Some examples of hyperarchimedean lattice-ordered groups

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Abstract. All $\ell$-groups shall be abelian. An $a$-extension of an $\ell$-group is an extension preserving the lattice of ideals; an $\ell$-group with no proper $a$-extension is called $a$-closed. A hyperarchimedean $\ell$-group is one for which each quotient is archimedean. This paper examines hyperarchimedean $\ell$-groups with unit and their $a$-extensions by means of the Yosida representation, focussing on several previously open problems. Paul Conrad asked in 1965: If $G$ is $a$-closed and $M$ is an ideal, is $G/M$ $a$-closed? And in 1972: If $G$ is a hyperarchimedean sub-$\ell$-group of a product of reals, is the $f$-ring which $G$ generates also hyperarchimedean? Marlow Anderson and Conrad asked in 1978 (refining the first question above): If $G$ is $a$-closed and $M$ is a minimal prime, is $G/M$ $a$-closed? If $G$ is $a$-closed and hyperarchimedean and $M$ is a prime, is $G/M$ isomorphic to the reals? Here, we introduce some techniques of $a$-extension and construct a several parameter family of examples. Adjusting the parameters provides answers “No” to the questions above.

1. Preliminaries. In the following, a lattice-ordered group, or an $\ell$-group, is an abelian group $(G,+)$ with a lattice order $\leq$ for which $a \leq b$ implies $a + c \leq b + c$ for all $c$. Moreover, $G^+ = \{g \in G \mid g \geq 0\}$ is the positive cone of $G$. We shall use the references [AF], [BKW], [D], and [LZ] for various aspects of $\ell$-group theory. We sketch some particular ideas which we need here.

An ideal in an $\ell$-group is a convex sub-$\ell$-group; these are the kernels of $\ell$-homomorphisms. The collection of all ideals in the $\ell$-group $G$ is denoted Idl$(G)$. Let $G$ be a sub-$\ell$-group of $H$; we write $G \leq H$. When the contraction map $\text{Idl}(H) \rightarrow \text{Idl}(G)$ is one-to-one and onto, $H$ is called an $a$-extension of $G$, and we write $G \leq_a H$.

PROPOSITION 1.1 ([C1, 2.1]). $G \leq_a H$ if and only if $G \leq H$ and for each $h \in H^+$ there is $g \in G^+$ such that $g \sim_a h$, i.e., $h \leq mg$ and $g \leq nh$ for some positive integers $m, n$.  

2000 Mathematics Subject Classification: 06F20, 54C40.
If $G \leq_a H$ implies $G = H$, then $G$ is called $a$-closed. When $G \leq_a H$ and $H$ is $a$-closed, $H$ is called an $a$-closure of $G$. The first systematic study of $a$-extensions is Conrad’s [C1]. A large literature has developed subsequently on this complicated subject; see especially [AF] and [D]. Two main themes, which this paper continues, are: non-uniqueness of $a$-closure, even in quite restricted contexts; what does “$a$-closed” mean?

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the natural numbers, the integers, the rational numbers, and the reals, respectively, as one sort of mathematical structure or another, which being clear from context.

Let $G$ be an $\ell$-group, and $u \in G^+$. An element $g \in G$ is infinitesimal with respect to $h$, written $g \ll h$, if $0 \leq ng \leq h$ for all $n \in \mathbb{N}$. $G$ is archimedean if $g \ll h$ implies $g = 0$. A positive element $u$ is a weak unit in $G$ if $|g| \land u = 0$ implies $g = 0$; and $u$ is a strong unit if $\langle u \rangle = G$, where $\langle u \rangle = \{g \in G \mid \exists n \in \mathbb{N} \ (|g| \leq nu)\}$ denotes the ideal in $G$ generated by $u$.

To illustrate: $\mathbb{R}$ is archimedean, and the class of archimedean $\ell$-groups is closed under product and sub-$\ell$-group formation. So, whenever $X$ is a topological space, which we always assume to be completely regular and Hausdorff (see [GJ]), the $\ell$-group $C(X)$ of continuous $\mathbb{R}$-valued functions is archimedean. A function $u \in C(X)^+$ is a weak unit if and only if $\text{coz}(u) = \{x \in X \mid u(x) \neq 0\}$ is dense in $X$; so the constant function $1$ is a weak unit. For compact $X$, $1$ is a strong unit since each $f \in C(X)$ is bounded.

The following is the classical first representation theorem of Yosida [Y], elaborated and augmented somewhat. Many useful variations and generalizations are discussed in [AF], [LZ], [HR] and [BH].

**Theorem 1.2 (Representation of objects).** Let $G$ be an archimedean $\ell$-group with strong unit $u$.

(a) The set of maximal ideals of $G$, with the hull-kernel topology, is a compact Hausdorff space, denoted $Y(G, u)$ or just $YG$.

(b) There is an $\ell$-isomorphism $G \cong \hat{G} \leq C(YG)$ for which $\hat{u} = 1$ and $\hat{G}$ separates the points of $YG$.

(c) If $G \cong \hat{G} \leq C(X)$ is an $\ell$-isomorphism with $X$ compact Hausdorff and $\hat{u} = 1$, then there is a continuous surjection $\tau : X \to YG$ for which $\hat{g} = \hat{g} \circ \tau$ for each $g \in G$. The group $\hat{G}$ separates points of $X$ if and only if $\tau$ is a homeomorphism.

(d) For each $p \in YG$, $M_p = \{g \mid \hat{g}(p) = 0\}$ is a maximal ideal, and each maximal ideal is of this form for unique $p$. For each $p$, we have $\hat{G}/M_p \approx \{\hat{g}(p) \mid g \in G\} \leq \mathbb{R}$.

**Theorem 1.3 (Representation of morphisms).** Let $(G, u)$ and $(H, v)$ be archimedean $\ell$-groups with strong units $u, v$, and let $\varphi : G \to H$ be an $\ell$-homomorphism with $\varphi(u) = v$. Define a function $Y\varphi : YG \leftarrow YH$ by: $(Y\varphi)(p) = q$ means $\varphi^{-1}(M_q) = M_p$. Then $Y\varphi$ is continuous and is the
unique map for which \( \tilde{\varphi}(g) = \tilde{g} \circ (Y \varphi) \) for each \( g \in G \). Further, \( \varphi \) is one-to-one if and only if \( Y \varphi \) is onto; if \( \varphi \) is onto, then \( Y \varphi \) is one-to-one.

**Corollary 1.4.** Suppose \( G \) is archimedean with strong unit \( u \), and \( G \leq _a H \). Then \( H \) is archimedean, \( u \) is a strong unit in \( H \), and \( YH = YG \).

For the rest of the paper, we shall identify each archimedean \( \ell \)-group with strong unit with its Yosida representation: \( 1 \in G \leq C(YG) \), and that notation means “\( G \) is an archimedean \( \ell \)-group with strong unit in its Yosida representation”.

For \( X \) a topological space, \( \text{clop}(X) \) denotes the Boolean algebra of clopen subsets of \( X \). For \( U \subseteq X \), \( \chi(U) \) denotes the characteristic function of \( U \) and \( g \chi(U) \) stands for the function on \( YG \) which vanishes off \( U \) and agrees with \( g \) on \( U \). The expression “\( g \chi(U) \in G \)” means there is \( h \in G \) which in the Yosida representation is \( g \chi(U) \). The following is simple but crucial.

**Proposition 1.5.** Suppose that \( 1 \in G \leq C(YG) \). Then \( g \chi(U) \in G \) for any \( U \in \text{clop}(YG) \) and \( g \in G \).

**Proof.** We first show that \( \chi(U) \in G \): Let \( \{ h < a \} \) stand for \( \{ x \in YG \mid h(x) < a \} \), and let \( U' = YG - U \). For each \( p \in U \) and each \( q \in U' \), choose \( g_{pq} \in G \) which is 2 at \( p \) and \( -1 \) at \( q \). (From Theorem 1.2, there is \( g \in G \) with \( g(p) = 1 \) and \( g(q) = 0 \). Let \( g_{pq} = 3g - 1 \).) Fixing \( p \), the set \( \{ g_{pq} < 0 \mid q \in U' \} \) covers \( U' \), so for some finite \( E \subseteq U' \), the set \( \{ g_{pq} < 0 \mid q \in E \} \) covers \( U' \). Then \( g_p = (\bigwedge_{q \in E} g_{pq}) \vee 0 \) is 2 at \( p \) and 0 on \( U' \). Now \( \{ g_p > 1 \mid p \in U \} \) covers \( U \), so for some finite \( F \subseteq U \), the set \( \{ g_p > 1 \mid p \in F \} \) covers \( U \). Then \( \chi(U) = (\bigvee_{p \in F} g_p) \land 1 \in G \).

If \( g \geq 0 \), there is \( n \) with \( g \leq n \), and then \( g \chi(U) = g \wedge n \chi(U) \in G \). Finally, \( g \chi(U) = (g \vee 0) \chi(U) - ((-g) \vee 0) \chi(U) \in G \). \( \blacksquare \)

**2. Hyperarchimedean \( \ell \)-groups with unit.** A hyperarchimedean \( \ell \)-group is one for which each quotient is archimedean. We abbreviate “hyperarchimedean” to “\( HA \)”. In this section we indicate some basic features of \( HA \) \( \ell \)-groups.

For \( G \) an abelian \( \ell \)-group, \( dG \) denotes the divisible hull. This is the unique divisible \( \ell \)-group containing \( G \) for which \( b \in dG \) implies there exists \( a \in G \) for which \( nb = a \) for some \( n \in \mathbb{N} \). For \( G \) archimedean with strong unit, viewed per Theorem 1.2 as \( 1 \in G \leq C(YG) \), it is easily seen that \( \{ rg \mid g \in G, \ r \in \mathbb{Q} \} \leq C(YG) \) is an explicit presentation of \( dG \).

**Proposition 2.1 ([C2]).** (a) If \( H \) is \( HA \) and \( G \leq H \), then \( G \) is \( HA \).
(b) If \( G \) is \( HA \) and \( G \leq _a H \), then \( H \) is \( HA \).
(c) If \( G \) is \( HA \), then \( dG \) is \( HA \).
(d) If \( G \) is \( HA \), then any weak unit in \( G \) is strong.

We turn to a discussion of the central role of “zero-sets”.
For \( f \in \mathbb{R}^X \), the zero-set of \( f \) is \( \text{Z}(f) = f^{-1}([0]) \) and the cozero-set of \( f \) is \( \text{coz}(f) = X - \text{Z}(f) \). If \( f \in C(X) \), then \( \text{Z}(f) \) is closed and \( \text{coz}(f) \) is open. For \( 1 \leq G \leq C(YG) \), let \( ZG = \{ Z(g) \mid g \in G \} \); note that Theorem 1.3 says \( \text{clop}(YG) \subseteq Z(G) \). For \( X \) a space, \( S(X) = \{ f \in C(X) \mid \text{range } f \text{ is finite} \} \subseteq C(X) \), and \( S(X, \mathbb{Z}) = \{ f \in S(X) \mid \text{range } f \subseteq \mathbb{Z} \} \subseteq S(X) \). Note that \( \mathbb{Z}S(X) = \mathbb{Z}S(X, \mathbb{Z}) = \text{clop}(X) \).

Let \( G \) be an \( \ell \)-group and \( g \in G^+ \). A component of \( g \) is an \( h \in G \) for which \( h \wedge (g - h) = 0 \). The following is immediate.

**Proposition 2.2.** Let \( 1 \leq G \leq C(YG) \), and let \( h \in G \). Then \( h \) is a component of \( 1 \) if and only if there is \( U \in \text{clop}(YG) \) for which \( h = \chi(U) \).

In view of [BKW], Theorem 1.2 and Proposition 2.1 we obtain the following characterization of HA groups.

**Theorem 2.3.** Let \( G \) be an abelian \( \ell \)-group with weak unit \( u \). The following are equivalent:

(a) \( G \) is HA.

(b) \( G \) is archimedean, \( u \) is a strong unit, and in the Yosida representation \( ZG = \text{clop}(YG) \).

(c) \( G \) is archimedean, \( u \) is a strong unit, and for some representation \( G \cong \tilde{G} \leq C(X) \) with \( X \) compact Hausdorff, \( \tilde{u} = 1 \), and \( Z\tilde{G} \subseteq \text{clop}(X) \).

(d) \( u \) is a strong unit, and for each \( g \in G^+ \) there is a pair \((\chi, n)\), where \( \chi \) is a component of \( u \) and \( n \in \mathbb{N} \), with \( ng \geq \chi \) and \( g \wedge (u - \chi) = 0 \).

(e) \( G \) is archimedean, and in the Yosida representation \( S(YG, \mathbb{Z}) \leq_a G \).

**Corollary 2.4.** (a) If \( G \) is HA with strong unit, then \( YG \) is zero-dimensional (i.e., \( \text{clop}(YG) \) is a base for the open sets).

(b) For \( X \) a space, \( S(X) \) is HA.

(c) Let \( G \) be an \( \ell \)-group. Then \( G \) is HA with strong unit if and only if there is a space \( Y \) with \( S(Y) \leq_a G \).

(d) For \( X \) a space, \( C(X) \) is HA if and only if the sub-\( \ell \)-group of bounded functions \( C^*(X) \) is HA if and only if \( X \) is finite.

(e) If \( G \) is HA with strong unit, then: \( G \) is a-closed if and only if \( (G \leq_a H \leq C(YG) \) implies \( H = G \)).

Some of Theorem 2.3 generalizes as follows. First, if \( G \) is archimedean with weak unit, in its Yosida representation (see [HR]), and \( g \in G^+ \) then \( Z(g) \) is open if and only if \( g \) has a \((\chi, n)\) per Theorem 2.3(c). Second, for \( G \) abelian with weak unit \( u \), and \( G^* = \langle u \rangle \), these are equivalent: \( G \) is archimedean and \( G^* \) is HA; \( G \) is archimedean and \( ZG = \text{clop}(YG) \); each
$g \in G^+$ has a $(\chi, n)$. These $\ell$-groups are called bounded away, and were introduced in the preprint [KM], and are discussed in [HKM].

This section concludes with the following useful results which show that if an $\ell$-group $G$ has $\ell$-group generators which seem hyperarchimedean, then $G$ actually is hyperarchimedean.

Let $H$ be an $\ell$-group, let $B$ be a subgroup of $H$, and let $\langle B \rangle$ denote the sub-$\ell$-group of $H$ generated by $B$. The elements of $\langle B \rangle$ are the elements of $H$ of the form $\bigwedge_k \bigvee_j b_{jk}$, for $b_{jk} \in B$ and finite index sets. (See [BKW].) The usual problem with analyzing $\langle B \rangle$ is the inscrutability of the expressions $\bigwedge \bigvee b_{jk}$.

PROPOSITION 2.5. Suppose $X$ is compact and zero-dimensional, and that $B$ is a point-separating subgroup of $C(X)$ with $1 \in B$ and $Z(b)$ open for all $b \in B$. Then $\langle B \rangle$ is $HA$ and $\langle B \rangle = \{ \sum_{i \in I} b_i(\chi(U_i)) \mid b_i \in B, U_i \in \text{clop}(X); I \text{ finite} \}$.

COROLLARY 2.6. Suppose $G$ is archimedean with strong unit: $1 \in G \leq C(YG)$, and suppose $b \in C(YG)$ has $Z(g + zb)$ open for each $g \in G$, $z \in \mathbb{Z}$. Then

$$\langle G + \mathbb{Z} \cdot b \rangle = G + \left\{ \sum_{i \in I} z_i b_i(\chi(U_i)) \mid z_i \in \mathbb{Z}, U_i \in \text{clop}(YG); I \text{ finite} \right\}.$$ 

$\langle G + \mathbb{Z} \cdot b \rangle$ is $HA$ and hence $G \leq \langle G + \mathbb{Z} \cdot b \rangle$. Moreover, if for each $U \in \text{clop}(YG)$, one of $b \chi(U), b \chi(YG - U)$ is in $G$, then $\langle G + \mathbb{Z} \cdot b \rangle = G + \mathbb{Z} \cdot b$; when $G$ is divisible, the converse holds.

The following lemma is necessary.

LEMMA 2.7. If $I$ is finite and $h = \sum \{ b_i(\chi(U_i)) \mid i \in I \}$, then for a finite set $\mathcal{E}$, there is a rewriting $h = \sum \{ b_E \chi(\mathcal{V}_E) \mid E \in \mathcal{E} \}$ with the sets $\mathcal{V}_E$ disjoint and non-empty and $X = \bigcup \{\mathcal{V}_E \mid E \in \mathcal{E} \}$.

Proof. For $E \subseteq I$, let $V_E = \bigcap \{U_i \mid i \in E\} \cap \bigcap \{X - U_i \mid i \notin E\}$, and let $\mathcal{E} = \{E \mid E \subseteq I, V_E \neq \emptyset\}$. Note that $V_{\emptyset} = X - \bigcup \{U_i \mid i \in E\}$. For $\emptyset \neq E \in \mathcal{E}$, let $b_E = \sum \{ b_i \mid i \in E \}$, and set $b_\emptyset = 0$. If $E \neq F$, say $i \in E$ but $i \notin F$, then $V_E \subseteq U_i$ while $V_F \subseteq X - U_i$, so $V_E \cap V_F = \emptyset$. For $p \in X$, let $E(p) = \{ i \in I \mid p \in U_i \}$. Then $V_{E(p)} \neq \emptyset$, and so $E(p) \in \mathcal{E}$, and

$$h(p) = \sum \{ b_i(p) \mid i \in E(p) \} = b_{E(p)} \chi(V_{E(p)})(p) = \sum \{ b_E \chi(\mathcal{V}_E) \mid E \in \mathcal{E} \}(p)$$

since the $V_E$‘s are disjoint. 

Proof of Proposition 2.5. Of course, $\langle B \rangle$ is archimedean with strong unit 1; by Theorem 1.2, $Y \langle B \rangle = X$ and the presentation of $\langle B \rangle$ is the Yosida representation. Let $B_0$ be the set of expressions $\sum b_i \chi(U_i)$. By Proposition 1.5, each $b \chi(u) \in \langle B \rangle$, so that $B_0 \subseteq \langle B \rangle$, and $B_0$ clearly is a group.
We shall show that $Z(h)$ is open for $h \in B_0$ and then that $B_0$ is an $\ell$-group. So $B_0 = \langle B \rangle$, and by Theorem 2.3, $\langle B \rangle$ is $HA$.

Now let $B_0 \ni h = \sum b E \chi(V_E)$ be written as in Lemma 2.7. Then $Z(h) = \bigcup \{Z(h) \cap V_E\} = \bigcup \{Z(b E) \cap V_E\}$, by disjointness. Since each $Z(b E)$ is supposed open, and each $V_E$ is open, $Z(h)$ is open.

For $B_0$ to be an $\ell$-group, it suffices that $h \in B_0$ imply $h \lor 0 \in B_0$ (see [BKW]). Again write $h = \sum b E \chi(V_E)$ as in Lemma 2.7. Disjointness implies $h \lor 0 = \sum (b E \chi(V_E) \lor 0)$ and $b E \chi(V_E) \lor 0 = (b \lor 0) \chi(E)$. So $h \lor 0 \in B_0$. $\blacksquare$

**Proof of Corollary 2.6.** For the first part, just apply Proposition 2.5 to $B = G + \mathbb{Z} \cdot b$. For the second part: Let $U' = YG - U$. Suppose one of $b \chi(U), b \chi(U')$ is in $G$ for all $U \in \text{clop}(YG)$. For all $U$, $b \chi(U) = g + z \cdot b$ for some $g \in G$, $z \in \mathbb{Z}$. If $b \chi(U) \in G$, use $z = 0$. If $b \chi(U') \in G$, then $B = b \chi(U) + b \chi(U')$, and we can use $z = -1$. This shows that any expression $g + \sum z_i b \chi(U_i)$ actually lies in $G + \mathbb{Z} \cdot b$. Conversely, suppose $\langle G + \mathbb{Z} \cdot b \rangle = G + \mathbb{Z} \cdot b$, and $U \in \text{clop}(YG)$. Suppose $b \chi(U) \in G$, so $b \chi(U) = g + zb$ with $z \neq 0$. Then $b \chi(U) = g + zb = g \chi(U) + z b \chi(U') + zb \chi(U) + zb \chi(U')$. For $x \in U'$, this equation becomes $0 = g(x) + zb(x)$. This shows $g \chi(U') + zb \chi(U') = 0$. Since $G$ is divisible, $z \neq 0$, and $g \chi(U') \in G$ by Proposition 1.5, we have $b \chi(U') \in G$. $\blacksquare$

It is not difficult to construct an $HA$ group such that $\langle G + \mathbb{Z} \cdot b \rangle \neq G + \mathbb{Z} \cdot b$; we omit this.

### 3. The $P$-groups.

$\alpha \mathbb{N} = \mathbb{N} \cup \{\alpha\}$ denotes the one-point compactification of the discrete space $\mathbb{N}$. In this section we describe all the groups $G$ with $S(\alpha \mathbb{N}, Z) \leq_a G$. This is a generalization of

**Example 3.1.** Let $C_{00} = \{f \in C(\alpha \mathbb{N}) : f \text{ vanishes on a neighborhood of } \alpha\}$. This is the weak product of countably many copies of $\mathbb{R}$, and $C_{00} \leq S(\alpha \mathbb{N})$. Let $b(n) = \pi + 1/n$, so $b \in C(\alpha \mathbb{N}) - S(\alpha \mathbb{N})$ and $b(\alpha) = \pi$. Write $\mathbb{R} = A \oplus \mathbb{Q} \pi$, as a direct sum of $\mathbb{Q}$-vector spaces.

(a) (P. Conrad [C2, 7.1]) Let $J = (C_{00} + \mathbb{Q}) + \mathbb{Q} \cdot b$. Then $S(\alpha \mathbb{N}, \mathbb{Z}) \leq_a J \not\subseteq S(\alpha \mathbb{N})$. Then $J$ is $HA$, cannot be represented as an $\ell$-group of step functions, and the vector lattice hull, $vJ$, is not $HA$.

(b) (M. Anderson and P. Conrad [AC, 4.1]) Let $K = (C_{00} + A) + \mathbb{Q} \cdot b$. Then $J \leq_a K$ and $K/M_p = \mathbb{R}$ for all $p \in \alpha \mathbb{N}$. So $K$ is $a$-closed, thus an $a$-closure of $J$. But $K$ is not a vector lattice. Note that, actually, [AC, p. 239] says “$K$ is the $a$-closure of $J$”. It can be seen from Proposition 3.5 below that $J$ has $2^e$ $a$-closures.

Some of these assertions about $J$ and $K$ are not particularly obvious. Some, not all, will be shown below. But we emphasize what is going on here:
The signature of the hyperarchimedean situation, which is the basic theme of this paper, is the interplay between the locally constant functions and the non-locally constant functions. This takes a rather simple form for the groups in Example 3.1 above, and indeed for general $G$ with $YG = \alpha \mathbb{N}$, as described in the rest of this section. For more general $G$, the interplay is highly visible, but more complicated and more difficult to describe. We quantify some of this at a rudimentary level:

For archimedean $G$ with strong unit and $YG = \alpha \mathbb{N}$, let

\[ L(G, \alpha) = \{ \delta \in \mathbb{R} | \exists g \in G \ (g = \delta \text{ on a nbhd of } \alpha) \}, \]

\[ nL(G, \alpha) = \{ \delta \in \mathbb{R} | \exists g \in G \ (g(\alpha) = \delta, \ g \text{ is not constant on any nbhd of } \alpha) \}. \]

Regarding Example 3.1, we have

\[ J/M_\alpha = \mathbb{Q} \oplus \mathbb{Q} \cdot \pi, \quad L(J, \alpha) = \mathbb{Q}, \quad nL(J, \alpha) = J/M_\alpha - \mathbb{Q}; \]

\[ K/M_\alpha = A \oplus \mathbb{Q} \cdot \pi = \mathbb{R}, \quad L(K, \alpha) = A, \quad nL(J, \alpha) = \mathbb{R} - A. \]

Here we have $L(J, \alpha) \cap nL(J, \alpha) = \emptyset$, likewise for $K$, and this is why the groups are hyperarchimedean; and $K/M_\alpha = \mathbb{R}$, and this is why $K$ is $a$-closed. These ideas will be articulated fully for $YG = \alpha \mathbb{N}$.

It will be helpful to keep these guidelines in mind as we proceed.

Suppose $S(\alpha \mathbb{N}, \mathbb{Z}) \leq S \leq S(\alpha \mathbb{N})$, let $F = S \cap C_{00}$ and $A = S/M_\alpha$. For each $s \in S$ and $n \in \mathbb{N}$, we have $s \chi(\{n\}) \in S$. Thus, $S/M_n = F/M_n$ for each $n \in \mathbb{N}$, for each $\delta \in A$ there is $n(\delta)$ such that $\delta \in F/M_n$ for $n \geq n(\delta)$, and $S = F + A$.

We prefer to start with $F$ and $A$ as initial data.

**Proposition 3.2.** Suppose $F \leq C_{00}, 1 \in A \leq \mathbb{R}$, and suppose that for each $\delta \in A$ there is $n(\delta)$ such that $\delta \in F/M_n$ for $n \geq n(\delta)$. Let $b \in C(\alpha \mathbb{N}) - S(\alpha \mathbb{N})$ with $b(n) \in F/M_n$ for each $n \in \mathbb{N}$. The following are equivalent:

(a) $b(\alpha) \notin dA = \mathbb{Q} \cdot A$.
(b) The sub-$\ell$-group of $C(\alpha \mathbb{N})$ generated by $A + \mathbb{Z} \cdot b$ is HA.
(c) $(F + A) + \mathbb{Z} \cdot b$ (or $(\mathbb{Q} \cdot F + \mathbb{Q}A) + \mathbb{Q} \cdot b$) is a sub-$\ell$-group of $C(\alpha \mathbb{N})$ which is HA.
(d) $(C_{00} + A) + \mathbb{Z} \cdot b$ (or $(C_{00} + \mathbb{Q} \cdot A) + \mathbb{Q} \cdot b$) is a sub-$\ell$-group of $C(\alpha \mathbb{N})$ which is HA.

**Proof.** (a)⇒(d). We use Corollary 2.6 with $G = C_{00} + A$. We first want to see that $Z(g + zb)$ is open for $g \in G$. This is obvious if $z = 0$. So suppose $z \neq 0$. Note that $g(\alpha) \in A$. Thus $g(\alpha) + zb(\alpha) \neq 0$ by (a), so $Z(g + zb)$ is a finite subset of $\mathbb{N}$, thus open in $\alpha \mathbb{N}$. Now using the second part of Corollary 2.6, consider $b\chi(U)$ for $U \in \text{clop}(\alpha \mathbb{N})$. Then $\alpha \notin U$ implies $U$ is finite, so $b\chi(U) \in C_{00} \subseteq G$, and $\alpha \in U$ implies $\alpha \mathbb{N} - U$ is finite, so $b\chi(\alpha \mathbb{N} - U) \in C_{00} \subseteq G$. The parenthetical part of (d) follows from Proposition 2.1(c).
We now use it to describe explicitly and from most points of view, without loss of generality.

(b)⇒(a). Let $G = \langle A + \mathbb{Z} \cdot b \rangle \leq C(\alpha N)$. By (b) and Theorem 2.3, $Z(\delta + zb)$ is open for each $z \in \mathbb{Z}$ and $\delta \in A$. As in (a)⇒(d), this means that if $z \neq 0$, then $\delta + zb(\alpha) \neq 0$, which says (a).

Proposition 3.2 is a straightforward generalization of Example 3.1(a). We now use it to describe explicitly all the divisible $HA$ groups, $G$, with strong unit for which $YG = \alpha N$. The “divisible” restriction is simplifying, and from most points of view, without loss of generality.

So consider Proposition 3.2 supposing $F$ and $A$ are divisible, and let

$$P = (F + A) + \mathbb{Q} \cdot b$$

be the $HA$ $\ell$-group in Proposition 3.2(c). Then $P \cap C_{00} = F$, $P \cap S(\alpha N) = F + A$, $P/M_n = F/M_n$ for each $n \in \mathbb{N}$, $P \cap S(\alpha N)/M_\alpha = A$ and $P/M_\alpha = A \oplus \mathbb{Q} \cdot b(\alpha)$ as a direct sum of $\mathbb{Q}$-vector spaces.

**Construction 3.3.** (a) Let $F$ and $A$ be as in Proposition 3.2 and divisible. Let $D \leq \mathbb{R}$ be divisible and $\mathbb{Q}$-linearly independent of $A$, i.e., $D \cap A = \{0\}$. Let $\varphi : D \to C(\alpha N)$ be a $\mathbb{Q}$-module homomorphism for which

(i) $\varphi(D) \cap S(\alpha N) = \{0\}$,
(ii) $\forall \delta \in D \forall n \in \mathbb{N} (\varphi(\delta)(n) \in F/M_n)$,
(iii) $\forall \delta \in D (\varphi(\delta)(\alpha) = \delta)$, which implies $\varphi$ is one-to-one.

Let $P(F, A, (D, \varphi)) = (F + A) + \varphi(D)$.

(b) Before proceeding, we take note that plenty of these exist: Given $F, A, D$, let $D_1$ be a $\mathbb{Q}$-basis for $D$. Then let $\varphi_1 : D_1 \to C(\alpha N) - S(\alpha N)$ be “constructed” like this: $A$ and each $F/M_n$ are topologically dense in $\mathbb{R}$, so given $\delta \in D_1$, there is a sequence $a_n \in A$ with $|\delta - a_n| < 1/n$. Let $n_1$ be the first integer such that $(n > n_1 \Rightarrow a_1 \in F/M_n)$, let $n_2$ be the first integer $> n_1$ such that $(n > n_2 \Rightarrow a_2 \in F/M_n)$, and so on. Now let $f_n = 1$ for $1 \leq n \leq n_1$, so these $f_n \in F/M_n$; let $f_n = a_n$ for $n_1 < n \leq n_2$, so these $f_n \in F/M_n$, and so on.

Thus, $f_n \in F/M_n$ for all $n$, and $f_n \to \delta$. Define $\varphi_1(\delta)(n) = f_n$ and $\varphi_1(\delta)(\alpha) = \delta$, and now extend $\varphi_1$ to $\varphi : D \to C(\alpha N)$ by $\mathbb{Q}$-linearity. Clearly $\varphi$ satisfies (ii) and (iii) above. Easy examples show that the preceding precautions are necessary to ensure (i), which we now show: Since $\varphi$ is 1-1, (i) just means that for all $\delta \in D - \{0\}$, $\varphi(\delta)$ is constant on no neighborhood of $\alpha$. But in fact there exists $n$ with $\varphi(\delta)(\alpha) = \delta$ if and only if $\delta = 0$, since $\varphi(\delta)(n) \in A$ and $A$ and $D$ are $\mathbb{Q}$-linearly independent.

**Theorem 3.4.** (I) $P = P(F, A, (D, \varphi))$ is a divisible hyperarchimedean $\ell$-group with strong unit 1 and $YP = \alpha N$, with the features:
$P \cap C_{00} = F$ and $P \cap S(\alpha N) = F + A$;

$P/M_n = F/M_n$ for all $n \in \mathbb{N}$ and $P \cap S(\alpha N)/M_\alpha = L(P, \alpha) = A$;

$P/M_\alpha = A \oplus D$ and $nL(P, \alpha) = P/M_\alpha - A = (D - \{0\}) + A$.

(II) If $G$ is any divisible hyperarchimedean $\ell$-group with strong unit, and $YG = \alpha N$, then there are $F, A, D, \varphi$ for which, in the Yosida representation of $G$, we have $G = P(F, A, (D, \varphi))$.

Note that in (II), given $G$, the sets $F$ and $A$ are determined by (I). But, by (I), $D$ only needs to have $A \oplus D = G/M_\alpha$, and situations $(D_1 \neq D_2$ with $A \oplus D_1 = A \oplus D_2)$ are common. Then of course, having fixed $D$, various $\varphi$ are possible. See below.

Proof. Note that $P(F, A, (D, \varphi)) = \bigcup\{(F + A) + b \mid b \in \varphi(D)\}$.

(I) Clearly, $1 \in P \subseteq C(\alpha N)$ and $P$ separates the points of $\alpha N$. If $0 \neq b = \varphi(\delta)$ for some $\delta \in D$, then $\delta \neq 0$ and $\varphi(\delta)(\alpha) \notin A$, so Proposition 3.2(a) holds. Thus Proposition 3.2(c) holds, and $(F + A) + \mathbb{Q} \cdot b$ is an $HA$ $\ell$-group with Yosida space $\alpha N$. Therefore, if $g_i = (f_i + a_i) + b_i \in P (i = 1, 2, b_i \in \varphi(D)$, etc.) then $Z(g_1)$ is open by Theorem 2.3, and $g_1 \vee 0 \in (F + A) + \mathbb{Q} \cdot b_1 \subseteq P$. Also, if $b_i = \varphi(\delta_i)$, then since $\varphi$ is $\mathbb{Q}$-linear, $b_0 = b_1 - b_2 = \varphi(\delta_1 - \delta_2)$, so that $g_1 - g_2 \in (F + A) + b_0 \subseteq P$. So $P$ is an $\ell$-group with open zero-sets, thus $HA$ by Theorem 2.3. By Theorem 1.2, $YP = \alpha N$, and the other claims for $P$ are clear.

(II) Let $G$ be given, let $F = G \cap C_{00}$ and $A = G \cap S(\alpha N)/M_\alpha = L(G, \alpha)$. Thus $nL(G, \alpha) = G/M_\alpha - A$.

Let $D_1$ be a subset of $G/M_\alpha$ maximal $\mathbb{Q}$-linearly independent of $A$, and let $\varphi_1 : D_1 \rightarrow G - S(\alpha N)$ be any choice function with $\varphi_1(\delta)(\alpha) = \delta$ for all $\delta \in D_1$. Let $D$ be the $\mathbb{Q}$-linear span of $D_1$ and let $\varphi : D \rightarrow G$ be the extension by $\mathbb{Q}$-linearity. This satisfies Construction 3.3(a)(i); i.e., the technicalities in Construction 3.3(b) are not needed here: If $\varphi(\delta) \neq 0$ then $0 \neq \delta = \varphi(\delta)(\alpha) \in G/M_\alpha - A$.

So let $P = P(F, A, (D, \varphi))$. Obviously, $P \subseteq G$. For the reverse, let $g \in G$. Our choice of $D$ makes $G/M_\alpha = A \oplus D$, so that $g(\alpha) = a + \delta$ for unique $a \in A$, $\delta \in D$. So there are $s \in G \cap S(\alpha N)$ and $b \in \varphi(D)$ with $a = s(\alpha)$ and $\delta = b(\alpha)$ and therefore $g(\alpha) = s(\alpha) + b(\alpha)$. Let $h = s + b \in (F + A) + b \subseteq P$. Since $h(\alpha) = g(\alpha)$, we have $h = g$ on a neighborhood $U$ of $\alpha$. Since $\alpha N - U$ is finite, $g - h \in C_{00} \cap G = F$. Thus $g = (g - h) + h \in (F + A) + b \subseteq P$.

It is easy to show the following.

**Proposition 3.5.** Let $P_i = P(F_i, A_i, (D_i, \varphi_i))$ for $i = 1, 2$ be as in Construction 3.3 and Theorem 3.4. These are equivalent:

(a) $P_1 \leq P_2$, so $P_1 \leq_a P_2$.
(b) $F_1 \leq F_2$, $A_1 \leq A_2$, and $P_1/M_\alpha \leq P_2/M_\alpha$. 


(c) \( F_1 \leq F_2, A_1 \leq A_2, \) and \( P_2 = P(F_2, A_2, (D_2', \varphi_2')) \) for some \( D_2' \supseteq D_1 \) and \( \varphi_2' \supseteq \varphi_1 \).

Proposition 3.5 makes it obvious how to construct all a-extensions of an \( \ell \)-group \( P = P(F, A, (D, \varphi)) \): enlarge \( F, A \) and \( D \) and then extend \( \varphi \). When these are not possible, \( P \) is a-closed:

**Corollary 3.6.** These are equivalent about \( P = P(F, A, (D, \varphi)) \):

(a) \( P \) is a-closed.
(b) \( F = C_{00} \) and \( P/M_\alpha = \mathbb{R} \).
(c) \( F = C_{00} \) and for any \( \mathbb{Q} \)-bases \( B \) for \( A \) and \( C \) for \( D \), \( B \cup C \) is a \( \mathbb{Q} \)-basis for \( \mathbb{R} \), i.e., a Hamel basis.

Proof. (b)\( \Leftrightarrow \) (c) since \( P/M_\alpha = A \oplus D \); (a)\( \Leftrightarrow \) (c) by inspection of Proposition 3.5. \( \blacksquare \)

In Corollary 3.6, (b)\( \Rightarrow \) (a) is a special case of a result from [AC]; (a)\( \Rightarrow \) (b) is also a consequence of topological properties of \( \alpha \mathbb{N} \). Corollary 3.6 and Theorem 3.4(I) give the following.

**Corollary 3.7.** Let \( A \) be a divisible subgroup of \( \mathbb{R} \) with \( 1 \in A \), and let \( D \) be any \( \mathbb{Q} \)-vector space complement of \( A \) in \( \mathbb{R} \). Then any \( P = P(C_{00}, A, (D, \varphi)) \) is a-closed and \( nL(P, \alpha) = \mathbb{R} - A \).

**Question 3.8** ([C2, p. 217, open question 4]). Suppose that \( G \) is a sub-\( \ell \)-group of a product \( H \) of reals which satisfies

\[ (*) \quad \text{if } 0 < g \in G, \text{ then } r < g < s \text{ for some } 0 < r, s \in \mathbb{R} \leq H. \]

Must the sub-\( \ell \)-ring of \( H \) generated by \( G \) be \( HA \)?

In general, \( (*) \) implies \( G \) is \( HA \). If \( G \) has strong unit, \( HA \) implies \( (*) \) for any such \( G \leq H \), as is easily seen from Sections 1 and 2 here.

The answer to Question 3.8 is “No”: Let \( P = P(C_{00}, \mathbb{Q}, (D, \varphi)) \) with \( D = \mathbb{Q} \cdot \sqrt{2} \). Here \( P \leq C(\alpha \mathbb{N}) \leq \mathbb{R}^{\mathbb{N}u\{\alpha\}} = H \), and the sub-\( \ell \)-ring of \( H \) generated by \( P \) is the sub-\( \ell \)-ring of \( C(\alpha \mathbb{N}) \) generated by \( P \). Call this \( gP \). By Theorem 2.3, \( gP \) is \( HA \) if and only if it has open zero-sets. But there is \( b = \varphi(\sqrt{2}) \in P \) so \( 2 - b^2 \in gP \), but \( Z(2 - b^2) \) is not open since \( b^2 \) cannot be \( 2 \) on a neighborhood of \( \alpha \).

4. The \( \Psi \)-groups. As noted in the Abstract, the following progressively more pointed questions have been asked by Conrad, and Anderson and Conrad.

**Question 4.1.** Suppose that \( G \) is a-closed.

1. Is \( G/I \) a-closed for every ideal \( I \)?
2. Is \( G/P \) a-closed for every minimal prime \( P \)?
(3) If $G$ is also HA, is $G/M$ a-closed for every maximal $M$, i.e., is $G/M = \mathbb{R}$?

Question (1) is from [C1, p. 153]; its converse is obviously true. Questions (2) and (3) are from [AC, p. 227]. The converse to (2) is Corollary I of [AC, p. 226], so the converse to (3) also holds.

We give the answer “No” to Question 4.1(3), so all these answers are “No”, taking aim at the issue by analyzing, for (divisible) HA $\ell$-groups with strong unit $G$, the mechanics of $a$-extendibility in terms of permissible enlargements of the $G/M_p$, one point $p$ at a time. We make another construction to produce various $a$-closed $G$ for which various $G/M_p$ are various proper subgroups of $\mathbb{R}$.

**Construction 4.2.** Let $X$ be an index set, and \( \{G_x \mid x \in X\} \) a set of archimedean $\ell$-groups with strong unit. Let $Y_x = YG_x$ for $x \in X$, and let $Y = \alpha \sum \{Y_x \mid x \in X\}$ be the one-point compactification of the disjoint union: $Y = \{\alpha\} \cup \sum Y_x$, in which $U \subseteq \sum Y_x$ is open if and only if each $U \cap Y_x$ is open in $Y_x$, and every neighborhood of $\alpha$ contains all but finitely many $Y_x$. Let $1 \in A \leq \mathbb{R}$, construed as constant functions on $Y$, or on any $Y_x$, and suppose that $A \leq G_x$ for each $x$ for simplicity. $\Psi = \Psi(\{G_x\}, A)$ denotes the $\ell$-subgroup of $C(Y)$ generated by the weak product $\prod \omega G_x$ and the constant functions from $A$.

It is easy to see that, for $f \in C(Y)$, we have $f \in \Psi$ if and only if there is a finite set $F \subseteq X$ and $a \in A$ such that $f|Y_x \in G_x$ for $x \in F$ and $f$ is constantly $a$ on $Y - \sum \{Y_x \mid x \in F\}$. Note that, for $g \in G_x$, the group $\Psi$ contains the function which is $g$ on $Y_x$ and 0 elsewhere; we denote this by $g\chi(Y_x)$. Clearly, for the divisible hull, $d\Psi = \Psi(\{dG_x\}, dA)$. Thus $\Psi = \prod \omega G_x + A \leq C(Y)$ is archimedean with strong unit 1, $Y\Psi = Y$ by Theorem 1.2(c), and evidently: for $p \in Y_x, \Psi/M_p = G_x/M_p$, and $\Psi/M_\alpha = A$. $\Psi(\{G_x\}, \mathbb{Z})$ is the “unital version” of $\prod \omega G_x$.

Note that the $\Psi$-groups are a partial generalization of the $P$-groups of §3: $\Psi(\{\mathbb{R}_n\}, A) = P(\prod \omega \mathbb{R}_n, A, (\emptyset, \emptyset))$. We could complete the generalization by adding a $(D, \varphi)$ in the data for $\Psi$ and we shall do that if a purpose develops.

Recall that a function $f \in C(X)$ is locally constant at $p \in X$ if there is a neighborhood of $p$ on which $f$ is constant. In the following definition, $\delta$ is always a real number. Recall also that given $G$ and $p \in YG$ we have $G/M_p = \{\delta \mid \exists g \in G \ (g(p) = \delta)\}$. We then define the analogous sets:

\[
L(G, p) = \{\delta \mid \exists g \in G \text{ locally constant at } p(g(p) = \delta)\},
\]

\[
nL(G, p) = \{\delta \mid \exists g \in G \not\text{ locally constant at } p(g(p) = \delta)\}
\]

\[
nL(G) = \bigcup\{nL(G, p) \mid p \in YG\}.
\]

Observe that $nL(\Psi) = \bigcup\{nL(G_x) \mid x \in X\}$. 

- (3) If $G$ is also HA, is $G/M$ a-closed for every maximal $M$, i.e., is $G/M = \mathbb{R}$?

- Question (1) is from [C1, p. 153]; its converse is obviously true. Questions (2) and (3) are from [AC, p. 227]. The converse to (2) is Corollary I of [AC, p. 226], so the converse to (3) also holds.

- We give the answer “No” to Question 4.1(3), so all these answers are “No”, taking aim at the issue by analyzing, for (divisible) HA $\ell$-groups with strong unit $G$, the mechanics of $a$-extendibility in terms of permissible enlargements of the $G/M_p$, one point $p$ at a time. We make another construction to produce various $a$-closed $G$ for which various $G/M_p$ are various proper subgroups of $\mathbb{R}$.

- **Construction 4.2.** Let $X$ be an index set, and \( \{G_x \mid x \in X\} \) a set of archimedean $\ell$-groups with strong unit. Let $Y_x = YG_x$ for $x \in X$, and let $Y = \alpha \sum \{Y_x \mid x \in X\}$ be the one-point compactification of the disjoint union: $Y = \{\alpha\} \cup \sum Y_x$, in which $U \subseteq \sum Y_x$ is open if and only if each $U \cap Y_x$ is open in $Y_x$, and every neighborhood of $\alpha$ contains all but finitely many $Y_x$. Let $1 \in A \leq \mathbb{R}$, construed as constant functions on $Y$, or on any $Y_x$, and suppose that $A \leq G_x$ for each $x$ for simplicity. $\Psi = \Psi(\{G_x\}, A)$ denotes the $\ell$-subgroup of $C(Y)$ generated by the weak product $\prod \omega G_x$ and the constant functions from $A$.

- It is easy to see that, for $f \in C(Y)$, we have $f \in \Psi$ if and only if there is a finite set $F \subseteq X$ and $a \in A$ such that $f|Y_x \in G_x$ for $x \in F$ and $f$ is constantly $a$ on $Y - \sum \{Y_x \mid x \in F\}$. Note that, for $g \in G_x$, the group $\Psi$ contains the function which is $g$ on $Y_x$ and 0 elsewhere; we denote this by $g\chi(Y_x)$. Clearly, for the divisible hull, $d\Psi = \Psi(\{dG_x\}, dA)$. Thus $\Psi = \prod \omega G_x + A \leq C(Y)$ is archimedean with strong unit 1, $Y\Psi = Y$ by Theorem 1.2(c), and evidently: for $p \in Y_x, \Psi/M_p = G_x/M_p$, and $\Psi/M_\alpha = A$. $\Psi(\{G_x\}, \mathbb{Z})$ is the “unital version” of $\prod \omega G_x$.

- Note that the $\Psi$-groups are a partial generalization of the $P$-groups of §3: $\Psi(\{\mathbb{R}_n\}, A) = P(\prod \omega \mathbb{R}_n, A, (\emptyset, \emptyset))$. We could complete the generalization by adding a $(D, \varphi)$ in the data for $\Psi$ and we shall do that if a purpose develops.

- Recall that a function $f \in C(X)$ is locally constant at $p \in X$ if there is a neighborhood of $p$ on which $f$ is constant. In the following definition, $\delta$ is always a real number. Recall also that given $G$ and $p \in YG$ we have $G/M_p = \{\delta \mid \exists g \in G \ (g(p) = \delta)\}$. We then define the analogous sets:

\[
L(G, p) = \{\delta \mid \exists g \in G \text{ locally constant at } p(g(p) = \delta)\},
\]

\[
nL(G, p) = \{\delta \mid \exists g \in G \not\text{ locally constant at } p(g(p) = \delta)\}
\]

\[
nL(G) = \bigcup\{nL(G, p) \mid p \in YG\}.
\]

- Observe that $nL(\Psi) = \bigcup\{nL(G_x) \mid x \in X\}$. 

**Hyperarchimedean lattice-ordered groups**
We establish criteria for $\Psi$ to be $HA$.

**Proposition 4.3.** $\Psi(\{G_x\}, A)$ is $HA$ if and only if for each $x \in X$, $G_x$ is $HA$ and $nL(G_x) \cap A = \emptyset$.

**Proof.** Suppose $\Psi$ is $HA$. Restriction $\Psi \ni f \mapsto f|Y_x \in G_x$ is a surjective homomorphism, so $G_x$ is $HA$. This implies $nL(G_x) \cap A = \emptyset$ since $A \subseteq L(G_x, p)$ for each $p \in Y_x$.

Conversely, if $g \in \Psi$, then $g = \sum \{g_x\chi(Y_x) \mid x \in F\} + a$ for a finite set $F$. If $a \neq 0$, then $Z(g) = \bigcup \{Z(g_x) \mid x \in F\}$ and this is open. If $a = 0$, then $Z(g) = \bigcup \{Z(g_x) \mid x \in F\} \cup (\bigcup \{Y_x \mid x \notin F\})$, which is open. By Corollary 2.6, $\Psi$ is $HA$.

The following outlines properties of $a$-extensions of $\Psi$, when it is $HA$, based on the action of adjoining certain values. Let

$$Ad(G, p) = \{\delta \mid \delta \notin G/M_p, \exists G \leq_a H (\delta \in H/M_p)\};$$

$$AdL(G, p) = \{\delta \mid \delta \notin G/M_p, \exists G \leq_a H (\delta \in L(H, p))\};$$

$$AdnL(G, p) = \{\delta \mid \delta \in G/M_p, \exists G \leq_a H (\delta \in nL(H, p))\}.$$

We now examine $Ad(G, p) = AdL(G, p) \cup AdnL(G, p)$, by examining the pieces separately.

**Theorem 4.4.** Let $p \in YG$, and let $\delta$ be a real number such that $\delta \notin G/M_p$. These are equivalent:

(a) $\delta \in AdL(G, p)$.

(b) There is $U \in \text{clop}(YG)$ with $p \in U$, and there is $G \leq_a H$ with $\delta\chi(U) \in H$.

(c) There is $U \in \text{clop}(YG)$ with $p \in U$ for which $\delta \notin nL(G, x)$ for each $x \in U$, i.e., $U \cap \{x \mid \delta \in nL(G, x)\} = \emptyset$.

**Proof.** (a)$\iff$(b) follows by noticing that, via Corollary 1.4, we get: $\delta \in L(G, p)$ if and only if there is $U \in \text{clop}(YG)$ with $p \in U$, and $\delta\chi(U) \in G$.

(a)$\implies$(c). If $G \leq_a H$ and $\delta \in L(H, p)$, then $\delta \notin nL(H, p) \supseteq nL(G, p)$ by the note above.

(c)$\implies$(b). Assuming (b), let $H = \langle G + Z \cdot \delta\chi(U)\rangle$. We use Corollary 2.6 to see that $G \leq_a H$. Consider $Z = Z(g + z\delta\chi(U))$. We have $Z = (Z \cap U) \cup (Z \cap (YG - U))$, and we show that each piece is open. The second, $Z \cap (YG - U) = Z(g) \cap (YG - U)$, is open since $Z(g)$ is open since $G$ is $HA$, and $U$ is closed. Now $Z \cap U = \{x \in U \mid g(x) = -z\delta\}$. So, if $x \in Z \cap U$, then $g$ is locally constant at $x$ since $\delta \notin nL(G, x)$ and $G$ is divisible, and there a is clopen $V \ni x$ with $g = -z\delta$ on $V$, so $V \subseteq Z \cap U$. Thus $Z \cap U$ is open.  

**Lemma 4.5.** Suppose $\Psi = \Psi(\{G_x\}, A)$ is $HA$. If $\Psi \leq_a H$, then for each $x$, $G_x \leq_a H|Y_x$. Conversely, if $G_x \leq_a H_x$ for each $x$, then $\Psi(\{G_x\}, A) \leq_a \Psi(\{H_x\}, A)$. 
Proof. The notation makes sense since $YH_x = YG_x$ for each $x$. The first assertion is because $G \leq_a H$ implies $G/I \cap G \leq_a H/I$ for each ideal $I$. Conversely, if $h = \sum \{h_x \chi(Y_x) \mid x \in F\} + a$, and for each $x \in F$ we have $g_x \sim_a h_x$, then $\sum \{g_x \chi(Y_x) \mid x \in F\} \sim_a h$.

**Proposition 4.6.** (a) Let $G$ be HA. Then $G$ is a-closed if and only if $Ad(G, p) = \emptyset$ for each $p \in YG$.

(b) Suppose $\Psi = \Psi(G_x, A)$ is HA. Then $\Psi$ is a-closed if and only if each $G_x$ is a-closed and $Ad(\Psi, a) = \emptyset$.

Proof. (a) “$\Rightarrow$” is clear. Conversely, let $G \leq_a H$ and $h \in H$. Since all $Ad(G, p)$ are empty, for each $p$ there is $g_p \in G$ with $g_p(p) = h(p)$. Since $H$ is HA, $h - g_p$ is constant on a clopen $U_p$ containing $p$, which means $h_\chi(U_p) = g_\chi(U_p) \in G$, by Theorem 1.3. By compactness, there is a finite $F$ with $\bigcup \{U_p \mid p \in F\} = YG$. Then $h = \bigvee_{p \in F}$ and $g_\chi(U_p) \in G$.

(b) By (a), $\Psi$ is a-closed if and only if $Ad(\Psi, p) = \emptyset$ for each $p \in Y$. For $p \in Y_x$, we have $Ad(\Psi, p) = Ad(G_x, p)$ using Lemma 4.5. The result follows.

We need the following well known lemma:

**Lemma 4.7.** $X$ be compact zero-dimensional. For $Z \subseteq X$, the following are equivalent: $Z \in ZC(X)$; $Z$ is a closed $G_\delta$-set; $Z$ is a countable intersection of clopen sets.

**Theorem 4.8.** Given $G, p \in YG$, and real $\delta \not\in G/M_p$:

(I) If $\delta \in AdnL(G, p)$, then there is a zero-set $Z$ with $p \in \partial Z$ with $\text{int} Z \cap \{x \mid \delta \in nL(G, x)\} = \emptyset$ and $\partial Z \cap \{x \mid \delta \in L(G, x)\} = \emptyset$.

(II) If there is a zero-set $Z$ with $p \in \partial Z$ with $\text{int} Z \cap \{x \mid \delta \in nL(G, x)\}$ $= \emptyset$ and $\partial Z \cap \{x \mid \delta \in L(G, x)\} = \emptyset$, then $\delta \in AdnL(G, p)$.

Proof. (I) Suppose $\delta \in AdnL(G, p)$, so $\delta \not\in G/M_p$ and there is $G \leq_a H$ with $h \in H$ for which $h(p) = \delta$ and $h$ is not locally constant at $p$. Let $Z = \{x \mid h(x) = \delta\}$. This is a zero-set and clearly for $x \in Z$:

(a) $x \in \text{int} Z$ if and only if $h$ is locally constant at $x$ and then $\delta \in L(H, x)$;

(b) $x \in \partial Z$ if and only if $h$ is not locally constant at $x$ and then $\delta \in nL(H, x)$.

By (b), $p \in \partial Z$. Since $nL(G, x) \subseteq nL(H, x)$ and $nL(H, x) \cap L(H, x) = \emptyset$, we see that $\text{int} Z \cap \{x \mid \delta \in nL(G, x)\} = \emptyset$ from (a). Since $L(G, x) \subseteq L(H, x)$ and $nL(H, x) \cap L(H, x) = \emptyset$, we get $\partial Z \cap \{x \mid \delta \in L(G, x)\} = \emptyset$ from (b).

(II) Let $Z$ be as described. By Lemma 4.7, $YG - Z = \bigcup U_n$ for $U_n \in \text{cl}(YG)$. Let $K_n = U_n - \bigcup_{i \leq n} U_i$. These are disjoint and clopen, $\bigcup K_n = YG - Z$ and $YG = \bigcup K_n \cup Z$. Let $(r_n)$ be a sequence of rational numbers with $r_n \to \delta$. Define $b \in C(YG)$ as $b(x) = r_n$ for $x \in K_n$ and $b(x) = \delta$ for $x \in Z$. Then $Z = \{x \mid b(x) = \delta\}$, and again for $x \in Z$:
(a) $x \in \text{int } Z$ if and only if $b$ is locally constant at $x$;
(b) $x \in \partial Z$ if and only if $b$ is not locally constant at $x$.

Let $H = (G + Z \cdot b)$. We shall use Corollary 2.6 to show that $H$ is $HA$ so that $G \leq_a H$. It follows that $\delta \in nL(H, p)$, so $\delta \in AdnL(G, p)$.

Let $g \in G$ and $z \in Z$. We show that $E = Z(g + zb)$ is open. Now $E = E_1 \cup E_2 \cup E_3$, where $E_1 = \bigcup (K_\alpha \cup E)$, $E_2 = \text{int } Z \cap E$, and $E_3 = \partial Z \cap E$. The set $E_1$ is open since $K_\alpha \cap E = K_\alpha \cap \{x \mid g(x) = -zr_\alpha\}$ is open. $E_2$ is open: since $\text{int } Z \cap \{x \mid \delta \in nL(G, x)\} = \emptyset$, it is also the case that $\text{int } Z \cap \{x \mid -z\delta \in nL(G, x)\} = \emptyset$ since $G$ is divisible. Thus, if $x \in E_2$ then there is a neighborhood $V$ of $x$ on which $g = -z\delta$ and $x \in V \cap \text{int } Z \subseteq E_2$. Finally, $E_3 = \emptyset$ since $\partial Z \cap \{x \mid \delta \in G/M_x\} = \emptyset$ and hence $\partial Z \cap \{x \mid -z\delta \in G/M_x\} = \emptyset$ since $G$ is divisible. ■

It is not difficult to construct examples illustrating the gap between the conditions in Theorem 4.8(I) and (II). We omit this.

So we can focus on $Ad(\Psi, \alpha)$. Keeping Theorem 4.4 in mind, note that if $\alpha \in U \subseteq Y$, then $U$ contains a clopen set containing $\alpha$ if and only if there is a finite $F$ such that $U \supseteq \bigcup \{Y_x \mid x \notin F\}$; $U$ contains a zero-set containing $\alpha$ if and only if there is countable $F$ with $U \supseteq \bigcup \{Y_x \mid x \notin F\}$ by Lemma 4.7.

**Proposition 4.9.** Suppose $\Psi = \Psi(G_x, A)$ is $HA$ with $X$ uncountable.

(a) $\delta \in AdL(\Psi, \alpha)$ if and only if there is finite a $F \subseteq X$ for which $x \notin F$ implies $\delta \notin nL(G_x)$.

(b) $\delta \in AdnL(\Psi, \alpha)$ if and only if there is a countable $F \subseteq X$ for which $x \notin F$ implies $\delta \notin nL(G_x)$.

Proof. (a) $nL(\Psi, \alpha) = \emptyset$ and for $p \in Y_x$, $nL(\Psi, p) = nL(G_x, p)$. So the condition says that $U = \bigcup \{Y_x \mid x \notin F\}$ satisfies Theorem 4.4(c).

(b) We make the following obvious, but useful, observation: If $G$ is $HA$ and $T \subseteq YG$ is closed, then $\text{int } T \cap \{x \mid \delta \in nL(G, x)\} = \emptyset$ if and only if $T \cap \{x \mid \delta \in nL(G, x)\} \subseteq \partial T$ and $\partial T \cap \{x \mid \delta \in L(G, x)\} = \emptyset$ if and only if $T \cap \{x \mid \delta \in L(G, x)\} \subseteq \text{int } T$.

Consider this observation for $Z$ a zero-set of $Y$ containing $\alpha$. Here $\partial Z = \{\alpha\}$, $L(\Psi, \alpha) = A$ and $nL(\Psi, \alpha) = \emptyset$. So for $\delta \notin A$, we see that $\partial Z \cap \{x \mid \delta \in \Psi/M_x\} = \emptyset$, thus $\partial Z \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$ and the conditions in the two parts of the observation each reduce to: $\text{int } Z \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$. The condition in (b) says that for $Z_0 = \bigcup \{Y_x \mid x \notin F\}$ we have $Z_0 \cap \{x \mid \delta \in nL(\Psi, x)\} = \emptyset$. Now apply Theorem 4.8. ■

**Corollary 4.10.** Let $\Psi = \Psi(G_x, A)$ be $HA$ with $X$ uncountable. Then $Ad(\Psi, \alpha) = \emptyset$ if and only if for each $\delta \notin A$ [for each countable $F \subseteq X$, there is $x \notin F$ with $\delta \in nL(G_x)$].
Proof. The condition bracketed is the negation of the condition in Proposition 4.9(b), which implies the negation of the condition in Proposition 4.9(a). ♦

Corollary 4.11. Let $G_0$ be HA, let $X$ be uncountable and for each $x \in X$ let $G_x = G_0$. Let $\Psi = \Psi(\{G_x\}, A)$. Then $\Psi$ is HA and $Ad(\Psi, \alpha) = \emptyset$ if and only if $\mathbb{R} - A = nL(G_0)$.

Such $G_0$ are given in Corollary 3.7.

Example 4.12. Let $A$ be any divisible subgroup of $\mathbb{R}$ containing 1. There is an $a$-closed $\Psi_A$ that is HA with strong unit, with a maximal ideal $M$ for which $\Psi_A/M = A$.

Such a $\Psi_A$ is a $\Psi$ in Corollary 4.11, with $M = M_\alpha$ using $G_0 = P(\{C_{00}\}$, $A, (D, \varphi)) \leq C(\alpha N)$ from §3, with a $\varphi$ for which $A \oplus D = \mathbb{R}$. Now $G_0$ is $a$-closed by Corollary 3.6, so $\Psi_A$ is $a$-closed by Proposition 4.6(b) and Corollary 4.11. As noted in Corollary 3.7, $nL(G_0, \alpha_0) = \mathbb{R} - A$ (writing $\alpha N = N \cup \{\alpha_0\}$ to avoid confusion).

For $\Psi_A$ in Example 4.12, $\Psi_A/M_p = \mathbb{R}$ for every $p \neq \alpha$, and $\Psi_A/M_\alpha = A$. Of course, more complicated situations can be constructed. We content ourselves with just one more level of complexity.

Example 4.13. Let $A$ be any divisible subgroup of $\mathbb{R}$ with $1 \in A$ and let $m$ be any uncountable cardinal number. Let $\alpha D(m)$ be the one-point compactification of the discrete space of cardinal $m$. There is an $a$-closed $\Psi = \Psi_{A,m}$, that is HA with strong unit, for which $Y \Psi$ contains a copy of $\alpha D(m)$ as a nowhere dense subset with $\Psi/M_p = A$ for each $p \in \alpha D(m)$:

Let $\Psi = \Psi(\{H_i \mid i < m\}, A)$ with each $H_i$ given by the $\Psi_A$ of Example 4.12. Here, $nL(H_i) = \mathbb{R} - A$ and since $nL(\Psi_A) = \mathbb{R} - A$, we see that $\Psi$ is $a$-closed by Proposition 4.6(b) and Corollary 4.10. We have $Y \Psi = \{\alpha\} \cup \sum YH_i$ and $\Psi/M_\alpha = A$. Also $YH_i = \{\alpha_i\} \cup \sum(\cdot)$ and $H_i/M_{\alpha_i} = A$. The desired copy of $\alpha D(m)$ is $\{\alpha\} \cup \{\alpha_i \mid i < m\}$.

With reference to “nowhere dense”, it is easy to show that: there is an $a$-closed $G$ for which $\{p \in YG \mid G/M_p \neq \mathbb{R}\}$ has interior if and only if there is $a$-closed $H$ for which $\{p \in YH \mid H/M_p = \mathbb{R}\} = \emptyset$ where $G$ and $H$ are HA with strong unit. We do not know if such $G, H$ exist.

References


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Received 7 March 2002;
in revised form 24 February 2004