FUNDAMENTA
MATHEMATICAE
203 (2009)

More on tie-points and homeomorphism in $\mathbb{N}^*$

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Abstract. A point x is a (bow) tie-point of a space X if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with x in its closure. We denote this as $X = A \underset{x}{\bowtie} B$ where A, B are the closed sets which have a unique common accumulation point x. Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$ (by Veličković and Shelah & Steprāns) and in the recent study (by Levy and Dow & Techanie) of precisely 2-to-1 maps on $\mathbb{N}^*$. In these cases the tie-points have been the unique fixed point of an involution on $\mathbb{N}^*$. One application of the results in this paper is the consistency of there being a 2-to-1 continuous image of $\mathbb{N}^*$ which is not a homeomorph of $\mathbb{N}^*$.

1. Introduction. A point x is a *tie-point* of a space X if there are closed sets A, B of X such that $X = A \cup B$, $\{x\} = A \cap B$ and x is an adherent point of both A and B. We let $X = A \underset{x}{\bowtie} B$ denote this relation and say that x is a tie-point *as witnessed by* A, B. Let $A \equiv_x B$ mean that there is a homeomorphism from A to B with x as a fixed point. If $X = A \underset{x}{\bowtie} B$ and $A \equiv_x B$, then there is an involution Φ of X (i.e. $\Phi^2 = \Phi$) such that $\{x\} = \mathrm{fix}(\Phi)$. In this case we will say that x is a *symmetric tie-point of* X.

Let Φ be a continuous function from $\mathbb{N}^*$ into $\mathbb{N}^*$. Of course, Φ^{-1} can be regarded as a function from the clopen subsets of $\mathbb{N}^*$ into the clopen subsets of $\mathbb{N}^*$. A function F from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ is a *lifting* of Φ if $F(a)^* = \Phi^{-1}(a^*)$ for all $a \subset \mathbb{N}$. A function h is said to *induce* F (and/or Φ) on I if $F(a) =^* h[a] = \{h(n) : n \in a\}$ for all $a \subset I$. The function F is said to be *trivial* on I if there is such a function h. Since the fixed point set of a trivial autohomeomorphism is clopen, a symmetric tie-point gives rise to a non-trivial autohomeomorphism. An ideal on $\mathbb{N}$ is a *P-ideal* if it is countably directed closed mod finite.

2000 *Mathematics Subject Classification*: 03A50, 54A25, 54D35.
Key words and phrases: automorphism, Stone–Čech, fixed points.
This is paper number B917 in the second author's personal listing.

If A and B are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \underset{x=y}{\bowtie} B$ denote the quotient space of $A \oplus B$ obtained by identifying x and y, and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

In this paper we establish the following theorem.

THEOREM 1.1. *It is consistent that* $\mathbb{N}^*$ *has symmetric tie-points* x, y *as witnessed by* A, B *and* A', B' *respectively such that* $\mathbb{N}^*$ *is not homeomorphic to the space* $A \underset{x=y}{\bowtie} A'$.

COROLLARY 1.2. *It is consistent that there is a* 2*-to-*1 *image of* $\mathbb{N}^*$ *which is not a homeomorph of* $\mathbb{N}^*$.

One can generalize the notion of tie-point and, for a point $x \in \mathbb{N}^*$, consider how many disjoint clopen subsets of $\mathbb{N}^* \setminus \{x\}$ (each accumulating to x) can be found. Let us say that a tie-point x of $\mathbb{N}^*$ satisfies $\tau(x) \geq n$ if $\mathbb{N}^* \setminus \{x\}$ can be partitioned into n disjoint clopen subsets each accumulating to x. Naturally, we will let $\tau(x) = n$ denote that $\tau(x) \geq n$ and $\tau(x) \not\geq n+1$. It follows easily from [2, 5.1] that each point x of character ω_1 in $\mathbb{N}^*$ is a tie-point and satisfies $\tau(x) \geq n$ for all n. Similarly each P-point of character ω_1 in $\mathbb{N}^*$ is a symmetric tie-point. We list several open questions in the final section.

THEOREM 1.3. *It is consistent that* $\mathbb{N}^*$ *has a tie-point* x *such that* $\tau(x) = 2$ *and* $\mathbb{N}^* = A \underset{x}{\bowtie} B$, *where neither* A *nor* B *is a homeomorph of* $\mathbb{N}^*$. *In addition, there are no symmetric tie-points.*

The following theorem of [10] provides an important equivalent condition for the triviality of autohomeomorphisms on $\mathbb{N}^*$ and it will allow us to utilize the results of Steprāns's paper [8].

LEMMA 1.4 (Veličković). *If* $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *is a lifting of an autohomeomorphism and there exist Borel functions* $\{\psi_n : n \in \omega\}$ *and a comeager set* $G \subset \mathcal{P}(\mathbb{N})$ *such that for every* $A \in G$ *there is* $n \in \omega$ *such that* $\psi_n(A) =^* F(A)$, *then* F *is trivial.*

This is Theorem 2 of [10] except that the strengthening to the case of a comeager set G is from [8, 2.1]. The topology on $\mathcal{P}(\mathbb{N})$ is the standard one induced by identifying each set $a \subset \mathbb{N}$ with its characteristic function $\chi_a \in 2^{\mathbb{N}}$. For a set $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ and a function F on $\mathcal{P}(\mathbb{N})$, let us say that $F{\restriction}\mathcal{C}$ is *σ-Borel* if there is sequence $\{\psi_n : n \in \omega\}$ of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for each $b \in \mathcal{C}$, there is an n such that $F(b) =^* \psi_n(b)$.

The following lemma is also implicit in [10, 1.3]:

LEMMA 1.5. *If* F *is a lifting of an autohomeomorphism of* $\mathbb{N}^*$ *and if* F *is trivial on each member of a* P*-ideal* $\mathcal{I}$ *for which* $F{\restriction}\mathcal{I}$ *is σ-Borel*, *then there is a function* h *which induces* F *on each member of* $\mathcal{I}$.

The following partial order $\mathbb{P}_2$ was introduced by Veličković in [10]. Our need for this poset is articulated in [8] where it is described as a poset which was introduced "to add a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ while doing as little else as possible—at least assuming PFA".

Definition 1.6. The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions $f : A \to B$ where

- $A \subseteq \omega$ and $B \subseteq \omega \setminus A$,
- for all $i \in \omega$ and $n \in \omega$, $f(i) \in 2^{n+1} \setminus 2^n$ if and only if $i \in 2^{n+1} \setminus 2^n$,
- $\limsup_{n\to\omega} |(2^{n+1} \setminus 2^n) \setminus (A \cup B)| = \omega$.

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

We define some trivial generalizations of $\mathbb{P}_2$. We use the notation $\mathbb{P}_2$ to signify that this poset introduces an involution of $\mathbb{N}^*$ because the condition $g = f \cup f^{-1}$ implies that $g^2 = g$. In the definition of $\mathbb{P}_2$ it is possible to suppress mention of A, B (which we do) and to have the poset $\mathbb{P}_2$ consist simply of the functions g (and A as $L_g = \{i \in \mathrm{dom}(g) : i < g(i)\}$, and B as $U_g = \{i \in \mathrm{dom}(g) : g(i) < i\}$).

Let $\mathbb{P}_1$ denote the poset we get if we omit mention of f consisting only of disjoint pairs (A, B), satisfying the growth condition in Definition 1.6, and extension is coordinatewise mod finite containment. For more consistent notation, we will instead represent the elements of $\mathbb{P}_1$ as partial functions into 2.

More generally, let $\mathbb{P}_l$ be similar to $\mathbb{P}_2$ except that we assume that conditions consist of functions g such that $\{i, g(i), g^2(i), \ldots, g^l(i)\}$ is contained in $l^{n+1} \setminus l^n$ and has precisely l elements for all $i \in \mathrm{dom}(g) \cap l^{n+1} \setminus l^n$.

The basic properties of $\mathbb{P}_2$ as defined by Veličković and treated by Shelah and Steprāns are also true of $\mathbb{P}_l$ for all $l \in \mathbb{N}$.

In particular, for example, the following is easily seen:

Proposition 1.7. *If* $L \subset \mathbb{N}$ *and* $\mathbb{P} = \prod_{l\in L} \mathbb{P}_l$ *(with full supports) and* G *is a* $\mathbb{P}$*-generic filter, then in* $V[G]$*, for each* $l \in L$*, there is a tie-point* $x_l \in \mathbb{N}^*$ *with* $\tau(x_l) \geq l$.

For the proof of Theorem 1.1 we use $\mathbb{P}_2 \times \mathbb{P}_2$ and for the proof of Theorem 1.3 we use $\mathbb{P}_1$. In any such $\mathbb{P}$ and $\vec{f}, \vec{g} \in \mathbb{P}$, we say that $\vec{f} = \langle f_l \rangle$ is an *n-preserving extension* of $\vec{g} = \langle g_l \rangle$, for an integer n, if for each coordinate l, $f_l \restriction n = g_l \restriction n$ and $f_l \supset g_l$. Also, if $\vec{s} = \langle s_l \rangle$ is a sequence of functions (usually with finite domain), then we define $\vec{s} \sqcup \vec{f}$ to be the sequence $\vec{g} = \langle g_l \rangle$ where, for each coordinate l,

$$g_l = s_l \sqcup f_l \equiv s_l \cup (f_l \restriction \mathrm{dom}(f_l) \setminus \mathrm{dom}(s_l)).$$

2. Preliminaries. Each poset $\mathbb{P}$ as above is $\aleph_1$-closed and, if PFA holds, $\aleph_2$-distributive (see [8, p. 4226]). In this paper we will restrict our study to finite products. The following partial order can be used to show that these products are $\aleph_2$-distributive.

DEFINITION 2.1. Let $\mathbb{P}$ be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Given $\mathfrak{F} \subset \mathbb{P}$, define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $g \in \mathbb{P}$ such that there is some $\vec{f} \in \mathfrak{F}$ such that $\vec{g} \equiv^* \vec{f}$. The ordering on $\mathbb{P}(\mathfrak{F})$ is coordinatewise $\supseteq$ as opposed to ${}^*\supseteq$ in $\mathbb{P}$.

If $\mathfrak{F}$ is downward directed (in fact it will be a descending sequence), then the forcing $\mathbb{P}(\mathfrak{F})$ introduces a tuple $\vec{f}$ such that $\vec{f} \leq \vec{f'}$ for all $\vec{f'} \in \mathfrak{F}$. Although $\vec{f}$ itself may not be a member of $\mathbb{P}$, it is simply because the domains of the component functions are too big. Following [6, 2.1], one must then use a σ-centered poset which will choose an appropriate sequence $\vec{f}^*$ of subfunctions of $\vec{f}$ which is a member of $\mathbb{P}$ and which is still below each member of $\mathfrak{F}$.

A strategic choice of the sequence $\mathfrak{F}$ will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc, but remarkably even more is true. Again we are lifting results from [6, 2.6] and [8, proof of Thm. 3.1] which introduced this innovative factoring of Veličković's original amoeba forcing poset and showed that it seems to preserve more properties. Let $\omega_2^{<\omega_1}$ denote the standard collapse which introduces a function from ω_1 onto ω_2.

A poset is said to be ω^ω-*bounding* if every new function in ω^ω is bounded by some ground model function.

LEMMA 2.2. *Let $\mathbb{P}$ be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. In the forcing extension, $V[H]$, by $\omega_2^{<\omega_1}$, there is a descending sequence $\mathfrak{F}$ from $\mathbb{P}$ which is $\mathbb{P}$-generic over V and for which $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω-bounding.*

It was also shown in [6] that $\mathfrak{F}$ can be chosen so that it, in addition, preserves that $\mathbb{R} \cap V$ is of second category. This is crucial for the proof of Lemma 2.3. We can manage with the ω^ω-bounding property because we are going to use Lemma 2.3. An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is said to be *dense* if each infinite subset of $\mathbb{N}$ contains a member of $\mathcal{I}$.

The following main result is extracted from [6] and [8, Theorem 3.3] which we record without proof.

LEMMA 2.3 (PFA). *Let $\mathbb{P}$ be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let $\dot{F}$ be a $\mathbb{P}$-name of a lifting of an autohomeomorphism of $\mathbb{N}^*$. Let $\mathfrak{F}$ and H be as in Lemma 2.2 and let F be the valuation of $\dot{F}$ by $\mathfrak{F}$. Then F is a lifting of an autohomeomorphism of $\mathbb{N}^*$ (in $V[H]$) and for any dense P-ideal $\mathcal{I}$ on $\mathbb{N}$ and for each $\mathbb{P}(\mathfrak{F})$-generic filter G, there is an $I \in \mathcal{I}$ such that $F{\restriction}(V[H] \cap [\mathbb{N} \setminus I]^\omega)$ is σ-Borel in the extension $V[H][G]$.*

Let $\mathfrak{F}$, $\dot{F}$, H and F be as in Lemma 2.3 and consider the situation in the forcing extension $V[H]$. Since $\mathfrak{F}$ is generic over V, we will see that for each $Y \in [\mathbb{N}]^\omega$, there is some $f \in \mathfrak{F}$ which decides if $\dot{F}\restriction[Y]^\omega$ is trivial. Also, the genericity of $\mathfrak{F}$ over V, and the fact that no new subsets of $\mathbb{N}$ are added, will ensure that if some f in $\mathfrak{F}$ forces that $\dot{F}\restriction[Y]^\omega$ is not trivial, then $F\restriction[Y]^\omega$ is also not trivial (in $V[H]$). We will assume all these properties of $\mathfrak{F}$ and F throughout the paper.

The following proposition is probably well-known but we do not have a reference.

Proposition 2.4. *Assume that $\mathbb{Q}$ is a ccc ω^ω-bounding poset and that x is an ultrafilter on $\mathbb{N}$. If G is a $\mathbb{Q}$-generic filter then there is no set $A \subset \mathbb{N}$ such that $A \setminus Y$ is finite for all $Y \in x$.*

Proof. Assume that $\{\dot{a}_n : n \in \omega\}$ are $\mathbb{Q}$-names of integers such that $1 \Vdash_{\mathbb{Q}}$ "$\dot{a}_n \geq n$". Let A denote the $\mathbb{Q}$-name such that $\Vdash_{\mathbb{Q}}$ "$A = \{\dot{a}_n : n \in \omega\}$". Since $\mathbb{Q}$ is ω^ω-bounding, there is some $q \in \mathbb{Q}$ and a sequence $\{n_k : k \in \omega\}$ in V such that $q \Vdash_{\mathbb{Q}}$ "$n_k \leq \dot{a}_i \leq n_{k+2}\ \forall i \in [n_k, n_{k+1})$". There is some $l \in 3$ such that $Y = \bigcup_k [n_{3k+l}, n_{3k+l+1})$ is a member of x. On the other hand, for each k, $q \Vdash_{\mathbb{Q}}$ "$A \cap [n_{3k+l+1}, n_{3k+l+3})$ is not empty". Therefore $q \not\Vdash_{\mathbb{Q}}$ "$A \setminus Y$ is finite". ∎

Another interesting and useful general lemma is the following.

Lemma 2.5. *Let $\mathbb{P}$ be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let H and $\mathfrak{F}$ be as in Lemma 2.2. Then for each $\mathbb{P}(\mathfrak{F})$-name $\dot{h} \in \mathbb{N}^\mathbb{N}$ there are an increasing sequence $n_0 < n_1 < \cdots$ of integers and a condition $\vec{f} \in \mathfrak{F}$ such that either*

(1) $\vec{f} \Vdash_{\mathbb{P}(\mathfrak{F})}$ *"$\dot{h}\restriction\bigcup\{[n_k, n_{k+1}) : k \in K\} \notin V$" for each infinite $K \subset \omega$, or*
(2) *for each $i \in [n_k, n_{k+1})$ and each $\vec{g} < \vec{f}$ such that $\vec{g}$ forces a value on $\dot{h}(i)$, $\langle f_l \cup (g_l\restriction[n_k, n_{k+1}))\rangle$ also forces a value on $\dot{h}(i)$.*

Furthermore, if $\vec{f}$ forces $\dot{h}$ to be finite-to-one, we can arrange that for each k and each $i \in [n_k, n_{k+1})$, $\vec{f}$ forces that $\dot{h}(i) \in [n_{k-1}, n_{k+2})$.

Proof. Fix any $\vec{f} \in \mathbb{P}$. Perform a standard fusion, as in [6, 2.4] or [8, 3.4], to find sequences $\{n_k : k \in \omega\} \subset \omega$ and $\{\vec{f^k} : k \in \omega\} \subset \mathbb{P}$ with the following properties. Each $\vec{f^k}$ is an n_k-preserving extension of $\vec{f^{k-1}}$. Let $j < n_k$ and let $\vec{s}, \vec{s^*} \in \mathbb{P}$ be such that, for each coordinate l of $\mathbb{P}$, $s_l \subset s^*_l$ and s_l has domain contained in n_k. If there is some n_k-preserving extension of $\vec{s} \sqcup \vec{f^k}$ which forces a value on $\dot{h}(j)$, then $\vec{s} \sqcup \vec{f^k}$ already does so. Further, if there is some integer $i \geq n_k$ for which $\vec{s^*} \sqcup \vec{f^k}$ has an n_k-preserving extension forcing a value on $\dot{h}(i)$ while $\vec{s} \sqcup \vec{f^k}$ does not, then there is such an integer below n_{k+1}. One also ensures that for each coordinate l there is an m such that

$n_k < 2^m < 2^{m+1} < n_{k+1}$, $[2^m, 2^{m+1}) \setminus \mathrm{dom}(f_l^k)$ has at least k elements (thus ensuring that the end result of the fusion will be a member of $\mathbb{P}$).

Let $\vec{f'}$ be the fusion and assume that $\underline{\vec{f}} < \vec{f'}$ and $K \in [\omega]^\omega$ are such that $\underline{\vec{f}}$ forces a value on $\dot{h}\restriction[n_k, n_{k+1})$ for all $k \in K$. We show that the second alternative then holds. By further extending $\underline{\vec{f}}$ we can assume that if $L = \{m : (\exists l)\ [n_m, n_{m+1}) \not\subset \mathrm{dom}(\underline{f}_l)\}$, then $K \cap [m, m']$ is not empty for all $m < m' \in L$.

Let $i \in [n_{m'}, n_{m'+1})$ and let $\vec{g} < \underline{\vec{f}}$ force a value on $\dot{h}(i)$. Assume that $\langle g_l\restriction[n_{m'}, n_{m'+1})\rangle \sqcup \underline{\vec{f}}$ does not force a value on $\dot{h}(i)$, and so has no $n_{m'+1}$-preserving extension which does.

Let m be the maximum member of $L \cap m'$ and choose $k \in K \cap [m, m']$. Set $\vec{s} = \langle f'_l\restriction n_k\rangle$ and $\vec{s}\,' = \langle \vec{g}_l\restriction n_k\rangle$. We note that i is a witness to the situation that $\vec{s}\,' \sqcup \vec{f}^{n_k}$ has an n_k-preserving extension to decide, while $\vec{s} \sqcup \vec{f}^{n_k}$ does not. Therefore, by construction, there should be some $j < n_{k+1}$ for which this is true. However, this is not the case since $\underline{\vec{f}}$ forces a value on $\dot{h}\restriction[n_k, n_{k+1})$.

If $\vec{f}$ forces that $\dot{h}$ is finite-to-one, then $\vec{f}^0$ could have been so chosen. In addition, since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding we may fix an increasing function $g \in \omega^\omega \cap V$ such that (if $\vec{f}$ forces $\dot{h}$ is finite-to-one) $\vec{f} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\{i, \dot{h}(i)\} \cup \dot{h}^{-1}(i) \subset g(i)$". The only change to the fusion is to additionally demand that n_{k+1} is chosen to be larger than $g(n_k)$ at each stage. ∎

3. The trivial ideal. In this section we establish a result that will guarantee that our autohomeomorphisms of $\mathbb{N}^*$ will be trivial on every member of a large P-ideal.

Lemma 3.1. *Let $\mathbb{P}$ be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$, let H and $\mathfrak{F}$ be as in Lemma 2.2, and let G be $\mathbb{P}(\mathfrak{F})$-generic over $V[H]$. Assume that $b \in V \cap [\mathbb{N}]^\omega$ is such that $F\restriction[V \cap [b]^\omega]$ is σ-Borel in $V[G]$. Then, in V, there is an increasing sequence $\{n_k : k \in \omega\} \subset \omega$ such that F is trivial on each $a \in [b]^\omega$ for which there is an $r \in \mathfrak{F}$ such that $a \subset \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(r)\}$.*

Proof. For notational convenience we will assume that $\mathbb{P}$ is simply a single member of $\{\mathbb{P}_l : l \in \mathbb{N}\}$. The modifications needed to handle a finite product are completely straightforward and will be omitted.

Fix names $\dot{\psi}_j$ ($j \in \omega$) for the Borel functions. Fix an appropriately large countable elementary submodel $M \prec H(\theta)$. For easier notation, we may just assume that b is actually $\mathbb{N}$. We will use the notation p with subscripts to refer to members of $\mathbb{P}$. For a finite set $t \subset \mathbb{N}$ and $n \in \mathbb{N}$, we will use $[t; n]$ to denote the clopen set $\{a \subset \mathbb{N} : a \cap n = t\}$.

We first want to show that we can assume that each $\dot\psi_j$ is actually continuous. As is well-known, each Borel function is continuous on a dense G_δ, hence we may fix a sequence $\{\dot U_n : n \in \omega\}$ of $\mathbb{P}(\mathfrak{F})$-names of a descending sequence of dense open sets such that each $\dot\psi_j$ is forced to be continuous on the intersection $\bigcap_n \dot U_n$. We perform a fusion sequence $\{p_k : k \in \omega\}$ (as in Lemma 3.1) which selects a sequence of intervals $\{[n_k, n_{k+1}) : k \in \omega\}$, and finite sets t_k contained in $[n_k, n_{k+1})$, so that (it is forced by p_k that) for each $s \subset n_k$, $[s \cup t_k; n_{k+1}]$ is a subset of $\dot U_k$. We deal with $F\restriction V \cap [\bigcup_k [n_{2k}, n_{2k+1})]^\omega$ (and by symmetry) with $F\restriction V \cap [\bigcup_k [n_{2k-1}, n_{2k})]^\omega$) by replacing, for $y \subset \bigcup_k [n_{2k}, n_{2k+1})$, $\dot\psi_j(y)$ with $\dot\psi_j(y \cup \bigcup_k t_{2k}) \setminus F(\bigcup_k t_{2k})$. Thus, we may simply assume that each $\dot\psi_j$ is continuous.

We perform another fusion sequence and produce a new sequence $\{n_i : i \in \omega\}$. This also is all done in M. For each i we will select a subset $f_i \subset [n_i, n_{i+1})$ and we are trying to imitate the "forcing a value" idea from [9]. That is, for each i and each $s \in n_i^{n_i}$ and $t \subset n_i$ and each $j \le i$, we arrange that if $s \sqcup p_i$ has an n_i-preserving extension which is able to *force a value* on $\dot\psi_j[t \cup f_i; n_{i+1}]\restriction n_i$ (meaning all $a \in \dot\psi_j[t \cup f_i; n_{i+1}]$ have the same intersection with n_i), then we do so (i.e. by possibly extending f_i or by extending $p_i\restriction[n_i, \infty)$). An additional requirement is to further finitely extend f_i, if possible, so that instead, there is some integer m (which will be made to be less than n_{i+1}) so that $s \sqcup p_i$ has no n_i-preserving extension and $f_i; n_i$ has no further finite extension $h; n'$, which will force a value on $\dot\psi_j[t \cup h; n']\restriction m$. As usual, we also ensure that for each i, there is a unique m_i such that $[2^{m_i}, 2^{m_i+1}) \subset [n_i, n_{i+1})$ and $\mathrm{dom}(p_i) \supset [n_i, n_{i+1}) \setminus [2^{m_i}, 2^{m_i+1})$ and $|[2^{m_i}, 2^{m_i+1}) \setminus \mathrm{dom}(p_i)| \ge i$.

For $e \in \{0, 1, 2\}$, let $f^e = \bigcup_i f_{3i+e}$. Choose a $p \in M$ such that p decides the value of $F(f^e)$ for each such e. We will focus on f^0 but the following argument can be repeated for f^1 and f^2.

We perform another fusion choosing $\{i_l : l \in \omega\} \subset \{3i : i \in \omega\}$ and conditions r_l. Again, with $n = n_{i_l}$, for each $j \le i_l$, $s \in n^n$, and $t \subset n$, we choose $r = r_l$ to be an n-preserving extension so that either $s \sqcup r$ has forced a value on $\dot\psi_j(t \cup f^0)$, or there is a $3i < i_{l+1}$ such that $s \sqcup r$ has no n-preserving extension which forces a value on $\dot\psi_j[(t \cup f^0) \cap n_{3i+1}; n_{3i+1}]\restriction n_{3i}$.

Let $r < p$ extend this final fusion sequence and be an $(M, \mathbb{P})$-generic condition. For each s, let $s \sqcup r \in G_s$ be some $\mathbb{P}(\mathfrak{F})$-filter which is generic over M. This gives us a countable family of Borel functions $\{\mathrm{val}_{G_s}(\dot\psi_j) : s \in \omega^{<\omega}, j \in \omega\}$ in $V[H]$ (and in V).

Let $L \subset \omega$ be any set of integers such that $[n_{i_l}, n_{i_{l+1}}) \subset \mathrm{dom}(r)$ for $l \in L$ and $L \cap \{l + 1 : l \in L\}$ is empty. Let $Y \subset \bigcup_{l \in L} [n_{i_l}, n_{i_{l+1}})$ be such that, in addition, $Y \cap [n_{3i}, n_{3i+1})$ (since we are using f^0) is empty for all i. To show that F is trivial (using Proposition 1.4) on $[Y]^\omega$, we prove that for each

$y \subset Y$, there are s, j such that

$$s \sqcup r \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}F(y) =^* \psi_j(y \cup f^0) \setminus F(f^0)\text{”} \quad .$$

It then follows, since $F(y)$ is an element of V, that $F(y) = \mathrm{val}_{G_s}(\dot{\psi}_j)(y \cup f^0) \setminus F(f^0)$. Since all this is taking place in $V[H]$ we find that $F\restriction[Y]^\omega$ is trivial.

Fix any $r_y < r$ which forces a value on $\dot{F}(y \cup f^0)$ and forces that this is equal to $\dot{\psi}_j(y \cup f^0)$ for some $j \in \omega$. Fix any $l_0 \in L$ such that $j < i_{l_0}$ and set $s = r_y\restriction n_{i_{l_0}}$. More generally, for each $l \in L$, let $s_l = r_y\restriction n_{i_l}$. The next three claims complete the proof that $s = s_0$ and j are as needed above.

Claim 1. *For each $l \in \omega \setminus (L \cup l_0)$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$.*

Proof. Let $t = (y \cup f^0) \cap n_{i_l}$. Note also that $(y \cup f^0) \cap n_{i_{l+1}}$ is equal to $(t \cup f^0) \cap n_{i_{l+1}}$ since $l \notin L$. By assumption and continuity of $\dot{\psi}_j$, r_y forces a value on $\dot{\psi}_j[(y \cup f^0) \cap n_{i_{l+1}}; n_{i_{l+1}}]\restriction n_{3i+1}$ for each $3i < i_{l+1}$. Therefore, $s_l \sqcup r_l$ did (does) have such an n_{i_l}-preserving extension to force values on $\dot{\psi}_j[(t \cup f^0) \cap n_{3i+1}; n_{3i+1}]\restriction n_{3i}$ for each $3i < i_{l+1}$. From this, it follows from the choice of r_l that $s_l \sqcup r$ does force a value on $\dot{\psi}_j(t \cup f^0)$. ∎

Claim 2. *For each $l \in \omega \setminus l_0$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$.*

Proof. By Claim 1, we may assume that $l \in L$ and so $l - 1 \notin L$. We know by Claim 1 that $s_{l-1} \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l - 1}) \cup f^0)$. But since $y \cap n_{i_l}$ is the same as $y \cap n_{i_{l-1}}$, it follows that $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$ since $s_{l-1} \sqcup r$ decides it. ∎

Claim 3. *For each $l \le l' \in \omega \setminus l_0$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_{l'}}) \cup f^0)$.*

Proof. We proceed by induction on l'. Assume the claim holds for l' and fails for $l' + 1$. Let l be maximal such that it fails for s_l. We know that $l \le l'$ by Claim 2. It follows that we may assume that $t = y \cap n_{i_{l'+1}} \ne y \cap n_{i_{l'}} = t'$, hence $l \in L$. When $r_{l'+1}$ was defined, it was asked if $((s_l \sqcup r)\restriction n_i) \sqcup r_{l'+1}$ had an n_i-preserving extension which forced a value on $\psi_j(t \cup f^0)$. Apparently the answer was no. But then, at stage $i = i_l$ in the $\langle p_i : i \in \omega\rangle$ fusion, it was asked if $t' \cup f_i$ had an extension for which $((s_l \cup r)\restriction n_i) \sqcup p_i$ did not have an n_i-preserving extension to decide arbitrarily far. Well it appears that $t \cup f^0$ is such an extension (note that $f_i \subset f^0$). In this case, $f_i; n_i$ were chosen so that it has no extension $h; n'$ for which $((s_l \cup r)\restriction n_i) \sqcup p_i$ has an n_i-preserving extension which will decide $\psi[t' \cup h; n']\restriction n_{i+1}$. But we do know that $s_l \sqcup r$ decides $\dot{\psi}_j(t' \cup f^0) = \dot{\psi}_j((y \cap n_{i_{l'}}) \cup f^0)$. Therefore at stage $i + 3$, $(s_l \sqcup r)\restriction n_{i+3}$ would have an n_{i+2}-preserving extension forcing a value on $\dot{\psi}_j[t \cup f_i \cup f_{i+3}; n_{i+3}]\restriction n_{i+2}$ (in fact, it would already do so). In particular, $t \cup f_i; n_{i+1}$ does have an extension, namely $t \cup f_i \cup f_{i+3}; n_{i+4}$, for which $(s_l \sqcup r)\restriction n_i \cup p_i$ does have an n_i-preserving extension forcing a value on $\dot{\psi}_j[t \cup f_i \cup f_{i+3}; n_{i+4}]\restriction n_{i+1}$. ∎

This ends the proof of Lemma 3.1. ∎

COROLLARY 3.2 (PFA). *Let* $\mathbb{P}$ *be a finite product of posets from the set* $\{\mathbb{P}_l : l \in \mathbb{N}\}$. *Let* $\dot{F}$ *be a* $\mathbb{P}$*-name of a lifting of an autohomeomorphism of* $\mathbb{N}^*$. *Let* H, $\mathfrak{F}$, F, P*-ideal* $\mathcal{I}$ *and* $I \in \mathcal{I}$ *be as in Lemma 2.2. Then there is an increasing sequence* $\{n_k : k \in \omega\} \subset \mathbb{N}$ *and a* $\mathbb{P}(\mathfrak{F})$*-name* $\dot{h}$ *for a function on* $\mathbb{N}$ *such that for each* $f \in \mathfrak{F}$ *and* $a = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(f)\}$, F *is trivial on* $a \setminus I$ *and* $\mathbb{P}(\mathfrak{F})$ *forces that* $\dot{h}\restriction(a \setminus I)$ *induces* F.

Proof. Let $\{n_k : k \in \omega\}$ be the sequence as constructed in Lemma 3.1. Let $\mathcal{J}$ denote the dense P-ideal consisting of all sets of the form $\bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(f)\}$ for some $f \in \mathfrak{F}$. Since there is a natural (and obvious) finite-to-one map sending the ideal $\mathcal{J}$ to an ultrafilter, it follows by Proposition 2.4 that $\mathcal{J}$ generates a dense P-ideal in the forcing extension by $\mathbb{P}(\mathfrak{F})$. By Lemma 2.2, we know that $F\restriction(V[H] \cap [\mathbb{N} \setminus I]^\omega)$ is σ-Borel. Let $\mathcal{J}^I$ be the ideal $\{J \setminus I : J \in \mathcal{J}\}$. By Lemma 3.1, F is trivial on J for each $J \in \mathcal{J}^I$. It then follows easily that, in the forcing extension by $\mathbb{P}(\mathfrak{F})$, $F\restriction\mathcal{J}^I$ is also σ-Borel. Finally, by Lemma 1.5, there is an $\dot{h}$ as required. ∎

4. Proof of Theorem 1.1

THEOREM 4.1 (PFA). *In the forcing extension by* $\mathbb{P} = \mathbb{P}_2 \times \mathbb{P}_2$, *there are symmetric tie-points* x, y *as witnessed by* A, B *and* C, D *respectively such that* $\mathbb{N}^*$ *is not homeomorphic to the space* $A \underset{x=y}{\bowtie} C$.

We briefly work in the forcing extension in order to select appropriate names. Let $G \subset \mathbb{P}_2 \times \mathbb{P}_2$ be a generic filter. The tie-point x as witnessed by A, B will be the one given canonically by the $\mathbb{P}_2$-generic filter consisting of the first coodinates of G (as per the notation following Definition 1.6). The tie-point y as witnessed by C, D will be given analogously by the second coordinates. More precisely the closed set A will be the closure of the union of the collection $\{L_f^* : (\exists g)\ (f, g) \in G\}$, while B will be the closure of the union of the collection $\{U_f^* : (\exists g)\ (f, g) \in G\}$. Of course, x is the ultrafilter (a P_{ω_2}-point) generated by the collection $\{\mathbb{N} \setminus \mathrm{dom}(f) : (\exists g)\ (f, g) \in G\}$.

Fix any enumeration $\{a_\alpha : \alpha \in \omega_2\}$ of a mod finite increasing cofinal chain in $\{L_f : (\exists g)\ (f, g) \in G\}$ and similarly $\{c_\alpha : \alpha \in \omega_2\}$ for $\{L_g : (\exists f)\ (f, g) \in G\}$. We may represent $A \underset{x=y}{\bowtie} C$ as a quotient of $(\mathbb{N} \times 2)^*$ in which, for each $\alpha \in \omega_2$, $(a_\alpha \times \{0\})^* \cup (c_\alpha \times \{1\})^*$ is mapped canonically to $a_\alpha^* \cup c_\alpha^*$ and the rest of the $(\mathbb{N} \times 2)^*$ is collapsed to a point. Assume there is a homeomorphism from this quotient space to $\mathbb{N}^*$ and let F be any lifting, i.e. we may assume that F is a function from $[\mathbb{N}]^\omega$ into $[\mathbb{N} \times 2]^\omega$ such that if we let $Z_\alpha = F^{-1}(a_\alpha \times \{0\} \cup c_\alpha \times \{1\})$ for each $\alpha \in \omega_2$, then $\{Z_\alpha : \alpha \in \omega_2\}$ forms the dual ideal $\mathcal{I}$ to an ultrafilter z.

Fix $\mathbb{P}$-names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathfrak{F} \subset \mathbb{P}$, the $\mathbb{P}$-name $\dot{h}$ and the sequence $\{n_k : k \in \omega\}$. There is no loss of generality in this proof to assume that the I mentioned in the statement of Corollary 3.2 is the empty set, and let $\mathcal{J}$ be the ideal as defined in the proof of Corollary 3.2. As we are working in $V[H]$, let us use λ to denote the ω_2 from V. For each $J \in \mathcal{J}$, there is a function h_J which induces F on J; h_J will be a function from J into $(a_\alpha \times \{0\} \cup (c_\alpha \times \{1\})$ for some $\alpha \in \lambda$.

We finish the proof by showing there is no such $\dot{h}$.

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $\vec{f_0} = (g_0, g_1) \in \mathfrak{F}$ satisfy:

(1) for each $i \in [n_k, n_{k+1})$, $\vec{f_0} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(i) \in ([0, n_{k+2}) \times 2)$",
(2) for each $i \in [n_k, n_{k+1})$, $\vec{f_0} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}^{-1}(\{i\} \times 2) \subset [0, n_{k+2})$",
(3) for each k and each $j \in \{0,1\}$ there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \operatorname{dom} g_j$ has at least k elements.

Choose any $(g'_0, g'_1) = \vec{f_1} < \vec{f_0}$ in $\mathfrak{F}$ such that $\mathbb{N} \setminus \operatorname{dom}(g'_0)$ is contained in $\bigcup_k [n_{6k+1}, n_{6k+2})$ and $\mathbb{N} \setminus \operatorname{dom}(g'_1) \subset \bigcup_k [n_{6k+4}, n_{6k+5})$. Next, choose any $\vec{f_2} < \vec{f_1}$ in $\mathfrak{F}$ and some $\alpha \in \lambda$ such that $\vec{f_2} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\operatorname{dom}(g'_0) \subset^* a_\alpha \cup g'_0[a_\alpha]$ and $\operatorname{dom}(g'_1) \subset^* c_\alpha \cup g'_1[c_\alpha]$". For each $\gamma \in \lambda$, note that $\vec{f_2} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$a_\gamma \setminus a_\alpha \subset^* \mathbb{N} \setminus \operatorname{dom}(g'_0)$" and similarly $\vec{f_2} \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$c_\gamma \setminus c_\alpha \subset^* \mathbb{N} \setminus \operatorname{dom}(g'_1)$".

Now consider the two disjoint sets: $Y_0 = \bigcup_k [n_{6k}, n_{6k+3})$ and its complement Y_1. Since z is the $\mathbb{P}$-name of an ultrafilter, by possibly extending $\vec{f_2} = (f_0, f_1)$ even more, we may assume there is some $\beta > \alpha$ such that (by symmetry) $\vec{f_2} \Vdash_{\mathbb{P}}$ "$Y_0 \subset^* Z_\beta$", in fact we may assume that $\vec{f_2} \Vdash_{\mathbb{P}}$ "$F(Y_0) \subset^* (L_{f_0} \times \{0\}) \cup (L_{f_1} \times \{1\})$".

Finally, let $\vec{f_3} = (f'_0, f'_1) < \vec{f_2}$ be chosen so that there is an infinite set $L \subset \mathbb{N}$ such that for $k \in L$, $[n_{6k+1}, n_{6k+2}) \subset \operatorname{dom}(f'_0)$ and $[n_{6k+1}, n_{6k+2}) \not\subset \operatorname{dom}(f_0)$. Set

$$y = \bigcup_{k \in L} [n_{6k+1}, n_{6k+2}) \cap L_{f'_0} \setminus L_{f_0}$$

and choose any $\vec{f_4} < \vec{f_3}$ and $\widetilde{y}$ such that $\vec{f_4}$ forces that $F(\widetilde{y}) = y \times \{0\}$. Since $\mathcal{J}$ is a dense ideal, we may fix any $J \in \mathcal{J}$ such that $J \cap \widetilde{y}$ is infinite.

It then follows that $\dot{h}[J \cap \widetilde{y}] =^* F(J \cap \widetilde{y}) \subset^* y \times \{0\}$, and so $J \cap \widetilde{y}$ is forced to be contained in $\bigcup_{k \in L} [n_{6k}, n_{6k+3})$ (by the assumption on the sequence of $\{n_k\}$'s). On the other hand, now that $J \cap \widetilde{y} \subset Y_0$, we have

$$F(J \cap \widetilde{y}) \subset^* F(Y_0) \cap (\mathbb{N} \times \{0\}) \subset^* L_{f_0} \times \{0\},$$

contradicting the fact that y is disjoint from L_{f_0}.

5. Proof of Theorem 1.3

THEOREM 5.1 (PFA). *In the forcing extension by* $\mathbb{P}_1$, *a tie-point* x *is introduced such that* $\tau(x) = 2$ *and with* $\mathbb{N}^* = A \underset{x}{\bowtie} B$, *neither* A *nor* B *is a homeomorph of* $\mathbb{N}^*$. *In addition, there is no involution* F *on* $\mathbb{N}^*$ *which has a unique fixed point, and so no tie-point is symmetric.*

We begin by proving that neither A nor B can be homeomorphic to $\mathbb{N}^*$. We proceed much as in the previous section. Let $G \subset \mathbb{P} = \mathbb{P}_1$ be a generic filter. The tie-point x as witnessed by A, B will be the one given canonically by the $\mathbb{P}$-generic filter. More precisely, the closed set A will be the closure of the union of the collection $\{(f^{-1}(0))^* : f \in G\}$, while B will be the closure of the union of the collection $\{f^{-1}(1) : f \in G\}$. Of course, x is the ultrafilter (a P_{ω_2}-point) generated by the collection $\{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$.

Assume there is an autohomeomorphism from $\mathbb{N}^*$ onto A and let F be any lifting, i.e. we may assume that there is an ultrafilter $z \in \mathbb{N}^*$ with dual ideal $\mathcal{I}$ such that F is a function from $\bigcup_{f\in G}[f^{-1}(0)]^\omega$ onto $\bigcup_{I\in\mathcal{I}}[I]^\omega$ such that for each $f \in G$, $F{\restriction}[f^{-1}(0)]^\omega$ is a lifting of a homeomorphism from $(f^{-1}(0))^*$ onto I_f^* for some $I_f \in \mathcal{I}$, and for each $I \in \mathcal{I}$, there is an $f \in G$ such that $I \subset^* I_f$.

Fix $\mathbb{P}$-names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathcal{F} \subset \mathbb{P}$, the $\mathbb{P}$-name $\dot{h}$ and the sequence $\{n_k : k \in \omega\}$. We obtain a contradiction by showing there can be no such $\dot{h}$.

There is no loss of generality in this proof to assume that the I mentioned in the statement of Corollary 3.2 is the empty set. The ideal denoted $\mathcal{J}$ as defined in the proof of Corollary 3.2 will now be generated by sets of the form $\bigcup\{[n_k, n_{k+1}) \cap f^{-1}(0) : [n_k, n_{k+1}) \subset \mathrm{dom}(f)\}$ for $f \in \mathfrak{F}$. It follows that the ideal $\{F(J) : J \in \mathcal{J}\}$ will be a dense ideal in $[\mathbb{N}]^\omega$. For each $J \in \mathcal{J}$, let h_J denote the function on J for which there is some $f_J \in \mathfrak{F}$ which forces that h_J induces F on J.

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $f_0 \in \mathfrak{F}$ satisfy:

(1) for each $i \in [n_k, n_{k+1})$, $f_0 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(i) \in [0, n_{k+2})$",
(2) for each $i \in [n_k, n_{k+1})$, $f_0 \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}^{-1}(i) \in [0, n_{k+2})$",
(3) for each k there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \mathrm{dom}\, f_0$ has at least k elements.

We need a significant strengthening of Lemma 2.5 which holds for $\mathbb{P} = \mathbb{P}_1$.

LEMMA 5.2. *Assume that* $\dot{h}$ *is a* $\mathbb{P}(\mathfrak{F})$*-name of a function from* $\mathbb{N}$ *into* $\mathbb{N}$. *Either there is an* $f \in \mathfrak{F}$ *such that* $f \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}{\restriction}\mathrm{dom}(f) \notin V$", *or there is an* $f \in \mathfrak{F}$ *and an increasing sequence* $m_1 < m_2 < \cdots$ *of integers such that* $\mathbb{N} \setminus \mathrm{dom}(f) = \bigcup_k S_k$ *where* $S_k \subset 2^{m_{k+1}} \setminus 2^{m_k}$ *and for each* $i \in S_k$ *the* *conditions* $f\cup\{(i,0)\}$ *and* $f\cup\{(i,1)\}$ *each force a value on* $\dot h(i)$. *Furthermore, if* f *forces* $\dot h$ *to be finite-to-one, we can arrange that for each* k *and each* $i\in[n_k,n_{k+1})$, *either* f *forces a value on* $\dot h(i)$, *or* f *forces that* $\dot h(i)\in[n_k,n_{k+1})$.

Proof. First we choose $f_0\in\mathfrak{F}$ and some increasing sequence $n_0<n_1<\cdots$ as in Lemma 2.5. We may choose, for each k, an m_k such that $n_k\le 2^{m_k}<2^{m_k+1}<n_{k+1}$ and $\lim_k|2^{m_k+1}\setminus(2^{m_k}\cup\mathrm{dom}(f_0))|=\infty$. For each k, let $S^0_k=2^{m_k+1}\setminus(2^{m_k}\cup\mathrm{dom}(f_0))$. By re-indexing we may assume that $|S^0_k|\ge k$, and we may arrange that $\mathbb{N}\setminus\mathrm{dom}(f_0)$ is equal to $\bigcup_k S^0_k$ and set $L_0=\mathbb{N}$. For each $k\in L_0$, let $i^0_k=\min S^0_k$ and choose any $f'_1<f_0$ such that (by definition of $\mathbb{P}$) $I_0=\{i^0_k:k\in L_0\}\subset(f'_1)^{-1}(0)$ and (by assumption on $\dot h$) f'_1 forces a value on $\dot h(i^0_k)$ for each $k\in L_0$. Set $f_1=f'_1\restriction(\mathbb{N}\setminus I_0)$ and for each $k\in L_0$, let $S^1_k=S^0_k\setminus(\{i^0_k\}\cup\mathrm{dom}(f_1))$. By further extending f_1 we may also assume that $f_1\cup\{(i^0_k,1)\}$ also forces a value on $\dot h(i^0_k)$. Choose $L_1\subset L_0$ such that $\lim_{k\in L_1}|S^1_k|=\infty$. Notice that each i^0_k is the minimum element of S^1_k. Again, we may extend f_1 and assume that $\mathbb{N}\setminus\mathrm{dom}(f_1)$ is equal to $\bigcup_{k\in L_1}S^1_k$. Suppose now we have some infinite L_j, some f_j, and for $k\in L_j$, an increasing sequence $\{i^0_k,i^1_k,\ldots,i^{j-1}_k\}\subset S^0_k$. Assume further that

$$S^j_k\cup\{i^l_k:l<j\}=S^0_k\setminus\mathrm{dom}(f_j)$$

and that $\lim_{k\in L_j}|S^j_k\setminus i^{j-1}_k|=\infty$. For each $k\in L_j$, let $i^j_k=\min(S^j_k\setminus\{i^l_k:l<j\})$. By a simple recursion of length 2^j, there is an $f_{j+1}<f_j$ such that, for each $k\in L_j$, $\{i^l_k:l\le j\}\subset S^0_k\setminus\mathrm{dom}(f_{j+1})$ and for each function s from $\{i^l_k:l\le j\}$ into 2, the condition $f_{j+1}\cup s$ forces a value on $\dot h(i^j_k)$. Again find $L_{j+1}\subset L_j$ so that $\lim_{k\in L_{j+1}}|S^{j+1}_k|=\infty$ (where $S^{j+1}_k=S^0_k\setminus\mathrm{dom}(f_{j+1})$) and extend f_{j+1} so that $\mathbb{N}\setminus\mathrm{dom}(f_{j+1})$ is equal to $\bigcup_{k\in L_{j+1}}S^{j+1}_k$.

We are half-way there. At the end of this fusion, the function $\bar f=\bigcup_j f_j$ is a member of $\mathbb{P}$ because for each j and $k\in L_{j+1}$, $2^{m_k+1}\setminus(2^{m_k}\cup\mathrm{dom}(\bar f))\supset\{i^0_k,\ldots,i^j_k\}$. For each k, let $\bar S_k=S^0_k\setminus\mathrm{dom}(\bar f)$; by possibly extending $\bar f$, we may again assume that there is some L such that $\lim_{k\in L}|\bar S_k|=\infty$. What we have proven about $\bar f$ is that for each $k\in L$ and each $i\in\bar S_k$ and each function s from $i\cap\bar S_k$ to 2, $\bar f\cup s\cup\{(i,0)\}$ and $\bar f\cup\{(i,1)\}$ each force a value on $\dot h(i)$. By the genericity of $\mathfrak{F}$, there must be such a condition as $\bar f$ in $\mathfrak{F}$.

To finish, simply repeat the same process as above except this time choose maximal values and work down the values in $\bar S_k$. That is, there will be an infinite set K and a condition $f^\dagger$ such that for each $k\in K$, there is a decreasing sequence $\{i^0_k,i^1_k,\ldots,i^{j_k}_k\}\subset\bar S_k\setminus\mathrm{dom}(f^\dagger)$ with $\lim_k\{j_k:k\in K\}=\infty$. These will have the property that for each $k\in K$ and $j\le j_k$ and each function $s:\{i^0_k,\ldots,i^{j-1}_k\}\to 2$, each of $f^\dagger\cup s\cup\{(i^j_k,0)\}$ and $f^\dagger\cup s\cup\{(i^j_k,1)\}$ will force a value on $\dot h(i^j_k)$.

Now we show that $f^\dagger \cup \{(i^j_k, e)\}$ ($e \in 2$) forces a value on $\dot{h}(i^j_k)$ as required. If it did not, then we could find extensions f_0, f_1 of $f^\dagger \cup \{(i^j_k, e)\}$ which force different values on $\dot{h}(i^j_k)$. Let $s_0 = f_0 \restriction S^k_0 \cap i^j_k$ and $s_1 = f_1 \restriction S^k_0 \setminus i^j_k$. Notice that $\bar{f} \cup s_0 \le f^\dagger \cup s_0$ forces a value (hence the same value as that forced by f_0) on $\dot{h}(i^j_k)$. This is also true for $f^\dagger \cup s_1$ in that it forces the same value on $\dot{h}(i^j_k)$ as that forced by f_1. The contradiction is that $\bar{f} \cup s_0$ and $f^\dagger \cup s_1$ force distinct values on $\dot{h}(i^j_k)$ although they have the common extension $f^\dagger \cup s_0 \cup s_1$. ∎

Returning to the proof of Theorem 5.1, we are ready to use Lemma 5.2 to show that forcing with $\mathbb{P}(\mathfrak{F})$ will not introduce undesirable functions h, analogously to the argument in Theorem 1.1. By Lemma 5.2, we have the condition $f_0 \in \mathfrak{F}$ and the sequence S_k ($k \in \mathbb{N}$) such that $\mathbb{N} \setminus \mathrm{dom}(f_0) = \bigcup_k S_k$ and that for each $i \in \bigcup_k S_k$, $f_0 \cup \{(i, 0)\}$ forces a value (call it $\bar{h}(i)$) on $\dot{h}(i)$. Therefore, $\bar{h}$ is a function with domain $\bigcup_k S_k$ in V. We may assume that $|S_k| \ge k$ for each k. It suffices to find a condition in $\mathbb{P}$ below f_0 which forces that there is some $J \in \mathcal{J}$ such that h_J is not extended by $\dot{h}$. It is useful to note that if $Y \subset \bigcup_k S_k$ is such that $\limsup |S_k \setminus Y|$ is infinite, then for any function $g \in 2^Y$, $f_0 \cup g \in \mathbb{P}$.

We first check that $\bar{h}$ is 1-to-1 on a cofinite set. If not, there is an infinite set of pairs $E_j \subset \bigcup_k S_k$, $\bar{h}[E_j]$ is a singleton and such that for each k, $S_k \cap \bigcup_j E_j$ has at most two elements. Let L denote the set of k for which S_k meets $\bigcup_j E_j$. By passing to a subcollection of the E_j's we may assume that L has infinite complement. Let g be the function with domain $\bigcup_j E_j$ which is constantly 0. Then $f_0 \cup g$ forces that $\dot{h}$ agrees with $\bar{h}$ on $\mathrm{dom}(g)$ and so is not 1-to-1. There is a further extension f_1 of $f_0 \cup g$ with the property that $S_k \subset \mathrm{dom}(f_1)$ for all $k \notin L$. Therefore, by virtue of f_1, there is some $J \in \mathcal{J}$ which contains $\bigcup_j E_j$. However, this is a contradiction, because apparently $h_J = \dot{h} \restriction J$ does not induce a homeomorphism on J^*.

But now that we know that $\bar{h}$ is 1-to-1 we can get a contradiction as follows. Let $f_1 < f_0$ be chosen so as to decide the value of $F(f_0^{-1}(0))$, and let Y denote this value. For each k, let $\bar{S}_k = S_k \setminus \mathrm{dom}(f_1)$ and let L be such that $\{|\bar{S}_k| : k \in L\}$ diverges to infinity. If $Y \cap \bar{h}[\bigcup_{k \in L} \bar{S}_k]$ is infinite, then there is an infinite set $L_0 \subset L$ (with $L \setminus L_0$ also infinite) such that for each $k \in L_0$, there is an $i_k \in \bar{S}_k$ such that $\bar{h}(i_k) \in Y$. Choose any $f_2 < f_1$ such that $f_2(i_k) = 0$ and $S_k \subset \mathrm{dom}(f_2)$ for all $k \in L_0$. It follows that there is a $J \in \mathcal{J}$ with $\bigcup_{k \in L_0} S_k \cap f_2^{-1}(0) \subset J$ and such that

$$f_2 \Vdash_{\mathbb{P}} \text{“}F(\{i_k : k \in L_0\}) \cap F(f_0^{-1}(0)) =^* h_J[\{i_k : k \in L_0\}] \cap Y \text{ is infinite”},$$

a contradiction since $\{i_k : k \in L_0\}$ is disjoint from $f^{-1}(0)$. On the other hand, let L be as above and L_0 any infinite-coinfinite subset. Fix any sequence $\{i_k : k \in L_0\}$ (with each $i_k \in \bar{S}_k$) and select $f_2 < f_1$ so that $f_2(i_k) = 1$ for all $k \in L_0$ and $\bigcup_{k \in L_0} S_k \subset \mathrm{dom}(f_2)$. Set $Y' = \bar{h}[\{i_k : k \in L_0\}]$. Since $\bar{h}$

is 1-to-1 it follows easily that $f_2 \Vdash_{\mathbb{P}}$ "$(\forall J \in \mathcal{J})\ F(J) \cap Y'$ is finite". This, of course, is also a contradiction.

Now we consider the possibility that $\tau(x) > 2$. It then follows that one of $A \setminus \{x\}$ or $B \setminus \{x\}$, say the former, can be partitioned into disjoint clopen non-compact sets. Therefore there is some sequence $\{c_\alpha : \alpha \in \omega_2\}$ of $\mathbb{P}$-names such that for each $\alpha < \beta \in \omega_2$, $c_\beta \subset a_\beta$ and $c_\beta \cap a_\alpha =^* c_\alpha$. In addition, for each $\alpha < \omega_2$ there must be a $\beta \in \omega_2$ such that $c_\beta \setminus a_\alpha$ and $a_\beta \setminus (c_\beta \cup a_\alpha)$ are both infinite.

In this case, we suppose that H and $\mathfrak{F}$ are chosen as in Lemma 2.2, and in the extension by H, let λ denote the ordinal ω_2 from V. In this model we will have a (λ, λ)-gap formed by the families $\{c_\alpha : \alpha \in \lambda\}$ and $\{a_\alpha \setminus c_\alpha : \alpha \in \lambda\}$. Assume that we can show that in the extension obtained by forcing with $\mathbb{P}(\mathfrak{F})$, there is no $C \subset \mathbb{N}$ such that $C \cap a_\alpha =^* c_\alpha$ for all $\alpha \in \lambda$. In other words, for any cofinal sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$, the collections $\{c_{\alpha_\xi}, a_{\alpha_\xi} \setminus c_{\alpha_\xi} : \xi \in \omega_1\}$ form an (ω_1, ω_1)-gap. There are well-known ccc posets Q_1 (see [1, 4.2]) which "freeze" the gap. What we mean here is that there is a family of ω_1-many dense subsets of the iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1$ such that if a filter meets them all, then the gap will remain a gap in any proper forcing extension. Finally, if we let Q_2 be the σ-centered poset mentioned after Definition 1.6, there is a filter (meeting ω_1-many dense subsets) on the proper iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1 * Q_2$ which introduces a condition $f \in \mathbb{P}$ which forces that c_λ will not exist.

Thus, we will have shown that $\tau(x) = 2$ once we show that there is no $\mathbb{P}(\mathfrak{F})$-name for a set C as above. Equivalently, we assume that $\dot{h}$ is a $\mathbb{P}(\mathfrak{F})$-name for the characteristic function of $\mathbb{N} \setminus C$, and derive a contradiction.

So, given our name $\dot{h}$, we repeat the steps above up to the point where we have f_0 and the sequence $\{S_k : k \in \mathbb{N}\}$ so that $f_0 \cup \{(i, 0)\}$ forces a value $\bar{h}(i)$ on $\dot{h}(i)$ for each $i \in \bigcup_k S_k$ and $\mathbb{N} \setminus \mathrm{dom}(f_0) = \bigcup_k S_k$.

Let $Y = \bar{h}^{-1}(0)$ and $Z = \bar{h}^{-1}(1)$. Since x is forced to be an ultrafilter, there is an $f_1 < f_0$ such that $\mathrm{dom}(f_1)$ contains one of Y or Z. If $\mathrm{dom}(f_1)$ contains Y, then f_1 forces that $\dot{h}[a_\beta \setminus \mathrm{dom}(f_1)] = 1$, and so $a_\beta \setminus \mathrm{dom}(f_1) \subset^* \mathbb{N} \setminus C$ for all $\beta \in \omega_2$. While if $\mathrm{dom}(f_1)$ contains Z, then f_1 forces that $\dot{h}[a_\beta \setminus \mathrm{dom}(f_1)] = 0$, and so $a_\beta \setminus \mathrm{dom}(f_1) \subset^* C$ for all $\beta \in \omega_2$. However, taking β so large that each of $c_\beta \setminus \mathrm{dom}(f_1)$ and $a_\beta \setminus (c_\beta \cup \mathrm{dom}(f_1))$ are infinite shows that no such $\dot{h}$ exists.

Finally, we show that there are no involutions on $\mathbb{N}^*$ which have a unique fixed point. Assume that Φ is such an involution and that y is the unique fixed point of Φ. Let F be an arbitrary lifting of Φ to $[\mathbb{N}]^\omega$. Let $\mathcal{I}$ denote the dual ideal to y). We first show that $\mathcal{I}$ is a P-ideal (i.e. that y is a P-point). For each $I \in \mathcal{I}$, $F(I)$ is also in $\mathcal{I}$ and $F(I \cup F(I)) =^* I \cup F(I)$. So we may let $\mathcal{Z}$ denote those $I \in \mathcal{I}$ such that $Z =^* F(Z)$. Given $Z \in \mathcal{Z}$, since

$\mathrm{fix}(\Phi) \cap Z^* = \emptyset$, there is a collection $\mathcal{Y} \subset [Z]^\omega$ such that $F(Y) \cap Y =^* \emptyset$ for each $Y \in \mathcal{Y}$, and such that Z^* is covered by $\{Y^* : Y \in \mathcal{Y}\}$. By compactness, we may assume that $\mathcal{Y} = \{Y_0, \dots, Y_n\}$ is finite. Set $Z_0 = Y_0 \cup F(Y_0)$. By induction, replace Y_k by $Y_k \setminus \bigcup_{j<k} Z_j$ and define $Z_k = Y_k \cup F(Y_k)$. Therefore $Y_Z = \bigcup_k Y_k$ satisfies $Y_Z \cap F(Y_Z) =^* \emptyset$ and $Z = Y_Z \cup F(Y_Z)$. This shows that for each $Z \in \mathcal{Z}$ there is a partition of $Z = Z^0 \cup Z^1$ such that $F(Z^0) =^* Z^1$. We can show y is a P-point. Indeed, if $\{Z_n = Z_n^0 \cup Z_n^1 : n \in \mathbb{N}\} \subset \mathcal{Z}$ are pairwise disjoint, then $y \notin \overline{\bigcup_n Z_n^*}$ since $F(\overline{\bigcup_n (Z_n^0)^*}) = \overline{\bigcup_n (Z_n^1)^*}$ and $\overline{\bigcup_n (Z_n^0)^*}$ is disjoint from $\overline{\bigcup_n (Z_n^1)^*}$.

Fix $\mathbb{P}$-names for F and the members of $\mathcal{I}$ and let H, $\mathcal{F}$, F and $\{n_k : k \in \omega\}$ be as given in Lemma 3.1. Also let $\mathcal{J}$ denote the ideal as defined in the proof of Corollary 3.2. Hence $J \in \mathcal{J}$ if there is an $f \in \mathcal{F}$ and $J \subset \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(f)\}$. It is again easily argued that the $I \in \mathcal{I}$ as specified in Corollary 3.2 can be assumed to be empty. For each $J \in \mathcal{J}$, let h_J be the function on J such that there is an $f \in \mathcal{F}$ which forces that h_J induces F on J. Let $\dot{h}$ be the $\mathbb{P}(\mathcal{F})$-name as given in Corollary 3.2. Since F is an involution with a unique fixed point, we may assume that $\dot{h}$ is forced to satisfy that $\dot{h}(\dot{h}(i)) = i \neq \dot{h}(i)$ for all i.

The rest of the proof depends on the following modification of Lemma 5.2.

Claim 4. *There is an* $f \in \mathfrak{F}$ *and a sequence of sets* $\{m_k, S_k, T_k : k \in \omega\}$ *and mappings* $\psi_k : T_k \to S_k$ *such that* $S_k \subset 2^{m_k+1} \setminus 2^{m_k} \subset [n_k, n_{k+1})$, $T_k \subset [n_k, n_{k+1})$, $\mathbb{N} \setminus \mathrm{dom}(f) = \bigcup_k S_k$, *and for each* k *and* $i \in S_k$ *and* $\bar{f} < f$, $\bar{f}$ *forces a value on* $\dot{h}{\restriction}\psi_k^{-1}(i)$ *iff* $i \in \mathrm{dom}(\bar{f})$.

Before proving the claim, let us show how this will finish the proof. For each $i \notin \mathrm{dom}(f)$, there are two functions h_i^0, h_i^1 with domain $\psi^{-1}(i)$ such that $f \cup \{\langle i, e\rangle\}$ forces that $h_i^e \subset \dot{h}$. Since, by assumption, for each $i \in \mathrm{dom}(f)$, f does not already force a value on $\dot{h}{\restriction}\psi^{-1}(i)$, we can choose $j_i \in \psi^{-1}(i)$ such that $v_i = h_i^0(j_i) \neq h_i^1(j_i) = w_i$. Note that, by our assumption on $\dot{h}$, it also follows that $\psi(v_i) = \psi(w_i) = i$.

Choose $g < f$ which forces a value, Y, on $F(\{j_i\}_{i \notin \mathrm{dom}(f)})$. Assume that $Y \cap \{v_i : i \notin \mathrm{dom}(g)\}$ is infinite. It follows easily that there is some $g^\dagger < g$ such that $Y \cap \{v_i : g^\dagger(i) = 1\}$ is infinite and let $J \subset \{i \in \mathrm{dom}(g^\dagger) : g^\dagger(i) = 1$ and $v_i \in Y\}$ be any infinite set such that $\{j_i\}_{i \in J}$ is in $\mathcal{J}$. However, this is a contradiction since

$$g^\dagger \Vdash_{\mathbb{P}} \text{“}F\text{”}(\{j_i : i \in J\}) =^* \dot{h}[\{j_i\}_{i \in J}] = \{w_i\}_{i \in I} \subset^* Y \cap \{v_i\}_{i \in J}.$$

The argument when $\{v_i : i \notin \mathrm{dom}(g)\} \setminus Y$ is infinite is similar.

Now we prove Claim 4. For any k, condition g, and $T \subset [n_k, n_{k+1})$ let $\mathrm{Orb}(T, g)$ denote the set $\{j : (\exists g' < g)(\exists t \in T)\ g' \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}(t) = j\text{”}\}$. Fix any k and the f as above that was selected from Lemma 2.5. Let $g_0 = f\restriction[n_k, n_{k+1})$ and assume, as we may, that $S^k_0 = [n_k, n_{k+1}) \setminus \mathrm{dom}(g_0)$ is contained in $[2^m, 2^{m+1})$ for some m. By a simple recursion much as in Lemma 5.2, we can choose increasing sequences $I_l = \{i_0, i_1, \ldots, i_{l-1}\} \subset S^k_0$ and extensions $g_l \supset g_{l-1} \supset \cdots \supset g_0$ so that $I_l \subset S^k_l = S^k_0 \setminus \mathrm{dom}(g_l)$ and $i_l = \min(S^k_l \setminus I_l)$. In addition, select sets $T(i_l) \subset [n_k, n_{k+1}) \setminus \bigcup_{j<l} \mathrm{Orb}(T(i_j), g_l)$ of minimum cardinality (at most 2^l) so that for each $s : S^k_l \to 2$ there is, if possible, a $t \in T(i_l)$ such that $s \cup g_l \cup \{(i_j, 0)\}$ and $s \cup g_l \cup \{(i_j, 1)\}$ force distinct values on $\dot{h}(t)$. Notice that $\mathrm{Orb}(\{t\}, g_l)$ has cardinality at most 2^l for each $t \in \bigcup_{j<l} T(i_j)$. We also require that for each $s : S^k_l \to 2$, each of $s \cup \{(i_l, 0)\} \cup g_{l+1}$ and $s \cup \{(i_l, 1)\} \cup g_{l+1}$ force a value on $\dot{h}\restriction \mathrm{Orb}(T(i_l), g_{l+1})$. If $T(i_l)$ is not empty, we can certainly ensure that for at least one $s : S^k_l \to 2$ and $t \in T(i_l)$, $s \cup \{(i_l, 1)\} \cup g_{l+1}$ forces a distinct value on $\dot{h}(t)$ from that forced by $s \cup \{(i_l, 0)\} \cup g_{l+1}$.

For each successive l, there is a recursion on k so that $f_l = f \cup \bigcup\{g^k_l : k \in \omega\}$ is a condition. If for each k, there is an $s^k : I^k_l \to 2$ for which no suitable $t \in [n_k, n_{k+1})$ can be chosen, then it is because the condition $g = f_k \cup \bigcup_k (s_k \cup g^k_j)$ forces a value on $\dot{h}(t)$ for all $t \notin \bigcup_k \bigcup_{j<l} \mathrm{Orb}(T(i^j_k), g)$. But if this were the case, then this condition would force a value on $\dot{h}(t)$ for all t.

After infinitely many steps, we may instead assume that (a new choice of) f simply has this property: for each k and each $S^k = [2^{m_k}, 2^{m_k+1}) \setminus \mathrm{dom}(f) = [n_k, n_{k+1}) \setminus \mathrm{dom}(f)$, there is a sequence $\{T(i) : i \in S^k\}$ of pairwise disjoint finite subsets of $[n_k, n_{k+1})$ such that for each $i \in S^k$ and each $s : S^k \cap i \to 2$, $s \cup \{(i, 0)\} \cup f$ and $s \cup \{(i, 1)\} \cup f$ each force a value on $\dot{h}\restriction T(i)$ while $s \cup f$ does not. (We do not need a superscript on the T's since they depend only on i and not on k.) We have also ensured that for $i \neq i'$, $\mathrm{Orb}(T(i), f)$ is disjoint from $\mathrm{Orb}(T(i'), f)$.

Now, much as in Lemma 5.2, we repeat the process but rather than choosing minimal members of S^k we choose maximal. A new trouble arises in this proof because of the sizes of the sets $T(i)$, while in Lemma 5.2, each $T(i)$ was just $\{i\}$. To overcome this, we will use the next claim.

Claim 5. *For each $f_1 < f$ and infinite $I \subset \mathbb{N} \setminus \mathrm{dom}(f_1)$ and $K \subset \mathbb{N}$ for which $\{|I \cap S^k| : k \in K\}$ diverges to infinity, there is an $f_2 < f_1$, $I' \subset I \setminus \mathrm{dom}(f_2)$, and $K' \subset K$ such that $\{|I' \cap S^k \setminus \mathrm{dom}(f_2)| : k \in K'\}$ diverges to infinity, and for all $i \in I'$, each of $f_2 \cup \{(i, 0)\}$ and $f_2 \cup \{(i, 1)\}$ force a value on $\dot{h}\restriction T(i)$.*

In order to not lose track of our progress, let us again defer the proof of Claim 5 and first finish the proof of Claim 4.

Let $K_0 \subset \omega$ be chosen so that $\{|S^k| : k \in K_0\}$ is strictly increasing. By Claim 5 there is an infinite $K_1 \subset K_0$ and an $f_1 < f$ so that for each $k \in K_1$, there is an $i_0^k \notin \mathrm{dom}(f_1)$ such that $f_1 \cup \{(i_0^k, 0)\}$ and $f_1 \cup \{(i_0^k, 1)\}$ each force a value on $\dot{h}\restriction T(i_0^k)$, and $|S^k \cap i_0^k| > |S^k|/2$.

By induction on $j > 0$, continue to choose $f_j < f_{j-1}$, $i_j^k \in S^k \cap i_{j-1}^k \setminus \mathrm{dom}(f_j)$ for all k in an infinite set $K_j \subset K_{j-1}$ such that the sequence $\{|(S^k \cap i_j^k) \setminus \mathrm{dom}(f_{j-1})| : k \in K_j\}$ diverges to infinity. We require that for each $k \in K_j$ and $s : \{i_l^k : l \leq j\} \to 2$, the condition $s \cup f_j$ forces a value on $\dot{h}\restriction T(i_j^k)$.

We find the sequence $\{i_j^k : k \in K_j\}$ by applying Claim 5 as follows. For each function $\psi : j \to 2$ and each $k \in K_{j-1}$, let s_ψ^k denote the function from $\{i_0^k, \ldots, i_{j-1}^k\}$ to 2 such that $s_\psi^k(i_{j'}^k) = \psi(j')$ for each $j' < j$. Start by applying Claim 5 with the f_1 in Claim 5 being $f_{j-1} \cup \bigcup_{k \in K_{j-1}} s_\psi^k$ for some fixed $\psi \in 2^j$, $K = K_{j-1}$ and, $I = \bigcup_{k \in K_{j-1}} S^k \setminus \mathrm{dom}(f_1)$. Simply apply Claim 5 recursively, each time swapping the values of the f_1 used so as to cycle through all the possible $\psi \in 2^j$. After these 2^j steps, each time shrinking the K' and the I' we can let f_j be the final condition denoted f_2 in Claim 5, K_j be the final set K' and let $\{i_j^k : k \in K_j\}$ be any selection from the final I' which has the additional property that $\{|S^k \cap i_j^k| : k \in K_j\}$ diverges to infinity.

What we have now is that for each $k \in K_j$ and each $\psi \in 2^j$, the conditions $s_\psi^k \cup f_j \cup \{(i_j^k, 0)\}$ and $s_\psi^k \cup f_j \cup \{(i_j^k, 1)\}$ each force a value on $\dot{h}\restriction T(i_j^k)$.

When this recursion finishes, let $\{k_j : j \in \omega\}$ be chosen so that $k_j \in K_j$ for each j. Set $\bar{f} \supset \bigcup_j f_j \restriction [n_{k_j}, n_{k_j+1})$. Note that for each j, $k = k_j$ and $l < j$, $i_l^k \notin \mathrm{dom}(\bar{f})$ and $\bar{f}$ is not forcing a value on $\dot{h}\restriction T(i_l^k)$. Assume $g < \bar{f}$, $k = k_j$ and $l < j$. Let $s = g\restriction(S^k \cap i_l^k)$ and $s' = g\restriction(S^k \setminus 1 + i_l^k)$.

There is a $t \in T(i_l^k)$ such that $s \cup f_j \cup \{(i_l^k, 0)\}$ and $s \cup f_j \cup \{(i_l^k, 1)\}$ force different values on $\dot{h}(t)$. Therefore if $i_l^k \notin \mathrm{dom}(g)$, then g cannot decide $\dot{h}(t)$. On the other hand, suppose i_l^k is in $\mathrm{dom}(g)$, let $e = g(i_l^k)$ and assume that there is a $t \in T(i_j^k)$ such that $\dot{h}(t)$ is not decided by g. Fix any $s_1 : \{i_j^k : j < l\} \to 2$ extending s' such that $s_1 \cup g$ forces a value on $\dot{h}(t)$ and let v be this value. Since g does not decide $\dot{h}(t) = v$, there is some $s_2 : S^k \cap i_j^k \to 2$ which extends s and forces a value distinct from v on $\dot{h}(t)$. This is a contradiction since $s_1 \cup s_2 \cup g$ is a condition.

We have shown that for each k_j and $l < j$, g forces a value on $\dot{h}\restriction T(i_l^k)$ iff $i_l^k \in \mathrm{dom}(g)$.

It remains to give the following proof.

Proof of Claim 5. Let f_1 and I be as in the statement of the claim and assume there is no such f_2 and I'. Let $\mathfrak{I} = \{I' \in [I]^\omega : \{|I' \cap S^k|\}_k \text{ is bounded}\}$ and $\mathfrak{I}^* = \{I' \in [I]^\omega : \{|I' \cap S^k|\}_k \text{ diverges to infinity}\}$. For each $I' \subset I$, let $K(I') = \{k : I' \cap S^k \neq \emptyset\}$. For any set I, let $\chi_0(I)$ (respectively $\chi_1(I)$) denote the function which is constantly 0 (respectively 1) on I.

Choose, if possible, $e \in \{0,1\}$ (say $e = 0$) and some pair $f_2 < f_1$ and $I_2 \subset I \setminus \mathrm{dom}(f_2)$ such that $I_2 \in \mathfrak{I}^*$ and $f_2 \cup \chi_0(I_2)$ forces a value on $\dot{h}\restriction T(i)$ for all $i \in I_2$. If no such e exists, then let $f_2 = f_1$ and $I_2 = I$. It now follows (in either case) that for any $f_3 < f_2$ and $I' \subset I_2$, the set of $i \in I'$ for which $f_3\restriction(\mathrm{dom}(f_3) \setminus I') \cup \chi_1(I')$ forces a value on $\dot{h}(i)$ is a member of $\mathfrak{I}$.

For each integer k, let $\mathcal{S}_k$ be the set of partial functions from S^k into 2. For integers l, k and condition g, let

$$\mathcal{S}(l,k,g) = \{s \in \mathcal{S}_k : g\restriction S^k \subset s \text{ and } |S^k \setminus \mathrm{dom}(s)| > l\} .$$

For $s \in \mathcal{S}(l,k,f_2 \cup \chi_1(I_2))$, let $I(s)$ be the set of $i \in I_2 \cap S^k$ such that $s \cup f_2$ forces a value on $\dot{h}\restriction T(i)$. Assume that for each l, $\{|I(s)| : s \in \bigcup_{k\in K(I_2)} \mathcal{S}(l,k,f_2 \cup \chi_1(I_2))\}$ is unbounded. We could then find an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ and corresponding $s(k_l) \in \mathcal{S}(l,k_l,f_2 \cup \chi_1(I_2))$ with $\{|I(s(k_l))| : l \in \omega\}$ diverging, in which case the condition $f_2 \cup \chi_1(I_2) \cup \bigcup_l s(k_l)$ would be guilty of forcing a value on $\dot{h}\restriction T(i)$ for each $i \in \bigcup_l I(s(k_l)) \in \mathfrak{I}^*$—a contradiction.

Therefore there is some l_0 such that for all $k \in K(I_2)$, and $s \in \mathcal{S}(l_0,k,f_2 \cup \chi_1(I_2))$, the set $I(s)$ has cardinality less than l_0. Now choose an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ so that $|I_2 \cap S^{k_l}|$ has cardinality greater than $l_0 + 2^{2^l}$. Choose any condition $f^\dagger < f_2 \cup \chi_1(I_2)$ so that $\mathbb{N} \setminus \mathrm{dom}(f^\dagger) \subset \bigcup_l S^{k_l}$ and $|S^{k_l} \setminus \mathrm{dom}(f^\dagger)| = l$ for all l. Notice that $\mathcal{S}(l_0,k_l,f^\dagger)$ has cardinality at most 2^{2^l}. For each l and $s \in \mathcal{S}(l_0,k_l,f^\dagger)$, choose an $i_s \in I_2 \cap S^{k_l}$ such that $s \cup f^\dagger$ does not force a value on $\dot{h}\restriction T(i_s)$. Ensure that the selection is such that $i_s \neq i_{s'}$ for distinct $s, s' \in \mathcal{S}(l_0,k_l,f^\dagger)$. Next, for each l and $s \in S(l_0,k_l,f^\dagger)$, choose $t_s \in T(i_s)$ and distinct u_s, w_s each with the property that there is some extension of $s \cup f^\dagger$ forcing $\dot{h}(t_s)$ to have that value.

We now define an ultrafilter in $\mathbb{N}^*$. For each g, let $X(g) = \{i_s : s \in \bigcup_l \mathcal{S}(l_0,k_l,g)\}$, $U(g) = \{u_s : i_s \in X(g)\}$, and $W(g) = \{w_s : i_s \in X(g)\}$. Let $z \in \mathbb{N}^*$ be any ultrafilter which extends the family $\{X(g) : g \in \mathfrak{F},\ g < f^\dagger\}$.

Since $U(f^\dagger) \cap W(f^\dagger)$ is empty, there is some $g < f^\dagger$ such that either $g \Vdash_{\mathbb{P}}$ "$U(f^\dagger) \notin z$" or $g \Vdash_{\mathbb{P}}$ "$W(f^\dagger) \notin z$" (by symmetry assume $U(f^\dagger) \notin z$). By possibly extending g, there is an $X \in x$ such that $g \Vdash_{\mathbb{P}}$ "$F(X) \cap U(f^\dagger) =^* \emptyset$". Since $X \cap X(g)$ is infinite we can choose an infinite set $L \subset \mathbb{N}$ such that for each $l \in L$, $s_l = g\restriction S^{k_l} \in \mathcal{S}(l_0,k_l,g)$ and $s_l \in X$. For each $l \in L$, let $s_l^* \in \mathcal{S}_l$ be chosen so that $s_l \subset s_l^*$ and $s_l^* \cup f^\dagger$ forces $\dot{h}(i_{s_l}) = u_{s_l}$. By genericity of $\mathfrak{F}$, there is a $g^\dagger < g$ such that $L' = \{l \in L : S^{k_l} \subset \mathrm{dom}(g^\dagger) \text{ and } s_l^\dagger \subset g^\dagger\restriction S^{k_l}\}$ is infinite. Since

$$g^\dagger \Vdash_{\mathbb{P}} \text{“}\{u_{s_l}\}_{l\in L'} = \dot{h}[\{i_{s_l}\}_{l\in L'}] =^* F(\{i_{s_l}\}_{l\in L'}) \subset^* \mathbb{N} \setminus U(f^\dagger)\text{”},$$

we have our contradiction.

This completes the proof of Claim 5. ∎

6. Questions

QUESTION 6.1. Assume PFA. If G is $\mathbb{P}_2$-generic, and $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is the generic tie-point introduced by $\mathbb{P}_2$, is it true that A is not homeomorphic to $\mathbb{N}^*$? Is it true that $\tau(x) = 2$? Is it true that each tie-point is a symmetric tie-point?

REMARK 1. The tie-point x_3 introduced by $\mathbb{P}_3$ does not satisfy $\tau(x_3) = 3$. This can be seen as follows. For each $f \in \mathbb{P}_3$, we can partition L_f into $\{i \in \mathrm{dom}(f) : i < f(i) < f^2(i)\}$ and $\{i \in \mathrm{dom}(f) : i < f^2(i) < f(i)\}$.

It seems then that the tie-points x_l introduced by $\mathbb{P}_l$ might be better characterized by the property that there is an autohomeomorphism F_l of $\mathbb{N}^*$ such that $\mathrm{fix}(F_l) = \{x_l\}$, and each $y \in \mathbb{N}^* \setminus \{x\}$ has an orbit of size l.

REMARK 2. A small modification to the poset $\mathbb{P}_2$ will result in a tie-point $\mathbb{N}^* = A \underset{x}{\bowtie} B$ such that A (hence the quotient space by the associated involution) is homeomorphic to $\mathbb{N}^*$. The modification is to build into the conditions a map from the pairs $\{i, f(i)\}$ into $\mathbb{N}$. A natural way to do this is to set $f \in \mathbb{P}_2^+$ if f is a 2-to-1 function such that for each n, f maps $\mathrm{dom}(f) \cap (2^{n+1} \setminus 2^n)$ into $2^n \setminus 2^{n-1}$, and again $\limsup_n |2^{n+1} \setminus (\mathrm{dom}(f) \cup 2^n)| = \infty$. $\mathbb{P}_2^+$ is ordered by almost containment. The generic filter introduces an ω_2-sequence $\{f_\alpha : \alpha \in \omega_2\}$ and two ultrafilters: $x \supset \{\mathbb{N} \setminus \mathrm{dom}(f_\alpha) : \alpha \in \omega_2\}$ and $z \supset \{\mathbb{N} \setminus \mathrm{range}(f_\alpha) : \alpha \in \omega_2\}$. For each α and $a_\alpha = \{i \in \mathrm{dom}(f_\alpha) : i = \min(f_\alpha^{-1}(f_\alpha(i))\}$, we set $A = \{x\} \cup \bigcup_\alpha a_\alpha^*$ and $B = \{x\} \cup \bigcup_\alpha (\mathrm{dom}(f_\alpha) \setminus a_\alpha)^*$; then $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is a symmetric tie-point. Finally, the map $F : A \to \mathbb{N}^*$ defined by $F(x) = z$ and $F{\restriction}A \setminus \{x\} = \bigcup_\alpha (f_\alpha)^*$ is a homeomorphism.

QUESTION 6.2. Assume PFA. If L is a finite subset of $\mathbb{N}$ and $\mathbb{P}_L = \prod\{\mathbb{P}_l : l \in L\}$, is it true in $V[G]$ that there is a finite upper bound to $\tau(x)$ for the tie-points x; and if $1 \notin L$, then every tie-point is a symmetric tie-point?

Acknowledgements. Research of the first author was supported by NSF grant No. NSF-DMS 20060114. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant no. 2002323), and by NSF grant No. NSF-DMS 0600940.

References

[1] J. E. Baumgartner, *Applications of the proper forcing axiom*, in: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 913–959.

[2] A. Dow, *Extending real-valued functions in* $\beta\kappa$, Fund. Math. 152 (1997), 21–41.

[3] A. Dow and G. Techanie, *Two-to-one continuous images of* $\mathbb{N}^*$, ibid. 186 (2005), 177–192.

[4] R. Levy, *The weight of certain images of* ω^*, Topology Appl. 153 (2006), 2272–2277.

[5] S. Shelah and J. Steprāns, *Non-trivial homeomorphisms of $\beta N \setminus N$ without the Continuum Hypothesis*, Fund. Math. 132 (1989), 135–141.

[6] —, —, *Somewhere trivial autohomeomorphisms*, J. London Math. Soc. (2) 49 (1994), 569–580.

[7] —, —, *Martin's axiom is consistent with the existence of nowhere trivial automorphisms*, Proc. Amer. Math. Soc. 130 (2002), 2097–2106.

[8] J. Steprāns, *The autohomeomorphism group of the Čech–Stone compactification of the integers*, Trans. Amer. Math. Soc. 355 (2003), 4223–4240.

[9] B. Veličković, *Definable automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$*, Proc. Amer. Math. Soc. 96 (1986), 130–135.

[10] —, *OCA and automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$*, Topology Appl. 49 (1993), 1–13.

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Received 28 August 2007;
in revised form 14 August 2008 and 3 January 2009