The Complex Stone–Weierstrass Property

by

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Abstract. The compact Hausdorff space X has the CSWP iff every subalgebra of $C(X, \mathbb{C})$ which separates points and contains the constant functions is dense in $C(X, \mathbb{C})$. Results of W. Rudin (1956) and Hoffman and Singer (1960) show that all scattered X have the CSWP and many non-scattered X fail the CSWP, but it was left open whether having the CSWP is just equivalent to being scattered.

Here, we prove some general facts about the CSWP; in particular we show that if X is a compact ordered space, then X has the CSWP iff X does not contain a copy of the Cantor set. This provides a class of non-scattered spaces with the CSWP. Among these is the double arrow space of Aleksandrov and Urysohn. The CSWP for this space implies a Stone–Weierstrass property for the complex regulated functions on the unit interval.

1. Introduction. All spaces discussed in this paper are Hausdorff.

DEFINITION 1.1. If X is compact, then $C(X) = C(X, \mathbb{C})$ is the algebra of continuous complex-valued functions on X, with the usual supremum norm. $\mathcal{A} \sqsubseteq C(X)$ means that $\mathcal{A}$ is a subalgebra of $C(X)$ which separates points and contains the constant functions.

If $\mathcal{A} \sqsubseteq C(X)$ is self-adjoint ($f \in \mathcal{A} \leftrightarrow \bar{f} \in \mathcal{A}$), then $\mathcal{A}$ is dense in $C(X)$ by the standard Stone–Weierstrass Theorem for real-valued functions.

DEFINITION 1.2. The compact space X has the *Complex Stone–Weierstrass Property* (*CSWP*) iff every $\mathcal{A} \sqsubseteq C(X)$ is dense in $C(X)$.

The CSWP may be considered a notion of "smallness" by the following easy

LEMMA 1.3. *If X is a closed subspace of the compact space Y, and Y has the CSWP, then X has the CSWP.*

Proof. If $\mathcal{A} \sqsubseteq C(X)$, let $\mathcal{A}' = \{f \in C(Y) : f{\upharpoonright}X \in \mathcal{A}\}$. If $\mathcal{A}'$ is dense in $C(Y)$, then $\mathcal{A}$ is dense in $C(X)$. ∎

Classical examples show that the CSWP can fail. As usual, let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{A} \sqsubseteq C(\overline{D})$ be the *disc algebra*; these are the functions in $C(\overline{D})$ which are holomorphic on D. Then $\mathcal{A}$ shows that $\overline{D}$ fails to have the CSWP, and $\mathcal{A}{\upharpoonright}\mathbb{T}$ shows that $\mathbb{T}$ fails to have the CSWP. Here, the restricted algebra $\mathcal{A}{\upharpoonright}\mathbb{T}$ is defined by:

DEFINITION 1.4. Given $\mathcal{A} \sqsubseteq C(X)$ and a closed subset $H \subseteq X$, let $\mathcal{A}{\upharpoonright}H = \{f{\upharpoonright}H : f \in \mathcal{A}\} \sqsubseteq C(H)$.

W. Rudin proved the following basic results about the CSWP:

THEOREM 1.5 (Rudin). *For all compact X:*

(1) *If X contains a copy of the Cantor set, then X does not have the CSWP.*
(2) *If X is scattered, then X has the CSWP.*

As usual, X is *scattered* if it has no perfect subsets; see also Definition 1.7 below. (1) is proved in [14] and (2) is proved in [15]. Of course, for (1), it is sufficient (by Lemma 1.3) to prove that the Cantor set itself fails the CSWP, and this is done by another use of algebras of holomorphic functions.

This theorem completely characterizes the CSWP for compact metric X, since for such spaces, X is scattered iff X does not contain a copy of the Cantor set. However, there are non-metrizable non-scattered compact spaces, such as $\beta\mathbb{N}$, which do not contain Cantor subsets.

We do not know a simple characterization of the CSWP which holds for all compact X, but by the results in this paper and earlier results of Hoffman and Singer [11], neither of the implications (1) and (2) reverses. (1) is not an "iff" since $\beta\mathbb{N}$ does not have the CSWP by [11]; see Section 6 for more on their example. (2) also is not an "iff" by the following result, which shows that (1) *is* an "iff" for a restricted class of spaces:

THEOREM 1.6. *Let X be totally ordered by $<$, and assume that X is compact in its order topology. Then X has the CSWP iff X does not contain a copy of the Cantor set.*

In particular, the double arrow space of Aleksandrov and Urysohn [1], which is not scattered, has the CSWP, since it does not contain a copy of the Cantor set. The double arrow space is usually obtained by replacing each point in $[0, 1]$ by a pair of neighboring points (see Section 4). It may also (see Lemma 6.2) be viewed as the maximal ideal space of the algebra of *regulated functions* on $\mathbb{T}$ or $[0, 1]$ (the functions whose left and right limits exist everywhere). With this identification, the CSWP for the double

arrow space may be interpreted as a statement about algebras of regulated functions (see Corollary 6.3).

Of course, one direction of Theorem 1.6 is contained in Theorem 1.5. We prove the other direction in the case that X is *separable* in Section 4, which also lists some elementary properties of compact ordered spaces. In particular (see Lemma 4.6), if such an X is separable and fails to contain a Cantor set, then it must be totally disconnected; equivalently (since X is compact), the subsets of X which are *clopen* (both closed and open) form a base for the topology. If H is clopen, then its characteristic function $\chi_H \in C(X)$ is an idempotent. Section 2 contains further results on idempotents; these results reduce the proof of the separable case of Theorem 1.6 to showing that for such X, every closed $\mathcal{A} \sqsubseteq C(X)$ contains a non-trivial idempotent.

The proof of Theorem 1.6, without the restriction that X be separable, is given in Section 5. This is actually quite easy, using the observation (Lemma 5.2) that if X fails the CSWP, then there is a $Y \subseteq X$ which also fails the CSWP such that Y is the (closed) support of a measure. In the case of compact ordered spaces, it is already known that the support of a measure is separable.

We remark that it is easy to construct a connected *non*-separable compact ordered space X which does not contain a Cantor set. It is curious that the proof of the CSWP for X proceeds via the study of idempotents, even though $C(X)$ itself has only the trivial idempotents.

Sections 3 considers $\mathcal{A}{\restriction}\ker(X)$, where $\ker(X)$ is defined by:

DEFINITION 1.7. For a topological space X:

(1) $Y \subseteq X$ is *perfect* iff Y is non-empty and closed and no point of Y is isolated in Y.
(2) X is *scattered* iff X has no perfect subsets.
(3) The *kernel*, $\ker(X)$, is $\emptyset$ if X is scattered, and the largest perfect subset of X otherwise.

In (3), one constructs the largest perfect subset by taking the closure of the union of all perfect subsets. For a compact non-scattered X, all idempotents of $\mathcal{A}{\restriction}\ker(X)$ extend to idempotents of $\mathcal{A}$ (see Lemma 3.2), and X has the CSWP iff $\ker(X)$ has the CSWP (see Corollary 3.7).

Some further remarks on these restrictions $\mathcal{A}{\restriction}H$: Our definition of the property "$\mathcal{A} \sqsubseteq C(X)$" did not require that $\mathcal{A}$ be closed in $C(X)$. Of course, to verify the CSWP for a given X, one need only consider closed $\mathcal{A}$. However, we shall frequently be studying restrictions of $\mathcal{A}$. Even if $\mathcal{A}$ is closed, $\mathcal{A}{\restriction}H$ need not be closed in $C(H)$. In fact, by Glicksberg [9], if $\mathcal{A} \sqsubseteq C(X)$ and $\mathcal{A}{\restriction}H$ is closed in $C(H)$ for all closed $H \subseteq X$, then $\mathcal{A} = C(X)$. Moreover, $\mathcal{A}{\restriction}H$ *is* closed in $C(H)$ if H contains the Shilov boundary (see Section 2), or if $H = \ker(X)$ (see Lemma 3.2).

If $\mathcal{A}$ is a commutative Banach algebra with a unit element, we shall use $\mathcal{M}(\mathcal{A})$ to denote the *maximal ideal space* or *spectrum*. Elements of $\mathcal{M}(\mathcal{A})$ may be viewed either as maximal ideals in $\mathcal{A}$, or as homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. The Gel'fand transform maps $\mathcal{A}$ into a subalgebra of $C(\mathcal{M}(\mathcal{A}))$. For more on these notions, see e.g. [8, 10, 18].

One may also define a "$\sqsubseteq$" relation between commutative unital Banach algebras: $\mathcal{A} \sqsubseteq \mathcal{B}$ iff $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and $\mathcal{A}$ separates the points of $\mathcal{M}(\mathcal{B})$ (mapping $\mathcal{A}, \mathcal{B}$ into subalgebras of $C(\mathcal{M}(\mathcal{B}))$). One might say that $\mathcal{B}$ has the CSWP iff $\mathcal{A} \sqsubseteq \mathcal{B}$ implies $\mathcal{A} = \mathcal{B}$. This is related to the Stone–Weierstrass Property (SWP) discussed by Katznelson and Rudin [12] (or see [16, §9.3]): $\mathcal{B}$ has the SWP iff every *self-adjoint* $\mathcal{A} \sqsubseteq \mathcal{B}$ equals $\mathcal{B}$.

2. Idempotents and restrictions. We begin with a useful criterion for telling whether $\mathcal{A}\restriction H$ is closed in $C(H)$.

DEFINITION 2.1. $\|f\|_H = \sup\{|f(x)| : x \in H\}$.

LEMMA 2.2. *Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$. Let H be a closed subset of X. Then the following are equivalent*:

(1) $\mathcal{A}\restriction H$ *is closed in* $C(H)$.
(2) *There is a finite constant $c \geq 1$ such that for all $f \in \mathcal{A}$, there is an $f^* \in \mathcal{A}$ with $f\restriction H = f^*\restriction H$ and $\|f^*\| \leq c\|f\|_H$.*

Proof. As pointed out in [9], (1)→(2) follows by applying the Open Mapping Theorem to the restriction map from $\mathcal{A}$ onto $\mathcal{A}\restriction H$.

For (2)→(1), we repeat one of the steps in the proof of the Open Mapping Theorem. Suppose that we have $f_n \in \mathcal{A}$ for $n \in \mathbb{N}$, with $f_n\restriction H$ converging in $C(H)$. We must find a $g \in \mathcal{A}$ such that $g\restriction H = \lim_n(f_n\restriction H)$. Without loss of generality, we may assume that each $\|f_{n+1} - f_n\|_H \leq 2^{-n}$. For each n, apply (2) and choose $k_n \in \mathcal{A}$ with $k_n\restriction H = (f_{n+1} - f_n)\restriction H$ and $\|k_n\| \leq c2^{-n}$. Let $g = f_0 + \sum_{n=0}^{\infty} k_n$. ∎

If $f \in C(X)$, then f is an *idempotent* iff $f^2 = f$. Note that f is an idempotent iff $f = \chi_H$ for some clopen $H \subseteq X$.

DEFINITION 2.3. Assume that $\mathcal{A} \sqsubseteq C(X)$ and $\mathcal{A}$ is closed.

- $\mathcal{B}_\mathcal{A}$ is the set of all clopen $H \subseteq X$ such that $\chi_H \in \mathcal{A}$.
- If $x, y \in X$, then $x \sim_\mathcal{A} y$ if $\chi_H(x) = \chi_H(y)$ for every $H \in \mathcal{B}_\mathcal{A}$. We delete the subscript $\mathcal{A}$ when it is clear from context.
- For $P \subseteq X$, $\widetilde{P} = \bigcap\{H \in \mathcal{B}_\mathcal{A} : P \subseteq H\}$.
- $P \subseteq X$ *factors through* $\sim$ iff P is a union of $\sim$-equivalence classes.

Some simple observations: $\mathcal{B}_\mathcal{A}$ is a boolean algebra, and each equivalence class is closed and factors through $\sim$. Furthermore, $\widetilde{P}$ is always closed; and if P is closed in X, then, by compactness, P factors through $\sim$ iff $P = \widetilde{P}$.

LEMMA 2.4. *Under the assumptions of Definition* 2.3, (1)→(2)→(3). *The three are equivalent when X is totally disconnected.*

(1) *Each equivalence class is a singleton.*
(2) $\mathcal{A} = C(X)$.
(3) $\mathcal{B}_\mathcal{A}$ *contains all clopen subsets of* $\mathcal{A}$.

Proof. For (1)→(2), let $\mathcal{A}_r$ be the set of all real-valued functions in $\mathcal{A}$. By (1), $\mathcal{A}_r$ separates points, so the usual Stone–Weierstrass Theorem implies that $\mathcal{A}_r = C(X, \mathbb{R})$, which implies that $\mathcal{A} = C(X, \mathbb{C})$. ∎

When $P = \widetilde{P}$, Lemma 2.6 below describes $\mathcal{B}_{\mathcal{A}\restriction P}$ in terms of $\mathcal{B}_\mathcal{A}$. As in [14, 11], we need the following lemma for producing elements of $\mathcal{B}_\mathcal{A}$:

LEMMA 2.5. *Suppose that $\mathcal{A} \sqsubseteq C(X)$ and $\mathcal{A}$ is closed, and suppose that $h \in \mathcal{A}$ and $b \in \mathbb{R} \setminus \mathrm{Re}(h(X))$. Then $\{x \in X : \mathrm{Re}(h(x)) < b\} \in \mathcal{B}_\mathcal{A}$.*

As usual, $\mathrm{Re}(z)$ denotes the real part of the complex number z; so $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$. Note that $H := \{x \in X : \mathrm{Re}(h(x)) < b\}$ is clearly clopen, and the result is trivial unless $\mathrm{Re}(h(X))$ contains elements from both $(-\infty, b)$ and (b, ∞). In that case, the lemma is easily proved using Runge's Theorem (see [17, §13]), which implies that if $F \subset \mathbb{C}$, F is compact, $\mathbb{C} \setminus F$ is connected, and f is holomorphic on a neighborhood of F, then f may be uniformly approximated by polynomials on F. Here, F is the union of two convex subsets, F_0, F_1, where $\mathrm{Re}(F_1) \subset (-\infty, b)$, $\mathrm{Re}(F_0) \subset (b, \infty)$, and $h(X) \subseteq F = F_0 \cup F_1$. Applying Runge's Theorem to the function which is 0 on F_0 and 1 on F_1, we see that for each $\varepsilon > 0$, there is a polynomial p_ε such that $\|\chi_H - p_\varepsilon \circ h\| < \varepsilon$, which implies that $\chi_H \in \mathcal{A}$.

It is easy to see that $\mathcal{A}\restriction H$ is closed in $C(H)$ for each $H \in \mathcal{B}_\mathcal{A}$. More generally,

LEMMA 2.6. *Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$, and assume that P is closed in X and factors through $\sim_\mathcal{A}$. Then*

(1) $\mathcal{A}\restriction P$ *is closed in* $C(P)$.
(2) $\mathcal{B}_{\mathcal{A}\restriction P} = \{H \cap P : H \in \mathcal{B}_\mathcal{A}\}$.
(3) *If P is an equivalence class, then $\mathcal{B}_{\mathcal{A}\restriction P} = \{\emptyset, P\}$.*

Proof. For (1): We verify (2) of Lemma 2.2, with $c = 2$. Fix $f \in \mathcal{A}$ with $f\restriction P$ not identically 0. Since $P = \widetilde{P}$, we may choose $K \in \mathcal{B}_\mathcal{A}$ with $P \subseteq K$ and $\|f\|_K \le 2\|f\|_P$. Now let $f^* = f \cdot \chi_K$. Then $f^* \in \mathcal{A}$ and $\|f^*\| \le 2\|f\|_P$, and $f\restriction P = f^*\restriction P$.

For (2): Fix $H \in \mathcal{B}_{\mathcal{A}\restriction P}$, and we produce an $\widehat{H} \in \mathcal{B}_\mathcal{A}$ with $H = \widehat{H} \cap P$. Fix $f \in \mathcal{A}$ such that $f\restriction P = \chi_H$. Since $P = \widetilde{P}$, we may choose $K \in \mathcal{B}_\mathcal{A}$ such that $\min(|f(x)|, |f(x) - 1|) < 1/4$ for all $x \in K$. Let $g = f \cdot \chi_K$. Then $1/2 \notin \mathrm{Re}(g(X))$, so, by Lemma 2.5, $\widehat{H} := \{x \in X : \mathrm{Re}(g(x)) > 1/2\} \in \mathcal{B}_\mathcal{A}$.

(3) is immediate from (2). ∎

Note that in (3), if the equivalence class P is not a singleton, then P cannot be scattered by Theorem 1.5. It may have isolated points (see Example 2.8 below), but if $Q = \ker(P)$, then $\mathcal{B}_{\mathcal{A}\upharpoonright Q} = \{\emptyset, Q\}$ (see Lemma 3.2). Our proof of Theorem 1.6 will focus on showing that for appropriate perfect Q, this cannot happen—i.e., $\mathcal{A}\upharpoonright Q$ must contain non-trivial idempotents.

We now define two important closed subsets of X associated with $\mathcal{A}$:

DEFINITION 2.7. Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$. Let H be a closed subset of X. Then:

- H is *essential* for $\mathcal{A}$ iff for all $f, g \in C(X)$ with $f\upharpoonright H = g\upharpoonright H$, we have $f \in \mathcal{A}$ iff $g \in \mathcal{A}$.
- $E(\mathcal{A})$ denotes the *essential set* for $\mathcal{A}$; this is the smallest closed set which is essential for $\mathcal{A}$.
- H is a *boundary* for $\mathcal{A}$ iff $\|f\|_H = \|f\|$ for all $f \in \mathcal{A}$.
- $\amalg(\mathcal{A})$ denotes the *Shilov boundary*; this is the smallest closed set which is a boundary for $\mathcal{A}$.

Bear [2] defined the notion of "essential" and proved the existence of a smallest essential set. The existence of a smallest boundary, $\amalg(\mathcal{A})$, is a classical result of Shilov; short proofs are given in [3, 8, 18]. If we fix X and let $\mathcal{A}$ vary, then as $\mathcal{A}$ increases, $E(\mathcal{A})$ decreases and $\amalg(\mathcal{A})$ increases. $E(\mathcal{A}) = \emptyset$ iff $\mathcal{A} = C(X)$. Furthermore, $\amalg(\mathcal{A})$ is never empty and $\amalg(C(X)) = X$. Proposition 6.7 contains some additional properties of $E(\mathcal{A})$ and $\amalg(\mathcal{A})$.

EXAMPLE 2.8. Let $\mathcal{A}_0 \sqsubseteq C(\overline{D})$ be the disc algebra. Let $X = \mathbb{T} \cup H \cup F$, where $H, F \subset \mathbb{C}$ are compact scattered, with $H \subset D$ and $F \cap \overline{D} = \emptyset$. Let $\mathcal{A}_1 = \mathcal{A}_0\upharpoonright(\mathbb{T} \cup H)$. Let $\mathcal{A} = \{g \in C(X) : g\upharpoonright(\mathbb{T} \cup H) \in \mathcal{A}_1\}$. Then the $\sim_{\mathcal{A}}$-equivalence classes are $P := \mathbb{T} \cup H$ and the singletons in F. Moreover, $\ker(P) = \ker(X) = \mathbb{T}$, $\mathcal{A}\upharpoonright\mathbb{T}$ is closed in $C(\mathbb{T})$, and $\mathcal{B}_{\mathcal{A}\upharpoonright\mathbb{T}} = \{\emptyset, \mathbb{T}\}$. Also, $E(\mathcal{A}) = \mathbb{T} \cup H$ and $\amalg(\mathcal{A}) = \mathbb{T} \cup F$. Finally, $\amalg(\mathcal{A}\upharpoonright E(\mathcal{A})) = \mathbb{T}$; the fact that $\amalg(\mathcal{A}\upharpoonright E(\mathcal{A}))$ is perfect holds in general; see Proposition 6.7.

The next lemma shows that the conclusions (1), (2) of Lemma 2.6 hold for many closed subsets of X which do not necessarily factor through $\sim_{\mathcal{A}}$:

LEMMA 2.9. *Assume that* $\mathcal{A} \sqsubseteq C(X)$ *and is closed in* $C(X)$, *and assume that* K *is closed in* X. *If* $\amalg(\mathcal{A}) \subseteq K$ *then* (1), (2) *below hold. If* $E(\mathcal{A}) \subseteq K$, *then* (1) *holds, and* (2) *holds when* X *is totally disconnected.*

(1) $\mathcal{A}\upharpoonright K$ *is closed in* $C(K)$.
(2) $\mathcal{B}_{\mathcal{A}\upharpoonright K} = \{H \cap K : H \in \mathcal{B}_{\mathcal{A}}\}$.

Proof. (1) is immediate from Lemma 2.2. For (2), when $\amalg(\mathcal{A}) \subseteq K$, fix $H \in \mathcal{B}_{\mathcal{A}\upharpoonright K}$, and then fix $f \in \mathcal{A}$ such that $f\upharpoonright K = \chi_H$. Then $f^2 - f$ is 0 on K and hence is 0 everywhere, so f is an idempotent. If $\widehat{H} = f^{-1}\{1\}$, then $\widehat{H} \in \mathcal{B}_{\mathcal{A}}$ and $H = \widehat{H} \cap K$. ∎

LEMMA 2.10. *Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$, with $\mathcal{A} \neq C(X)$. Then $\mathcal{A} {\restriction} E(\mathcal{A}) \neq C(E(\mathcal{A}))$ and $\mathcal{A} {\restriction} \mathrm{III}(\mathcal{A}) \neq C(\mathrm{III}(\mathcal{A}))$.*

Proof. The first statement is clear from the definition of $E(\mathcal{A})$. The second is clear unless $\mathrm{III}(\mathcal{A}) \neq X$, so fix $p \in X \setminus \mathrm{III}(\mathcal{A})$. Define $\varphi : \mathcal{A} {\restriction} \mathrm{III}(\mathcal{A}) \to \mathbb{C}$ by $\varphi(f {\restriction} \mathrm{III}(\mathcal{A})) = f(p)$. Then $\varphi \in \mathcal{M}(\mathcal{A} {\restriction} \mathrm{III}(\mathcal{A}))$ and φ differs from all point evaluations, so $\mathcal{A} {\restriction} \mathrm{III}(\mathcal{A}) \neq C(\mathrm{III}(\mathcal{A}))$. ∎

The next two lemmas relate $\mathrm{III}(\mathcal{A})$ and $E(\mathcal{A})$ to the equivalence classes:

LEMMA 2.11. *Let $\mathcal{A} \sqsubseteq C(X)$ be closed. Then $\mathrm{III}(\mathcal{A})$ meets every $\sim_{\mathcal{A}}$-equivalence class.*

LEMMA 2.12. *Let $\mathcal{A} \sqsubseteq C(X)$ be closed. Let H be closed in X such that $\{p\}$ is an $\sim_{\mathcal{A}}$-equivalence class for all $p \in X \setminus H$. Then $E(\mathcal{A}) \subseteq H$.*

3. Restricting to the kernel. We consider the restrictions of $\mathcal{A} \sqsubseteq C(X)$ to $K = \ker(X)$ and to $\widetilde{K}$ (see Definition 2.3). In Example 2.8, $\widetilde{\widetilde{K}}$ properly contains K (i.e., K does not factor through $\sim$), and K is neither a boundary nor essential. Nevertheless, the next two lemmas show that the conclusions (1), (2) of Lemmas 2.6 and 2.9 hold for K:

LEMMA 3.1. *Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$. Assume that X is not scattered and let $K = \ker(X)$. Then $\mathrm{III}(\mathcal{A} {\restriction} \widetilde{K}) \subseteq K$.*

Proof. If not, then K is not a boundary for $\mathcal{A} {\restriction} \widetilde{K}$, so fix $f \in \mathcal{A}$ such that $\|f\|_K < \|f\|_{\widetilde{K}}$. Multiplying by a constant, we may assume that $\|f\|_K \leq 1$ and $f(p) = 1 + 2\varepsilon > 1$ for some $p \in \widetilde{K}$. Let $W \subset X$ be clopen with $K \subseteq W$ and $f(W) \subseteq B(0; 1 + \varepsilon)$, so that $\mathrm{Re}(f(W)) \subseteq (1 - \varepsilon, 1 + \varepsilon)$. Since $X \setminus W$ is scattered and compact, so is $\mathrm{Re}(f(X \setminus W))$, so we can fix $b \in (1 + \varepsilon, 1 + 2\varepsilon)$ such that $b \notin \mathrm{Re}(f(X))$. Let $H = \{x \in X : \mathrm{Re}(f(x)) < b\}$. Then $K \subseteq H$ and $p \notin H$, and by Lemma 2.5, $H \in \mathcal{B}_{\mathcal{A}}$, contradicting $p \in \widetilde{K}$. ∎

LEMMA 3.2. *Assume that $\mathcal{A} \sqsubseteq C(X)$ and is closed in $C(X)$, and let $K = \ker(X)$. Then*

(1) $\mathcal{A} {\restriction} K$ *is closed in* $C(K)$.
(2) $\mathcal{B}_{\mathcal{A} {\restriction} K} = \{H \cap K : H \in \mathcal{B}_{\mathcal{A}}\}$.

Proof. Use Lemmas 3.1, 2.6, and 2.9. ∎

DEFINITION 3.3. The compact space X has the *NTIP* iff every closed $\mathcal{A} \sqsubseteq C(X)$ has a non-trivial idempotent (i.e., of the form χ_H with $\emptyset \subsetneq H \subsetneq X$).

So, the NTIP is trivially false of connected spaces. If X is not connected, then the CSWP implies the NTIP. There are totally disconnected X which have the NTIP but not the CSWP; see Example 6.6. Lemma 3.2 and Theorem 1.5 imply immediately:

LEMMA 3.4. *If X is compact scattered and $|X| \geq 2$, then X has the NTIP. For any compact non-scattered X, if $\ker(X)$ has the NTIP then X has the NTIP.*

The converse is false even for totally disconnected X; see Example 6.6.

LEMMA 3.5. *Assume that X is compact and every perfect subset of X has the NTIP. Then X has the CSWP.*

Proof. Let $\mathcal{A} \sqsubseteq C(X)$ be closed but not all of $C(X)$. Then by Lemma 2.4, there must be a $\sim_{\mathcal{A}}$-equivalence class P which is not a singleton. By Lemma 2.6, $\mathcal{B}_{\mathcal{A} \restriction P} = \{\emptyset, P\}$, so P does not have the NTIP, so P is not scattered. If $Q = \ker(P)$, then Q is perfect and does not have the NTIP by Lemma 3.4. ∎

LEMMA 3.6. *Let $\mathcal{A} \sqsubseteq C(X)$, assume that X is not scattered, and let $K = \ker(X)$. If $\mathcal{A} \restriction K$ is dense in $C(K)$, then $\mathcal{A}$ is dense in $C(X)$.*

Proof. Without loss of generality, $\mathcal{A}$ is closed in $C(X)$, so $\mathcal{A} \restriction K = C(K)$ by Lemma 3.2. By Lemma 3.1, we have $\text{Ш}(\mathcal{A} \restriction \widetilde{K}) \subseteq K$, so that $\mathcal{A} \restriction \widetilde{K} = C(\widetilde{K})$ by Lemma 2.10, which implies that $\widetilde{K} = K$. It now follows by Rudin's Theorem 1.5 that each point in $X \setminus K$ is a $\sim$-equivalence class, so $E(\mathcal{A}) \subseteq K$ by Lemma 2.12. This, plus $\mathcal{A} \restriction K = C(K)$, implies that $\mathcal{A} = C(X)$. ∎

COROLLARY 3.7. *X has the CSWP iff $\ker(X)$ has the CSWP.*

4. Compact ordered spaces. A *linearly ordered topological space*, or *LOTS*, is a topological space X with a total order $<$ such that the topology on X is the order topology generated by $<$. We begin with some notation and well known facts about such spaces.

DEFINITION 4.1. In any set X totally ordered by $<$:

- 0_X and 1_X (or $0, 1$, when X is clear from context) denote, respectively, the first and last elements of X (when these elements exist).
- $x^- \lessdot x^+$ holds iff $x^- < x^+$ and there are no points of X strictly between x^- and x^+. In that case, we call $\{x^-, x^+\}$ a *neighboring pair*, and say that x^- and x^+ are *neighbors*.

If X is a compact LOTS with no isolated points, then the points 0 and 1 exist and have no neighbors; and each $x \in X \setminus \{0, 1\}$ has either no neighbors (when it is a limit point from the left and the right), or exactly one neighbor (when it is a limit point from one side and not the other).

The standard unit interval in the reals is connected, so there are no neighboring pairs, but one can form examples with neighboring pairs by doubling points in the interval:

DEFINITION 4.2. If $E \subseteq (0,1) \subseteq \mathbb{R}$, then $\mathbb{D}(E) = [0,1] \times \{0\} \cup E \times \{1\}$, ordered lexicographically.

Equivalently, we form $\mathbb{D}(E)$ by doubling every x in E (with x^- represented by $(x,0)$ and x^+ represented by $(x,1)$), and not doubling the points of $[0,1] \setminus E$. We never double 0 or 1, since that would create an isolated point.

PROPOSITION 4.3. *For every $E \subseteq (0,1) \subseteq \mathbb{R}$, the space $\mathbb{D}(E)$ is a compact separable LOTS with no isolated points. $\mathbb{D}(E)$ contains a copy of the Cantor set iff in the reals, $(0,1) \setminus E$ contains a copy of the Cantor set.*

The *double arrow space* is $\mathbb{D}((0,1))$. The proof of Theorem 1.6 will involve only spaces of the form $\mathbb{D}(E)$, by the following four lemmas. The first is from Lutzer and Bennett [13].

LEMMA 4.4. *If X is a separable LOTS and $H \subseteq X$, then H is separable and Lindelöf in its relative topology inherited from X.*

The relative topology on H is not in general the LOTS topology induced by the order restricted to H, but

LEMMA 4.5. *If X is a compact LOTS and H is a closed subset of H, then the relative topology and the order topology agree on H.*

LEMMA 4.6. *If X is a compact separable LOTS and X does not contain a copy of the Cantor set, then X is totally disconnected.*

Proof. If $a < b$ and $[a, b] \subseteq X$ is connected, then $[a, b]$ is isomorphic to a closed interval in $\mathbb{R}$, and hence contains a Cantor subset. ∎

PROPOSITION 4.7. *If X is a compact separable LOTS with no isolated points, then X is isomorphic to $\mathbb{D}(E)$ for some $E \subseteq (0,1) \subseteq \mathbb{R}$.*

To prove Theorem 1.6, we need to show that every compact separable LOTS Y which does not contain a copy of the Cantor set has the CSWP. By Lemma 3.5, it is enough to show that if X is any perfect subset of Y, then X has the NTIP. By Lemmas 4.4 and 4.5, X is also a compact separable LOTS and does not contain a Cantor subset. By Lemma 4.6, X and Y are totally disconnected. Propositions 4.3 and 4.7 are not used in the proof of Theorem 1.6, but they characterize the spaces to which Theorem 1.6 applies.

Lemma 4.15 will establish the NTIP for X. It is proved by analyzing step functions. In general, $\sigma \in C(X)$ is a *step function* if $\sigma(X)$ is finite. If X is compact and totally disconnected, then the step functions are dense in $C(X)$. If X is also a LOTS, then there is a simple description of such step functions:

DEFINITION 4.8. If X is a compact totally disconnected LOTS with no isolated points and $S \subseteq X$, then:

- A *step function with endpoints in S* is a function of the form $\sum_{i=0}^{n} z_i \chi_{I_i}$, where $n \geq 0$, each $z_i \in \mathbb{C}$, and each $I_i = [a_i^+, a_{i+1}^-]$ is a clopen interval in X, with $0 = a_0^+ < a_1^- \lessdot a_1^+ < \cdots < a_n^- \lessdot a_n^+ < a_{n+1}^- = 1$, where $a_0^+, a_1^-, a_1^+, \ldots, a_n^-, a_n^+, a_{n+1}^- \in S$.
- S is *nice* iff S is of the form $\{0,1\} \cup \bigcup_\alpha \{x_\alpha^-, x_\alpha^+\}$, where each $x_\alpha^- \lessdot x_\alpha^+$.
- $\mathsf{SF}(S)$ is the set of all step functions with endpoints in S.

When studying step functions with endpoints in S, we consider only nice S because only $0, 1$, and neighboring pairs get used as endpoints. Note that $\mathsf{SF}(S)$ is always a subalgebra of $C(X)$. $\mathsf{SF}(\{0,1\})$ is the set of constant functions (the $n = 0$ case of Definition 4.8). $\mathsf{SF}(S)$ is dense in $C(X)$ in the case that S contains all neighboring pairs. However, if S omits some neighboring pair $\{x^-, x^+\}$, then $\chi_{[0,x^-]}$ is a step function which has distance $1/2$ from $\mathsf{SF}(S)$.

DEFINITION 4.9. For X a compact LOTS, $f \in C(X)$ and $\varepsilon > 0$,

$$\mathsf{JMP}_\varepsilon(f) = \{0,1\} \cup \bigcup \{\{x^-, x^+\} : x^- \lessdot x^+ \ \& \ |f(x^-) - f(x^+)| \geq \varepsilon\},$$

$$\mathsf{JMP}(f) = \bigcup \{\mathsf{JMP}_\varepsilon : \varepsilon > 0\}.$$

So, $\mathsf{JMP}(f)$ contains all neighboring pairs at which f *jumps*. By compactness of X and continuity of f,

LEMMA 4.10. *Each* $\mathsf{JMP}_\varepsilon(f)$ *is finite, and* $\mathsf{JMP}(f)$ *is countable.*

Now, suppose that the X of Definition 4.8 is separable but not second countable. Then we may choose a nice S with S countable and dense in X, but there are uncountably many neighboring pairs, so $\mathsf{SF}(S)$ will not be dense in $C(X)$. Specifically, say $f \in C(X)$, $\varepsilon > 0$, and $x^\pm \in \mathsf{JMP}_{2\varepsilon}(f) \setminus S$. Then there is a step function σ with $\|f - \sigma\| < \varepsilon$, but the endpoints $a_i^\pm$ will not all lie in S, since $x^\pm$ is among the $a_i^\pm$. However, we can choose $b_i^\pm \in S$ with $a_i^+ < b_i^- \lessdot b_i^+ < a_{i+1}^-$, and then "describe" σ just using points in S as follows:

DEFINITION 4.11. If X is a compact totally disconnected LOTS with no isolated points and $S \subseteq X$, then a *step function descriptor from S* is a finite array of the form

$$\Delta = \begin{pmatrix} z_0 & z_1 & \cdots\cdots & z_n \\ b_0^-, b_0^+ & b_1^-, b_1^+ & \cdots\cdots & b_n^-, b_n^+ \end{pmatrix},$$

where $0 < b_0^- \lessdot b_0^+ < b_1^- \lessdot b_1^+ < \cdots < b_n^- \lessdot b_n^+ < 1$ in X, with each $b_i^\pm \in S$ and each $z_i \in \mathbb{C}$. Then $\mathsf{STEP}(\Delta)$ is the set of all step functions σ which *meet this description* in the sense that $\sigma = \sum_{i=0}^{n} z_i \chi_{I_i}$ for some $I_i = [a_i^+, a_{i+1}^-]$,

where

$$0 = a_0^+ < b_0^- \lessdot b_0^+ < a_1^- \lessdot a_1^+ < b_1^- \lessdot b_1^+$$
$$< \cdots < a_n^- \lessdot a_n^+ < b_n^- \lessdot b_n^+ < a_{n+1}^- = 1.$$

Note that the endpoints of σ are not required to be in S.

LEMMA 4.12. *Suppose Δ is as in Definition 4.11, $\sigma, \tau \in \mathsf{STEP}(\Delta)$, $\varepsilon > 0$, and $f, g \in C(X)$ with $\|f - \sigma\| < \varepsilon$ and $\|g - \tau\| < \varepsilon$. Let $h = f - g$. Then*

$$h(X) \subseteq B(0; 2\varepsilon) \cup \bigcup_{i < n} B(w_i; 2\varepsilon),$$

where each w_i is either $z_i - z_{i+1}$ or $z_{i+1} - z_i$.

With appropriate values of the z_i, and ε small enough, this will force $\mathrm{Re}(h(X))$ to be disconnected, so that we can apply Lemma 2.5 to get a non-trivial idempotent in $\mathcal{A}$. We shall prove that $\mathrm{Re}(h(X))$ is disconnected by using:

LEMMA 4.13. *Assume $r \geq 1$, $F \subseteq \mathbb{C}$, $F \cap B(0; 1) \neq \emptyset$, $F \cap B(w_0; 1/(3r)) \neq \emptyset$, and $F \subseteq B(0; 1) \cup \bigcup_{k < r} B(w_k; 1/(3r))$ for some $w_0, \ldots, w_{r-1} \in \mathbb{C}$ with $w_0 = \pm 2$. Then $\mathrm{Re}(F)$ is not connected.*

We shall get $h = f - g \in \mathcal{A}$ by applying the following lemma:

LEMMA 4.14. *Assume that X is a compact separable totally disconnected LOTS with no isolated points and $\mathcal{A} \sqsubseteq C(X)$. Then there is a countable dense nice $S \subseteq X$ such that for all step function descriptors Δ from S and all $\varepsilon > 0$:*

(∗)   *If there is a $\sigma \in \mathsf{STEP}(\Delta)$ and an $f \in \mathcal{A}$ with $\|f - \sigma\| < \varepsilon$, then there is a step function $\tau \in \mathsf{STEP}(\Delta)$ with endpoints in S and some $g \in \mathcal{A}$ with $\|g - \tau\| < \varepsilon$.*

Proof. Call a step function descriptor Δ *rational* if the z_i from Definition 4.11 all have rational real and imaginary parts. Observe that in proving the lemma, it is sufficient to obtain (∗) for all rational ε and all rational Δ. Let $S_0 \subseteq X$ be nice and countable and dense. Then we obtain an increasing chain $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$ as follows:

Given S_n, and given a rational step function descriptor Δ from S_n and a rational $\varepsilon > 0$: If there exists a $\sigma \in \mathsf{STEP}(\Delta)$ and an $f \in \mathcal{A}$ with $\|f - \sigma\| < \varepsilon$, then let $\sigma_{\Delta,\varepsilon}^n \in \mathsf{STEP}(\Delta)$ and $f_{\Delta,\varepsilon}^n \in \mathcal{A}$ be some such σ, f. Let S_{n+1} be S_n together with all the endpoints of all the $\sigma_{\Delta,\varepsilon}^n$.

Let $S = \bigcup_{n \in \mathbb{N}} S_n$. ∎

Putting these together, we get:

LEMMA 4.15. *Assume that X is a compact totally disconnected LOTS which is separable but not second countable. Then X has the NTIP.*

Proof. Note that by Lemma 4.4, $\ker(X)$ is also separable but not second countable, and it is sufficient to prove that $\ker(X)$ has the NTIP by Lemma 3.4. Thus, we may assume that X is perfect.

Let $\mathcal{A} \sqsubseteq C(X)$ be closed. First, fix $S \subseteq X$ as in Lemma 4.14. Then, since X is not second countable, we can fix a neighboring pair $q^- \prec q^+$ with $q^\pm \notin S$. Then fix an $f \in \mathcal{A}$ with $f(q^+) = 1$ and $f(q^-) = -1$.

Let $R = \mathsf{JMP}_{1/3}(f)$, which is finite by Lemma 4.10, and let $|R| = 2r + 2$; that is, R contains 0 and 1 plus r neighboring pairs. Then $r \geq 1$, since $q^\pm \in R$.

Fix a step function σ with $\|f - \sigma\| < 1/(6r)$. Say $\sigma = \sum_{i=0}^n z_i \chi_{I_i}$, where each $z_i \in \mathbb{C}$, each $I_i = [a_i^+, a_{i+1}^-]$, and $0 = a_0^+ < a_1^- \prec a_1^+ < \cdots < a_n^- \prec a_n^+ < a_{n+1}^- = 1$. Then $n \geq r$, since the $a_i^\pm$ include in particular the neighboring pairs from R.

Choose the $b_i^\pm \in S$ and Δ as in Definition 4.11 so that $\sigma \in \mathsf{STEP}(\Delta)$, and then apply $(*)$ from Lemma 4.14 and choose $\tau \in \mathsf{STEP}(\Delta)$ with endpoints in S and $g \in \mathcal{A}$ with $\|g - \tau\| < 1/(6r)$. We now have

$$0 = a_0^+ < b_0^- \prec b_0^+ < a_1^- \prec a_1^+ < b_1^+ \prec b_1^+$$
$$< \cdots < a_n^- \prec a_n^+ < b_n^+ \prec b_n^+ < a_{n+1}^- = 1,$$
$$0 = c_0^+ < b_0^- \prec b_0^+ < c_1^- \prec c_1^+ < b_1^+ \prec b_1^+$$
$$< \cdots < c_n^- \prec c_n^+ < b_n^+ \prec b_n^+ < c_{n+1}^- = 1,$$

where the $c_i^\pm \in S$ and $\tau = \sum_{i=0}^n z_i \chi_{J_i}$, with each $J_i = [c_i^+, c_{i+1}^-]$. Let $h = f - g$. By Lemma 4.12, we have

$$F := h(X) \subseteq B(0; 1/(3r)) \cup \bigcup_{i<n} B(w_i; 1/(3r)),$$

where each w_i is either $z_i - z_{i+1}$ or $z_{i+1} - z_i$.

Next, note that $|w_i| \leq 2/3$ for all but at most r indices i: this holds because $\|f - \sigma\| < 1/6$, so if $|w_i| > 2/3$, then $|f(a_i^-) - f(a_i^+)| > 1/3$, so that $a_i^\pm$ is one of the r neighboring pairs in R.

Now, $F \cap B(0; 1) \neq \emptyset$ because $h(0) \in B(0; 1)$. Also, one of the $a_i^\pm$ is the pair $q^\pm$, and this $w_i = \pm 2$. Since this $a_i^\pm = q^\pm \neq S$, while $c_i^\pm \in S$, we get $F \cap B(w_i; 1/(3r)) \neq \emptyset$.

Furthermore, whenever $|w_i| \leq 2/3$, we have $B(w_i; 1/(3r)) \subseteq B(0; 1)$, and this holds for all but at most r of the indices i. It now follows by Lemma 4.13 that $\mathrm{Re}(h(X))$ is disconnected, so that by Lemma 2.5, $\mathcal{A}$ contains a non-trivial idempotent. ∎

This lets us prove Theorem 1.6 in the separable case:

LEMMA 4.16. *Let X be totally ordered by $<$, and assume that X is compact and separable in its order topology. Then X has the CSWP iff X does not contain a copy of the Cantor set.*

Proof. To prove that X has the CSWP, it is sufficient, by Lemma 3.5, to prove that every perfect $P \subseteq X$ has the NTIP, and this holds by Lemma 4.15. ∎

The reader familiar with elementary submodel techniques (see Dow [6]) will note that one could simplify somewhat the proof of Lemma 4.15, at the expense of introducing some new concepts into the proof. Specifically, if θ is a suitably large regular cardinal and $\mathcal{A}, X \in M \prec H(\theta)$, with M countable, then we can simply define S in Lemma 4.15 to be $M \cap X$, and omit Lemma 4.14 entirely.

5. Measures and supports

DEFINITION 5.1. If μ is a regular complex Borel measure on the compact space X, then $|\mu|$ denotes its total variation, and $\operatorname{supt}(\mu) = \operatorname{supt}(|\mu|)$ denotes its (closed) support; that is, $\operatorname{supt}(\mu) = X \setminus \bigcup \{U \subseteq X : U \text{ is open and } |\mu|(U) = 0\}$.

LEMMA 5.2. *Assume that X is compact and that $\operatorname{supt}(\mu)$ has the CSWP for all regular Borel measures μ. Then X has the CSWP.*

Proof. Suppose we have $\mathcal{A} \sqsubseteq C(X)$ with $\mathcal{A}$ closed in $C(X)$ and $\mathcal{A} \neq C(X)$. Just viewing $\mathcal{A}$ and $C(X)$ as Banach spaces, fix $\varphi \in C(X)^*$ such that $\varphi(f) = 0$ for all $f \in \mathcal{A}$, but such that $\varphi(g) \neq 0$ for some $g \in C(X)$. Let μ be a regular complex Borel measure such that $\varphi(f) = \int f \, d\mu$ for all $f \in C(X)$. Let $H = \operatorname{supt}(\mu)$.

Then $\int_H h \, d\mu = 0$ for all $h \in \mathcal{A}{\restriction}H$, but $\int_H (g{\restriction}H) \, d\mu = \varphi(g) \neq 0$, so $\mathcal{A}{\restriction}H$ is not dense in $C(H)$. Thus, H does not have the CSWP. ∎

In the case of a LOTS, it is easy to see (by a minor modification of the proof of Theorem 1.0 in [7]):

LEMMA 5.3. *Assume that X is a compact LOTS and μ is a regular Borel measure on X. Then $\operatorname{supt}(\mu)$ is separable.*

Proof of Theorem 1.6. Immediate from Lemmas 4.16, 5.2, and 5.3. ∎

6. Remarks and examples. Much is known about the maximal ideal space, $\mathcal{M}(\mathcal{A})$, for some standard commutative unital Banach algebras $\mathcal{A}$. For example, it is well known that $\mathcal{M}(L^\infty(\mathbb{T}))$ is $\operatorname{st}(\mathcal{B})$, where $\mathcal{B}$ is the measure algebra of $\mathbb{T}$ (the measurable sets modulo the null sets) and $\operatorname{st}(\mathcal{B})$ denotes its Stone space.

Hoffman and Singer [11] (see also [8, 10]) discuss $\mathcal{M}(H^\infty)$, where H^∞ consists of those elements of $L^\infty(\mathbb{T})$ whose negative Fourier coefficients vanish. One may view H^∞ as a closed subalgebra of $C(\mathcal{M}(H^\infty))$ via the Gel'fand Transform (which is an isometry in this case). [11] shows, among other things, that with this identification, $\text{Ш}(H^\infty)$ is homeomorphic to

$\mathcal{M}(L^\infty(\mathbb{T})) = \operatorname{st}(\mathcal{B})$, and hence $H^\infty \sqsubseteq C(\operatorname{st}(\mathcal{B}))$, establishing that $\operatorname{st}(\mathcal{B})$ fails to have the CSWP. Since each of $\operatorname{st}(\mathcal{B})$ and $\beta\mathbb{N}$ contains the other, it follows, as mentioned in the Introduction, that $\beta\mathbb{N}$ (and hence every compact space containing $\beta\mathbb{N}$) fails to have the CSWP. Also, $\mathcal{M}(H^\infty)$ contains an open copy of D, and by [11], $\mathcal{M}(H^\infty)$ properly contains $D \cup \text{Ш}(H^\infty)$. By Carleson's Corona Theorem (see [8]), D is dense in $\mathcal{M}(H^\infty)$; hence, $E(H^\infty) = \mathcal{M}(H^\infty)$, since $E(H^\infty) \supseteq D$ follows from that fact that each function in H^∞ is holomorphic on D.

We also have $C(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. That is, with the natural identification of $f \in C(\mathbb{T})$ with its equivalence class $[f] \in L^\infty(\mathbb{T})$, we identify $C(\mathbb{T})$ with a closed subalgebra of $L^\infty(\mathbb{T})$.

Between $C(\mathbb{T})$ and $L^\infty(\mathbb{T})$, there is the algebra of functions which have everywhere a left and right limit. Note that the usual notion of left and right limits make sense on the circle. If $g : \mathbb{T} \to \mathbb{C}$ and $z = e^{i\theta} \in \mathbb{T}$, then by $\lim_{w \to z+} g(w)$ we mean $\lim_{\varphi \to \theta+} g(e^{i\varphi})$ and by $\lim_{w \to z-} g(w)$ we mean $\lim_{\varphi \to \theta-} g(e^{i\varphi})$.

DEFINITION 6.1. $J(\mathbb{T})$ is the set of all $g : \mathbb{T} \to \mathbb{C}$ such that for all z: $\lim_{w \to z+} g(w)$ and $\lim_{w \to z-} g(w)$ exist and

$$g(z) = (\lim_{w \to z+} g(w) + \lim_{w \to z-} g(w))/2.$$

We require $g(z)$ to be the average of its left and right limits so that if $g, h \in J(\mathbb{T})$ and $g \neq h$, then $\{z : g(z) \neq h(z)\}$ contains an interval, so g, h define different elements of L^∞. We now have $C(\mathbb{T}) \subset J(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Note that if $f \in J(\mathbb{T})$, then the left and right limits of f must be equal except at a countable set of points. $J(\mathbb{T})$ is the closure (in the supremum norm) of the algebra of piecewise continuous functions; these are the $f \in J(\mathbb{T})$ whose left and right limits are equal except at a finite set of points. We can identify $\mathcal{M}(J(\mathbb{T}))$ as the double arrow space by the following, due essentially to Berberian [4]:

LEMMA 6.2. $J(\mathbb{T})$ is isometric with $C(\mathbb{D}((0,1)))$, so that $\mathcal{M}(J(\mathbb{T}))$ is homeomorphic to $\mathbb{D}((0,1))$.

Proof. Define $\Psi : C(\mathbb{D}((0,1))) \to J(\mathbb{T})$ so that for $f \in C(\mathbb{D}((0,1)))$ and $z \in \mathbb{T}$, we compute $(\Psi(f))(z)$ as follows: Let $z = e^{2\pi i x}$, with $x \in [0,1]$. If $x \in (0,1)$, we have $x^\pm \in \mathbb{D}((0,1)))$, and we let $(\Psi(f))(z) = (f(x^+) + f(x^-))/2$. If $x \in \{0,1\}$, so that $z = 1$, we let $(\Psi(f))(1) = (f(0) + f(1))/2$. ∎

Thus, by Theorem 1.6,

COROLLARY 6.3. *Suppose that $\mathcal{A}$ is a subalgebra of $J(\mathbb{T})$ which contains the constant functions and separates the points of $\mathbb{T}$, and assume that for each $z \in \mathbb{T}$, some function in $\mathcal{A}$ has a discontinuity at z. Then $\mathcal{A}$ is dense in $J(\mathbb{T})$.*

In particular, $J(\mathbb{T}) \cap H^\infty = C(\mathbb{T}) \cap H^\infty$, since otherwise, by rotational symmetry, $J(\mathbb{T}) \cap H^\infty$ would contradict Corollary 6.3. However, this special case of 6.3 can be seen directly by integrating with the Poisson kernel; in fact, by Zalcman [19], if $f \in H^\infty$ is essentially discontinuous at a point $z \in \mathbb{T}$, then its essential range at z is uncountable, so in particular it cannot have a simple jump discontinuity.

$J(\mathbb{T})$ is closely related to the algebra $R_0(\mathbb{T}) \supset J(\mathbb{T})$ of *regulated* functions; these have left and right limits everywhere, but need not satisfy the requirement that $g(z)$ be the average of its left and right limits. $R_0(\mathbb{T})$ is a Banach algebra with the usual supremum norm; it cannot be identified as a subalgebra of $L^\infty(\mathbb{T})$. Because $g(z)$ can have an arbitrary value, each point evaluation is an element of the maximal ideal space, so that $\mathcal{M}(R_0(\mathbb{T}))$ contains $2^{\aleph_0}$ isolated points. It can be obtained by replacing each $z \in \mathbb{T}$ by a *triple* of points, $\{z^-, z_0, z^+\}$, where z_0 corresponds to evaluation at z, and z^-, z^+ correspond to the left and right limits. $\mathcal{M}(R_0(\mathbb{T}))$ is discussed in Berberian [4] and Blatter [5]. Since $\mathcal{M}(R_0(\mathbb{T}))$ has the CSWP, one can state an analogue to Corollary 6.3 for $R_0(\mathbb{T})$.

Usually, regulated functions are considered to be defined on an interval $[a, b] \subseteq \mathbb{R}$, rather than on $\mathbb{T}$, but this leads to essentially the same theory.

The following lemma and corollary describe a class of spaces with the NTIP:

LEMMA 6.4. *Suppose that the compact X is not second countable but* $\ker(X)$ *is second countable. Then X has the NTIP.*

Proof. If $\ker(X)$ is empty, then X is scattered, so X has the CSWP by Rudin's Theorem 1.5(2). So, assume that X is not scattered.

Let $\mathcal{A} \sqsubseteq C(X)$ be closed. Then $\mathcal{A}{\restriction}\ker(X)$ is closed by Lemma 3.2. Furthermore, $\mathcal{A}$ is not separable (since it separates points in X, which is not second countable), while $\mathcal{A}{\restriction}\ker(X)$ is separable, so the restriction map from $\mathcal{A}$ to $\mathcal{A}{\restriction}\ker(X)$ is not 1-1. Fix a non-zero $f \in \mathcal{A}$ such that its restriction to $\ker(X)$ is zero. We may assume that $\mathrm{Re}(f(X))$ contains more than just 0. But also, $\mathrm{Re}(f(X))$ is scattered (since it is a continuous image of the compact scattered $X/\ker(X)$), and hence $\mathrm{Re}(f(X))$ is disconnected, so that $\mathcal{A}$ contains a non-trivial idempotent by Lemma 2.5. ∎

COROLLARY 6.5. *Suppose that X is totally disconnected, with $\ker(X)$ second countable and non-empty. Then X has the NTIP iff X is not second countable.*

Proof. One direction is by Lemma 6.4. For the other direction: Note that $\ker(X)$ is homeomorphic to the Cantor set. Assume that X is second countable, and follow [14]: We may assume that $X \subset \mathbb{C} \subset S^2$, the Riemann sphere, and that all non-empty (relatively) open subsets of $\ker(X)$ have pos-

itive Lebesgue measure. Let $\mathcal{A} \sqsubseteq C(S^2)$ consist of the functions holomorphic on $S^2 \setminus \ker(X)$. Then $f(\ker(X)) = f(X) = f(S^2)$ for all $f \in \mathcal{A}$, so all $f(X)$ are connected. ∎

EXAMPLE 6.6. There are compact totally disconnected X, Y such that:

(1) X has the NTIP but $\ker(X)$ does not have the NTIP.
(2) Y has the NTIP but not the CSWP.

Proof. For (1), use Corollary 6.5 with any X such that X is not second countable but $\ker(X)$ is the Cantor set; for example, X can be the Aleksandrov duplicate of the Cantor set.

For (2), Y can be the same X. Or, Y can be the disjoint sum of the Cantor set and the double arrow space. This is a compact LOTS and has the NTIP by Lemma 4.15. ∎

Some additional properties of the Shilov boundary and the essential set are given by:

PROPOSITION 6.7. *Assume that* $\mathcal{A} \sqsubseteq C(X)$ *and* $\mathcal{A}$ *is closed in* $C(X)$. *Then*

(1) $E(\mathcal{A}) \cup \text{Ш}(A) = X$.
(2) $E(\mathcal{A}) \cap \text{Ш}(A) \neq \emptyset$ *unless* $E(\mathcal{A}) = \emptyset$ (*equivalently, unless* $\mathcal{A} = C(X)$).
(3) *If* $p \in \text{Ш}(\mathcal{A})$ *and is isolated in* $\text{Ш}(\mathcal{A})$, *then* p *is isolated in* X *and* $\chi_{\{p\}} \in \mathcal{A}$, *so* $p \neq E(\mathcal{A})$.
(4) *If* $E(\mathcal{A}) \neq \emptyset$, *then* $\text{Ш}(\mathcal{A} \restriction E(\mathcal{A}))$ *is perfect*.

Proof. For (1), if $p \notin E(\mathcal{A}) \cup \text{Ш}(A)$, fix $f \in C(X)$ with $f(E(\mathcal{A}) \cup \text{Ш}(A)) = \{0\}$ and $f(p) = 1$. Then $f \in \mathcal{A}$ because $E(\mathcal{A})$ is essential, but f contradicts the fact that $\text{Ш}(\mathcal{A})$ is a boundary.

For (2), if $E(\mathcal{A}) \cap \text{Ш}(\mathcal{A}) = \emptyset$ and $E(\mathcal{A}) \neq \emptyset$, fix $f \in C(X)$ with $f(E(\mathcal{A})) = \{1\}$ and $f(\text{Ш}(\mathcal{A})) = \{0\}$. Then f yields a contradiction as in (1).

For (3), assume that p is an isolated point of $\text{Ш}(\mathcal{A})$. Then $H := \text{Ш}(\mathcal{A}) \setminus \{p\}$ is closed and is not a boundary, so fix $f \in \mathcal{A}$ such that $\|f\| = 1$ but $\|f\|_H < 1$. Then $|f(p)| = 1$, since $\text{Ш}(\mathcal{A})$ is a boundary; multiplying by a constant, we may assume that $f(p) = 1$. Then $f^n \to \chi_{\{p\}}$ on $\text{Ш}(\mathcal{A})$ as $n \to \infty$. Since $\mathcal{A} \restriction \text{Ш}(\mathcal{A})$ is closed in $C(\text{Ш}(\mathcal{A}))$ (see Lemma 2.9), there is a $g \in \mathcal{A}$ such that $g \restriction \text{Ш}(\mathcal{A}) = \chi_{\{p\}}$. Then $g^2 - g$ is 0 on $\text{Ш}(\mathcal{A})$, and hence everywhere, so $g = \chi_K$ for some clopen K, with $p \in K$ and $H \cap K = \emptyset$. If $K = \{p\}$, then $\chi_{\{p\}} \in \mathcal{A}$. If $K \neq \{p\}$, fix $q \in K \setminus \{p\}$, and then fix $h \in \mathcal{A}$ with $h(q) = 1$ and $h(p) = 0$; then $h \cdot \chi_K$ contradicts the fact that $H \cup \{p\}$ is a boundary.

For (4), note that $E(\mathcal{A} \restriction E(\mathcal{A})) = E(\mathcal{A})$, so that applying (3) to $\mathcal{A} \restriction E(\mathcal{A})$ shows that no $p \in \text{Ш}(\mathcal{A} \restriction E(\mathcal{A}))$ is isolated in $\text{Ш}(\mathcal{A} \restriction E(\mathcal{A}))$. ∎

(4) immediately implies Rudin's Theorem 1.5(2); also, (4) yields another proof of Lemma 3.6 (using Lemmas 2.9 and 2.10).

References

[1] P. S. Alexandroff [P. S. Aleksandrov] et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch. Amsterdam 14 (1929), 1–96.

[2] H. S. Bear, *Complex function algebras*, Trans. Amer. Math. Soc. 90 (1959), 383–393.

[3] —, *The Silov boundary for a linear space of continuous functions*, Amer. Math. Monthly 68 (1961), 483–485.

[4] S. K. Berberian, *The character space of the algebra of regulated functions*, Pacific J. Math. 74 (1978), 15–36.

[5] J. Blatter, *The Gel'fand space of the Banach algebra of Riemann integrable functions*, in: Approximation Theory, III, Academic Press, 1980, 229–232.

[6] A. Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. 13 (1988), 17–72.

[7] M. Džamonja and K. Kunen, *Properties of the class of measure separable compact spaces*, Fund. Math. 147 (1995), 261–277.

[8] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[9] I. Glicksberg, *Function algebras with closed restrictions*, Proc. Amer. Math. Soc. 14 (1963), 158–161.

[10] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, 1962.

[11] K. Hoffman and I. M. Singer, *Maximal algebras of continuous functions*, Acta Math. 103 (1960), 217–241.

[12] Y. Katznelson and W. Rudin, *The Stone–Weierstrass property in Banach algebras*, Pacific J. Math. 11 (1961), 253–265.

[13] D. J. Lutzer and H. R. Bennett, *Separability, the countable chain condition and the Lindelöf property in linearly orderable spaces*, Proc. Amer. Math. Soc. 23 (1969), 664–667.

[14] W. Rudin, *Subalgebras of spaces of continuous functions*, ibid. 7 (1956), 825–830.

[15] —, *Continuous functions on compact spaces without perfect subsets*, ibid. 8 (1956), 39–42.

[16] W. Rudin, *Fourier Analysis on Groups*, Interscience, 1962.

[17] —, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.

[18] —, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.

[19] L. Zalcman, *The essential range of a function of class $H^\infty + C$*, Michigan Math. J. 19 (1972), 385–386.

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