More on tie-points and homeomorphism in $\mathbb{N}^*$

by

Alan Dow (Charlotte, NC) and
Saharon Shelah (Piscataway, NJ, and Jerusalem)

Abstract. A point $x$ is a (bow) tie-point of a space $X$ if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with $x$ in its closure. We denote this as $X = A \bowtie x B$ where $A, B$ are the closed sets which have a unique common accumulation point $x$. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$ (by Veličković and Shelah & Steprāns) and in the recent study (by Levy and Dow & Techanie) of precisely 2-to-1 maps on $\mathbb{N}^*$. In these cases the tie-points have been the unique fixed point of an involution on $\mathbb{N}^*$. One application of the results in this paper is the consistency of there being a 2-to-1 continuous image of $\mathbb{N}^*$ which is not a homeomorph of $\mathbb{N}^*$.

1. Introduction. A point $x$ is a tie-point of a space $X$ if there are closed sets $A, B$ of $X$ such that $X = A \cup B$, $\{x\} = A \cap B$ and $x$ is an adherent point of both $A$ and $B$. We let $X = A \bowtie x B$ denote this relation and say that $x$ is a tie-point as witnessed by $A, B$. Let $A \equiv_x B$ mean that there is a homeomorphism from $A$ to $B$ with $x$ as a fixed point. If $X = A \bowtie x B$ and $A \equiv_x B$, then there is an involution $\Phi$ of $X$ (i.e. $\Phi^2 = \Phi$) such that $\{x\} = \text{fix}(\Phi)$. In this case we will say that $x$ is a symmetric tie-point of $X$.

Let $\Phi$ be a continuous function from $\mathbb{N}^*$ into $\mathbb{N}^*$. Of course, $\Phi^{-1}$ can be regarded as a function from the clopen subsets of $\mathbb{N}^*$ into the clopen subsets of $\mathbb{N}^*$. A function $F$ from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ is a lifting of $\Phi$ if $F(a)^* = \Phi^{-1}(a^*)$ for all $a \subset \mathbb{N}$. A function $h$ is said to induce $F$ (and/or $\Phi$) on $I$ if $F(a) =^* h[a] = \{h(n) : n \in a\}$ for all $a \subset I$. The function $F$ is said to be trivial on $I$ if there is such a function $h$. Since the fixed point set of a trivial autohomeomorphism is clopen, a symmetric tie-point gives rise to a non-trivial autohomeomorphism. An ideal on $\mathbb{N}$ is a $P$-ideal if it is countably directed closed mod finite.

2000 Mathematics Subject Classification: 03A50, 54A25, 54D35.

Key words and phrases: automorphism, Stone–Čech, fixed points.

This is paper number B917 in the second author’s personal listing.

DOI: 10.4064/fm203-3-1 [191] © Instytut Matematyczny PAN, 2009
If $A$ and $B$ are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \cong_{x=y} B$ denote the quotient space of $A \oplus B$ obtained by identifying $x$ and $y$, and let $xy$ denote the collapsed point. Clearly the point $xy$ is a tie-point of this space.

In this paper we establish the following theorem.

**Theorem 1.1.** It is consistent that $\mathbb{N}^*$ has symmetric tie-points $x, y$ as witnessed by $A, B$ and $A', B'$ respectively such that $\mathbb{N}^*$ is not homeomorphic to the space $A \cong_{x=y} A'$.

**Corollary 1.2.** It is consistent that there is a 2-to-1 image of $\mathbb{N}^*$ which is not a homeomorph of $\mathbb{N}^*$.

One can generalize the notion of tie-point and, for a point $x \in \mathbb{N}^*$, consider how many disjoint clopen subsets of $\mathbb{N}^* \setminus \{x\}$ (each accumulating to $x$) can be found. Let us say that a tie-point $x$ of $\mathbb{N}^*$ satisfies $\tau(x) \geq n$ if $\mathbb{N}^* \setminus \{x\}$ can be partitioned into $n$ disjoint clopen subsets each accumulating to $x$. Naturally, we will let $\tau(x) = n$ denote that $\tau(x) \geq n$ and $\tau(x) \not\geq n+1$. It follows easily from [2, 5.1] that each point $x$ of character $\omega_1$ in $\mathbb{N}^*$ is a tie-point and satisfies $\tau(x) \geq n$ for all $n$. Similarly each $P$-point of character $\omega_1$ in $\mathbb{N}^*$ is a symmetric tie-point. We list several open questions in the final section.

**Theorem 1.3.** It is consistent that $\mathbb{N}^*$ has a tie-point $x$ such that $\tau(x) = 2$ and $\mathbb{N}^* = A \cong_{x=y} B$, where neither $A$ nor $B$ is a homeomorph of $\mathbb{N}^*$. In addition, there are no symmetric tie-points.

The following theorem of [10] provides an important equivalent condition for the triviality of autohomeomorphisms on $\mathbb{N}^*$ and it will allow us to utilize the results of Steprāns’s paper [8].

**Lemma 1.4 (Veličković).** If $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a lifting of an autohomeomorphism and there exist Borel functions $\{\psi_n : n \in \omega\}$ and a comeager set $G \subset \mathcal{P}(\mathbb{N})$ such that for every $A \in G$ there is $n \in \omega$ such that $\psi_n(A) =^* F(A)$, then $F$ is trivial.

This is Theorem 2 of [10] except that the strengthening to the case of a comeager set $G$ is from [8, 2.1]. The topology on $\mathcal{P}(\mathbb{N})$ is the standard one induced by identifying each set $a \subset \mathbb{N}$ with its characteristic function $\chi_a \in 2^{\mathbb{N}}$. For a set $C \subset \mathcal{P}(\mathbb{N})$ and a function $F$ on $\mathcal{P}(\mathbb{N})$, let us say that $F|C$ is $\sigma$-Borel if there is sequence $\{\psi_n : n \in \omega\}$ of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for each $b \in C$, there is an $n$ such that $F(b) =^* \psi_n(b)$.

The following lemma is also implicit in [10, 1.3]:

**Lemma 1.5.** If $F$ is a lifting of an autohomeomorphism of $\mathbb{N}^*$ and if $F$ is trivial on each member of a $P$-ideal $\mathcal{I}$ for which $F|\mathcal{I}$ is $\sigma$-Borel, then there is a function $h$ which induces $F$ on each member of $\mathcal{I}$. 

The following partial order $\mathbb{P}_2$ was introduced by Veličković in [10]. Our need for this poset is articulated in [8] where it is described as a poset which was introduced “to add a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/\mathbb{N}^{<\aleph_0}$ while doing as little else as possible—at least assuming PFA”.

**Definition 1.6.** The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions $f : A \to B$ where

- $A \subseteq \omega$ and $B \subseteq \omega \setminus A$,
- for all $i \in \omega$ and $n \in \omega$, $f(i) \in 2^{n+1} \setminus 2^n$ if and only if $i \in 2^{n+1} \setminus 2^n$,
- $\lim \sup_{n \to \infty} |(2^{n+1} \setminus 2^n) \setminus (A \cup B)| = \omega$.

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

We define some trivial generalizations of $\mathbb{P}_2$. We use the notation $\mathbb{P}_2$ to signify that this poset introduces an involution of $\mathbb{N}^*$ because the condition $g = f \cup f^{-1}$ implies that $g^2 = g$. In the definition of $\mathbb{P}_2$ it is possible to suppress mention of $A, B$ (which we do) and to have the poset $\mathbb{P}_2$ consist simply of the functions $g$ (and $A$ as $L_g = \{i \in \text{dom}(g) : i < g(i)\}$, and $B$ as $U_g = \{i \in \text{dom}(g) : g(i) < i\}$).

Let $\mathbb{P}_1$ denote the poset we get if we omit mention of $f$ consisting only of disjoint pairs $(A, B)$, satisfying the growth condition in Definition 1.6, and extension is coordinatewise mod finite containment. For more consistent notation, we will instead represent the elements of $\mathbb{P}_1$ as partial functions into 2.

More generally, let $\mathbb{P}_l$ be similar to $\mathbb{P}_2$ except that we assume that conditions consist of functions $g$ such that $\{i, g(i), g^2(i), \ldots, g^l(i)\}$ is contained in $l^{n+1} \setminus l^n$ and has precisely $l$ elements for all $i \in \text{dom}(g) \cap l^{n+1} \setminus l^n$.

The basic properties of $\mathbb{P}_2$ as defined by Veličković and treated by Shelah and Steprāns are also true of $\mathbb{P}_l$ for all $l \in \mathbb{N}$.

In particular, for example, the following is easily seen:

**Proposition 1.7.** If $L \subset \mathbb{N}$ and $\mathbb{P} = \prod_{l \in L} \mathbb{P}_l$ (with full supports) and $G$ is a $\mathbb{P}$-generic filter, then in $V[G]$, for each $l \in L$, there is a tie-point $x_l \in \mathbb{N}^*$ with $\tau(x_l) \geq l$.

For the proof of Theorem 1.1 we use $\mathbb{P}_2 \times \mathbb{P}_2$ and for the proof of Theorem 1.3 we use $\mathbb{P}_1$. In any such $\mathbb{P}$ and $\bar{f}, \bar{g} \in \mathbb{P}$, we say that $\bar{f} = (f_l)$ is an $n$-preserving extension of $\bar{g} = (g_l)$, for an integer $n$, if for each coordinate $l$, $f_l|n = g_l|n$ and $f_l \supset g_l$. Also, if $\bar{s} = (s_l)$ is a sequence of functions (usually with finite domain), then we define $\bar{s} \cup \bar{f}$ to be the sequence $\bar{g} = (g_l)$ where, for each coordinate $l$,

$$g_l = s_l \cup f_l \equiv s_l \cup (f_l|\text{dom}(f_l) \setminus \text{dom}(s_l)).$$
2. Preliminaries. Each poset $\mathbb{P}$ as above is $\aleph_1$-closed and, if PFA holds, $\aleph_2$-distributive (see [8, p. 4226]). In this paper we will restrict our study to finite products. The following partial order can be used to show that these products are $\aleph_2$-distributive.

**Definition 2.1.** Let $\mathbb{P}$ be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Given $F \subset \mathbb{P}$, define $\mathbb{P}(F)$ to be the partial order consisting of all $g \in \mathbb{P}$ such that there is some $f \in F$ such that $g \equiv^* f$. The ordering on $\mathbb{P}(F)$ is coordinatewise $\supseteq$ as opposed to $^* \supseteq$ in $\mathbb{P}$.

If $F$ is downward directed (in fact it will be a descending sequence), then the forcing $\mathbb{P}(F)$ introduces a tuple $f$ such that $f \leq f'$ for all $f' \in F$. Although $f$ itself may not be a member of $\mathbb{P}$, it is simply because the domains of the component functions are too big. Following [6, 2.1], one must then use a $\sigma$-centered poset which will choose an appropriate sequence $f^*$ of subfunctions of $f$ which is a member of $\mathbb{P}$ and which is still below each member of $F$.

A strategic choice of the sequence $F$ will ensure that $\mathbb{P}(F)$ is ccc, but remarkably even more is true. Again we are lifting results from [6, 2.6] and [8, proof of Thm. 3.1] which introduced this innovative factoring of Veličković’s original amoeba forcing poset and showed that it seems to preserve more properties. Let $\omega_2^{<\omega_1}$ denote the standard collapse which introduces a function from $\omega_1$ onto $\omega_2$.

A poset is said to be $\omega^\omega$-bounding if every new function in $\omega^\omega$ is bounded by some ground model function.

**Lemma 2.2.** Let $\mathbb{P}$ be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. In the forcing extension $V[H]$, by $\omega_2^{<\omega_1}$, there is a descending sequence $F$ from $\mathbb{P}$ which is $\mathbb{P}$-generic over $V$ and for which $\mathbb{P}(F)$ is ccc and $\omega^\omega$-bounding.

It was also shown in [6] that $F$ can be chosen so that it, in addition, preserves that $\mathbb{R} \cap V$ is of second category. This is crucial for the proof of Lemma 2.3. We can manage with the $\omega^\omega$-bounding property because we are going to use Lemma 2.3. An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is said to be dense if each infinite subset of $\mathbb{N}$ contains a member of $\mathcal{I}$.

The following main result is extracted from [6] and [8, Theorem 3.3] which we record without proof.

**Lemma 2.3 (PFA).** Let $\mathbb{P}$ be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let $\check{F}$ be a $\mathbb{P}$-name of a lifting of an autohomeomorphism of $\mathbb{N}^*$. Let $\check{F}$ and $H$ be as in Lemma 2.2 and let $F$ be the valuation of $\check{F}$ by $\check{F}$. Then $F$ is a lifting of an autohomeomorphism of $\mathbb{N}^*$ (in $V[H]$) and for any dense $\mathbb{P}$-ideal $\mathcal{I}$ on $\mathbb{N}$ and for each $\mathbb{P}(\check{F})$-generic filter $G$, there is an $I \in \mathcal{I}$ such that $F|\mathcal{I}((V[H] \cap [\mathbb{N} \setminus I]^{\omega}))$ is $\sigma$-Borel in the extension $V[H][G]$. 
Let $\mathcal{F}$, $\hat{F}$, $H$ and $F$ be as in Lemma 2.3 and consider the situation in the forcing extension $V[H]$. Since $\mathcal{F}$ is generic over $V$, we will see that for each $Y \in [\mathbb{N}]^{\omega}$, there is some $f \in \mathcal{F}$ which decides if $\hat{F}|[Y]^{\omega}$ is trivial. Also, the genericity of $\mathcal{F}$ over $V$, and the fact that no new subsets of $\mathbb{N}$ are added, will ensure that if some $f$ in $\mathcal{F}$ forces that $\hat{F}|[Y]^{\omega}$ is not trivial, then $F|[Y]^{\omega}$ is also not trivial (in $V[H]$). We will assume all these properties of $\mathcal{F}$ and $F$ throughout the paper.

The following proposition is probably well-known but we do not have a reference.

**Proposition 2.4.** Assume that $\mathcal{Q}$ is a ccc $\omega^{\omega}$-bounding poset and that $x$ is an ultrafilter on $\mathbb{N}$. If $G$ is a $\mathcal{Q}$-generic filter then there is no set $A \subset \mathbb{N}$ such that $A \setminus Y$ is finite for all $Y \in x$.

**Proof.** Assume that $\{\dot{a}_n : n \in \omega\}$ are $\mathcal{Q}$-names of integers such that $1 \forces_{\mathcal{Q}} \dot{a}_n \geq n$. Let $A$ denote the $\mathcal{Q}$-name such that $\forces_{\mathcal{Q}} A = \{\dot{a}_n : n \in \omega\}$.

Since $\mathcal{Q}$ is $\omega^{\omega}$-bounding, there is some $q \in \mathcal{Q}$ and a sequence $\{n_k : k \in \omega\}$ in $V$ such that $q \forces_{\mathcal{Q}} n_k \leq \dot{a}_i \leq n_{k+2}$ $\forall i \in [n_k, n_{k+1})$. There is some $l \in 3$ such that $Y = \bigcup_k [n_{3k+l}, n_{3k+l+1})$ is a member of $x$. On the other hand, for each $k$, $q \forces_{\mathcal{Q}} A \cap [n_{3k+l+1}, n_{3k+l+3})$ is not empty. Therefore $q \not\forces_{\mathcal{Q}} \text{ "} A \setminus Y \text{ is finite".} \quad \blacksquare$

Another interesting and useful general lemma is the following.

**Lemma 2.5.** Let $\mathbb{P}$ be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let $H$ and $\mathcal{F}$ be as in Lemma 2.2. Then for each $\mathbb{P}(\mathcal{F})$-name $\dot{h} \in \mathbb{N}^{\mathbb{N}}$ there are an increasing sequence $n_0 < n_1 < \cdots$ of integers and a condition $\check{f} \in \mathcal{F}$ such that either

1. $\check{f} \forces_{\mathbb{P}(\mathcal{F})} \dot{h} \upharpoonright \bigcup\{[n_k, n_{k+1}) : k \in K\} \notin V$ for each infinite $K \subset \omega$, or
2. for each $i \in [n_k, n_{k+1})$ and each $\check{g} < \check{f}$ such that $\check{g}$ forces a value on $h(i)$, $(f_I \cup (g_I[n_k, n_{k+1})))$ also forces a value on $h(i)$.

Furthermore, if $\check{f}$ forces $\dot{h}$ to be finite-to-one, we can arrange that for each $k$ and each $i \in [n_k, n_{k+1})$, $\check{f}$ forces that $\dot{h}(i) \in [n_{k-1}, n_{k+2})$.

**Proof.** Fix any $\check{f} \in \mathbb{P}$. Perform a standard fusion, as in [6, 2.4] or [8, 3.4], to find sequences $\{n_k : k \in \omega\} \subset \omega$ and $\{\check{f}_k : k \in \omega\} \subset \mathbb{P}$ with the following properties. Each $\check{f}_k$ is an $n_k$-preserving extension of $\check{f}_k^{-1}$. Let $j < n_k$ and let $\check{s}, \check{s}^* \in \mathbb{P}$ be such that, for each coordinate $l$ of $\check{f}$, $s_l \subset s^*_l$ and $s_l$ has domain contained in $n_k$. If there is some $n_k$-preserving extension of $\check{s} \upharpoonright \check{f}_k$ which forces a value on $h(j)$, then $\check{s} \upharpoonright \check{f}_k$ already does so. Further, if there is some integer $i \geq n_k$ for which $\check{s}^* \upharpoonright \check{f}_k$ has an $n_k$-preserving extension forcing a value on $h(i)$ while $\check{s} \upharpoonright \check{f}_k$ does not, then there is such an integer below $n_{k+1}$. One also ensures that for each coordinate $l$ there is an $m$ such that
\[ n_k < 2^m < 2^{m+1} < n_{k+1}, \ [2^m, 2^{m+1}) \ \ominus \ \text{dom}(f^k_i) \text{ has at least } k \text{ elements (thus ensuring that the end result of the fusion will be a member of } \mathbb{P}). \]

Let \( \vec{f} \) be the fusion and assume that \( \vec{f} < \vec{f}^0 \) and \( K \in [\omega]^\omega \) are such that \( \vec{f} \) forces a value on \( \hat{h}[n_k, n_{k+1}) \) for all \( k \in K \). We show that the second alternative then holds. By further extending \( \vec{f} \) we can assume that if \( L = \{m : (\exists l) \ [n_m, n_{m+1}) \ \ominus \ \text{dom}(f_i)\} \), then \( K \cap [m, m'] \) is not empty for all \( m < m' \in L \).

Let \( i \in [n_{m'}, n_{m'+1}) \) and let \( \vec{g} < \vec{f} \) force a value on \( \hat{h}(i) \). Assume that \( \langle g_l, [n_{m'}, n_{m'+1}) \rangle \cup \vec{f} \) does not force a value on \( \hat{h}(i) \), and so has no \( n_{m'+1} \)-preserving extension which does.

Let \( m \) be the maximum number of \( L \cap m' \) and choose \( k \in K \cap [m, m'] \). Set \( \vec{s} = (f^i_j | n_k) \) and \( \vec{s}' = (g_i | n_k) \). We note that \( i \) is a witness to the situation that \( \vec{s}' \cup f^{n_k} \) has an \( n_k \)-preserving extension to decide, while \( \vec{s} \cup f^{n_k} \) does not. Therefore, by construction, there should be some \( j < n_{k+1} \) for which this is true. However, this is not the case since \( \vec{f} \) forces a value on \( \hat{h}[n_k, n_{k+1}) \).

If \( \vec{f} \) forces that \( \hat{h} \) is finite-to-one, then \( \vec{f}^0 \) could have been so chosen. In addition, since \( \mathbb{P}(S) \) is \( \omega^\omega \)-bounding we may fix an increasing function \( g \in \omega^\omega \cap V \) such that (if \( \vec{f} \) forces \( \hat{h} \) is finite-to-one) \( \vec{f} \Vdash_{\mathbb{P}(S)} \{ i, \hat{h}(i) \} \cup \hat{h}^{-1}(i) \subset g(i) \). The only change to the fusion is to additionally demand that \( n_{k+1} \) is chosen to be larger than \( g(n_k) \) at each stage. ■

3. The trivial ideal. In this section we establish a result that will guarantee that our autohomeomorphisms of \( \mathbb{N}^* \) will be trivial on every member of a large \( P \)-ideal.

**Lemma 3.1.** Let \( \mathbb{P} \) be a finite product of posets from the set \( \{\mathbb{P}_l : l \in \mathbb{N}\} \), let \( H \) and \( S \) be as in Lemma 2.2, and let \( G \) be \( \mathbb{P}(S) \)-generic over \( V[H] \). Assume that \( b \in V \cap [\omega]^\omega \) is such that \( F[V \cap [b]^\omega] \) is \( \sigma \)-Borel in \( V[G] \). Then, in \( V \), there is an increasing sequence \( \{n_k : k \in \omega\} \subset \omega \) such that \( F \) is trivial on each \( a \in [b]^\omega \) for which there is an \( r \in S \) such that \( a \subset \bigcup\{[n_{k}, n_{k+1}) : [n_k, n_{k+1}) \subset \text{dom}(r)\} \).

**Proof.** For notational convenience we will assume that \( \mathbb{P} \) is simply a single member of \( \{\mathbb{P}_l : l \in \mathbb{N}\} \). The modifications needed to handle a finite product are completely straightforward and will be omitted.

Fix names \( \psi_j \) \( (j \in \omega) \) for the Borel functions. Fix an appropriately large countable elementary submodel \( M \prec H(\theta) \). For easier notation, we may just assume that \( b \) is actually \( \mathbb{N} \). We will use the notation \( p \) with subscripts to refer to members of \( \mathbb{P} \). For a finite set \( t \subset \mathbb{N} \) and \( n \in \mathbb{N} \), we will use \( [t; n] \) to denote the clopen set \( \{a \subset \mathbb{N} : a \cap n = t\} \).
We first want to show that we can assume that each \( \psi_j \) is actually continuous. As is well-known, each Borel function is continuous on a dense \( G_δ \), hence we may fix a sequence \( \{ U_n : n \in \omega \} \) of \( \mathcal{P}(\mathcal{F}) \)-names of a descending sequence of dense open sets such that each \( \psi_j \) is forced to be continuous on the intersection \( \bigcap_n U_n \). We perform a fusion sequence \( \{ p_k : k \in \omega \} \) (as in Lemma 3.1) which selects a sequence of intervals \( \{ [n_k, n_{k+1}) : k \in \omega \} \), and finite sets \( t_k \) contained in \( [n_k, n_{k+1}) \), so that (it is forced by \( p_k \)) that for each \( s \subset n_k \), \( s \cup t_k \) is a subset of \( U_k \). We deal with \( F|V \cap \bigcup_k [n_{2k}, n_{2k+1}] \) (and by symmetry) with \( F|V \cap [\bigcup_k [n_{2k-1}, n_{2k}] \) by replacing, for \( y \subset \bigcup_k [n_{2k}, n_{2k+1}] \), \( \dot{\psi}_j(y) \) with \( \dot{\psi}_j(y \cup [n_k t_{2k}) \setminus F([n_k t_{2k}) \). Thus, we may simply assume that each \( \dot{\psi}_j \) is continuous.

We perform another fusion sequence and produce a new sequence \( \{ n_i : i \in \omega \} \). This also is all done in \( M \). For each \( i \) we will select a subset \( f_i \subset [n_i, n_{i+1}) \) and we are trying to imitate the “forcing a value” idea from [9]. That is, for each \( i \) and each \( s \subset n_i \) and \( t \subset n_i \) and each \( j \leq i \), we arrange that if \( s \cup p_i \) has an \( n_i \)-preserving extension which is able to force a value on \( \dot{\psi}_j[t \cup f_i; n_i+1] | n_i \) (meaning all \( a \in \dot{\psi}_j[t \cup f_i; n_i+1] \) have the same intersection with \( n_i \)), then we do so (i.e. by possibly extending \( f_i \) or by extending \( p_i \) or \( [n_i, \omega] \)). An additional requirement is to further finitely extend \( f_i \), if possible, so that instead, there is some integer \( m \) (which will be made to be less than \( n_i+1 \)) so that \( s \cup p_i \) has no \( n_i \)-preserving extension and \( f_i; n_i \) has no further finite extension \( h; n' \), which will force a value on \( \dot{\psi}_j[t \cup h; n'] | m \). As usual, we also ensure that for each \( i \), there is a unique \( m_i \) such that \( [2^{m_i}, 2^{m_i+1}] \subset [n_i, n_i+1) \) and \( dom(p_i) \supset [n_i, n_i+1) \setminus [2^{m_i}, 2^{m_i+1}] \) and \( [2^{m_i}, 2^{m_i+1}] \setminus dom(p_i) \geq i \).

For \( e \in \{ 0, 1, 2 \} \), let \( f^e = \bigcup_i f_{3i+e} \). Choose a \( p \in M \) such that \( p \) decides the value of \( F(f^e) \) for each such \( e \). We will focus on \( f^0 \) but the following argument can be repeated for \( f^1 \) and \( f^2 \).

We perform another fusion choosing \( \{ i_l : l \in \omega \} \subset \{ 3i : i \in \omega \} \) and conditions \( r_l \). Again, with \( n = n_{i_l} \), for each \( j \leq i_l \), \( s \in n^n \), and \( t \subset n \), we choose \( r = r_l \) to be an \( n \)-preserving extension so that either \( s \cup r \) has forced a value on \( \dot{\psi}_j([t \cup f^0) , \) or there is a \( 3i < i_l+1 \) such that \( s \cup r \) has no \( n \)-preserving extension which forces a value on \( \dot{\psi}_j([t \cup f^0) \cap [n_{3i+1}; n_{3i+1}] | n_{3i} \).

Let \( r \subset p \) extend this final fusion sequence and be an \( (M, \mathcal{P}) \)-generic condition. For each \( s \), let \( s \cup r \in G_s \) be some \( \mathcal{P}(\mathcal{F}) \)-filter which is generic over \( M \). This gives us a countable family of Borel functions \( \{ \text{val}_G_s(\psi_j) : s \in \omega^{<\omega} , j \in \omega \} \) in \( V[H] \) (and in \( V \)).

Let \( L \subset \omega \) be any set of integers such that \( [n_{i_l}, n_{i_l+1}) \subset dom(r) \) for \( l \in L \) and \( L \cap \{ l+1 : l \in L \} \) is empty. Let \( Y \subset \bigcup_{l \in L} [n_{i_l}, n_{i_l+1}) \) be such that, in addition, \( Y \cap [n_{3i}, n_{3i+1}) \) (since we are using \( f^0 \)) is empty for all \( i \). To show that \( F \) is trivial (using Proposition 1.4) on \( [Y]^{\omega} \), we prove that for each
y ⊂ Y, there are s, j such that

$$s ∪ r \Vdash_{P(\emptyset)} "F(y) = * \psi_j(y ∪ f^0) \setminus F(f^0)".$$  

It then follows, since $F(y)$ is an element of V, that $F(y) = \text{val}_{G_s}(\psi_j)(y ∪ f^0) \setminus F(f^0)$. Since all this is taking place in $V[H]$ we find that $F[I][Y]^\omega$ is trivial.

Fix any $r_y < r$ which forces a value on $\dot{F}(y ∪ f^0)$ and forces that this is equal to $\psi_j(y ∪ f^0)$ for some $j ∈ ω$. Fix any $l_0 ∈ L$ such that $j < i_{l_0}$ and set $s = r_y|n_{i_{l_0}}$. More generally, for each $l ∈ L$, let $s_l = r_y|n_{i_l}$. The next three claims complete the proof that $s = s_0$ and $j$ are as needed above.

**Claim 1.** For each $l ∈ ω \setminus (L ∪ l_0)$, $s_l ∪ r$ decides $\psi_j((y ∩ n_{i_l}) ∪ f^0)$.

**Proof.** Let $t = (y ∪ f^0) ∩ n_{i_l}$. Note also that $(y ∪ f^0) ∩ n_{i_{l+1}}$ is equal to $(t ∪ f^0) ∩ n_{i_{l+1}}$ since $l ∉ L$. By assumption and continuity of $ψ_j$, $r_i$ forces a value on $\psi_j[(y ∪ f^0) ∩ n_{i_{l+1}}; n_{i_{l+1}}]|n_{3i_{l+1}}$ for each $3i < i_{l+1}$. Therefore, $s_l ∪ r_l$ did (does) have such an $n_{i_l}$-preserving extension to force values on $\psi_j[(t ∪ f^0) ∩ n_{3i_{l+1}}; 3i_{l+1}]|n_{3i}$ for each $3i < i_{l+1}$. From this, it follows from the choice of $r_l$ that $s_l ∪ r$ does force a value on $\dot{\psi}_j(t ∪ f^0)$.

**Claim 2.** For each $l ∈ ω \setminus l_0$, $s_l ∪ r$ decides $\psi_j((y ∩ n_{i_l}) ∪ f^0)$.

**Proof.** By Claim 1, we may assume that $l ∈ L$ and so $l – 1 ∉ L$. We know by Claim 1 that $s_{l-1} ∪ r$ decides $\psi_j((y ∩ n_{i_{l-1}}) ∪ f^0)$. But since $y ∩ n_{i_l}$ is the same as $y ∩ n_{i_{l-1}}$, it follows that $s_{l-1} ∪ r$ decides $\psi_j((y ∩ n_{i_l}) ∪ f^0)$ since $s_{l-1} ∪ r$ decides it.

**Claim 3.** For each $l ≤ l’ ∈ ω \setminus l_0$, $s_l ∪ r$ decides $\psi_j((y ∩ n_{i_l'}) ∪ f^0)$.

**Proof.** We proceed by induction on $l’$. Assume the claim holds for $l’$ and fails for $l’ + 1$. Let $l$ be maximal such that it fails for $s_l$. We know that $l ≤ l’$ by Claim 2. It follows that we may assume that $t = y ∩ n_{i_{l’+1}} ≠ y ∩ n_{i_{l’}} = t’$, hence $l ∈ L$. When $r_{l’+1}$ was defined, it was asked if $((s_l ∪ r)|n_{i_l}) ∪ r_{l’+1}$ had an $n_{i_l}$-preserving extension which forced a value on $\psi_j(t ∪ f^0)$. Apparently the answer was no. But then, at stage $i = i_l$ in the $\langle p_i : i ∈ ω \rangle$ fusion, it was asked if $t’ ∪ f_i$ had an extension for which $((s_l ∪ r)|n_{i_l}) ∪ p_i$ did not have an $n_{i_l}$-preserving extension to decide arbitrarily far. Well it appears that $t ∪ f^0$ is such an extension (note that $f_i ⊂ f^0$). In this case, $f_i; n_{i_l}$ were chosen so that it has no extension $h; n’$ for which $((s_l ∪ r)|n_{i_l}) ∪ p_i$ has an $n_{i_l}$-preserving extension which will decide $\psi_j(t’ ∪ h; n’) \setminus n_{i_{l+1}}$. But we do know that $s_l ∪ r$ decides $\dot{\psi}_j(t’ ∪ f^0) = \psi_j((y ∩ n_{i_{l’}}) ∪ f^0)$. Therefore at stage $i + 3$, $(s_l ∪ r)|n_{i_{l+3}}$ would have an $n_{i_{l+3}}$-preserving extension forcing a value on $\psi_j[t ∪ f_i ∪ f_{i+3}; n_{i+3}]|n_{i+2}$ (in fact, it would already do so). In particular, $t ∪ f_i; n_{i+1}$ does have an extension, namely $t ∪ f_i ∪ f_{i+3}; n_{i+4}$, for which $(s_l ∪ r)|n_{i_l} ∪ p_i$ does have an $n_{i_l}$-preserving extension forcing a value on $\dot{\psi}_j[t ∪ f_i ∪ f_{i+3}; n_{i+4}]|n_{i+1}$.
This ends the proof of Lemma 3.1. □

**Corollary 3.2 (PFA).** Let $\mathbb{P}$ be a finite product of posets from the set \(\{\mathbb{P}_i : i \in \mathbb{N}\}\). Let $\dot{F}$ be a $\mathbb{P}$-name of a lifting of an autohomeomorphism of $\mathbb{N}^*$. Let $H, \dot{G}, F$, $P$-ideal $I$ and $I' \in \mathcal{I}$ be as in Lemma 2.2. Then there is an increasing sequence \(\{n_k : k \in \omega\} \subseteq \mathbb{N}\) and a $\mathbb{P}(\dot{G})$-name $\dot{h}$ for a function on $\mathbb{N}$ such that for each $f \in \dot{G}$ and $a = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subseteq \text{dom}(f)\}$, $F$ is trivial on $a \setminus I$ and $\mathbb{P}(\dot{G})$ forces that $\dot{h}|(a \setminus I)$ induces $F$.

**Proof.** Let \(\{n_k : k \in \omega\}\) be the sequence as constructed in Lemma 3.1. Let $\mathcal{J}$ denote the dense $P$-ideal consisting of all sets of the form $\bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subseteq \text{dom}(f)\}$ for some $f \in \dot{G}$. Since there is a natural (and obvious) finite-to-one map sending the ideal $\mathcal{J}$ to an ultrafilter, it follows by Proposition 2.4 that $\mathcal{J}$ generates a dense $P$-ideal in the forcing extension by $\mathbb{P}(\dot{G})$. By Lemma 2.2, we know that $F|((V[H] \cap [\mathbb{N} \setminus I]_\omega)$ is $\sigma$-Borel. Let $\mathcal{J}'$ be the ideal \(J \setminus I : J \in \mathcal{J}\). By Lemma 3.1, $F$ is trivial on $J$ for each $J \in \mathcal{J}'$. It then follows easily that, in the forcing extension by $\mathbb{P}(\dot{G})$, $F|\mathcal{J}'$ is also $\sigma$-Borel. Finally, by Lemma 1.5, there is an $\dot{h}$ as required. □

### 4. Proof of Theorem 1.1

**Theorem 4.1 (PFA).** In the forcing extension by $\mathbb{P} = \mathbb{P}_2 \times \mathbb{P}_2$, there are symmetric tie-points $x, y$ as witnessed by $A, B$ and $C, D$ respectively such that $\mathbb{N}^*$ is not homeomorphic to the space $A \rangle_{x=y} C$.

We briefly work in the forcing extension in order to select appropriate names. Let $G \subseteq \mathbb{P}_2 \times \mathbb{P}_2$ be a generic filter. The tie-point $x$ as witnessed by $A, B$ will be the one given canonically by the $\mathbb{P}_2$-generic filter consisting of the first coordinates of $G$ (as per the notation following Definition 1.6). The tie-point $y$ as witnessed by $C, D$ will be given analogously by the second coordinates. More precisely the closed set $A$ will be the closure of the union of the collection \(\{L^*_f : (\exists g) (f, g) \in G\}\), while $B$ will be the closure of the union of the collection \(\{U^*_f : (\exists g) (f, g) \in G\}\). Of course, $x$ is the ultrafilter (a $P_{\omega_2}$-point) generated by the collection \(\{\mathbb{N} \setminus \text{dom}(f) : (\exists g) (f, g) \in G\}\).

Fix any enumeration \(\{a_\alpha : \alpha \in \omega_2\}\) of a mod finite increasing cofinal chain in \(\{L^*_f : (\exists g) (f, g) \in G\}\) and similarly \(\{c_\alpha : \alpha \in \omega_2\}\) for \(\{L^*_g : (\exists f) (f, g) \in G\}\). We may represent $A \rangle_{x=y} C$ as a quotient of \((\mathbb{N} \times 2)^*\) in which, for each $\alpha \in \omega_2$, \((a_\alpha \times \{0\})^* \cup (c_\alpha \times \{1\})^*\) is mapped canonically to $a^*_\alpha \cup c^*_\alpha$ and the rest of the $(\mathbb{N} \times 2)^*$ is collapsed to a point. Assume there is a homeomorphism from this quotient space to $\mathbb{N}^*$ and let $F$ be any lifting, i.e. we may assume that $F$ is a function from $[\mathbb{N}]^\omega$ into $[\mathbb{N} \times 2]^\omega$ such that if we let $Z_\alpha = F^{-1}(a_\alpha \times \{0\} \cup c_\alpha \times \{1\})$ for each $\alpha \in \omega_2$, then $\{Z_\alpha : \alpha \in \omega_2\}$ forms the dual ideal $\mathcal{I}$ to an ultrafilter $z$. 
Fix $\mathbb{P}$-names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathcal{F} \subset \mathbb{P}$, the $\mathbb{P}$-name $\dot{h}$ and the sequence $\{n_k : k \in \omega\}$. There is no loss of generality in this proof to assume that the $I$ mentioned in the statement of Corollary 3.2 is the empty set, and let $\mathcal{J}$ be the ideal as defined in the proof of Corollary 3.2. As we are working in $V[H]$, let us use $\lambda$ to denote the $\omega_2$ from $V$. For each $J \in \mathcal{J}$, there is a function $h_J$ which induces $F$ on $J$; $h_J$ will be a function from $J$ into $(a_\alpha \times \{0\} \cup (c_\alpha \times \{1\})$ for some $\alpha \in \lambda$.

We finish the proof by showing there is no such $h$.

Since $\mathbb{P}(\mathcal{F})$ is $\omega^\omega$-bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $\vec{f}_0 = (g_0, g_1) \in \mathcal{F}$ satisfy:

(1) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathcal{F})} "h(i) \in ([0, n_{k+2}) \times 2)"$,
(2) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathcal{F})} "h^{-1}(\{i\} \times 2) \subset [0, n_{k+2})"$,
(3) for each $k$ and each $j \in \{0, 1\}$ there is an $m$ such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \text{dom}(g_j)$ has at least $k$ elements.

Choose any $(g_0', g_1') = \vec{f}_1 < \vec{f}_0$ in $\mathcal{F}$ such that $\mathbb{N} \setminus \text{dom}(g_0')$ is contained in $\bigcup_k [n_{6k+1}, n_{6k+2})$ and $\mathbb{N} \setminus \text{dom}(g_1') \subset \bigcup_k [n_{6k+4}, n_{6k+5})$. Next, choose any $\vec{f}_2 < \vec{f}_1$ in $\mathcal{F}$ and some $\alpha \in \lambda$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathcal{F})} "\text{dom}(g_0') \subset a_\alpha \cup \text{dom}(g_0') \subset \text{dom}(g_1') \subset c_\alpha \cup \text{dom}(g_1')"$. For each $\gamma \in \lambda$, note that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathcal{F})} "a_\gamma \setminus a_\alpha \subset \mathbb{N} \setminus \text{dom}(g_0')"$ and similarly $\vec{f}_2 \Vdash_{\mathbb{P}(\mathcal{F})} "c_\gamma \setminus c_\alpha \subset \mathbb{N} \setminus \text{dom}(g_1')"$.

Now consider the two disjoint sets: $Y_0 = \bigcup_k [n_{6k}, n_{6k+3})$ and its complement $Y_1$. Since $z$ is the $\mathbb{P}$-name of an ultrafilter, by possibly extending $\vec{f}_2 = (f_0, f_1)$ even more, we may assume there is some $\beta > \alpha$ such that (by symmetry) $\vec{f}_2 \Vdash_{\mathbb{P}} "\text{dom}(g_0') \subset Z_\beta"$, in fact we may assume that $\vec{f}_2 \Vdash_{\mathbb{P}} "\text{dom}(g_0') \subset (L_{f_0} \times \{0\}) \cup (L_{f_1} \times \{1\})"$.

Finally, let $\vec{f}_3 = (f_0', f_1') < \vec{f}_2$ be chosen so that there is an infinite set $L \subset \mathbb{N}$ such that for $k \in L$, $[n_{6k+1}, n_{6k+2}) \subset \text{dom}(f_0')$ and $[n_{6k+1}, n_{6k+2}) \not\subset \text{dom}(f_0)$. Set

$$y = \bigcup_{k \in L} [n_{6k+1}, n_{6k+2}) \cap L_{f_0} \setminus L_{f_0}$$

and choose any $\vec{f}_4 < \vec{f}_3$ and $\tilde{y}$ such that $\vec{f}_4$ forces that $F(\tilde{y}) = y \times \{0\}$. Since $\mathcal{J}$ is a dense ideal, we may fix any $J \in \mathcal{J}$ such that $J \cap \tilde{y}$ is infinite.

It then follows that $h[J \cap \tilde{y}] = \ast F(J \cap \tilde{y}) \subset \ast y \times \{0\}$, and so $J \cap \tilde{y}$ is forced to be contained in $\bigcup_{k \in L} [n_{6k}, n_{6k+3})$ (by the assumption on the sequence of $\{n_k\}$’s). On the other hand, now that $J \cap \tilde{y} \subset Y_0$, we have

$$F(J \cap \tilde{y}) \subset \ast F(Y_0) \cap (\mathbb{N} \times \{0\}) \subset \ast L_{f_0} \times \{0\},$$

contradicting the fact that $y$ is disjoint from $L_{f_0}$. 
5. Proof of Theorem 1.3

**Theorem 5.1 (PFA).** In the forcing extension by $\mathbb{P}_1$, a tie-point $x$ is introduced such that $\tau(x) = 2$ and with $\mathbb{N}^* = A \bowtie B$, neither $A$ nor $B$ is a homeomorph of $\mathbb{N}^*$. In addition, there is no involution $F$ on $\mathbb{N}^*$ which has a unique fixed point, and so no tie-point is symmetric.

We begin by proving that neither $A$ nor $B$ can be homeomorphic to $\mathbb{N}^*$. We proceed much as in the previous section. Let $G \subset \mathbb{P} = \mathbb{P}_1$ be a generic filter. The tie-point $x$ as witnessed by $A, B$ will be the one given canonically by the $\mathbb{P}$-generic filter. More precisely, the closed set $A$ will be the closure of the union of the collection $\{(f^{-1}(0))^* : f \in G\}$, while $B$ will be the closure of the union of the collection $\{f^{-1}(1) : f \in G\}$. Of course, $x$ is the ultrafilter (a $P_{\omega_2}$-point) generated by the collection $\{\mathbb{N} \setminus \text{dom}(f) : f \in G\}$.

Assume there is an autohomeomorphism from $\mathbb{N}^*$ onto $A$ and let $F$ be any lifting, i.e. we may assume that there is an ultrafilter $z \in \mathbb{N}^*$ with dual ideal $\mathcal{I}$ such that $F$ is a function from $\bigcup_{f \in G} [f^{-1}(0)]^\omega$ onto $\bigcup_{I \in \mathcal{I}} [I]^\omega$ such that for each $f \in G$, $F|_{[f^{-1}(0)]^\omega}$ is a lifting of a homeomorphism from $(f^{-1}(0))^*$ onto $I_f^*$ for some $I_f \in \mathcal{I}$, and for each $I \in \mathcal{I}$, there is an $f \in G$ such that $I \subset^* I_f$.

Fix $\mathbb{P}$-names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathcal{F} \subset \mathbb{P}$, the $\mathbb{P}$-name $\dot{h}$ and the sequence $\{n_k : k \in \omega\}$. We obtain a contradiction by showing there can be no such $\dot{h}$.

There is no loss of generality in this proof to assume that the $I$ mentioned in the statement of Corollary 3.2 is the empty set. The ideal denoted $\mathcal{J}$ as defined in the proof of Corollary 3.2 will now be generated by sets of the form $\bigcup\{(n_k, n_{k+1}) \cap f^{-1}(0) : [n_k, n_{k+1}) \subset \text{dom}(f)\}$ for $f \in \mathcal{F}$. It follows that the ideal $\{F(J) : J \in \mathcal{J}\}$ will be a dense ideal in $[\mathbb{N}]^\omega$. For each $J \in \mathcal{J}$, let $h_J$ denote the function on $J$ for which there is some $f_J \in \mathcal{F}$ which forces that $h_J$ induces $F$ on $J$.

Since $\mathbb{P}(\mathcal{F})$ is $\omega^\omega$-bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $f_0 \in \mathcal{F}$ satisfy:

1. for each $i \in [n_k, n_{k+1})$, $f_0 \models_{\mathbb{P}(\mathcal{F})} \langle h(i) \rangle \in [0, n_{k+2}]$,
2. for each $i \in [n_k, n_{k+1})$, $f_0 \models_{\mathbb{P}(\mathcal{F})} \langle h^{-1}(i) \rangle \in [0, n_{k+2}]$,
3. for each $k$ there is an $m$ such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \text{dom} f_0$ has at least $k$ elements.

We need a significant strengthening of Lemma 2.5 which holds for $\mathbb{P} = \mathbb{P}_1$.

**Lemma 5.2.** Assume that $\dot{h}$ is a $\mathbb{P}(\mathcal{F})$-name of a function from $\mathbb{N}$ into $\mathbb{N}$. Either there is an $f \in \mathcal{F}$ such that $f \models_{\mathbb{P}(\mathcal{F})} \langle h|\text{dom}(f) \notin V \rangle$, or there is an $f \in \mathcal{F}$ and an increasing sequence $m_1 < m_2 < \cdots$ of integers such that $\mathbb{N} \setminus \text{dom}(f) = \bigcup_k S_k$ where $S_k \subset 2^{m_{k+1}} \setminus 2^{m_k}$ and for each $i \in S_k$ the
conditions \( f \cup \{(i, 0)\} \) and \( f \cup \{(i, 1)\} \) each force a value on \( \dot{h}(i) \). Furthermore, if \( f \) forces \( \dot{h} \) to be finite-to-one, we can arrange that for each \( k \) and each \( i \in [n_k, n_{k+1}) \), either \( f \) forces a value on \( \dot{h}(i) \), or \( f \) forces that \( \dot{h}(i) \in [n_k, n_{k+1}) \).

**Proof.** First we choose \( f_0 \in \mathcal{F} \) and some increasing sequence \( n_0 < n_1 < \cdots \) as in Lemma 2.5. We may choose, for each \( k \), an \( m_k \) such that \( n_k \leq 2^{m_k} < 2^{m_k+1} < n_{k+1} \) and \( \lim_k |2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(f_0))| = \infty \). For each \( k \), let \( S_k^0 = 2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(f_0)) \). By re-indexing we may assume that \( |S_k^0| \geq k \), and we may arrange that \( \mathbb{N} \setminus \text{dom}(f_0) \) is equal to \( \bigcup_k S_k^0 \) and set \( L_0 = \mathbb{N} \).

For each \( k \in L_0 \), let \( i_k^0 = \min S_k^0 \) and choose any \( f'_1 < f_0 \) such that (by definition of \( \mathbb{P} \)) \( I_0 = \{ i_k^0 : k \in L_0 \} \subset (f'_1)^{-1}(0) \) and (by assumption on \( \dot{h} \)) \( f'_1 \) forces a value on \( \dot{h}(i_k^0) \) for each \( k \in L_0 \). Set \( f_1 = f'_1 \upharpoonright (\mathbb{N} \setminus I_0) \) and for each \( k \in L_0 \), let \( S_k^1 = S_k^0 \setminus \{ (i_k^0) \} \cup \text{dom}(f_1) \). By further extending \( f_1 \) we may also assume that \( \bigcup_{k \leq L_1} S_k^1 = \infty \). Notice that each \( i_k^0 \) is the minimum element of \( S_k^1 \). Again, we may extend \( f_1 \) and assume that \( \mathbb{N} \setminus \text{dom}(f_1) \) is equal to \( \bigcup_{k \in L_1} S_k^1 \). Suppose now we have some infinite \( L_j \), some \( f_j \), and for \( k \in L_j \), an increasing sequence \( \{ i_k^0, i_k^1, \ldots, i_k^{j-1} \} \subset S_k^0 \). Assume further that

\[
S_k^1 \cup \{ i_k^l : l < j \} = S_k^0 \setminus \text{dom}(f_j)
\]

and that \( \lim_{k \in L_j} |S_k^1 \setminus i_k^{j-1}| = \infty \). For each \( k \in L_j \), let \( i_k^j = \min(S_k^1 \setminus \{ i_k^l : l < j \}) \). By a simple recursion of length \( 2^j \), there is an \( f_{j+1} < f_j \) such that, for each \( k \in L_j \), \( \{ i_k^l : l \leq j \} \subset S_k^0 \setminus \text{dom}(f_{j+1}) \) and for each function \( s \) from \( \{ i_k^l : l \leq j \} \) into \( 2 \), the condition \( f_{j+1} \cup s \) forces a value on \( \dot{h}(i_k^l) \). Again find \( L_{j+1} \subset L_j \) so that \( \lim_{k \in L_{j+1}} |S_k^{j+1}| = \infty \) (where \( S_k^{j+1} = S_k \setminus \text{dom}(f_{j+1}) \)) and extend \( f_{j+1} \) so that \( \mathbb{N} \setminus \text{dom}(f_{j+1}) \) is equal to \( \bigcup_{k \in L_{j+1}} S_k^{j+1} \).

We are half-way there. At the end of this fusion, the function \( \bar{f} = \bigcup f_j \) is a member of \( \mathbb{P} \) because for each \( j \) and \( k \in L_{j+1} \), \( 2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(\bar{f})) \) contains \( \{ i_k^0, i_k^1, \ldots, i_k^j \} \). For each \( k \), let \( \bar{S}_k = S_k \setminus \text{dom}(\bar{f}) \); by possibly extending \( \bar{f} \), we may again assume that there is some \( L \) such that \( \lim_{k \in L} |\bar{S}_k| = \infty \). What we have proven about \( \bar{f} \) is that for each \( k \in L \) and each \( i \in \bar{S}_k \) and each function \( s \) from \( i \cap \bar{S}_k \) to \( 2 \), \( \bar{f} \cup s \cup \{ (i, 0) \} \) and \( \bar{f} \cup \{ (i, 1) \} \) each force a value on \( \dot{h}(i) \). By the genericity of \( \mathcal{F} \), there must be such a condition as \( \bar{f} \) in \( \mathcal{F} \).

To finish, simply repeat the same process as above except this time choose maximal values and work down the values in \( \bar{S}_k \). That is, there will be an infinite set \( K \) and a condition \( f^\dagger \) such that for each \( k \in K \), there is a decreasing sequence \( \{ i_k^0, i_k^1, \ldots, i_k^j \} \subset \bar{S}_k \setminus \text{dom}(f^\dagger) \) with \( \lim_k \{ j_k : k \in K \} = \infty \). These will have the property that for each \( k \in K \) and \( j \leq j_k \) and each function \( s : \{ i_k^0, \ldots, i_k^{j-1} \} \to 2 \), each of \( f^\dagger \cup s \cup \{ (i_k^j, 0) \} \) and \( f^\dagger \cup s \cup \{ (i_k^j, 1) \} \) will force a value on \( \dot{h}(i_k^j) \).
Now we show that $f^\dagger \cup \{(i^*_k,e)\}$ ($e \in 2$) forces a value on $\bar{h}(i^*_k)$ as required. If it did not, then we could find extensions $f_0$, $f_1$ of $f^\dagger \cup \{(i^*_k,e)\}$ which force different values on $\bar{h}(i^*_k)$. Let $s_0 = f_0|_{S^*_0 \cap i^*_k}$ and $s_1 = f_1|_{S^*_0 \setminus i^*_k}$. Notice that $\bar{f} \cup s_0 \leq f^\dagger \cup s_0 \leq f^\dagger \cup s_1$ forces a value (hence the same value as that forced by $f_0$) on $\bar{h}(i^*_k)$. This is also true for $f^\dagger \cup s_1$ in that it forces the same value on $\bar{h}(i^*_k)$ as that forced by $f_1$. The contradiction is that $\bar{f} \cup s_0$ and $f^\dagger \cup s_1$ force distinct values on $\bar{h}(i^*_k)$ although they have the common extension $f^\dagger \cup s_0 \cup s_1$.

Returning to the proof of Theorem 5.1, we are ready to use Lemma 5.2 to show that forcing with $P(\mathfrak{F})$ will not introduce undesirable functions $h$, analogously to the argument in Theorem 1.1. By Lemma 5.2, we have the condition $f_0 \in \mathfrak{F}$ and the sequence $S_k (k \in \mathbb{N})$ such that $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$ and that for each $i \in \bigcup_k S_k$, $f_0 \cup \{(i,0)\}$ forces a value (call it $\bar{h}(i)$) on $h(i)$. Therefore, $h$ is a function with domain $\bigcup_k S_k$ in $V$. We may assume that $|S_k| \geq k$ for each $k$. It suffices to find a condition in $\mathbb{P}$ below $f_0$ which forces that there is some $J \in \mathcal{J}$ such that $h,J$ is not extended by $\bar{h}$. It is useful to note that if $Y \subset \bigcup_k S_k$ is such that $\limsup |S_k \setminus Y|$ is infinite, then for any function $g \in 2^Y$, $f_0 \cup g \in \mathbb{P}$.

We first check that $\bar{h}$ is 1-to-1 on a cofinite set. If not, there is an infinite set of pairs $E_j \subset \bigcup_k S_k$, $\bar{h}[E_j]$ is a singleton and such that for each $k$, $S_k \cap \bigcup_j E_j$ has at most two elements. Let $L$ denote the set of $k$ for which $S_k$ meets $\bigcup_j E_j$. By passing to a subcollection of the $E_j$’s we may assume that $L$ has infinite complement. Let $g$ be the function with domain $\bigcup_j E_j$ which is constantly 0. Then $f_0 \cup g$ forces that $\bar{h}$ agrees with $\bar{h}$ on dom$(g)$ and so is not 1-to-1. There is a further extension $f_1$ of $f_0 \cup g$ with the property that $S_k \subset \text{dom}(f_1)$ for all $k \notin L$. Therefore, by virtue of $f_1$, there is some $J \in \mathcal{J}$ which contains $\bigcup_j E_j$. However, this is a contradiction, because apparently $h,J = \bar{h}|J$ does not induce a homeomorphism on $J^*$.

But now that we know that $\bar{h}$ is 1-to-1 we can get a contradiction as follows. Let $f_1 < f_0$ be chosen so as to decide the value of $F(f_0^{-1}(0))$, and let $Y$ denote this value. For each $k$, let $\bar{S}_k = S_k \setminus \text{dom}(f_1)$ and let $L$ be such that $|\bar{S}_k| : k \in L$ diverges to infinity. If $Y \cap \bar{h}[\bigcup_{k \in L} \bar{S}_k]$ is infinite, then there is an infinite set $L_0 \subset L$ (with $L \setminus L_0$ also infinite) such that for each $k \in L_0$, there is an $i_k \in \bar{S}_k$ such that $\bar{h}(i_k) \in Y$. Choose any $f_2 < f_1$ such that $f_2(i_k) = 0$ and $S_k \subset \text{dom}(f_2)$ for all $k \in L_0$. It follows that there is a $J \in \mathcal{J}$ with $\bigcup_{k \in L_0} S_k \cap f_2^{-1}(0) \subset J$ and such that $f_2 \Vdash_{\mathbb{P}} \text{"}F(\{i_k : k \in L_0\}) \cap F(f_0^{-1}(0)) = h,J[\{i_k : k \in L_0\}] \cap Y \text{ is infinite"	extquoteright}$. a contradiction since $\{i_k : k \in L_0\}$ is disjoint from $f^{-1}(0)$. On the other hand, let $L$ be as above and $L_0$ any infinite-cofinite subset. Fix any sequence $\{i_k : k \in L_0\}$ (with each $i_k \in \bar{S}_k$) and select $f_2 < f_1$ so that $f_2(i_k) = 1$ for all $k \in L_0$ and $\bigcup_{k \in L_0} S_k \subset \text{dom}(f_2)$. Set $Y' = \bar{h}[\{i_k : k \in L_0\}]$. Since $\bar{h}$
is 1-to-1 it follows easily that $f_2 \Vdash \forall J \in \mathcal{J} \ F(J) \cap Y$ is finite”. This, of course, is also a contradiction.

Now we consider the possibility that $\tau(x) > 2$. It then follows that one of $A \setminus \{x\}$ or $B \setminus \{x\}$, say the former, can be partitioned into disjoint clopen non-compact sets. Therefore there is some sequence $\{c_\alpha : \alpha \in \omega_2\}$ of $\mathbb{P}$-names such that for each $\alpha < \beta \in \omega_2$, $c_\beta \subset a_\beta$ and $c_\beta \cap a_\alpha = \ast c_\alpha$. In addition, for each $\alpha < \omega_2$ there must be a $\beta \in \omega_2$ such that $c_\beta \setminus a_\alpha$ and $a_\beta \setminus (c_\beta \cup a_\alpha)$ are both infinite.

In this case, we suppose that $H$ and $\mathfrak{F}$ are chosen as in Lemma 2.2, and in the extension by $H$, let $\lambda$ denote the ordinal $\omega_2$ from $V$. In this model we will have a $(\lambda, \lambda)$-gap formed by the families $\{c_\alpha : \alpha \in \lambda\}$ and $\{a_\alpha \setminus c_\alpha : \alpha \in \lambda\}$. Assume that we can show that in the extension obtained by forcing with $\mathbb{P}(\mathfrak{F})$, there is no $C \subset \mathbb{N}$ such that $C \cap a_\alpha = \ast c_\alpha$ for all $\alpha \in \lambda$. In other words, for any cofinal sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$, the collections $\{c_{\alpha_\xi}, a_{\alpha_\xi} \setminus c_{\alpha_\xi} : \xi \in \omega_1\}$ form an $(\omega_1, \omega_1)$-gap. There are well-known ccc posets $Q_1$ (see [1, 4.2]) which “freeze” the gap. What we mean here is that there is a family of $\omega_1$-many dense subsets of the iteration $\omega_2^{<\omega_1} \ast \mathbb{P}(\mathcal{F}) \ast Q_1$ such that if a filter meets them all, then the gap will remain a gap in any proper forcing extension. Finally, if we let $Q_2$ be the $\sigma$-centered poset mentioned after Definition 1.6, there is a filter (meeting $\omega_1$-many dense subsets) on the proper iteration $\omega_2^{<\omega_1} \ast \mathbb{P}(\mathcal{F}) \ast Q_1 \ast Q_2$ which introduces a condition $f \in \mathbb{P}$ which forces that $c_\lambda$ will not exist.

Thus, we will have shown that $\tau(x) = 2$ once we show that there is no $\mathbb{P}(\mathfrak{F})$-name for a set $C$ as above. Equivalently, we assume that $\dot{h}$ is a $\mathbb{P}(\mathfrak{F})$-name for the characteristic function of $\mathbb{N} \setminus C$, and derive a contradiction.

So, given our name $\dot{h}$, we repeat the steps above up to the point where we have $f_0$ and the sequence $\{S_k : k \in \mathbb{N}\}$ so that $f_0 \cup \{(i, 0)\}$ forces a value $\bar{h}(i)$ on $\dot{h}(i)$ for each $i \in \bigcup_k S_k$ and $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$.

Let $Y = \bar{h}^{-1}(0)$ and $Z = \bar{h}^{-1}(1)$. Since $x$ is forced to be an ultrafilter, there is an $f_1 < f_0$ such that $\text{dom}(f_1)$ contains one of $Y$ or $Z$. If $\text{dom}(f_1)$ contains $Y$, then $f_1$ forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 1$, and so $a_\beta \setminus \text{dom}(f_1) \subset^* \mathbb{N} \setminus C$ for all $\beta \in \omega_2$. While if $\text{dom}(f_1)$ contains $Z$, then $f_1$ forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 0$, and so $a_\beta \setminus \text{dom}(f_1) \subset^* C$ for all $\beta \in \omega_2$. However, taking $\beta$ so large that each of $c_\beta \setminus \text{dom}(f_1)$ and $a_\beta \setminus (c_\beta \cup \text{dom}(f_1))$ are infinite shows that no such $\dot{h}$ exists.

Finally, we show that there are no involutions on $\mathbb{N}^*$ which have a unique fixed point. Assume that $\Phi$ is such an involution and that $y$ is the unique fixed point of $\Phi$. Let $F$ be an arbitrary lifting of $\Phi$ to $[\mathbb{N}]^\omega$. Let $\mathcal{I}$ denote the dual ideal to $y$. We first show that $\mathcal{I}$ is a $P$-ideal (i.e. that $y$ is a $P$-point). For each $I \in \mathcal{I}$, $F(I)$ is also in $\mathcal{I}$ and $F(I \cup F(I)) = \ast I \cup F(I)$. So we may let $Z$ denote those $I \in \mathcal{I}$ such that $Z = \ast F(Z)$. Given $Z \in Z$, since
we may assume that each $\psi$ is an involution with a unique fixed point, we may assume that $\dot{\Psi}$ induces $F$ on $Y$. Let $Y = Z_0 \cup \bigcup_{j<k} Z_j$ and define $Z_k = Y_k \cup F(Y_k)$. Therefore $Y_Z = \bigcup_k Y_k$ satisfies $Y_Z \cap F(Y_Z) = * \emptyset$ and $Z = Y_Z \cup F(Y_Z)$. This shows that for each $Z \in Z$ there is a partition of $Z = Z^0 \cup Z^1$ such that $F(Z^0) = * Z^1$. We can show $y$ is a $P$-point. Indeed, if $\{Z_n = Z^0_n \cup Z^1_n : n \in \mathbb{N}\} \subset Z$ are pairwise disjoint, then $y \notin \bigcup_n Z^*_n$ since $F(\bigcup_n (Z^0_n)^*) = \bigcup_n (Z^1_n)^*$ and $\bigcup_n (Z^0_n)^*$ is disjoint from $\bigcup_n (Z^1_n)^*$.

Fix $\mathbb{P}$-names for $F$ and the members of $I$ and let $H$, $F$, $F$ and $\{n_k : k \in \omega\}$ be as given in Lemma 3.1. Also let $J$ denote the ideal as defined in the proof of Corollary 3.2. Hence $J \in J$ if there is an $f \in F$ and $J \subset \bigcup\{n_k,n_{k+1} : [n_k,n_{k+1}) \subset \text{dom}(f)\}$. It is again easily argued that the $I \in I$ as specified in Corollary 3.2 can be assumed to be empty. For each $J \in J$, let $h_J$ be the function on $J$ such that there is an $f \in F$ which forces that $h_J$ induces $F$ on $J$. Let $\dot{h}$ be the $\mathbb{P}(F)$-name as given in Corollary 3.2. Since $F$ is an involution with a unique fixed point, we may assume that $\dot{h}$ is forced to satisfy that $\dot{h}(\dot{h}(i)) = i \neq h(i)$ for all $i$.

The rest of the proof depends on the following modification of Lemma 5.2.

**Claim 4.** There is an $f \in \mathcal{F}$ and a sequence of sets $\{m_k,S_k,T_k : k \in \omega\}$ and mappings $\psi_k : T_k \rightarrow S_k$ such that $S_k \subset 2^{m_k+1} \setminus 2^m \subset [n_k,n_{k+1})$, $T_k \subset [n_k,n_{k+1})$, $\mathbb{N} \setminus \text{dom}(f) = \bigcup_k S_k$, and for each $k$ and $i \in S_k$ and $\tilde{f} < f$, $\tilde{f}$ forces a value on $\dot{h} | \psi_k^{-1}(i)$ iff $i \in \text{dom}(f)$.

Before proving the claim, let us show how this will finish the proof. For each $i \notin \text{dom}(f)$, there are two functions $h^0_i, h^1_i$ with domain $\psi^{-1}(i)$ such that $f \cup \{(i,e)\}$ forces that $h_i \subset \dot{h}$. Since, by assumption, for each $i \in \text{dom}(f)$, $f$ does not already force a value on $\dot{h} | \psi^{-1}(i)$, we can choose $j_i \in \psi^{-1}(i)$ such that $v_i = h^0_i(j_i) \neq h^1_i(j_i) = w_i$. Note that, by our assumption on $\dot{h}$, it also follows that $\psi(v_i) = \psi(w_i) = i$.

Choose $g < f$ which forces a value, $Y$, on $F(\{j_i\}_{i \notin \text{dom}(f)})$. Assume that $Y \cap \{v_i : i \notin \text{dom}(g)\}$ is infinite. It follows easily that there is some $g^\dagger < g$ such that $Y \cap \{v_i : g^\dagger(i) = 1\}$ is infinite and let $J \subset \{i \in \text{dom}(g^\dagger) : g^\dagger(i) = 1\}$ and $v_i \in Y$ be any infinite set such that $\{j_i\}_{i \in J}$ is in $J$. However, this is a contradiction since

$$g^\dagger \models \s“F”(\{j_i : i \in J\}) = * (\dot{h}|_{\{j_i\}_{i \in J}} = \{w_i\}_{i \in I} \subset Y \cap \{v_i\}_{i \in J}$$

The argument when $\{v_i : i \notin \text{dom}(g)\} \setminus Y$ is infinite is similar.

Now we prove Claim 4. For any $k$, condition $g$, and $T \subset [n_k,n_{k+1})$ let $\text{ Orb}(T,g)$ denote the set $\{j : (\exists g' < g)(\exists t \in T) g' \models \s“F”(\{j_i : i \in J\}) = \{w_i\}_{i \in I} \subset Y \cap \{v_i\}_{i \in J}\}$. Fix
any \( k \) and the \( f \) as above that was selected from Lemma 2.5. Let \( g_0 = f \upharpoonright [n_k, n_{k+1}] \) and assume, as we may, that \( S_0^k = [n_k, n_{k+1}] \setminus \text{dom}(g_0) \) is contained in \([2^m, 2^{m+1}]\) for some \( m \). By a simple recursion much as in Lemma 5.2, we can choose increasing sequences \( I_l = \{i_0, i_1, \ldots, i_{l-1}\} \subset S_0^k \) and extensions \( g_l \supset g_{l-1} \supset \cdots \supset g_0 \) so that \( I_l \subset S_l^k = S_0^k \setminus \text{dom}(g_l) \) and \( i_l = \min(S_l^k \setminus I_l) \). In addition, select sets \( T(i_l) \subset [n_k, n_{k+1}] \setminus \bigcup_{j<l} \text{Orb}(T(i_j), g_l) \) of minimum cardinality (at most \( 2^l \)) so that for each \( s : S_l^k \rightarrow 2 \) there is, if possible, a \( t \in T(i_l) \) such that \( s \cup g_l \cup \{(i_j, 0)\} \) and \( s \cup g_l \cup \{(i_j, 1)\} \) force distinct values on \( \hat{h}(t) \). Notice that \( \text{Orb}\{\{t\}, g_l\} \) has cardinality at most \( 2^l \) for each \( t \in \bigcup_{j<l} T(i_j) \). We also require that for each \( s : S_l^k \rightarrow 2 \), each of \( s \cup \{(i_l, 0)\} \cup g_{l+1} \) and \( s \cup \{(i_l, 1)\} \cup g_{l+1} \) force a value on \( \hat{h}(t) \) from that forced by \( s \cup \{(i_l, 0)\} \cup g_{l+1} \).

For each successive \( l \), there is a recursion on \( k \) so that \( f_l = f \cup \bigcup\{g_l^k : k \in \omega\} \) is a condition. If for each \( k \), there is an \( s^k : I_l^k \rightarrow 2 \) for which no suitable \( t \in [n_k, n_{k+1}] \) can be chosen, then it is because the condition \( g = f_k \cup \bigcup_k (s_k^l \cup g_k^l) \) forces a value on \( \hat{h}(t) \) for all \( t \notin \bigcup_k \bigcup_{j<l} \text{Orb}(T(i_j^l), g) \). But if this were the case, then this condition would force a value on \( \hat{h}(t) \) for all \( t \).

After infinitely many steps, we may instead assume that (a new choice of) \( f \) simply has this property: for each \( k \) and each \( S^k = [2^{m_k}, 2^{m_k+1}] \setminus \text{dom}(f) = [n_k, n_{k+1}] \setminus \text{dom}(f) \), there is a sequence \( \{T(i) : i \in S^k\} \) of pairwise disjoint finite subsets of \([n_k, n_{k+1}]\) such that for each \( i \in S^k \) and each \( s : S^k \cap i \rightarrow 2 \), \( s \cup \{(i, 0)\} \cup f \) and \( s \cup \{(i, 1)\} \cup f \) each force a value on \( \hat{h}(T(i)) \) while \( s \cup f \) does not. (We do not need a superscript on the \( T \)'s since they depend only on \( i \) and not on \( k \).) We have also ensured that for \( i \neq i' \), \( \text{Orb}(T(i), f) \) is disjoint from \( \text{Orb}(T(i'), f) \).

Now, much as in Lemma 5.2, we repeat the process but rather than choosing minimal members of \( S^k \) we choose maximal. A new trouble arises in this proof because of the sizes of the sets \( T(i) \), while in Lemma 5.2, each \( T(i) \) was just \( \{i\} \). To overcome this, we will use the next claim.

**Claim 5.** For each \( f_1 < f \) and infinite \( I \subset \mathbb{N} \setminus \text{dom}(f_1) \) and \( K \subset \mathbb{N} \) for which \( \{|I \cap S^k| : k \in K\} \) diverges to infinity, there is an \( f_2 < f_1, I' \subset I \setminus \text{dom}(f_2) \), and \( K' \subset K \) such that \( \{|I' \cap S^k \setminus \text{dom}(f_2)| : k \in K'\} \) diverges to infinity, and for all \( i \in I' \), each of \( f_2 \cup \{(i, 0)\} \) and \( f_2 \cup \{(i, 1)\} \) force a value on \( \hat{h}(T(i)) \).

In order to not lose track of our progress, let us again defer the proof of Claim 5 and first finish the proof of Claim 4.

Let \( K_0 \subset \omega \) be chosen so that \( \{|S^k| : k \in K_0\} \) is strictly increasing. By Claim 5 there is an infinite \( K_1 \subset K_0 \) and an \( f_1 < f \) so that for each \( k \in K_1 \),
there is an $i^k_l \notin \text{dom}(f_1)$ such that $f_1 \cup \{(i^k_l, 0)\}$ and $f_1 \cup \{(i^k_l, 1)\}$ each force a value on $\hat{h}|T(i^k_j)$ and $|S^k \cap i^k_l| > |S^k|/2$.

By induction on $j > 0$, continue to choose $f_j < f_{j-1}$, $i^k_j \in S^k \cap i^k_{j-1} \setminus \text{dom}(f_j)$ for all $k$ in an infinite set $K_j \subset K_{j-1}$ such that the sequence $\{|(S^k \cap i^k_j) \setminus \text{dom}(f_j)| : k \in K_j\}$ diverges to infinity. We require that for each $k \in K_j$ and $s : \{i^k_l : l \leq j\} \to 2$, the condition $s \cup f_j$ forces a value on $\hat{h}|T(i^k_j)$.

We find the sequence $\{i^k_j : k \in K_j\}$ by applying Claim 5 as follows. For each function $\psi : j \to 2$ and each $k \in K_{j-1}$, let $s^k_\psi$ denote the function from $\{i^k_0, \ldots, i^k_{j-1}\}$ to $2$ such that $s^k_\psi(i^k_j) = \psi(j')$ for each $j' < j$. Start by applying Claim 5 with the $f_1$ in Claim 5 being $f_{j-1} \cup \bigcup_{k \in K_{j-1}} s^k_\psi$ for some fixed $\psi \in 2^j$, $K = K_{j-1}$ and, $I = \bigcup_{k \in K_{j-1}} S^k \setminus \text{dom}(f_1)$. Simply apply Claim 5 recursively, each time swapping the values of the $f_1$ used so as to cycle through all the possible $\psi \in 2^j$. After these $2^j$ steps, each time shrinking the $K'$ and the $I'$ we can let $f_j$ be the final condition denoted $f_2$ in Claim 5, $K_j$ be the final set $K'$ and let $\{i^k_j : k \in K_j\}$ be any selection from the final $I'$ which has the additional property that $\{|S^k \cap i^k_j| : k \in K_j\}$ diverges to infinity.

What we have now is that for each $k \in K_j$ and each $\psi \in 2^j$, the conditions $s^k_\psi \cup f_j \cup \{(i^k_j, 0)\}$ and $s^k_\psi \cup f_j \cup \{(i^k_j, 1)\}$ each force a value on $\hat{h}|T(i^k_j)$.

When this recursion finishes, let $\{k_j : j \in \omega\}$ be chosen so that $k_j \in K_j$ for each $j$. Set $\tilde{f} = \bigcup_j f_j|\{n_{k_j}, n_{k_j+1}\}$. Note that for each $j$, $k = k_j$ and $l < j$, $\tilde{f} \cup f_j$ forces a value on $\hat{h}|T(i^k_j)$. Assume $g < \tilde{f}$, $k = k_j$ and $l < j$. Let $s = g|(S^k \setminus 1 + i^k_l)$ and $s' = g|(S^k \setminus 1 + i^k_j)$.

There is a $t \in T(i^k_l)$ such that $s \cup f_j \cup \{(i^k_l, 0)\}$ and $s \cup f_j \cup \{(i^k_l, 1)\}$ force different values on $\hat{h}(t)$. Therefore if $i^k_l \notin \text{dom}(g)$, then $g$ cannot decide $\hat{h}(t)$. On the other hand, suppose $i^k_l$ is in $\text{dom}(g)$, let $e = g(i^k_l)$ and assume that there is a $t \in T(i^k_l)$ such that $\hat{h}(t)$ is not decided by $g$. Fix any $s_1 : \{i^k_j : j < l\} \to 2$ extending $s'$ such that $s_1 \cup g$ forces a value on $\hat{h}(t)$ and let $v$ be this value. Since $g$ does not decide $\hat{h}(t) = v$, there is some $s_2 : S^k \setminus i^k_j \to 2$ which extends $s$ and forces a value distinct from $v$ on $\hat{h}(t)$. This is a contradiction since $s_1 \cup s_2 \cup g$ is a condition.

We have shown that for each $k_j$ and $l < j$, $g$ forces a value on $\hat{h}|T(i^k_l)$ iff $i^k_l \in \text{dom}(g)$.

It remains to give the following proof.

**Proof of Claim 5.** Let $f_1$ and $I$ be as in the statement of the claim and assume there is no such $f_2$ and $I'$. Let $\mathcal{I} = \{I' \subset [I]^{\omega} : \{|I' \cap S^k|\} \text{ is bounded}\}$ and $\mathcal{I}^* = \{I' \subset [I]^{\omega} : \{|I' \cap S^k|\} \text{ diverges to infinity}\}$. For each $I' \subset I$, let $K(I') = \{k : I' \cap S^k \neq \emptyset\}$. For any set $I$, let $\chi_0(I)$ (respectively $\chi_1(I)$) denote the function which is constantly $0$ (respectively $1$) on $I$. 


Choose, if possible, $e \in \{0, 1\}$ (say $e = 0$) and some pair $f_2 < f_1$ and $I_2 \subset I \setminus \text{dom}(f_2)$ such that $I_2 \in \mathcal{I}^*$ and $f_2 \cup \chi_0(I_2)$ forces a value on $\hat{h}(T(i))$ for all $i \in I_2$. If no such $e$ exists, then let $f_2 = f_1$ and $I_2 = I$. It now follows (in either case) that for any $f_3 < f_2$ and $I' \subset I_2$, the set of $i \in I'$ for which $f_3 \setminus (\text{dom}(f_3) \setminus I') \cup \chi_1(I')$ forces a value on $\hat{h}(i)$ is a member of $\mathcal{I}$.

For each integer $k$, let $S_k$ be the set of partial functions from $S^k$ into 2. For integers $l, k$ and condition $g$, let

$$S(l, k, g) = \{s \in S_k : g|S^k \subset s \text{ and } |S^k \setminus \text{dom}(s)| > l\}.$$ 

For $s \in S(l, k, f_2 \cup \chi_1(I_2))$, let $I(s)$ be the set of $i \in I_2 \cap S_k$ such that $s \cup f_2$ forces a value on $\hat{h}(T(i))$. Assume that for each $l$, $\{|I(s)| : s \in \bigcup_{k \in K(I_2)} S(l, k, f_2 \cup \chi_1(I_2))\}$ is unbounded. We could then find an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ and corresponding $s(k_l) \in S(l, k_l, f_2 \cup \chi_1(I_2))$ with $\{|I(s(k_l))| : l \in \omega\}$ diverging, in which case the condition $f_2 \cup \chi_1(I_2) \cup \bigcup_{l} s(k_l)$ would be guilty of forcing a value on $\hat{h}(T(i))$ for each $i \in \bigcup_l I(s(k_l)) \in \mathcal{I}^*$—a contradiction.

Therefore there is some $l_0$ such that for all $k \in K(I_2)$, and $s \in S(l_0, k, f_2 \cup \chi_1(I_2))$, the set $I(s)$ has cardinality less than $l_0$. Now choose an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ so that $|I_2 \cap S_k| \leq l_0 + 2^{2^l}$. Choose any condition $f^+ < f_2 \cup \chi_1(I_2)$ so that $\mathbb{N} \setminus \text{dom}(f^+) \subset \bigcup_l S_k$ and $|S_k \setminus \text{dom}(f^+)| = l$ for all $l$. Notice that $S(l_0, k_l, f^+)$ has cardinality at most $2^{2^l}$. For each $l$ and $s \in S(l_0, k_l, f^+)$, choose an $i_s \in I_2 \cap S_k$ such that $s \cup f^+$ does not force a value on $\hat{h}(T(i_s))$. Ensure that the selection is such that $i_s \neq i_{s'}$ for distinct $s, s' \in S(l_0, k_l, f^+)$. Next, for each $l$ and $s \in S(l_0, k_l, f^+)$, choose $t_s \in T(i_s)$ and distinct $u_s, w_s$ each with the property that there is some extension of $s \cup f^+$ forcing $\hat{h}(t_s)$ to have that value.

We now define an ultrafilter in $\mathbb{N}^*$. For each $g$, let $X(g) = \{i_s : s \in \bigcup_l S(l_0, k_l, g)\}$, $U(g) = \{u_s : i_s \in X(g)\}$, and $W(g) = \{w_s : i_s \in X(g)\}$. Let $z \in \mathbb{N}^*$ be any ultrafilter which extends the family $\{X(g) : g \in \mathfrak{F}, g < f^+\}$.

Since $U(f^+) \cap W(f^+)$ is empty, there is some $g < f^+$ such that either $g \not\vdash \text{"}U(f^+) \notin z\text{"}$ or $g \not\vdash \text{"}W(f^+) \notin z\text{"}$ (by symmetry assume $U(f^+) \notin z$). By possibly extending $g$, there is an $X \in x$ such that $g \not\vdash \text{"}F(X) \cap U(f^+) = \emptyset\text{"}$. Since $X \in X(g)$ is infinite we can choose an infinite set $L \subset \mathbb{N}$ such that for each $l \in L$, $s_l = g|S_k \in S(l_0, k_l, g)$ and $s_l \in X$. For each $l \in L$, let $s^*_l \in S_l$ be chosen so that $s_l \subset s^*_l$ and $s^*_l \cup f^+$ forces $\hat{h}(i_{s_l}) = u_{s_l}$. By genericity of $\mathfrak{F}$, there is a $g^+ < g$ such that $L' = \{l \in L : S_k \subset \text{dom}(g^+)\}$ and $s^*_l \subset g^+|S_k$ is infinite. Since

$$g^+ \not\vdash \text{"}\{u_{s_l}\}_{l \in L'} = \hat{h}([i_{s_l}]_{l \in L'}) = \ast F([i_{s_l}]_{l \in L'}) \subset \ast \mathbb{N} \setminus U(f^+)\text{"},$$

we have our contradiction.

This completes the proof of Claim 5. $\blacksquare$
6. Questions

**Question 6.1.** Assume PFA. If $G$ is $\mathbb{P}_2$-generic, and $\mathbb{N}^* = A \bowtie^x B$ is the generic tie-point introduced by $\mathbb{P}_2$, is it true that $A$ is not homeomorphic to $\mathbb{N}^*$? Is it true that $\tau(x) = 2$? Is it true that each tie-point is a symmetric tie-point?

**Remark 1.** The tie-point $x_3$ introduced by $\mathbb{P}_3$ does not satisfy $\tau(x_3) = 3$. This can be seen as follows. For each $f \in \mathbb{P}_3$, we can partition $L_f$ into $\{i \in \text{dom}(f) : i < f(i) < f^2(i)\}$ and $\{i \in \text{dom}(f) : i < f^2(i) < f(i)\}$.

It seems then that the tie-points $x_l$ introduced by $\mathbb{P}_l$ might be better characterized by the property that there is an autohomeomorphism $F_l$ of $\mathbb{N}^*$ such that $\text{fix}(F_l) = \{x_l\}$, and each $y \in \mathbb{N}^* \setminus \{x\}$ has an orbit of size $l$.

**Remark 2.** A small modification to the poset $\mathbb{P}_2$ will result in a tie-point $\mathbb{N}^* = A \bowtie^x B$ such that $A$ (hence the quotient space by the associated involution) is homeomorphic to $\mathbb{N}^*$. The modification is to build into the conditions a map from the pairs $\{i, f(i)\}$ into $\mathbb{N}$. A natural way to do this is to set $f \in \mathbb{P}_2^+$ if $f$ is a 2-to-1 function such that for each $n$, $f$ maps $\text{dom}(f) \cap (2^{n+1} \setminus 2^n)$ into $2^n \setminus 2^{n-1}$, and again $\limsup_n 2^{n+1} \setminus (\text{dom}(f) \cup 2^n)$ is order by almost containment. The generic filter introduces an $\omega_2$-sequence $\{f_\alpha : \alpha \in \omega_2\}$ and two ultrafilters: $x \supset \{\mathbb{N} \setminus \text{dom}(f_\alpha) : \alpha \in \omega_2\}$ and $z \supset \{\mathbb{N} \setminus \text{range}(f_\alpha) : \alpha \in \omega_2\}$. For each $\alpha$ and $a_\alpha = \{i \in \text{dom}(f_\alpha) : i = \min(f_\alpha^{-1}(f_\alpha(i)))\}$, we set $A = \{x\} \cup \bigcup_\alpha a_\alpha^*$ and $B = \{x\} \cup \bigcup_\alpha (\text{dom}(f_\alpha) \setminus a_\alpha)^*$; then $\mathbb{N}^* = A \bowtie^x B$ is a symmetric tie-point. Finally, the map $F : A \to \mathbb{N}^*$ defined by $F(x) = z$ and $F' A \setminus \{x\} = \bigcup_\alpha (f_\alpha)^*$ is a homeomorphism.

**Question 6.2.** Assume PFA. If $L$ is a finite subset of $\mathbb{N}$ and $\mathbb{P}_L = \prod\{\mathbb{P}_l : l \in L\}$, is it true in $V[G]$ that there is a finite upper bound to $\tau(x)$ for the tie-points $x$; and if $1 \notin L$, then every tie-point is a symmetric tie-point?

**Acknowledgements.** Research of the first author was supported by NSF grant No. NSF-DMS 20060114. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant no. 2002323), and by NSF grant No. NSF-DMS 0600940.

**References**


—, —, Martin’s axiom is consistent with the existence of nowhere trivial automorphisms, Proc. Amer. Math. Soc. 130 (2002), 2097–2106.


University of North Carolina at Charlotte
Charlotte, NC 28223, U.S.A.
E-mail: adow@uncc.edu

Department of Mathematics
Rutgers University
Hill Center
Piscataway, NJ 08854-8019, U.S.A.
E-mail: shelah@math.rutgers.edu

Current address:
Institute of Mathematics
Hebrew University
Givat Ram, Jerusalem 91904, Israel

Received 28 August 2007;
in revised form 14 August 2008 and 3 January 2009