\alpha\mathbb{N})$ be any choice function with $\varphi_1(\delta)(\alpha) = \delta$ for all $\delta \in D_1$. Let D be the $\mathbb{Q}$-linear span of D_1 and let $\varphi : D \to G$ be the extension by $\mathbb{Q}$-linearity. This satisfies Construction 3.3(a)(i); i.e., the technicalities in Construction 3.3(b) are not needed here: If $\varphi(\delta) \neq 0$ then $0 \neq \delta = \varphi(\delta)(\alpha) \in G/M_\alpha - A$.

So let $P = P(F, A, (D, \varphi))$. Obviously, $P \subseteq G$. For the reverse, let $g \in G$. Our choice of D makes $G/M_\alpha = A \oplus D$, so that $g(\alpha) = a + \delta$ for unique $a \in A$, $\delta \in D$. So there are $s \in G \cap S(\alpha\mathbb{N})$ and $b \in \varphi(D)$ with $a = s(\alpha)$ and $\delta = b(\alpha)$ and therefore $g(\alpha) = s(\alpha) + b(\alpha)$. Let $h = s + b \in (F + A) + b \subseteq P$. Since $h(\alpha) = g(\alpha)$, we have $h = g$ on a neighborhood U of α. Since $\alpha\mathbb{N} - U$ is finite, $g - h \in C_{00} \cap G = F$. Thus $g = (g - h) + h \in (F + A) + b \subseteq P$. ∎

It is easy to show the following.

Proposition 3.5. *Let $P_i = P(F_i, A_i, (D_i, \varphi_i))$ for $i = 1, 2$ be as in Construction 3.3 and Theorem 3.4. These are equivalent:*

(a) *$P_1 \leq P_2$, so $P_1 \leq_a P_2$.*
(b) *$F_1 \leq F_2$, $A_1 \leq A_2$, and $P_1/M_\alpha \leq P_2/M_\alpha$.*

(c) $F_1 \le F_2$, $A_1 \le A_2$, *and* $P_2 = P(F_2, A_2, (D'_2, \varphi'_2))$ *for some* $D'_2 \supseteq D_1$ *and* $\varphi'_2 \supseteq \varphi_1$.

Proposition 3.5 makes it obvious how to construct all a-extensions of an ℓ-group $P = P(F, A, (D, \varphi))$: enlarge F, A and D and then extend φ. When these are not possible, P is a-closed:

COROLLARY 3.6. *These are equivalent about* $P = P(F, A, (D, \varphi))$:

(a) P *is* a*-closed.*
(b) $F = C_{00}$ *and* $P/M_\alpha = \mathbb{R}$.
(c) $F = C_{00}$ *and for any* $\mathbb{Q}$*-bases* B *for* A *and* C *for* D, $B \cup C$ *is a* $\mathbb{Q}$*-basis for* $\mathbb{R}$, *i.e., a Hamel basis.*

Proof. (b)⇔(c) since $P/M_\alpha = A \oplus D$; (a)⇔(c) by inspection of Proposition 3.5. ■

In Corollary 3.6, (b)⇒(a) is a special case of a result from [AC]; (a)⇒(b) is also a consequence of topological properties of $\alpha\mathbb{N}$. Corollary 3.6 and Theorem 3.4(I) give the following.

COROLLARY 3.7. *Let* A *be a divisible subgroup of* $\mathbb{R}$ *with* $1 \in A$, *and let* D *be any* $\mathbb{Q}$*-vector space complement of* A *in* $\mathbb{R}$. *Then any* $P = P(C_{00}, A, (D, \varphi))$ *is* a*-closed and* $nL(P, \alpha) = \mathbb{R} - A$.

QUESTION 3.8 ([C2, p. 217, open question 4]). *Suppose that* G *is a sub-*ℓ*-group of a product* H *of reals which satisfies*

(∗) *if* $0 < g \in G$, *then* $r < g < s$ *for some* $0 < r, s \in \mathbb{R} \le H$.

Must the sub-ℓ*-ring of* H *generated by* G *be* HA?

In general, (∗) implies G is HA. If G has strong unit, HA implies (∗) for any such $G \le H$, as is easily seen from Sections 1 and 2 here.

The answer to Question 3.8 is "No": Let $P = P(C_{00}, \mathbb{Q}, (D, \varphi))$ with $D = \mathbb{Q} \cdot \sqrt{2}$. Here $P \le C(\alpha\mathbb{N}) \le \mathbb{R}^{\mathbb{N} \cup \{\alpha\}} = H$, and the sub-$\ell$-ring of H generated by P is the sub-ℓ-ring of $C(\alpha\mathbb{N})$ generated by P. Call this ϱP. By Theorem 2.3, ϱP is HA if and only if it has open zero-sets. But there is $b = \varphi(\sqrt{2}) \in P$ so $2 - b^2 \in \varrho P$, but $Z(2 - b^2)$ is not open since b^2 cannot be 2 on a neighborhood of α.

4. The Ψ-groups. As noted in the Abstract, the following progressively more pointed questions have been asked by Conrad, and Anderson and Conrad.

QUESTION 4.1. *Suppose that* G *is* a*-closed.*

(1) *Is* G/I a*-closed for every ideal* I?
(2) *Is* G/P a*-closed for every minimal prime* P?

(3) *If G is also HA, is G/M a-closed for every maximal M, i.e., is $G/M = \mathbb{R}$?*

Question (1) is from [C1, p. 153]; its converse is obviously true. Questions (2) and (3) are from [AC, p. 227]. The converse to (2) is Corollary I of [AC, p. 226], so the converse to (3) also holds.

We give the answer "No" to Question 4.1(3), so all these answers are "No", taking aim at the issue by analyzing, for (divisible) HA ℓ-groups with strong unit G, the mechanics of a-extendibility in terms of permissible enlargements of the G/M_p, one point p at a time. We make another construction to produce various a-closed G for which various G/M_p are various proper subgroups of $\mathbb{R}$.

CONSTRUCTION 4.2. Let X be an index set, and $\{G_x \mid x \in X\}$ a set of archimedean ℓ-groups with strong unit. Let $Y_x = YG_x$ for $x \in X$, and let $Y = \alpha \sum \{Y_x \mid x \in X\}$ be the one-point compactification of the disjoint union: $Y = \{\alpha\} \cup \sum Y_x$, in which $U \subseteq \sum Y_x$ is open if and only if each $U \cap Y_x$ is open in Y_x, and every neighborhood of α contains all but finitely many Y_x. Let $1 \in A \leq \mathbb{R}$, construed as constant functions on Y, or on any Y_x, and suppose that $A \leq G_x$ for each x for simplicity. $\Psi = \Psi(\{G_x\}, A)$ denotes the ℓ-subgroup of $C(Y)$ generated by the weak product $\prod^{\omega} G_x$ and the constant functions from A.

It is easy to see that, for $f \in C(Y)$, we have $f \in \Psi$ if and only if there is a finite set $F \subseteq X$ and $a \in A$ such that $f|Y_x \in G_x$ for $x \in F$ and f is constantly a on $Y - \sum\{Y_x \mid x \in F\}$. Note that, for $g \in G_x$, the group Ψ contains the function which is g on Y_x and 0 elsewhere; we denote this by $g\chi(Y_x)$. Clearly, for the divisible hull, $d\Psi = \Psi(\{dG_x\}, dA)$. Thus $\Psi = \prod^{\omega} G_x + A \leq C(Y)$ is archimedean with strong unit 1, $Y\Psi = Y$ by Theorem 1.2(c), and evidently: for $p \in Y_x$, $\Psi/M_p = G_x/M_p$, and $\Psi/M_\alpha = A$.

$\Psi(\{G_x\}, \mathbb{Z})$ is the "unital version" of $\prod^{\omega} G_x$.

Note that the Ψ-groups are a partial generalization of the P-groups of §3: $\Psi(\{\mathbb{R}_n\}, A) = P(\prod^{\omega} \mathbb{R}_n, A, (\emptyset, \emptyset))$. We could complete the generalization by adding a (D, φ) in the data for Ψ and we shall do that if a purpose develops.

Recall that a function $f \in C(X)$ is *locally constant at* $p \in X$ if there is a neighborhood of p on which f is constant. In the following definition, δ is always a real number. Recall also that given G and $p \in YG$ we have $G/M_p = \{\delta \mid \exists g \in G\ (g(p) = \delta)\}$. We then define the analogous sets:

$$L(G,p) = \{\delta \mid \exists g \in G \text{ locally constant at } p(g(p) = \delta)\},$$
$$nL(G,p) = \{\delta \mid \exists g \in G \text{ not locally constant at } p(g(p) = \delta)\}$$
$$nL(G) = \bigcup \{nL(G,p) \mid p \in YG\}.$$

Observe that $nL(\Psi) = \bigcup\{nL(G_x) \mid x \in X\}$.

We establish criteria for Ψ to be HA.

PROPOSITION 4.3. *$\Psi(\{G_x\}, A)$ is HA if and only if for each $x \in X$, G_x is HA and $nL(G_x) \cap A = \emptyset$.*

Proof. Suppose Ψ is HA. Restriction $\Psi \ni f \mapsto f|Y_x \in G_x$ is a surjective homomorphism, so G_x is HA. This implies $nL(G_x) \cap A = \emptyset$ since $A \subseteq L(G_x, p)$ for each $p \in Y_x$.

Conversely, if $g \in \Psi$, then $g = \sum\{g_x \chi(Y_x) \mid x \in F\} + a$ for a finite set F. If $a \neq 0$, then $Z(g) = \bigcup\{Z(g_x) \mid x \in F\}$ and this is open. If $a = 0$, then $Z(g) = \bigcup\{Z(g_x) \mid x \in F\} \cup (\bigcup\{Y_x \mid x \notin F\})$, which is open. By Corollary 2.6, Ψ is HA. ∎

The following outlines properties of a-extensions of Ψ, when it is HA, based on the action of adjoining certain values. Let

$$Ad(G,p) = \{\delta \mid \delta \notin G/M_p,\ \exists G \leq_a H\ (\delta \in H/M_p)\},$$
$$AdL(G,p) = \{\delta \mid \delta \notin G/M_p,\ \exists G \leq_a H\ (\delta \in L(H,p))\},$$
$$AdnL(G,p) = \{\delta \mid \delta \in G/M_p,\ \exists G \leq_a H\ (\delta \in nL(H,p))\}.$$

We now examine $Ad(G,p) = AdL(G,p) \cup AdnL(G,p)$, by examining the pieces separately.

THEOREM 4.4. *Let $p \in YG$, and let δ be a real number such that $\delta \notin G/M_p$. These are equivalent:*

(a) *$\delta \in AdL(G,p)$.*
(b) *There is $U \in \operatorname{clop}(YG)$ with $p \in U$, and there is $G \leq_a H$ with $\delta\chi(U) \in H$.*
(c) *There is $U \in \operatorname{clop}(YG)$ with $p \in U$ for which $\delta \notin nL(G,x)$ for each $x \in U$, i.e., $U \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$.*

Proof. (a)⇔(b) follows by noticing that, *via* Corollary 1.4, we get: $\delta \in L(G,p)$ if and only if there is $U \in \operatorname{clop}(YG)$ with $p \in U$, and $\delta\chi(U) \in G$.

(a)⇒(c). If $G \leq_a H$ and $\delta \in L(H,p)$, then $\delta \notin nL(H,p) \supseteq nL(G,p)$ by the note above.

(c)⇒(b). Assuming (b), let $H = \langle G + \mathbb{Z} \cdot \delta\chi(U)\rangle$. We use Corollary 2.6 to see that $G \leq_a H$. Consider $Z = Z(g + z\delta\chi(U))$. We have $Z = (Z \cap U) \cup (Z \cap (YG - U))$, and we show that each piece is open. The second, $Z \cap (YG - U) = Z(g) \cap (YG - U)$, is open since $Z(g)$ is open since G is HA, and U is closed. Now $Z \cap U = \{x \in U \mid g(x) = -z\delta\}$. So, if $x \in Z \cap U$, then g is locally constant at x since $\delta \notin nL(G,x)$ and G is divisible, and there a is clopen $V \ni x$ with $g = -z\delta$ on V, so $V \subseteq Z \cap U$. Thus $Z \cap U$ is open. ∎

LEMMA 4.5. *Suppose $\Psi = \Psi(\{G_x\}, A)$ is HA. If $\Psi \leq_a H$, then for each x, $G_x \leq_a H|Y_x$. Conversely, if $G_x \leq_a H_x$ for each x, then $\Psi(\{G_x\}, A) \leq_a \Psi(\{H_x\}, A)$.*

Proof. The notation makes sense since $YH_x = YG_x$ for each x. The first assertion is because $G \leq_a H$ implies $G/I \cap G \leq_a H/I$ for each ideal I. Conversely, if $h = \sum\{h_x\chi(Y_x) \mid x \in F\} + a$, and for each $x \in F$ we have $g_x \sim_a h_x$, then $\sum\{g_x|\chi(Y_x) \mid x \in F\} \sim_a h$. ■

PROPOSITION 4.6. (a) *Let G be HA. Then G is a-closed if and only if $Ad(G,p) = \emptyset$ for each $p \in YG$.*

(b) *Suppose $\Psi = \Psi(\{G_x\}, A)$ is HA. Then Ψ is a-closed if and only if each G_x is a-closed and $Ad(\Psi, \alpha) = \emptyset$.*

Proof. (a) "⇒" is clear. Conversely, let $G \leq_a H$ and $h \in H$. Since all $Ad(G,p)$ are empty, for each p there is $g_p \in G$ with $g_p(p) = h(p)$. Since H is HA, $h - g_p$ is constant on a clopen U_p containing p, which means $h\chi(U_p) = g\chi(U_p) \in G$, by Theorem 1.3. By compactness, there is a finite F with $\bigcup\{U_p \mid p \in F\} = YG$. Then $h = \bigvee_{p\in F}$ and $g\chi(U_p) \in G$.

(b) By (a), Ψ is a-closed if and only if $Ad(\Psi, p) = \emptyset$ for each $p \in Y$. For $p \in Y_x$, we have $Ad(\Psi, p) = Ad(G_x, p)$ using Lemma 4.5. The result follows. ■

We need the following well known lemma:

LEMMA 4.7. *X be compact zero-dimensional. For $Z \subseteq X$, the following are equivalent*: *$Z \in \mathcal{Z}C(X)$*; *Z is a closed G_δ-set*; *Z is a countable intersection of clopen sets.*

THEOREM 4.8. *Given $G, p \in YG$, and real $\delta \notin G/M_p$*:

(I) *If $\delta \in AdnL(G,p)$, then there is a zero-set Z with $p \in \partial Z$ with* $\operatorname{int} Z \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$ *and* $\partial Z \cap \{x \mid \delta \in L(G,x)\} = \emptyset$.

(II) *If there is a zero-set Z with $p \in \partial Z$ with* $\operatorname{int} Z \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$ *and* $\partial Z \cap \{x \mid \delta \in G/M_x\} = \emptyset$, *then $\delta \in AdnL(G,p)$.*

Proof. (I) Suppose $\delta \in AdnL(G,p)$, so $\delta \notin G/M_p$ and there is $G \leq_a H$ with $h \in H$ for which $h(p) = \delta$ and h is not locally constant at p. Let $Z = \{x \mid h(x) = \delta\}$. This is a zero-set and clearly for $x \in Z$:

(a) $x \in \operatorname{int} Z$ if and only if h is locally constant at x and then $\delta \in L(H,x)$;

(b) $x \in \partial Z$ if and only if h is not locally constant at x and then $\delta \in nL(H,x)$.

By (b), $p \in \partial Z$. Since $nL(G,x) \subseteq nL(H,x)$ and $nL(H,x) \cap L(H,x) = \emptyset$, we see that $\operatorname{int} Z \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$ from (a). Since $L(G,x) \subseteq L(H,x)$ and $nL(H,x) \cap L(H,x) = \emptyset$, we get $\partial Z \cap \{x \mid \delta \in L(G,x)\} = \emptyset$ from (b).

(II) Let Z be as described. By Lemma 4.7, $YG - Z = \bigcup_n U_n$ for $U_n \in \operatorname{clop}(YG)$. Let $K_n = U_n - \bigcup_{i<n} U_i$. These are disjoint and clopen, $\bigcup K_n = YG - Z$ and $YG = \bigcup K_n \cup Z$. Let (r_n) be a sequence of rational numbers with $r_n \to \delta$. Define $b \in C(YG)$ as $b(x) = r_n$ for $x \in K_n$ and $b(x) = \delta$ for $x \in Z$. Then $Z = \{x \mid b(x) = \delta\}$, and again for $x \in Z$:

(a) $x \in \operatorname{int} Z$ if and only if b is locally constant at x;
(b) $x \in \partial Z$ if and only if b is not locally constant at x.

Let $H = \langle G + \mathbb{Z} \cdot b \rangle$. We shall use Corollary 2.6 to show that H is HA so that $G \leq_a H$. It follows that $\delta \in nL(H,p)$, so $\delta \in AdnL(G,p)$.

Let $g \in G$ and $z \in \mathbb{Z}$. We show that $E = Z(g + zb)$ is open. Now $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \bigcup (K_n \cup E)$, $E_2 = \operatorname{int} Z \cap E$, and $E_3 = \partial Z \cap E$. The set E_1 is open since $K_n \cap E = K_n \cap \{x \mid g(x) = -zr_n\}$ is open. E_2 is open: since $\operatorname{int} Z \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$, it is also the case that $\operatorname{int} Z \cap \{x \mid -z\delta \in nL(G,x)\} = \emptyset$ since G is divisible. Thus, if $x \in E_2$ then there is a neighborhood V of x on which $g = -z\delta$ and $x \in V \cap \operatorname{int} Z \subseteq E_2$. Finally, $E_3 = \emptyset$ since $\partial Z \cap \{x \mid \delta \in G/M_x\} = \emptyset$ and hence $\partial Z \cap \{x \mid -z\delta \in G/M_x\} = \emptyset$ since G is divisible. ■

It is not difficult to construct examples illustrating the gap between the conditions in Theorem 4.8(I) and (II). We omit this.

So we can focus on $Ad(\Psi, \alpha)$. Keeping Theorem 4.4 in mind, note that if $\alpha \in U \subseteq Y$, then U contains a clopen set containing α if and only if there is a finite F such that $U \supseteq \bigcup \{Y_x \mid x \notin F\}$; U contains a zero-set containing α if and only if there is countable F with $U \supseteq \bigcup \{Y_x \mid x \notin F\}$ by Lemma 4.7.

Proposition 4.9. *Suppose $\Psi = \Psi(\{G_x\}, A)$ is HA with X uncountable.*

(a) *$\delta \in AdL(\Psi, \alpha)$ if and only if there is finite a $F \subseteq X$ for which $x \notin F$ implies $\delta \notin nL(G_x)$.*
(b) *$\delta \in AdnL(\Psi, \alpha)$ if and only if there is a countable $F \subseteq X$ for which $x \notin F$ implies $\delta \notin nL(G_x)$.*

Proof. (a) $nL(\Psi, \alpha) = \emptyset$ and for $p \in Y_x$, $nL(\Psi, p) = nL(G_x, p)$. So the condition says that $U = \bigcup \{Y_x \mid x \notin F\}$ satisfies Theorem 4.4(c).

(b) We make the following obvious, but useful, observation: If G is HA and $T \subseteq YG$ is closed, then $\operatorname{int} T \cap \{x \mid \delta \in nL(G,x)\} = \emptyset$ if and only if $T \cap \{x \mid \delta \in nL(G,x)\} \subseteq \partial T$ and $\partial T \cap \{x \mid \delta \in L(G,x)\} = \emptyset$ if and only if $T \cap \{x \mid \delta \in L(G,x)\} \subseteq \operatorname{int} T$.

Consider this observation for Z a zero-set of Y containing α. Here $\partial Z = \{\alpha\}$, $L(\Psi, \alpha) = A$ and $nL(\Psi, \alpha) = \emptyset$. So for $\delta \notin A$, we see that $\partial Z \cap \{x \mid \delta \in \Psi/M_x\} = \emptyset$, thus $\partial Z \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$ and the conditions in the two parts of the observation each reduce to: $\operatorname{int} Z \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$. The condition in (b) says that for $Z_0 = \bigcup \{Y_x \mid x \notin F\}$ we have $Z_0 \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$. Now apply Theorem 4.8. ■

Corollary 4.10. *Let $\Psi = \Psi(\{G_x\}, A)$ be HA with X uncountable. Then $Ad(\Psi, \alpha) = \emptyset$ if and only if for each $\delta \notin A$ [for each countable $F \subseteq X$, there is $x \notin F$ with $\delta \in nL(G_x)$].*

Proof. The condition bracketed is the negation of the condition in Proposition 4.9(b), which implies the negation of the condition in Proposition 4.9(a). ■

COROLLARY 4.11. *Let G_0 be HA, let X be uncountable and for each $x \in X$ let $G_x = G_0$. Let $\Psi = \Psi(\{G_x\}, A)$. Then Ψ is HA and $Ad(\Psi, \alpha) = \emptyset$ if and only if $\mathbb{R} - A = nL(G_0)$.*

Such G_0 are given in Corollary 3.7.

EXAMPLE 4.12. Let A be any divisible subgroup of $\mathbb{R}$ containing 1. There is an a-closed Ψ_A that is HA with strong unit, with a maximal ideal M for which $\Psi_A/M = A$.

Such a Ψ_A is a Ψ in Corollary 4.11, with $M = M_\alpha$ using $G_0 = P(\{C_{00}\}, A, (D, \varphi)) \leq C(\alpha\mathbb{N})$ from §3, with a φ for which $A \oplus D = \mathbb{R}$. Now G_0 is a-closed by Corollary 3.6, so Ψ_A is a-closed by Proposition 4.6(b) and Corollary 4.11. As noted in Corollary 3.7, $nL(G_0, \alpha_0) = \mathbb{R} - A$ (writing $\alpha\mathbb{N} = \mathbb{N} \cup \{\alpha_0\}$ to avoid confusion).

For Ψ_A in Example 4.12, $\Psi_A/M_p = \mathbb{R}$ for every $p \neq \alpha$, and $\Psi_A/M_\alpha = A$. Of course, more complicated situations can be constructed. We content ourselves with just one more level of complexity.

EXAMPLE 4.13. Let A be any divisible subgroup of $\mathbb{R}$ with $1 \in A$ and let m be any uncountable cardinal number. Let $\alpha D(m)$ be the one-point compactification of the discrete space of cardinal m. There is an a-closed $\Psi = \Psi_{A,m}$, that is HA with strong unit, for which $Y\Psi$ contains a copy of $\alpha D(m)$ as a nowhere dense subset with $\Psi/M_p = A$ for each $p \in \alpha D(m)$:

Let $\Psi = \Psi(\{H_i \mid i < m\}, A)$ with each H_i given by the Ψ_A of Example 4.12. Here, $nL(H_i) = \mathbb{R} - A$ and since $nL(\Psi_A) = \mathbb{R} - A$, we see that Ψ is a-closed by Proposition 4.6(b) and Corollary 4.10. We have $Y\Psi = \{\alpha\} \cup \sum YH_i$ and $\Psi/M_\alpha = A$. Also $YH_i = \{\alpha_i\} \cup \sum(\cdot)$ and $H_i/M_{\alpha_i} = A$. The desired copy of $\alpha D(m)$ is $\{\alpha\} \cup \{\alpha_i \mid i < m\}$.

With reference to "nowhere dense", it is easy to show that: there is an a-closed G for which $\{p \in YG \mid G/M_p \neq \mathbb{R}\}$ has interior if and only if there is a-closed H for which $\{p \in YH \mid H/M_p = \mathbb{R}\} = \emptyset$ where G and H are HA with strong unit. We do not know if such G, H exist.

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Department of Mathematics and Computer Science
Wesleyan University
Middletown, CT 06459, U.S.A.
E-mail: ahager@wesleyan.edu

Department of Mathematics
Lafayette College
Easton, PA 18042, U.S.A.
E-mail: kimberc@lafayette.edu

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