Fixed points on Klein bottle fiber bundles over the circle

by

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Abstract. The main purpose of this work is to study fixed points of fiber-preserving maps over the circle S^1 for spaces which are fiber bundles over S^1 and the fiber is the Klein bottle K. We classify all such maps which can be deformed fiberwise to a fixed point free map. The similar problem for torus fiber bundles over S^1 has been solved recently.

Introduction. Given a fiber bundle $E \to B$ and a fiber-preserving map $f : E \to E$ over B, the question whether f can be deformed over B (by a fiberwise homotopy) to a fixed point free map has been considered by many authors (see for example [Do74], [FH81], [Go87]). In [FH81], Fadell and Husseini showed that the above problem can be stated in terms of obstructions (including higher ones). This was done under the hypothesis that the base space, the total space and the fiber F are manifolds, and the dimension of F is greater than or equal to 3. The case where the fiber has dimension 2 was considered in [GPV04], where a few generalities were discussed and the fixed point problem over B as defined above was completely solved for any torus fiber bundles over the circle S^1. In the present work we study the fixed point problem over B for Klein bottle fiber bundles over S^1.

Recall that a Klein bottle bundle over S^1 has as total space the mapping torus $M(\phi)$ where $\phi : K \to K$ is a homeomorphism. A relevant step in solving the problem is to determine, for each fiber bundle $M(\phi) \to S^1$, the set of homotopy classes of maps f over S^1 such that f restricted to the fiber can be deformed to a fixed point free map. This is done in Theorem 2.4. The main result of the paper is Theorem 6.26, which gives a classification of the homotopy classes of fiber-preserving maps given by Theorem 2.4 which can be deformed over S^1 to a fixed point free map. Our method is to study solutions of a system of equations in a free group either by providing an explicit solution or by considering the system in some quotients of this group.

[263]

The paper is organized in six sections. In Section 1 we give some results about the Klein bottle, K-bundles and an algebraic formulation of our main question. In Section 2 we classify all K-bundles over S^1 and bundle maps which have the property that their restriction to a fiber can be deformed to a fixed point free map (Theorem 2.4). In Section 3 we compute the fundamental group of the K-bundle $M(\phi) \times_{S^1} M(\phi)$ with suitable base points. In Section 4 we compute the fundamental group of the K-bundle $M(\phi) \times_{S^1} M(\phi) - \Delta$ where Δ is the diagonal. In Section 5 we give necessary and sufficient conditions for the existence of a lifting in the algebraic diagram (1.1); this existence is equivalent to the map being fiberwise deformable to a fixed point free map (Theorem 5.1). Then we reduce the cases to be analyzed (Corollary 5.4). In Section 6 we derive necessary conditions for the relevant system of equations on a free group F to have a solution, by looking at the system on some quotients of F. In certain cases we construct explicit solutions. Then we derive the main result, which is Theorem 6.26.

1. Preliminaries

1.1. *The algebraic problem.* Let $f : E \to E$ be a fiber-preserving map over S^1 where E is a fiber bundle over S^1 with fiber a surface S. From Corollary 1.3 in [GPV04] we know that all such bundles are of the form $S \to M(\phi) \xrightarrow{p} S^1$ where $\phi : S \to S$ is a homeomorphism and $M(\phi)$ is the quotient of $S \times I$ by the relation $(x, 0) \sim (\phi(x), 1)$.

Let $M(\phi) \times_{S^1} M(\phi)$ be the pullback of $p : M(\phi) \to S^1$ along $p : M(\phi) \to S^1$, and $p_i : M(\phi) \times_{S^1} M(\phi) \to M(\phi)$, $i = 1, 2$, the projections to the first and the second coordinates, respectively. The fixed point problem for a fiber-preserving map f over S^1 is equivalent to the algebraic problem given by Proposition 1.6 from [GPV04]. So we must compute the groups and homomorphisms in the following diagram:

(1.1)

$$
\begin{array}{c}
1 \\
\downarrow \\
\pi_1(\mathcal{F}) \simeq \pi_2(S, S - x_0) \\
\downarrow {\scriptstyle j_{2\#} \circ \partial_2} \\
\pi_1(E(M(\phi) \times_{S^1} M(\phi) - \Delta)) \simeq \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta) \\
\downarrow {\scriptstyle q_\#} \\
\pi_1(M(\phi)) \xrightarrow{(1,f)_\#} \pi_1(M(\phi) \times_{S^1} M(\phi)) \\
\downarrow \\
1
\end{array}
$$

where S is the Klein bottle and the homomorphisms are defined at the beginning of Section 5. The group $\pi_2(S, S - x_0)$ is well known and Propositions 1.7 and 1.8 of [GPV04] tell us how to compute the groups $\pi_1(M(\phi))$ and $\pi_1(M(\phi) \times_{S^1} M(\phi))$, respectively. To compute $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta)$ we will use the proposition below, which holds for any closed surface S different from S^2 and $\mathbb{R}P^2$.

PROPOSITION 1.1. *The fundamental group* $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta)$ *is isomorphic to the semidirect product* $\pi_1(S \times S - \Delta) \rtimes_\theta \mathbb{Z}$ *for some action* θ.

Proof. The fiber bundle $S \times S - \Delta \to M(\phi) \times_{S^1} M(\phi) - \Delta \xrightarrow{p \circ p_2|} S^1$ provides the short exact sequence

$$1 \to \pi_1(S \times S - \Delta) \to \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta) \xrightarrow{p \circ p_2|_\#} \mathbb{Z} \to 1,$$

and the result follows since $\mathbb{Z}$ is free. ∎

1.2. *Generators of* $\pi_1(K)$ *and the Nielsen number of a continuous map* $f : K \to K$. Let $\pi_1(K) = \langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} = 1 \rangle$ be the well known presentation of the fundamental group of the Klein bottle.

If $f : K \to K$ is a continuous map then $f_\# : \pi_1(K) \to \pi_1(K)$ is a homomorphism of the form:

- Type 1: $f_\#(\alpha) = 1$, $f_\#(\beta) = \alpha^p \beta^{2q}$,
- Type 2: $f_\#(\alpha) = \alpha^r$, $f_\#(\beta) = \alpha^p \beta^{2q+1}$.

In some situations we will distinguish the two types, but not always.

To compute the Nielsen number of a map $f : K \to K$ given by $f_\#(\alpha) = \alpha^r$, $f_\#(\beta) = \alpha^s \beta^t$ we have the following result of [DJ93].

THEOREM 1.2. *The Nielsen number of the above map* $f : K \to K$ *is*

$$N(f) = |t - 1| \max\{1, |r|\}.$$

COROLLARY 1.3. *If* $f : K \to K$ *is a continuous map so that* $f_\# : \pi_1(K) \to \pi_1(K)$ *is of type 2, then*

$$N(f) = \begin{cases} |2rq| & \text{if } r \neq 0, \\ |2q| & \text{if } r = 0. \end{cases}$$

2. The Klein bottle case—preliminary reductions. We use some homeomorphisms of the Klein bottle to describe all K-bundles over S^1 up to fiberwise isotopy.

Let us consider in $\mathbb{R}^2$ the equivalence relation generated by the relations $(x, y) \sim (x, y + 1)$ and $(x, y) \sim (x + 1, 1 - y)$. The quotient space is K and the equivalence class of $(x, y) \in \mathbb{R}^2$ is denoted by $[(x, y)] \in K$.

Let ϕ be a homeomorphism of K so that $\phi([(0, 0)]) = [(0, 0)] = x_2$.

As in Section 1, let $M(\phi)$ be the quotient space of $K \times [0, 1]$, where we identify $([(x, y)], 0)$ with $(\phi[(x, y)], 1)$. The class of $([(x, y)], t)$ in the quotient

is denoted by $\langle[(x,y)], t\rangle$. The space $M(\phi)$ is a fiber bundle over the circle S^1, where the fiber is the Klein bottle. The projection map $p : M(\phi) \to S^1$ is given by $p(\langle[(x,y)], t\rangle) = \langle t\rangle \in [0,1]/_{0 \simeq 1} \simeq S^1$.

We denote by $f_s(r,t) : \alpha \mapsto \alpha^r$, $\beta \mapsto \alpha^s \beta^t$ the homomorphism on $\pi_1(K)$ induced by the restriction of f to K, and by $\phi_p(\epsilon, \eta) : \alpha \mapsto \alpha^\epsilon$, $\beta \mapsto \alpha^p \beta^\eta$ the isomorphism of $\pi_1(K)$ induced by the homeomorphism ϕ. Since $\phi_p(\epsilon, \eta)$ is an isomorphism it follows that $\epsilon = \pm 1$ and $\eta = \pm 1$.

PROPOSITION 2.1.

(1) $\pi_1(M(\phi_p(\epsilon, \eta)), \mathbf{0}) = \langle \alpha, \beta, c_0 \mid \alpha\beta\alpha\beta^{-1} = 1, c_0\alpha c_0^{-1} = \alpha^\epsilon, c_0\beta c_0^{-1} = \alpha^p \beta^\eta \rangle$, where $\mathbf{0} = \langle[(0,0)], 0\rangle$.

(2) *There are four isotopy classes of homeomorphisms of K where a set of representatives is given by* $\{\phi_0(1,1), \phi_1(1,1), \phi_0(1,-1), \phi_1(1,-1)\}$.

(3) *For any homeomorphism $\phi : K \to K$, $M(\phi)$ is homeomorphic over S^1 to $M(\phi_p(1, \eta))$ where $\phi_p(1, \eta)$ is given by (1). Further, the $M(\phi_p(1, \eta))$'s are not homeomorphic for two different $\phi_p(1, \eta)$ given in (2).*

Proof. (1) This follows from Proposition 1.7 in [GPV04].

(2) It is a straightforward calculation to show that the number of conjugacy classes of isomorphisms of $\pi_1(K)$ is four and a set of representatives is given by $\{\phi_0(1,1), \phi_1(1,1), \phi_0(1,-1), \phi_1(1,-1)\}$. Since the surface K is a $K(\pi, 1)$ we can identify these classes with the isotopy classes of K.

(3) The first part follows from Proposition 1.2 in [GPV04]. The last part follows from the fact that the spaces have nonisomorphic fundamental groups (given before the proposition). ∎

PROPOSITION 2.2. *If f is deformable to a fixed point free map over S^1 then the Nielsen number of f restricted to the fiber is zero, which implies that $f_s(r,t)$ is of the form $f_s(r,1)$.*

Proof. The first part is clear. That $f_s(r,t)$ is of the form $f_s(r,1)$ follows from Corollary 1.3. ∎

Let us denote by $f_s(r, t, c_1, c_2) : \pi_1(M(\phi_q(1, \eta))) \to \pi_1(M(\phi_q(1, \eta)))$ the homomorphism $\alpha \mapsto \alpha^r$, $\beta \mapsto \alpha^s \beta^t$ and $c_0 \mapsto \alpha^{c_1} \beta^{c_2} c_0$. Consequently, r, s, t, c_1, c_2 must satisfy certain equations as a result of the relations in the group.

PROPOSITION 2.3. *Let $f : M(\phi_q(1, \eta)) \to M(\phi_q(1, \eta))$ be a map over S^1, where $q \in \{0, 1\}$ and $\eta = \pm 1$. If the Nielsen number of f restricted to the fiber is zero then $f_\# : \pi_1(M(\phi_q(1, \eta))) \to \pi_1(M(\phi_q(1, \eta)))$ has the form $f_s(r, 1, c_1, c_2)$ where*

(i) $(-1)^{c_2} r = r$,

(ii) $2c_1 = s[1 - (-1)^{c_2}] + q[r - (-1)^{c_2}]$.

Conversely, for each homomorphism $f_s(r, 1, c_1, c_2)$: $\pi_1(M(\phi_q(1, \eta))) \to \pi_1(M (\phi_q(1, \eta)))$, *with* $q \in \{0, 1\}$ *and* $\eta = \pm 1$, *where* (r, s, c_1, c_2, q) *satisfy the conditions* (i) *and* (ii) *above, there is a map* $f : M(\phi_q(1, \eta)) \to M(\phi_q(1, \eta))$ *over* S^1 *such that* $f_\# = f_s(r, 1, c_1, c_2)$ *and the Nielsen number of* f *restricted to the fiber is zero.*

Proof. Since $f : M(\phi_q(1, \eta)) \to M(\phi_q(1, \eta))$ is a map over S^1 we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(K) & \longrightarrow & \pi_1(M(\phi)) & \longrightarrow & \pi_1(S^1) & \longrightarrow & 1 \\
\downarrow & & (f|_K)_\# \downarrow & & f_\# \downarrow & & \text{id} \downarrow & & \downarrow \\
1 & \longrightarrow & \pi_1(K) & \longrightarrow & \pi_1(M(\phi)) & \longrightarrow & \pi_1(S^1) & \longrightarrow & 1
\end{array}
$$

Because $f|_K$ has Nielsen number zero, $(f|_K)_\# = f_s(r, 1)$ for some $r, s \in \mathbb{Z}$. Therefore $f_\# = f_s(r, 1, c_1, c_2)$.

Now we will show (i) and (ii).

Because $\pi_1(M(\phi_q(1, \eta))) = \langle \alpha, \beta, c_0 \mid \alpha\beta\alpha\beta^{-1} = 1, c_0\alpha c_0^{-1} = \alpha, c_0\beta c_0^{-1} = \alpha^q \beta^\eta \rangle$ by Proposition 2.1, we must have $f_\#(c_0\alpha c_0^{-1}) = f_\#(\alpha)$ and $f_\#(c_0\beta c_0^{-1}) = f_\#(\alpha^q\beta^\eta)$. Using

$$
(\alpha^r\beta^s)^t = \begin{cases} \alpha^{\frac{t}{2}[r(1+(-1)^s)]}\beta^{st} & \text{if } t \text{ is even,} \\ \alpha^{\frac{t-1}{2}[r(1+(-1)^s)]+r}\beta^{st} & \text{if } t \text{ is odd,} \end{cases}
$$

we have $(-1)^{c_2}r = r$ and $2c_1 = s[1 - (-1)^{c_2}] + q[r - (-1)^{c_2}]$. This finishes the proof of the first part.

For the converse we first observe that $p_\# \circ f_s(r, 1, c_1, c_2) = p_\#$. Because all spaces are $K(\pi, 1)$ there exists $g : M(\phi_q(1, \eta)) \to M(\phi_q(1, \eta))$ and a homotopy $H : M(\phi_q(1, \eta)) \times I, x_1 \times I) \to (S^1, 1)$ so that $H(x, 0) = p \circ g(x)$ and $H(x, 1) = p(x)$ and $g_\# = f_s(r, 1, c_1, c_2)$. The map $G : (x_1 \times I \cup M(\phi_q(1, \eta)) \times 0, x_1 \times I) \to (M(\phi_q(1, \eta)), x_2)$ defined by $G(x, 0) = g(x)$ and $G(x_1 \times I) = x_2$ makes the diagram

$$
\begin{array}{ccc}
(x_1 \times I \cup M(\phi_q(1, \eta)) \times 0, x_1 \times I) & \xrightarrow{\ G\ } & (M(\phi_q(1, \eta)), x_2) \\
i \downarrow & \overset{L}{\nearrow} & \downarrow p \\
(M(\phi_q(1, \eta)) \times I, x_1 \times I) & \xrightarrow{\ \ H\ \ } & (S^1, 1)
\end{array}
$$

commutative. Since $p : (M(\phi_q(1, \eta)), x_2) \to (S^1, 1)$ is a fiber bundle it follows that there exists $L : (M(\phi_q(1, \eta)) \times I, x_1 \times I) \to (M(\phi_q(1, \eta)), x_2)$ such that $p \circ L = H$ and $f = L(\ , 1) : (M(\phi_q(1, \eta)), x_1) \to (M(\phi_q(1, \eta)), x_2)$ is over S^1, and the induced homomorphism on the fundamental group is $f_s(r, 1, c_1, c_2)$; indeed, $p \circ f = p \circ L(\ , 1) = H(\ , 1) = p$ and $f_\# = L(\ , 0)_\# = g_\# = f_s(r, 1, c_1, c_2)$. ∎

Then we have

THEOREM 2.4. *Let* $f : M(\phi_q(1, \eta)) \to M(\phi_q(1, \eta))$ *be a map over* S^1 , *where* $q \in \{0, 1\}$ *and* $\eta = \pm 1$. *If the Nielsen number of* f *restricted to the fiber is zero then* $f_\# : \pi_1(M(\phi_q(1, \eta))) \to \pi_1(M(\phi_q(1, \eta)))$ *is given by the table*:

Case I $\phi_0(1,1)$	I.1:	$f_s(r, 1, 0, 2k) : \alpha \mapsto \alpha^r,\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \beta^{2k}c_0$	$r, s, k \in \mathbb{Z}$
	I.2:	$f_s(0, 1, s, 2k+1) : \alpha \mapsto 1,\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \alpha^s\beta^{2k+1}c_0$	$s, k \in \mathbb{Z}$
Case II $\phi_1(1,1)$		$f_s(2r+1, 1, r, 2k) : \alpha \mapsto \alpha^{2r+1},\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \alpha^r\beta^{2k}c_0$	$r, s, k \in \mathbb{Z}$
Case III $\phi_0(1,-1)$	III.1:	$f_s(r, 1, 0, 2k) : \alpha \mapsto \alpha^r,\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \beta^{2k}c_0$	$r, s, k \in \mathbb{Z}$
	III.2:	$f_s(0, 1, s, 2k+1) : \alpha \mapsto 1,\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \alpha^s\beta^{2k+1}c_0$	$s, k \in \mathbb{Z}$
Case IV $\phi_1(1,-1)$		$f_s(2r+1, 1, r, 2k) : \alpha \mapsto \alpha^{2r+1},\ \beta \mapsto \alpha^s\beta,\ c_0 \mapsto \alpha^r\beta^{2k}c_0$	$r, s, k \in \mathbb{Z}$

Proof. By Proposition 2.3, we have $2c_1 = s[1 - (-1)^{c_2}] + q[r - (-1)^{c_2}]$ and $(-1)^{c_2}r = r$. Using these equations for various values of $\eta \in \{1, -1\}$ and $q \in \{0, 1\}$ yields the result. ∎

3. Calculation of $\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_i, 0\rangle))$, $i = 1, 2$. Recall that $x_2 = [(0, 0)]$ and set $x_1 = [(0, q)]$ for q small. Later we will choose ϕ such that also $\phi(x_1) = x_1$. We have the short exact sequence

$$1 \to \pi_1(K, x_i) \xrightarrow{l_\#} \pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_i, 0\rangle)) \xrightarrow{(p_1)_\#} \pi_1(M(\phi), \langle x_2, 0\rangle) \to 1$$

where $l_\#$ is the homomorphism induced by the map $l : K \to M(\phi) \times_{S^1} M(\phi)$ given by $x \mapsto (\langle x_2, 0\rangle, \langle x, 0\rangle)$, and $(p_1)_\#$ is induced by the map $p_1 : M(\phi) \times_{S^1} M(\phi) \to M(\phi)$ given by $(\langle x, t\rangle, \langle y, t\rangle) \mapsto \langle x, t\rangle$.

We will define an explicit set of generators for $\pi_1(K)$. For this we start by choosing a set of elements of $\pi_1(K - x)$ which will also be used in the next section. Let $\varrho_{11}, \varrho_{12}, B_{12}$ and $\varrho_{21}, \varrho_{22}$ be the elements of $\pi_1(K - x_2, x_1)$ and $\pi_1(K - x_1, x_2)$, as defined in [S69], respectively. See Figures 1 and 2.
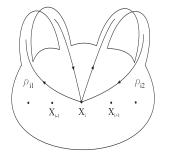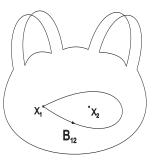

Fig. 1. The braids ϱ_{i1} and ϱ_{i1} Fig. 2. The braid B_{12}

By abuse of notation denote also by ϱ_{ij} the element of $\pi_1(K)$ which is the image of the ϱ_{ij} defined above under the map induced by the inclusion $K - x \hookrightarrow K$.

We consider the presentations $\pi_1(K, x_i) = \langle a_i, b_i \mid a_i b_i a_i b_i^{-1} = 1 \rangle$ where $a_i = \varrho_{i1}\varrho_{i2}$ and $b_i = \varrho_{i2}^{-1}$ and $\pi_1(M(\phi), \langle x_2, 0 \rangle) = \langle \alpha, \beta, c_0 \mid \alpha\beta\alpha\beta^{-1} = 1,$ $c_0\alpha c_0^{-1} = \alpha,\ c_0\beta c_0^{-1} = \alpha^p\beta^\eta) \rangle$ where $\eta = \pm 1$ and $p \in \{0, 1\}$.

Denote by $\alpha_1, \beta_1, c_{01}, u_1, v_1$ the homotopy classes of the loops given respectively by the pairs of loops $(\alpha(t), \langle x_1, 0 \rangle)$, $(\beta(t), \langle x_1, 0 \rangle)$, $(c_0(t), \langle x_1, t \rangle)$, $(\langle x_2, 0 \rangle, a_1(t))$, $(\langle x_2, 0 \rangle, b_1(t))$; and $\alpha_2, \beta_2, c_{02}, u_2, v_2$ the homotopy classes of the loops given respectively by the pairs of loops $(\alpha(t), \langle x_2, 0 \rangle)$, $(\beta(t), \langle x_2, 0 \rangle)$, $(c_0(t), c_0(t))$, $(\langle x_2, 0 \rangle, \alpha(t))$, $(\langle x_2, 0 \rangle, \beta(t))$.

THEOREM 3.1. *Let* $\phi_p(1, \eta)$ *be one of the four cases given by Theorem 2.4 and* $\alpha_i, \beta_i, c_{0i}, u_i, v_i$ *the elements in* $\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0 \rangle, \langle x_i, 0 \rangle))$. *Then* $\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0 \rangle, \langle x_i, 0 \rangle)) = \langle \alpha_i, \beta_i, c_{0i}, u_i, v_i \mid u_i v_i u_i v_i^{-1} = 1,$ $\alpha_i\beta_i\alpha_i\beta_i^{-1} = 1,\ c_{0i}\alpha_i c_{0i}^{-1}\alpha_i^{-1} = 1,\ c_{0i}\beta_i c_{0i}^{-1}\beta_i^{-\eta}\alpha_i^{-p} = 1,\ \alpha_i u_i \alpha_i^{-1} = u_i,$ $\alpha_i v_i \alpha_i^{-1} = v_i,\ \beta_i u_i \beta_i^{-1} = u_i,\ \beta_i v_i \beta_i^{-1} = v_i,\ c_{0i} u_i c_{0i}^{-1} = u_i,\ c_{0i} v_i c_{0i}^{-1} = u_i^p v_i^\eta \rangle$.

Proof. The result follows promptly from Proposition 1.8 of [GPV04]. Also it is a corollary of Theorem 4.1 by letting B be the trivial element in the presentation given by that theorem. ∎

4. Calculation of $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$. As before we have $x_2 = [(0, 0)]$, $x_1 = [(0, q)]$, and $\phi : K \to K$ is a homeomorphism such that $\phi(x_2) = x_2$ and $\phi(x_1) = x_1$.

In order to compute $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$ we consider the fiber bundle

$$(K - x_2, x_1) \xrightarrow{j_2} (M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle)) \xrightarrow{p_{1|}} (M(\phi), \langle x_2, 0 \rangle)$$

where $j_2 : K - x_2 \to M(\phi) \times_{S^1} M(\phi) - \Delta$ is given by $y \mapsto (\langle x_2, 0 \rangle, \langle y, 0 \rangle)$ and $p_{1|} : M(\phi) \times_{S^1} M(\phi) - \Delta \to M(\phi)$ is given by $(\langle x, t \rangle, \langle y, t \rangle)$ (with $y \neq x$) $\mapsto \langle x, t \rangle$. The homotopy sequence of this bundle is

$$1 \to \pi_1(K - x_2, x_1) \xrightarrow{j_{2\#}} \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle)) \xrightarrow{(p_{1|})_\#} \pi_1(M(\phi), \langle x_2, 0 \rangle) \to 1.$$

We consider the presentations $\pi_1(K - x_2, x_1) = \langle \bar{a}, \bar{b} \rangle$ (the free group on the set $\{\bar{a}, \bar{b}\}$) where $\bar{a} = \varrho_{11}\varrho_{12}$ and $\bar{b} = \varrho_{12}^{-1}$, and $\pi_1(M(\phi), \langle x_2, 0 \rangle) = \langle \alpha, \beta, c_0 \mid$ $\alpha\beta\alpha\beta^{-1} = 1, c_0\alpha c_0^{-1} = \alpha, c_0\beta c_0^{-1} = \alpha^p\beta^\eta \rangle$ where $\eta = \pm 1$, $p \in \{0, 1\}$. The ϱ_{ij} were defined in the previous section, and $\alpha, \beta, c_0 : I \to M(\phi)$ are given by $\alpha = \langle \varrho_{21}\varrho_{22}, 0 \rangle$, $\beta = \langle \varrho_{22}^{-1}, 0 \rangle$ and $c_0 = \langle x_2, t \rangle$. Finally, consider the elements $\tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b} : I \to M(\phi) \times_{S^1} M(\phi) - \Delta$ given by $\tilde{\alpha} = (\alpha, \langle x_1, 0 \rangle)$, $\tilde{\beta} = (\beta, \langle x_1, 0 \rangle)$, $\tilde{c}_0 = (c_0, \langle x_1, t \rangle)$, $\tilde{a} = (\langle x_2, 0 \rangle, \langle \bar{a}, 0 \rangle)$ and $\tilde{b} = (\langle x_2, 0 \rangle, \langle \bar{b}, 0 \rangle)$.

According to [Jo76, Chap. 13, Theorem 1], there is a presentation

$\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle)) = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b} \mid \tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = p_1(\tilde{a}, \tilde{b}),$

$\quad \tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} \tilde{\alpha}^{-\epsilon} = p_2(\tilde{a}, \tilde{b}), \ \tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} \tilde{\beta}^{-\eta} \tilde{\alpha}^{-p} = p_3(\tilde{a}, \tilde{b}), \ \tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1} = p_4(\tilde{a}, \tilde{b}),$

$\quad \tilde{\alpha}\tilde{b}\tilde{\alpha}^{-1} = p_5(\tilde{a}, \tilde{b}), \ \tilde{\beta}\tilde{a}\tilde{\beta}^{-1} = p_6(\tilde{a}, \tilde{b}), \ \tilde{\beta}\tilde{b}\tilde{\beta}^{-1} = p_7(\tilde{a}, \tilde{b}), \ \tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = p_8(\tilde{a}, \tilde{b}),$

$$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = p_9(\tilde{a}, \tilde{b})\rangle$$

where $p_j(\tilde{a}, \tilde{b})$, $j = 1, \ldots, 9$, are words in $\tilde{a}$ and $\tilde{b}$.

THEOREM 4.1. *Let $\phi_p(1, \eta)$ be one of the four cases given by Theorem 2.4 and $\tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b}$ the elements in $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$ defined above. Then*

$\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$

$\quad = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b} \mid \tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = p_1(\tilde{a}, \tilde{b}), \ \tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} \tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}),$

$\quad\quad \tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} \tilde{\beta}^{-\eta} \tilde{\alpha}^{-p} = p_3(\tilde{a}, \tilde{b}), \ \tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1} = p_4(\tilde{a}, \tilde{b}), \ \tilde{\alpha}\tilde{b}\tilde{\alpha}^{-1} = p_5(\tilde{a}, \tilde{b}),$

$\quad\quad \tilde{\beta}\tilde{a}\tilde{\beta}^{-1} = p_6(\tilde{a}, \tilde{b}), \ \tilde{\beta}\tilde{b}\tilde{\beta}^{-1} = p_7(\tilde{a}, \tilde{b}), \ \tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = p_8(\tilde{a}, \tilde{b}), \ \tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = p_9(\tilde{a}, \tilde{b})\rangle$

where $p_j(\tilde{a}, \tilde{b})$, $j = 1, \ldots, 9$, are words in $\tilde{a}$ and $\tilde{b}$ given by the tables

(4.1)

$\tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = B = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}$	
$\tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1} = B\tilde{a}B^{-1}$	$\tilde{\beta}\tilde{a}\tilde{\beta}^{-1} = \tilde{b}^{-1}\tilde{a}^{-1}\tilde{b}$
$\tilde{\alpha}\tilde{b}\tilde{\alpha}^{-1} = B(\tilde{a}^{-1}\tilde{b}\tilde{a}^{-1})B^{-1}$	$\tilde{\beta}\tilde{b}\tilde{\beta}^{-1} = \tilde{b}^{-1}(B\tilde{b})\tilde{b}$

and

(4.2)

Case I	$\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} = \tilde{\alpha}$	$\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} = \tilde{\beta}$
$\phi_0(1, 1)$	$\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = \tilde{a}$	$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = \tilde{b}$
Case II	$\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} = \tilde{\alpha}$	$\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} = B^{-1}\tilde{\alpha}\tilde{\beta}$
$\phi_1(1, 1)$	$\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = \tilde{a}$	$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = \tilde{b}\tilde{a}^{-1}$
Case III	$\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} = B^{-1}\tilde{\alpha}$	$\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} = \tilde{\beta}^{-1}$
$\phi_0(1, -1)$	$\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = \tilde{a}B^{-1}$	$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = B\tilde{b}^{-1}B^{-1}$
Case IV	$\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} = B^{-1}\tilde{\alpha}$	$\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} = \tilde{\alpha}\tilde{\beta}^{-1}$
$\phi_1(1, -1)$	$\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = \tilde{a}B^{-1}$	$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = B\tilde{b}^{-1}\tilde{a}^{-1}$

Proof. To simplify we set $\bar{\alpha} = \varrho_{21}\varrho_{22}$ and $\bar{\beta} = \varrho_{22}^{-1}$. Since $(\bar{\alpha}\bar{\beta}\bar{\alpha}\bar{\beta}^{-1}, x_1)$ $= (\varrho_{21}\varrho_{22}\varrho_{22}^{-1}\varrho_{21}\varrho_{22}\varrho_{22}, x_1) = (\varrho_{21}^2\varrho_{22}^2, x_1)$ is homotopic to $(x_2, \varrho_{11}^2\varrho_{12}^2) = (x_2, \bar{a}\bar{b}\bar{a}\bar{b}^{-1})$ it follows that

$$\tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = (\alpha\beta\alpha\beta^{-1}, \langle x_1, 0 \rangle) = (\langle \bar{\alpha}\bar{\beta}\bar{\alpha}\bar{\beta}^{-1}, 0 \rangle, \langle x_1, 0 \rangle)$$

$$= (\langle x_2, 0 \rangle, \langle \bar{a}\bar{b}\bar{a}\bar{b}^{-1}, 0 \rangle) = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}.$$

We set $B = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}$. Then $\tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = B$.

In order to compute $\tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1}, \tilde{\alpha}\tilde{b}\tilde{\alpha}^{-1}, \tilde{\beta}\tilde{a}\tilde{\beta}^{-1}$ and $\tilde{\beta}\tilde{b}\tilde{\beta}^{-1}$ as elements of the kernel we consider the presentation $\pi_1(K, *) = \langle \varrho_1, \varrho_2 \mid \varrho_1^2\varrho_2^2 = 1 \rangle$.

Since the short exact sequence

$$1 \to \pi_1(K \times K - \Delta_K, (x_2, x_1)) \overset{i_\#}{\to} \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

$$\xrightarrow{p \circ p_{1|\#}} \pi_1(S^1, \langle 0\rangle) \simeq \mathbb{Z} \to 1$$

splits we have $\tilde{\alpha} = i_\#(\hat{\alpha}), \tilde{\beta} = i_\#(\hat{\beta}), \tilde{a} = i_\#(\hat{a})$ and $\tilde{b} = i_\#(\hat{b})$ where $\hat{\alpha}, \hat{\beta}, \hat{a}$ and $\hat{b}$ are the generators of $\pi_1(K \times K - \Delta_K, (x_2, x_1))$.

If we denote by $j_\# : \pi_1(K - x_2, x_1) \to \pi_1(K, x_1)$ the homomorphism induced by the inclusion $j : K - x_2 \to K$, and by $k_\# : \pi_1(K - x_1, x_2) \to \pi_1(K, x_2)$ the homomorphism induced by the inclusion $k : K - x_1 \to K$ then $\bar{a} = \varrho_{11}\varrho_{12}, \bar{b} = \varrho_{12}^{-1}, \bar{\alpha} = \varrho_{21}\varrho_{22}$ and $\bar{\beta} = \varrho_{22}^{-1}$ where $\varrho_{11}, \varrho_{12}$ are the generators of $\pi_1(K - x_2, x_1)$ such that $j_\#(\varrho_{11}) = \varrho_1$ and $j_\#(\varrho_{12}) = \varrho_2$, and $\varrho_{21}, \varrho_{22}$ are the generators of $\pi_1(K - x_1, x_2)$ such that $k_\#(\varrho_{21}) = \varrho_1$ and $k_\#(\varrho_{22}) = \varrho_2$.

Also, if we denote by $(i_2)_\# : \pi_1(K - x_2, x_1) \to \pi_1(K \times K - \Delta_K, (x_2, x_1))$ the homomorphism induced by $i_2 : K - x_2 \to K \times K - \Delta_K$ given by $i_2(x) = (x_2, x)$, and by $(i_1)_\#$ the homomorphism induced by $i_1 : K - x_1 \to K \times K - \Delta_K$ given by $i_1(x) = (x, x_1)$, then $\hat{a} = (i_2)_\#(\varrho_{11}\varrho_{12}), \hat{b} = (i_2)_\#(\varrho_{12}^{-1}),$ $\hat{\alpha} = (i_1)_\#(\varrho_{21}\varrho_{22})$ and $\hat{\beta} = (i_1)_\#(\varrho_{22}^{-1})$, or equivalently, $\hat{a} = (i_2)_\#(\bar{a}),$ $\hat{b} = (i_2)_\#(\bar{b}), \hat{\alpha} = (i_1)_\#(\bar{\alpha})$ and $\hat{\beta} = (i_1)_\#(\bar{\beta})$ where $\bar{a}, \bar{b}, \bar{\alpha}$ and $\bar{\beta}$ are the generators of $\pi_1(K - x_2, x_1)$ and $\pi_1(K - x_1, x_2)$, respectively.

Let B_{12} be as given in Figure 2 of Section 3, and $B_{21} = B_{12}^{-1}$.

Using [S69] and the convention that the product cd of two elements in π_1 is the class of a representative of c followed by a representative of d we obtain the following presentation for $\pi_1(K \times K - \Delta_K, (x_2, x_1))$:

$$B_{12} = \varrho_{11}^2\varrho_{12}^2, \qquad\qquad B_{21}^{-1} = \varrho_{21}^2\varrho_{22}^2,$$

$$\varrho_{21}\varrho_{11}\varrho_{21}^{-1} = \varrho_{11}B_{12}^{-1}, \qquad \varrho_{21}\varrho_{12}\varrho_{21}^{-1} = B_{12}\varrho_{11}^{-1}B_{12}\varrho_{11}\varrho_{12}\varrho_{11}^{-1}B_{12}^{-1}\varrho_{11}B_{12}^{-1},$$

$$\varrho_{21}B_{12}\varrho_{21}^{-1} = B_{12}\varrho_{11}^{-1}B_{12}^{-1}\varrho_{11}B_{12}^{-1}, \quad \varrho_{22}\varrho_{11}\varrho_{22}^{-1} = \varrho_{11},$$

$$\varrho_{22}\varrho_{12}\varrho_{22}^{-1} = \varrho_{12}B_{12}^{-1}, \qquad \varrho_{22}B_{12}\varrho_{22}^{-1} = B_{12}\varrho_{12}^{-1}B_{12}^{-1}\varrho_{12}B_{12}^{-1},$$

$$\varrho_{21}^{-1}\varrho_{11}\varrho_{21} = \varrho_{11}^2B_{12}^{-1}\varrho_{11}^{-1}, \qquad \varrho_{21}^{-1}\varrho_{12}\varrho_{21} = \varrho_{11}B_{12}\varrho_{11}^{-1}B_{12}\varrho_{12}B_{12}^{-1}\varrho_{11}B_{12}^{-1}\varrho_{11}^{-1},$$

$$\varrho_{21}^{-1}B_{12}\varrho_{21} = \varrho_{11}B_{12}^{-1}\varrho_{11}^{-1}, \qquad \varrho_{22}^{-1}\varrho_{11}\varrho_{22} = \varrho_{11},$$

$$\varrho_{22}^{-1}\varrho_{12}\varrho_{22} = \varrho_{12}^2B_{12}^{-1}\varrho_{12}^{-1}, \qquad \varrho_{22}^{-1}B_{12}\varrho_{22} = \varrho_{12}B_{12}^{-1}\varrho_{12}^{-1}.$$

We have $\hat{\alpha}\hat{a}\hat{\alpha}^{-1} = (i_1)_\#(\varrho_{21}\varrho_{22})(i_2)_\#(\varrho_{11}\varrho_{12})(i_1)_\#(\varrho_{22}^{-1}\varrho_{21}^{-1})$. Using the relations above we obtain $\varrho_{21}(\varrho_{22}(\varrho_{11}\varrho_{12})\varrho_{22}^{-1})\varrho_{21}^{-1} = \varrho_{21}(\varrho_{11}\varrho_{12}B_{12}^{-1})\varrho_{21}^{-1} = B_{12}\varrho_{11}\varrho_{12} B_{12}^{-1} = \hat{B}\hat{a}\hat{B}^{-1}$. Therefore $\tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1} = B\tilde{a}B^{-1}$.

Similarly we obtain all the other relations of the table (4.1).

The fiber bundle

$$(K \times K - \Delta_K, (x_2, x_1)) \xrightarrow{i} (M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle)) \xrightarrow{p \circ p_{1|}} S^1,$$

where $i(x, y) = (\langle x, 0 \rangle, \langle y, 0 \rangle)$, provides the short exact sequence

$$1 \to \pi_1(K \times K - \Delta_K, (x_2, x_1)) \xrightarrow{i_\#} \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$$
$$\xrightarrow{p \circ p_{1|\#}} \pi_1(S^1, \langle 0 \rangle) \simeq \mathbb{Z} \to 1,$$

which splits since the quotient is $\mathbb{Z}$. The action of $\mathbb{Z} \to \mathrm{Aut}(\pi_1(K \times K - \Delta_K, (x_2, x_1)))$ which comes from the section s_0 is given by $c \cdot \varrho = c\varrho c^{-1} = (\phi \times \phi)_{|\#}(\varrho)$, where $c = p \circ p_{1|\#}\langle s_0 \rangle$ is the generator of $\pi_1(S^1)$, hence $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$ is isomorphic to the semidirect product $\pi_1(K \times K - \Delta_K, (x_2, x_1)) \rtimes \mathbb{Z}$.

Using the homeomorphism $h : M(\phi) \times_{S^1} M(\phi) \to M(\phi \times \phi)$ over S^1 given by $h(\langle x, t \rangle, \langle y, t \rangle) = \langle x, y, t \rangle$ we obtain by restriction the homeomorphism $h_| : M(\phi) \times_{S^1} M(\phi) - \Delta \to M(\phi \times \phi) - h(\Delta)$ over S^1.

In $M(\phi \times \phi) - h(\Delta)$ the loops $\tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b}$ can be seen as $\tilde{\alpha} = \langle \bar{\alpha}, x_1, 0 \rangle$, $\tilde{\beta} = \langle \bar{\beta}, x_1, 0 \rangle$, $\tilde{c}_0 = \langle x_2, x_1, t \rangle$, $\tilde{a} = \langle x_2, \bar{a}, 0 \rangle$ and $\tilde{b} = \langle x_2, \bar{b}, 0 \rangle$, and they can be viewed as classes of the representative loops in $K \times K - \Delta_K$ of natural form.

So the loops $\tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b}$ can be interpreted as follows: $\tilde{a}, \tilde{b}$ are elements in the second copy K of $K \times K$ and $\tilde{\alpha}, \tilde{\beta}$ are elements in the first copy K of $K \times K$. So

$$\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1} \tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}),$$

$$\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1} \tilde{\beta}^{-\eta} \tilde{\alpha}^{-p} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta}^{-\eta}\tilde{\alpha}^{-p} = p_3(\tilde{a}, \tilde{b}),$$

$$\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}),$$

$$\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}).$$

Now we are going to calculate

$$(\phi \times \phi)_{|\#} : \pi_1(K \times K - \Delta_K, (x_2, x_1)) \to \pi_1(K \times K - \Delta_K, (x_2, x_1)).$$

For this:

(i) We consider the commutative diagrams

$$
\begin{array}{ccc}
\pi_1(K - x_2, x_1) & \xrightarrow{(\phi_|)_\#} & \pi_1(K - x_2, x_1) \\
{\scriptstyle j_\#} \downarrow & & \downarrow {\scriptstyle j_\#} \\
\pi_1(K, x_1) & \xrightarrow{\phi_\#} & \pi_1(K, x_1)
\end{array}
$$

and

$$
\begin{array}{ccc}
\pi_1(K - x_2, x_1) & \xrightarrow{\quad (\phi_|)_\# \quad} & \pi_1(K - x_2, x_1) \\
{\scriptstyle (i_2)_\#}\downarrow & & \downarrow{\scriptstyle (i_2)_\#} \\
\pi_1(K \times K - \Delta_K, (x_2, x_1)) & \xrightarrow{(\phi \times \phi)_{|\#}} & \pi_1(K \times K - \Delta_K, (x_2, x_1)) \\
{\scriptstyle (q_{1|})_\#}\downarrow & & \downarrow{\scriptstyle (q_{1|})_\#} \\
\pi_1(K, x_2) & \xrightarrow{\quad \phi_\# \quad} & \pi_1(K, x_2)
\end{array}
$$

where $j_\#$ is induced by the inclusion $j : K - x_2 \to K$, $(i_2)_\#$ is induced by $i_2 : K - x_2 \to K \times K - \Delta_K$ given by $i_2(x) = (x_2, x)$, and $(q_{1|})_\#$ is induced by $q_1 : K \times K - \Delta_K \to K$ given by $q_{1|}(x, y) = x$.

(ii) We denote by $\hat{a}, \hat{b}, \hat{\alpha}$ and $\hat{\beta}$ the generators of $\pi_1(K \times K - \Delta_K, (x_2, x_1))$ such that $i_\#(\hat{a}) = \tilde{a}$, $i_\#(\hat{b}) = \tilde{b}$, $i_\#(\hat{\alpha}) = \tilde{\alpha}$, $i_\#(\hat{\beta}) = \tilde{\beta}$ and $i_{2\#}(\bar{a}) = \hat{a}$, $i_{2\#}(\bar{b}) = \hat{b}$, $i_{1\#}(\bar{\alpha}) = \hat{\alpha}$, $i_{1\#}(\bar{\beta}) = \hat{\beta}$.

(iii) We consider the identities

$$
\tilde{c}_0 \tilde{\alpha}(\tilde{c}_0^{-1} \tilde{a} \tilde{c}_0) \tilde{\alpha}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1}) \tilde{a}(\tilde{c}_0 \tilde{\alpha}^{-1} \tilde{c}_0^{-1}),
$$
$$
\tilde{c}_0 \tilde{\alpha}(\tilde{c}_0^{-1} \tilde{b} \tilde{c}_0) \tilde{\alpha}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1}) \tilde{b}(\tilde{c}_0 \tilde{\alpha}^{-1} \tilde{c}_0^{-1}),
$$
$$
\tilde{c}_0 \tilde{\beta}(\tilde{c}_0^{-1} \tilde{a} \tilde{c}_0) \tilde{\beta}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1}) \tilde{a}(\tilde{c}_0 \tilde{\beta}^{-1} \tilde{c}_0^{-1}),
$$
$$
\tilde{c}_0 \tilde{\beta}(\tilde{c}_0^{-1} \tilde{b} \tilde{c}_0) \tilde{\beta}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\beta} \tilde{c}_0^{-1}) \tilde{b}(\tilde{c}_0 \tilde{\beta}^{-1} \tilde{c}_0^{-1}).
$$

(iv) We write $\pi_1(K, *) = \langle \bar{a}, \bar{b} \mid \bar{a}\bar{b}\bar{a}\bar{b}^{-1} = 1 \rangle$ and $\pi_1(K - x, *) = \langle a, b \rangle$. If $w_a, w_b \in \pi_1(K - x, *)$ are such that $w_a w_b w_a w_b^{-1} = abab^{-1}$ or $w_a w_b w_a \, w_b^{-1} = ba^{-1}b^{-1}a^{-1}$ then by Theorem 1.1 of [Sko87] there exists a homeomorphism $\phi : K \to K$ such that $(\phi_|)_\#$ maps $a \mapsto w_a$ and $b \mapsto w_b$.

Now we are going to use the above diagrams in each of the four cases.

CASE I: $\phi_0(1, 1)$. In this case let $\phi : K \to K$ be the identity. We have $(\phi_|)_\# : \bar{a} \mapsto \bar{a}$, $\bar{b} \mapsto \bar{b}$ and so $(\phi \times \phi)_{|\#} : \hat{a} \mapsto \hat{a}$, $\hat{b} \mapsto \hat{b}$. Therefore $\tilde{c}_0 \tilde{a} \tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}) = (\phi_|)_\#(\tilde{a}) = \tilde{a}$ and $\tilde{c}_0 \tilde{b} \tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}) = \tilde{b}$. Since in this case $\tilde{c}_0^{-1} \tilde{b} \tilde{c}_0 = \tilde{b}$ and $\tilde{c}_0^{-1} \tilde{a} \tilde{c}_0 = \tilde{a}$ it follows from the identities $\tilde{c}_0 \tilde{\alpha}(\tilde{c}_0^{-1} \tilde{a} \tilde{c}_0) \tilde{\alpha}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1}) \tilde{a}(\tilde{c}_0 \tilde{\alpha}^{-1} \tilde{c}_0^{-1})$ and $\tilde{c}_0 \tilde{\alpha}(\tilde{c}_0^{-1} \tilde{b} \tilde{c}_0) \tilde{\alpha}^{-1} \tilde{c}_0^{-1} = (\tilde{c}_0 \tilde{\alpha} \tilde{c}_0^{-1}) \tilde{b}(\tilde{c}_0 \tilde{\alpha}^{-1} \tilde{c}_0^{-1})$ that $p_2(\tilde{a}, \tilde{b})$ must satisfy $B\tilde{a}B^{-1} = p_2(\tilde{a}, \tilde{b}) B\tilde{a}B^{-1} p_2(\tilde{a}, \tilde{b})^{-1}$ and $B\tilde{a}^{-1}(\tilde{b}\tilde{a}^{-2})\tilde{a}B^{-1} = p_2(\tilde{a}, \tilde{b}) B\tilde{a}^{-1}(\tilde{b}\tilde{a}^{-2})\tilde{a}B^{-1} p_2(\tilde{a}, \tilde{b})^{-1}$. So $p_2(\tilde{a}, \tilde{b})$ commutes with the subgroup generated by $B\tilde{a}B^{-1}$ and $B\tilde{a}^{-1}(\tilde{b}\tilde{a}^{-2}) \cdot \tilde{a}B^{-1}$. Suppose that $p_2(\tilde{a}, \tilde{b}) \neq 1$. From Exercise 7 in Section 1.4 of [MKS66] we know that some nontrivial power of $B\tilde{a}B^{-1}$ is also a power of $p_2(\tilde{a}, \tilde{b})$. Similarly for $B\tilde{a}^{-1}(\tilde{b}\tilde{a}^{-2})\tilde{a}B^{-1}$. By Exercise 4 in Section 1.4 of [MKS66]

the two elements $B\tilde{a}B^{-1}$ and $B\tilde{a}^{-1}(\tilde{b}\tilde{a}^{-2})\tilde{a}B^{-1}$ commute, so they generate a cyclic subgroup. But the subgroup generated by $B\tilde{a}B^{-1}$ and $B\tilde{a}^{-1}$ $\cdot (\tilde{b}\tilde{a}^{-2})\tilde{a}B^{-1}$ cannot be cyclic since it is not cyclic in the abelianization. So we get $p_2(\tilde{a}, \tilde{b}) = 1$. Also, from the identities $\tilde{c}_0\tilde{\beta}(\tilde{c}_0^{-1}\tilde{a}\tilde{c}_0)\tilde{\beta}^{-1}\tilde{c}_0^{-1} = (\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1})\tilde{a}(\tilde{c}_0\tilde{\beta}^{-1}\tilde{c}_0^{-1})$ and $\tilde{c}_0\tilde{\beta}(\tilde{c}_0^{-1}\tilde{b}\tilde{c}_0)\tilde{\beta}^{-1}\tilde{c}_0^{-1} = (\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1})\tilde{b}(\tilde{c}_0\tilde{\beta}^{-1}\tilde{c}_0^{-1})$ we find that $p_3(\tilde{a}, \tilde{b})$ must satisfy $\tilde{b}^{-1}\tilde{a}^{-1}\tilde{b} = p_3(\tilde{a}, \tilde{b})\tilde{b}^{-1}\tilde{a}^{-1}\tilde{b} + p_3(\tilde{a}, \tilde{b})^{-1}$ and $\tilde{b}^{-1}(B\tilde{b})\tilde{b} = p_3(\tilde{a}, \tilde{b})\tilde{b}^{-1}(B\tilde{b})\tilde{b}p_3(\tilde{a}, \tilde{b})^{-1}$. So $p_3(\tilde{a}, \tilde{b}) = 1$ is the unique solution. Therefore $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}) = 1$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}^{-1} = p_3(\tilde{a}, \tilde{b}) = 1$. As $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1}$ $= (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-1} = 1$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta}^{-1} = 1$ it follows that $(\phi \times \phi)_{|\#}(\tilde{\alpha}) = \tilde{\alpha}$ and $(\phi \times \phi)_{|\#}(\tilde{\beta}) = \tilde{\beta}$.

Case II: $\phi_1(1, 1)$. By (iv) we can take ϕ such that $(\phi_|)_\# : \bar{a} \mapsto \bar{a}, \bar{b} \mapsto \bar{b}\bar{a}^{-1}$. Now, by diagrams in (i) we have $\phi_\# = \phi_1(1, 1)$ and $(\phi \times \phi)_{|\#} : \hat{a} \mapsto \hat{a}, \hat{b} \mapsto \hat{b}\hat{a}^{-1}$. So $\tilde{c}_0\tilde{a}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}) = \tilde{a}$ and $\tilde{c}_0\tilde{b}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}) = \tilde{b}\tilde{a}^{-1}$. Now, as in Case I, it follows from the identities (iii) that $p_2(\tilde{a}, \tilde{b}) = 1$ and $p_3(\tilde{a}, \tilde{b}) = B^{-1}$ are the unique solutions. Therefore $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}) = 1$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}^{-1}\tilde{\alpha}^{-1} = p_3(\tilde{a}, \tilde{b}) = B^{-1}$. As $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-1} = 1$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}^{-1}\tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta}^{-1}\tilde{\alpha}^{-1} = B^{-1}$ where $B = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}$ it follows that $(\phi \times \phi)_{|\#}(\tilde{\alpha}) = \tilde{\alpha}$ and $(\phi \times \phi)_{|\#}(\tilde{\beta}) = B^{-1}\tilde{\alpha}\tilde{\beta}$.

Case III: $\phi_0(1, -1)$. By (iv) we can take ϕ such that $(\phi_|)_\# : \bar{a} \mapsto \bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}\bar{a}^{-1} = \bar{a}\bar{E}^{-1}, \bar{b} \mapsto \bar{a}\bar{b}\bar{a}\bar{b}^{-1}\bar{a}^{-1}\bar{b}^{-1}\bar{a}^{-1} = \bar{E}\bar{b}^{-1}\bar{E}^{-1}$ where $\bar{E} = \bar{a}\bar{b}\bar{a}\bar{b}^{-1}$. Now, by diagrams in (i) we have $\phi_\# = \phi_0(1, -1)$ and $(\phi \times \phi)_{|\#} : \hat{a} \mapsto \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1}\hat{a}^{-1} = \hat{a}\hat{E}^{-1}, \hat{b} \mapsto \hat{a}\hat{b}\hat{a}\hat{b}^{-1}\hat{a}^{-1}\hat{b}^{-1}\hat{a}^{-1} = \hat{E}\hat{b}^{-1}\hat{E}^{-1}$ where $\hat{E} = \hat{a}\hat{b}\hat{a}\hat{b}^{-1}$. Therefore $\tilde{c}_0\tilde{a}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}) = \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a}^{-1} = \tilde{a}B^{-1}$ and $\tilde{c}_0\tilde{b}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}) = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a}^{-1} = B\tilde{b}^{-1}B^{-1}$ where $B = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1}$. Now, as in Case I, it follows from the identities (iii) that $p_2(\tilde{a}, \tilde{b}) = B^{-1}$ and $p_3(\tilde{a}, \tilde{b}) = 1$ are the unique solutions. Therefore $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}) = B^{-1}$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta} = p_3(\tilde{a}, \tilde{b}) = 1$. As $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-1} = B^{-1}$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta} = 1$ it follows that $(\phi \times \phi)_{|\#}(\tilde{\alpha}) = B^{-1}\tilde{\alpha}$ and $(\phi \times \phi)_{|\#}(\tilde{\beta}) = \tilde{\beta}^{-1}$.

Case IV: $\phi_1(1, -1)$. By (iv) we can take ϕ such that $(\phi_|)_\# : \bar{a} \mapsto \bar{a}\bar{E}^{-1}, \bar{b} \mapsto \bar{E}\bar{b}^{-1}\bar{a}^{-1}$ where $\bar{E} = \bar{a}\bar{b}\bar{a}\bar{b}^{-1}$. Now, by diagrams in (i) we have $\phi_\# = \phi_1(1, -1)$ and $(\phi \times \phi)_{|\#} : \hat{a} \mapsto \hat{a}\hat{E}^{-1}, \hat{b} \mapsto \hat{E}\hat{b}^{-1}\hat{a}^{-1}$ where $\hat{E} = \hat{a}\hat{b}\hat{a}\hat{b}^{-1}$. Therefore $\tilde{c}_0\tilde{a}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}) = \tilde{a}B^{-1}$ and $\tilde{c}_0\tilde{b}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}) = B\tilde{b}^{-1}\tilde{a}^{-1}$. So, it follows from the identities (iii) that $p_2(\tilde{a}, \tilde{b}) = B^{-1}$ and $p_3(\tilde{a}, \tilde{b}) = 1$ are the unique solutions. Therefore $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = p_2(\tilde{a}, \tilde{b}) = B^{-1}$ and $\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}\tilde{\alpha}^{-1} = p_3(\tilde{a}, \tilde{b}) = 1$. As $\tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-1} = B^{-1}$ and

$\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}\tilde{\alpha}^{-1} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta}\tilde{\alpha}^{-1} = 1$ it follows that $(\phi \times \phi)_{|\#}(\tilde{\alpha}) = B^{-1}\tilde{\alpha}$ and $(\phi \times \phi)_{|\#}(\tilde{\beta}) = \tilde{\alpha}\tilde{\beta}^{-1}$. This settlesall the cases from the table (4.2). ∎

REMARK 4.1. We observe that every word $p(\tilde{a}, \tilde{b})$ can be written as a word in $w = \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a}^{-1}$ and $v = \tilde{a}\tilde{b}$ because $\tilde{a} = v^{-1}w^{-1}v$ and $\tilde{b} = v^{-1}wv^2$. For the next sections it will be convenient to calculate the conjugates of the generators v and w by $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{c}_0$. They are given in the tables (4.3) and (4.4) below.

(4.3)

$\tilde{\alpha}v\tilde{\alpha}^{-1} = wvw$	$\tilde{\alpha}^{-1}v\tilde{\alpha} = w^{-1}vw^{-1}$
$\tilde{\alpha}w\tilde{\alpha}^{-1} = w$	$\tilde{\alpha}^{-1}w\tilde{\alpha} = w$
$\tilde{\alpha}B\tilde{\alpha}^{-1} = \tilde{\alpha}(w^{-1}v^{-1}w^{-1}v)\tilde{\alpha}^{-1}$	$\tilde{\alpha}^{-1}B\tilde{\alpha} = \tilde{\alpha}^{-1}(w^{-1}v^{-1}w^{-1}v)\tilde{\alpha}$
$= w^{-1}Bw$	$= wBw^{-1}$
$\tilde{\beta}v\tilde{\beta}^{-1} = v$	$\tilde{\beta}^{-1}v\tilde{\beta} = v$
$\tilde{\beta}w\tilde{\beta}^{-1} = v^{-1}w^{-1}v$	$\tilde{\beta}^{-1}w\tilde{\beta} = vw^{-1}v^{-1}$
$\tilde{\beta}B\tilde{\beta}^{-1} = v^{-1}wB^{-1}w^{-1}v$	$\tilde{\beta}^{-1}B\tilde{\beta} = vwB^{-1}w^{-1}v^{-1}$

(4.4)

Case I	$\tilde{c}_0v\tilde{c}_0^{-1} = v$	$\tilde{c}_0w\tilde{c}_0^{-1} = w$	$\tilde{c}_0B\tilde{c}_0^{-1} = B$
$\phi_0(1,1)$	$\tilde{c}_0^{-1}v\tilde{c}_0 = v$	$\tilde{c}_0^{-1}w\tilde{c}_0 = w$	$\tilde{c}_0^{-1}B\tilde{c}_0 = B$
Case II	$\tilde{c}_0v\tilde{c}_0^{-1} = wv$	$\tilde{c}_0w\tilde{c}_0^{-1} = w$	$\tilde{c}_0B\tilde{c}_0^{-1} = B$
$\phi_1(1,1)$	$\tilde{c}_0^{-1}v\tilde{c}_0 = w^{-1}v$	$\tilde{c}_0^{-1}w\tilde{c}_0 = w$	$\tilde{c}_0^{-1}B\tilde{c}_0 = B$
Case III	$\tilde{c}_0v\tilde{c}_0^{-1} = v^{-1}$	$\tilde{c}_0w\tilde{c}_0^{-1} = v^{-1}w^{-1}v$	$\tilde{c}_0B\tilde{c}_0^{-1} = B^{-1}$
$\phi_0(1,-1)$	$\tilde{c}_0^{-1}v\tilde{c}_0 = v^{-1}$	$\tilde{c}_0^{-1}w\tilde{c}_0 = v^{-1}w^{-1}v$	$\tilde{c}_0^{-1}B\tilde{c}_0 = B^{-1}$
Case IV	$\tilde{c}_0v\tilde{c}_0^{-1} = v^{-1}w^{-1}$	$\tilde{c}_0w\tilde{c}_0^{-1} = v^{-1}w^{-1}v$	$\tilde{c}_0B\tilde{c}_0^{-1} = B^{-1}$
$\phi_1(1,-1)$	$\tilde{c}_0^{-1}v\tilde{c}_0 = v^{-1}w$	$\tilde{c}_0^{-1}w\tilde{c}_0 = v^{-1}w^{-1}v$	$\tilde{c}_0^{-1}B\tilde{c}_0 = B^{-1}$

5. The lifting problem and some reductions. Now we study the existence of the lift in the algebraic diagram (1.1) given in Subsection 1.1.

The homomorphism $j_{2\#}$ is the induced by the map

$$j_2 : (K - x_2, x_1) \to (M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$$

given by $y \mapsto (\langle x_2, 0 \rangle, \langle y, 0 \rangle)$. The homomorphism ∂_2 is the connecting homomorphism of the homotopy exact sequence of the pair $(K, K - x_2)$,

$$1 \to \pi_2(K, K - x_2, x_1) \xrightarrow{\partial_2} \pi_1(K - x_2, x_1) \xrightarrow{j_\pi} \pi_1(K, x_1) \to 1$$

and the homomorphism

$$\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$$
$$\downarrow q_\#$$
$$\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0 \rangle, \langle x_2, 0 \rangle))$$

is given by the composition

$$\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

$$\kappa \downarrow$$

$$\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

$$\bar{\nu} \downarrow \simeq$$

$$\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_2, 0\rangle)).$$

Here $\bar{\nu}(\eta) = \nu^{-1}\eta\nu$ where $\nu : I \to M(\phi) \times_{S^1} M(\phi)$ is given by $\nu(t) = (\langle x_2, 0\rangle, \langle \sigma(t), 0\rangle)$ with $\sigma : I \to K$ a path joining x_1 to x_2 and κ is the homomorphism given by $\tilde{\alpha} \mapsto \alpha_1$, $\tilde{\beta} \mapsto \beta_1$, $\tilde{c}_0 \mapsto c_{01}$, $\tilde{a} \mapsto u_1$, $\tilde{b} \mapsto v_1$. In order to compute $\bar{\nu}^{-1} \circ (1, f)_\#$ we will compute the homomorphism $(1, f)_\#$ and the homomorphism $\bar{\nu}^{-1}$ of the above diagram, where $f_\# = f_s(r, 1, c_1, c_2)$ is given in the table from Theorem 2.4. From Sections 1, 3 and 4 we have

$$\pi_1(M(\phi), \langle x_2, 0\rangle) = \langle \alpha, \beta, c_0 \mid \alpha\beta\alpha\beta^{-1} = 1, c_0\alpha c_0^{-1} = \alpha, c_0\beta c_0^{-1} = \alpha^p\beta^\eta \rangle,$$

$$\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_2, 0\rangle)) = \langle \alpha_2, \beta_2, c_{02}, u_2, v_2 \mid u_2 v_2 u_2 v_2^{-1} = 1,$$

$$\alpha_2\beta_2\alpha_2\beta_2^{-1} = 1, \ c_{02}\alpha_2 c_{02}^{-1}\alpha_2^{-1} = 1, \ c_{02}\beta_2 c_{02}^{-1}\beta_2^{-\eta}\alpha_2^{-p} = 1, \ \alpha_2 u_2\alpha_2^{-1} = u_2,$$

$$\alpha_2 v_2\alpha_2^{-1} = v_2, \ \beta_2 u_2\beta_2^{-1} = u_2, \ \beta_2 v_2\beta_2^{-1} = v_2, \ c_{02}u_2 c_{02}^{-1} = u_2^\epsilon, \ c_{02}v_2 c_{02}^{-1} = u_2^p v_2^\eta \rangle,$$

$$\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

$$= \langle \tilde{\alpha}, \tilde{\beta}, \tilde{c}_0, \tilde{a}, \tilde{b} \mid \tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = \tilde{a}\tilde{b}\tilde{a}\tilde{b}^{-1} = B, \ \tilde{c}_0\tilde{\alpha}\tilde{c}_0^{-1}\tilde{\alpha}^{-\epsilon} = (\phi \times \phi)_{|\#}(\tilde{\alpha})\tilde{\alpha}^{-\epsilon},$$

$$\tilde{c}_0\tilde{\beta}\tilde{c}_0^{-1}\tilde{\beta}^{-\eta}\tilde{\alpha}^{-p} = (\phi \times \phi)_{|\#}(\tilde{\beta})\tilde{\beta}^{-\eta}\tilde{\alpha}^{-p}, \ \tilde{\alpha}\tilde{a}\tilde{\alpha}^{-1} = B\tilde{a}B^{-1},$$

$$\tilde{\alpha}\tilde{b}\tilde{\alpha}^{-1} = B\tilde{a}^{-1}\tilde{b}\tilde{a}^{-1}B^{-1}, \ \tilde{\beta}\tilde{a}\tilde{\beta}^{-1} = \tilde{b}^{-1}\tilde{a}^{-1}\tilde{b}, \ \tilde{\beta}\tilde{b}\tilde{\beta}^{-1} = \tilde{b}^{-1}(B\tilde{b})\tilde{b},$$

$$\tilde{c}_0\tilde{a}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{a}), \ \tilde{c}_0\tilde{b}\tilde{c}_0^{-1} = (\phi \times \phi)_{|\#}(\tilde{b}) \rangle$$

and

$$\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_1, 0\rangle)) = \langle \alpha_1, \beta_1, c_{01}, u_1, v_1 \mid u_1 v_1 u_1 v_1^{-1} = 1,$$

$$\alpha_1\beta_1\alpha_1\beta_1^{-1} = 1, \ c_{01}\alpha_1 c_{01}^{-1}\alpha_1^{-\epsilon} = 1, \ c_{01}\beta_1 c_{01}^{-1}\beta_1^{-\eta}\alpha_1^{-p} = 1, \ \alpha_1 u_1\alpha_1^{-1} = u_1,$$

$$\alpha_1 v_1\alpha_1^{-1} = v_1, \ \beta_1 u_1\beta_1^{-1} = u_1, \ \beta_1 v_1\beta_1^{-1} = v_1, \ c_{01}u_1 c_{01}^{-1} = u_1^\epsilon, \ c_{01}v_1 c_{01}^{-1} = u_1^p v_1^\eta \rangle,$$

where $p \in \{0, 1\}$, $\epsilon = \pm 1$ and $\eta = \pm 1$.

We have $(1, f)_\#(\langle \alpha(t)\rangle) = \langle (1, f) \circ \alpha(t)\rangle = \langle (\alpha(t), f \circ \alpha(t))\rangle$. But $(\alpha(t), f \circ \alpha(t)) \sim_{\partial I} (\alpha(t), \langle x_2, 0\rangle) * (\langle x_2, 0\rangle, f \circ \alpha(t))$.

Since $f_{\#}(\alpha) = \alpha^r$ it follows that

$$\begin{aligned}
(1,f)_{\#}(\langle\alpha\rangle) &= \langle(1,f)\circ\alpha\rangle = \langle(\alpha, f\circ\alpha)\rangle \\
&= \langle(\alpha, \langle x_2, 0\rangle)\rangle\langle(\langle x_2, 0\rangle, f\circ\alpha)\rangle \\
&= \langle(\alpha, \langle x_2, 0\rangle)\rangle\langle(\langle x_2, 0\rangle, \alpha^r)\rangle = \alpha_2 u_2^r.
\end{aligned}$$

Similarly, as $f_{\#}(\beta) = \alpha^s\beta$ we have $(1,f)_{\#}(\langle\beta\rangle) = \beta_2 u_2^s v_2$. Also, $(1,f)_{\#}(\langle c_0\rangle)$ $= \langle(1,f)\circ c_0\rangle = \langle(c_0, f\circ c_0)\rangle$. But, since $f_{\#}(c_0) = \alpha^{c_1}\beta^{c_2}c_0$ we have $(c_0, f\circ c_0)$ $\simeq_{\partial I} (\langle x_2, 0\rangle, \alpha^{c_1}) * (\langle x_2, 0\rangle, \beta^{c_2}) * (c_0, c_0)$. So $(1,f)_{\#}(c_0) = u_2^{c_1} v_2^{c_2} c_{02}$. Summarizing, in all cases, if $f_{\#} = f_s(r, 1, c_1, c_2)$ then $(1,f)_{\#}$ is given by $\alpha \mapsto \alpha_2 u_2^r$, $\beta \mapsto \beta_2 u_2^s v_2$ and $c_0 \mapsto u_2^{c_1} v_2^{c_2} c_{02}$.

For

$$\bar{\nu} : \ \pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_1, 0\rangle)) \to \pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_2, 0\rangle))$$

we have

$$\begin{aligned}
\bar{\nu}(\alpha_1) = \nu^{-1}\alpha_1\nu &= (\langle x_2, 0\rangle, \langle\sigma^{-1}, 0\rangle)(\alpha, \langle x_1, 0\rangle)(\langle x_2, 0\rangle, \langle\sigma, 0\rangle) \\
&= (\alpha, \langle x_2, 0\rangle) = \alpha_2.
\end{aligned}$$

Similarly $\bar{\nu}(\beta_1) = \beta_2$, $\bar{\nu}(c_{01}) = c_{02}$, $\bar{\nu}(u_1) = u_2$ and $\bar{\nu}(v_1) = v_2$.

Therefore, in all cases, if $f_{\#} = f_s(r, 1, c_1, c_2)$ then

$$\bar{\nu}^{-1} \circ (1,f)_{\#} : \pi_1(M(\phi), \langle x_2, 0\rangle) \to \pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

is given by $\alpha \mapsto \alpha_1 u_1^r$, $\beta \mapsto \beta_1 u_1^s v_1$ and $c_0 \mapsto u_1^{c_1} v_1^{c_2} c_{01}$.

The next theorem describes the homomorphisms of diagram (1.1).

THEOREM 5.1. *Let $f_s(r, 1, c_1, c_2)$ be one of the four cases given by Theorem 2.4, α, β, c_0 loops in $M(\phi)$ based at $\langle x_2, 0\rangle$ given respectively by $\alpha = \langle\varrho_{21}\varrho_{22}, 0\rangle$, $\beta = \langle\varrho_{22}^{-1}, 0\rangle$, $c_0(t) = \langle x_2, t\rangle$, and $\bar{a} = \varrho_{11}\varrho_{12}$, $\bar{b} = \varrho_{12}^{-1}$ loops in $K - x_2$ based at x_1. Then:*

(1) *The homomorphism*

$$(1,f)_{\#} : \pi_1(M(\phi), \langle x_2, 0\rangle) \to \pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0\rangle, \langle x_2, 0\rangle))$$

is given by

$$\alpha \mapsto \alpha_2 u_2^r, \qquad \beta \mapsto \beta_2 u_2^s v_2, \qquad c_0 \mapsto u_2^{c_1} v_2^{c_2} c_{02}.$$

(2) *The homomorphism $\bar{\nu}$ is given by $\bar{\nu}(\eta) = \nu^{-1}\eta\nu$ where $\nu : I \to M(\phi) \times_{S^1} M(\phi)$ is given by $\nu(t) = (\langle x_2, 0\rangle, \langle\sigma(t), 0\rangle)$ with $\sigma : I \to K$ a path joining x_1 to x_2, and the homomorphism κ is given by $\tilde{\alpha} \mapsto \alpha_1$, $\tilde{\beta} \mapsto \beta_1$, $\tilde{c}_0 \mapsto c_{01}$, $\tilde{a} \mapsto u_1$ and $\tilde{b} \mapsto v_1$.*

(3) *The lifting homomorphism*

$$\psi : \pi_1(M(\phi), \langle x_2, 0\rangle) \to \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$$

exists if and only if we can find elements $Z_1, Z_2, Z_3 \in \pi_2(K, K - x_2, x_1)$ and $A, F, C \in \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0\rangle, \langle x_1, 0\rangle))$

such that

$$\psi : \begin{cases} \alpha \mapsto Z_1 A & \text{with } \kappa(A) = \alpha_1 u_1^r \text{ if } f_\#(\alpha) = \alpha^r, \\ \beta \mapsto Z_2 F & \text{with } \kappa(F) = \beta_1 u_1^s v_1 \text{ if } f_\#(\beta) = \alpha^s \beta, \\ c_0 \mapsto Z_3 C & \text{with } \kappa(C) = u_1^{c_1} v_1^{c_2} c_{01} \text{ if } f_\#(c_0) = \alpha^{c_1} \beta^{c_2} c_0, \end{cases}$$

and it must satisfy $\psi(\alpha\beta\alpha\beta^{-1}) = 1$, $\psi(c_0\alpha c_0^{-1}\alpha^{-\epsilon}) = 1$ *and* $\psi(c_0\beta c_0^{-1}\beta^{-\eta}\alpha^{-p}) = 1$ *where* $p \in \{0,1\}$, $\epsilon = \pm 1$ *and* $\eta = \pm 1$.

Proof. (1) and (2) follow from the considerations before.

Therefore it is enough to prove (3). First we observe that if $\psi(\alpha) = x$ and $f_\#(\alpha) = \alpha^r$ then $\kappa(x) = \bar{\nu}^{-1} \circ (1, f)_\#(\alpha) = \alpha_1 u_1^r$. On the other hand, if $\kappa(A) = \alpha_1 u_1^r$ then $xA^{-1} = Z_1$, where $Z_1 \in \pi_2(K, K - x_2, x_1) = $ kernel of $\pi_1(K - x_2, x_1) \to \pi_1(K, x_1)$, and the result follows. Similarly we argue for β and c_0. Now the equalities $\psi(\alpha\beta\alpha\beta^{-1}) = 1$, $\psi(c_0\alpha c_0^{-1}\alpha^{-\epsilon}) = 1$ and $\psi(c_0\beta c_0^{-1}\beta^{-\eta}\alpha^{-p}) = 1$ follow from the relations in $\pi_1(M(\phi), \langle x_2, 0 \rangle)$. ∎

Now we will derive the equations that Z_1, Z_2, Z_3, A, F, C must satisfy in order that there exists a lifting

$$\psi : \pi_1(M(\phi), \langle x_2, 0 \rangle) \to \pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$$

in each of the four cases given by Theorem 2.4.

From the theorem above, for $f_\# = f_s(r, 1, c_1, c_2)$, ψ is of the form

$$\begin{cases} \alpha \mapsto Z_1 A & \text{with } \kappa(A) = \alpha_1 u_1^r, \\ \beta \mapsto Z_2 F & \text{with } \kappa(F) = \beta_1 u_1^s v_1, \\ c_0 \mapsto Z_3 C & \text{with } \kappa(C) = u_1^{c_1} v_1^{c_2} c_{01}, \end{cases}$$

and it must satisfy $\psi(\alpha\beta\alpha\beta^{-1}) = 1$, $\psi(c_0\alpha c_0^{-1}\alpha^{-\epsilon}) = 1$ and $\psi(c_0\beta c_0^{-1}\beta^{-\eta}\alpha^{-p}) = 1$ where $p \in \{0,1\}$, $\epsilon = \pm 1$ and $\eta = \pm 1$. Hence, there exists a lift ψ if and only if the system of equations

$$(I) \quad \begin{cases} Z_1(AZ_2A^{-1})(AFAF^{-1})(FA^{-1}Z_1AF^{-1})Z_2^{-1} = 1, \\ Z_3(CZ_1C^{-1})(CAC^{-1}A^{-1})(AZ_3^{-1}A^{-1})Z_1^{-1} = 1, \\ Z_3(CZ_2C^{-1})(CFC^{-1}F^{-\eta}A^{-p})(A^pF^\eta Z_3^{-1}F^{-\eta}A^{-p}) \\ \quad \cdot (A^pF^{(\eta-1)/2}Z_2^{-\eta}F^{(1-\eta)/2}A^{-p})Z_1^{-p} = 1 \end{cases}$$

has a solution in Z_1, Z_2, Z_3.

We refer to (I) as the *system generated by the input data* $(A, F, C; r, s, (c_1, c_2))$ where $\kappa(A) = \alpha_1 u_1^r$, $\kappa(F) = \beta_1 u_1^s v_1$ and $\kappa(C) = u_1^{c_1} v_1^{c_2} c_{01}$.

Each equation in (I) involves a product where all the factors are conjugates of Z_i, except one. The factor which is not a conjugate of Z_i is called the *constant term* of the equation; these are $AFAF^{-1}$, $CAC^{-1}A^{-1}$ and $CFC^{-1}F^{-\eta}A^{-p}$, respectively. Since the variables Z_j belong to the normal

subgroup $\pi_2(K, K - x_2, x_1)$ of $\pi_1(M(\phi) \times_{S^1} M(\phi) - \Delta, (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$, all factors (including the constant terms) belong to $\pi_2(K, K - x_2, x_1)$.

PROPOSITION 5.2. *There is a solution of the system generated by the input data* $(A_1, F_1, C_1; r, s, (c_1, c_2))$ *if and only if there exist solutions for all systems generated by any input data* $(A, F, C; r, s, (c_1, c_2))$.

Proof. Suppose that X_1, X_2, X_3 is a solution for the system generated by $(A_1, F_1, C_1; r, s, (c_1, c_2))$. As $\kappa(A_1) = \kappa(A)$, $\kappa(F_1) = \kappa(F)$ and $\kappa(C_1) = \kappa(C)$ we can write $A_1 = Y_1 A$, $F_1 = Y_2 F$ and $C_1 = Y_3 C$ for some $Y_j \in \pi_2(K, K - x_2, x_1)$. When we substitute these in the above system we obtain

$$\begin{cases} X_1 Y_1 A X_2 Y_2 F X_1 Y_1 A F^{-1} Y_2^{-1} X_2^{-1} = 1, \\ X_3 Y_3 C X_1 Y_1 A C^{-1} Y_3^{-1} X_3^{-1} A^{-1} Y_1^{-1} X_1^{-1} = 1, \\ X_3 Y_3 C X_2 Y_2 F C^{-1} Y_3^{-1} X_3^{-1} (F^{-1} Y_2^{-1} X_2^{-1})^\eta (A^{-1} Y_1^{-1} X_1^{-1})^p = 1. \end{cases}$$

Therefore $Z_1 = X_1 Y_1$, $Z_2 = X_2 Y_2$, $Z_3 = X_3 Y_3$ is a solution of the system generated by the input data $(A, F, C; r, s, (c_1, c_2))$. The converse is immediate. ∎

Conjugating the equations of the system by a word q in $\tilde{a}$ and $\tilde{b}$ (or v and w) we find that $\kappa(q) = u_1^m v_1^n$ and so we obtain

THEOREM 5.3. *Let* $\phi = \phi_p(1, \eta)$ *be one of the four cases given by Theorem 2.4. If* q *denotes a word so that* $\kappa(q) = u_1^m v_1^n$ *then conjugating by* q *the equations of the system* (I) *generated by the input data* $(A, F, C; r, s, (c_1, c_2))$, *we obtain a new system generated by*

$$\left(A', F', C'; (-1)^n r, 2m + (-1)^n s, \right.$$

$$\left. \left((1 - (-1)^{c_2 + (1-\eta)n}) m + (-1)^{c_2 + n} \left(\frac{1 - (-1)^n}{2} \right) p + (-1)^n c_1, (1 - \eta)n + c_2 \right) \right)$$

where $A' = qAq^{-1}$, $F' = qFq^{-1}$ *and* $C' = qCq^{-1}$. *Moreover, if* (Z_1, Z_2, Z_3) *is a solution of the system* (I) *then* $(Z_1' = qZ_1q^{-1}, Z_2' = qZ_2q^{-1}, Z_3' = qZ_3q^{-1})$ *is a solution of the new system given above.*

Proof. The final part is straightforward. For the first part it is sufficient to calculate $\kappa(A')$, $\kappa(F')$ and $\kappa(C')$.

Using the relations in $\pi_1(M(\phi) \times_{S^1} M(\phi), (\langle x_2, 0 \rangle, \langle x_1, 0 \rangle))$ we obtain

$$\kappa(A') = \kappa(qAq^{-1}) = u_1^m v_1^n \alpha_1 u_1^r v_1^{-n} u_1^{-m} = \alpha_1 \alpha_1^{-1} u_1^m v_1^n \alpha_1 u_1^r v_1^{-n} u_1^{-m}$$

$$= \alpha_1 u_1^m v_1^n u_1^r v_1^{-n} u_1^{-m} = \alpha_1 u_1^m u_1^{(-1)^n r} v_1^n v_1^{-n} u_1^{-m} = \alpha_1 u_1^{(-1)^n r}.$$

Similarly $\kappa(F') = \beta_1 u_1^{2m + (-1)^n s} v_1$.

For the system (I) we have

$$\kappa(C') = \kappa(qCq^{-1}) = u_1^m v_1^n u_1^{c_1} v_1^{c_2} c_{01} v_1^{-n} u_1^{-m}$$

$$= u_1^m v_1^n u_1^{c_1} v_1^{c_2} c_{01} v_1^{-n} u_1^{-m} c_{01}^{-1} c_{01}$$

$$= u_1^{[1-(-1)^{c_2+(1-\eta)n}]m+[(-1)^{n+c_2}(\frac{1-(-1)^n}{2})]p+(-1)^n c_1} v_1^{c_2+(1-\eta)n} c_{01}. \quad \blacksquare$$

REMARK 5.1. As a consequence of the above theorem the existence of a lifting for $f_s(r, 1, c_1, 2k)$ or $f_s(r, 1, s, 2k+1)$ is equivalent to the existence of a lifting for $f_{2m+(-1)^n s}((-1)^n r, 1, -(\frac{1-(-1)^n}{2})p + (-1)^n c_1, (1-\eta)n + 2k)$ or $f_{2m+(-1)^n s}((-1)^n r, 1, 2m + (\frac{1-(-1)^n}{2})p + (-1)^n s, (1-\eta)n + 2k + 1)$ respectively.

Using the results above we can reduce the cases to be analyzed.

COROLLARY 5.4. *In order to study the problem of existence of solution of the system generated by the input data* $(A, F, C; r, s, (c_1, c_2))$ *it suffices to solve the problem for input data given by the homomorphisms* $f_s(r, 1, c_1, c_2)$ *listed in the following table*:

Case I $\phi_0(1,1)$	I.1: $f_s(r, 1, 0, 2k) : \alpha \mapsto \alpha^r, \beta \mapsto \alpha^s \beta, c_0 \mapsto \beta^{2k} c_0,\ r \geq 0, s \in \{0,1\}, k \in \mathbb{Z}$ I.2: $f_s(0, 1, s, 2k+1) : \alpha \mapsto 1, \beta \mapsto \alpha^s \beta, c_0 \mapsto \alpha^s \beta^{2k+1} c_0,\ s \in \{0,1\}, k \in \mathbb{Z}$
Case II $\phi_1(1,1)$	$f_s(2r+1, 1, r, 2k) : \alpha \mapsto \alpha^{2r+1}, \beta \mapsto \alpha^s \beta, c_0 \mapsto \alpha^r \beta^{2k} c_0,\ r \geq 0, s \in \{0,1\}, k \in \mathbb{Z}$
Case III $\phi_0(1,-1)$	III.1: $f_s(r, 1, 0, 2k) : \alpha \mapsto \alpha^r, \beta \mapsto \alpha^s \beta, c_0 \mapsto \beta^{2k} c_0,\ r \geq 0, s \in \{0,1\}, k \in \{0,1\}$ III.2: $f_s(0, 1, s, 1) : \alpha \mapsto 1, \beta \mapsto \alpha^s \beta, c_0 \mapsto \alpha^s \beta c_0,\ s \in \{0,1\}$
Case IV $\phi_1(1,-1)$	$f_s(2r+1, 1, r, 0) : \alpha \mapsto \alpha^{2r+1}, \beta \mapsto \alpha^s \beta, c_0 \mapsto \alpha^r c_0,\ r \in \mathbb{Z}, s \in \{0,1\}$

Proof. Let $\mathcal{S}$ be the system generated by $(A, F, C; r, s, (c_1, c_2))$ where $f_s(r, 1, c_1, c_2)$ is given by the table of Theorem 2.4. From Theorem 5.3, by straightforward calculation in each of the four cases, it follows that there is a system $\mathcal{S}'$, generated by $(A', F', C'; r', s', (c_1', c_2'))$ where $f_{s'}(r', 1, c_1', c_2')$ belongs to the table in the statement of the corollary, such that $\mathcal{S}$ has a solution if and only if $\mathcal{S}'$ has a solution.

Now from Proposition 5.2 there is a solution for $\mathcal{S}'$ if and only if there exist solutions for all systems $\mathcal{S}''$ determined by $f_{s'}(r', 1, c_1', c_2')$. So the result follows. $\blacksquare$

6. The systems on some quotients and the main classification theorem. The system (I) given after Theorem 5.1 is on $\pi_2(K, K - x_2, x_1) = \pi_2$. Now, we will look at the equations of that system on some quotients of the abelianized π_2, which is $(\pi_2)_{\text{ab}} = \pi_2/[\pi_2, \pi_2]$. Whenever one of those equations on the abelianization $(\pi_2)_{\text{ab}}$ has no solution, we can infer that the

initial system has no solution. On the other hand, if the system has a solution on the abelianization we will try to find a solution on π_2 itself. In order to decide whether the equations on the abelianization $(\pi_2)_{ab}$ have no solution, we project the system to $\mathbb{Z}$ using the augmentation homomorphism $\mathcal{E} : (\pi_2)_{ab} \to \mathbb{Z}$ and study if the corresponding equation on $\mathbb{Z}$ has a solution or not. We will show that one system has a solution if and only if the other does.

We recall that the group $\pi_1(K, x_1)$ acts on $\pi_2/[\pi_2, \pi_2] = (\pi_2)_{ab}$ since for each $\xi \in \pi_1(K - x_2, x_1)$ we have the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_2(K, K - x_2, x_1) & \xrightarrow{\partial_2} & \pi_1(K - x_2, x_1) & \xrightarrow{j_\pi} & \pi_1(K, x_1) & \longrightarrow & 1 \\
& & \downarrow{\tau'_\xi} & & \downarrow{\tau_\xi} & & \downarrow{\tau_\eta} & & \\
1 & \longrightarrow & \pi_2(K, K - x_2, x_1) & \xrightarrow{\partial_2} & \pi_1(K - x_2, x_1) & \xrightarrow{j_\pi} & \pi_1(K, x_1) & \longrightarrow & 1
\end{array}
$$

where $\eta = j_\pi(\xi)$.

We note that π_2 is the kernel of the map $j_\pi : \langle \bar{w}, \bar{v} \rangle = \pi_1(K - x_2, x_1) \to \pi_1(K, x_1) = \langle \bar{w}, \bar{v} \mid \bar{w}^{-1}\bar{v}^{-1}\bar{w}^{-1}\bar{v} = 1 \rangle$ and we set $B = w^{-1}v^{-1}w^{-1}v$ where w, v are as in Remark 4.1.

THEOREM 6.1.

(1) *There is an isomorphism* $(\pi_2)_{ab} \cong \mathbb{Z}[\pi_1(K)]$, *where* $\mathbb{Z}[\pi_1(K)]$ *is the group ring of the fundamental group of the Klein bottle.*

(2) *The abelianization homomorphism* $\mathcal{A} : \pi_2 \to (\pi_2)_{ab}$ *sends the element* $p(w, v)B(p(w, v))^{-1}$ *to* $1.[p(\bar{w}, \bar{v})]$ *where* $[p(\bar{w}, \bar{v})] = j_\pi(p(\bar{w}, \bar{v}))$.

Proof. This follows from [Ly50, end of page 650]. ∎

Let $\mathcal{A} : \pi_2(K, K - x_2, x_1) = \pi_2 \to \pi_2/[\pi_2, \pi_2] = (\pi_2)_{ab}$ be the projection. Then $\mathcal{A}(p(w, v)B^t(p(w, v))^{-1}) = t.(\bar{w}^x \bar{v}^y)$ where $p(w, v)B^t p(w, v)^{-1} = j_{2\#}(p(\bar{w}, \bar{v})\bar{E}^t(p(\bar{w}, \bar{v}))^{-1})$ and $j_\pi(p(\bar{w}, \bar{v})) = \bar{w}^x \bar{v}^y$ for some $x, y \in \mathbb{Z}$, where $p(w, v)$ and $p(\bar{w}, \bar{v})$ denote the same words in w, v and $\bar{w}, \bar{v}$ respectively. Now we define the $\pi_1(K, x_1)$-homomorphism $\mathcal{E} : (\pi_2)_{ab} \to \mathbb{Z}$ by $\mathcal{E}(t.(\bar{w}^x \bar{v}^y)) = t$.

In the first subsection we study the system on the abelianization and in the second subsection we prove the classification result.

6.1. *The system of equations on quotients of* $(\pi_2)_{ab} \cong \mathbb{Z}[\pi_1(K)]$. In this subsection we will calculate $AFAF^{-1}$, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1})$ and $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-p})$, $p = 0, 1$, where $AFAF^{-1}$, $CAC^{-1}A^{-1}$ and $CFC^{-1}FA^{-p}$ are the constant terms of the equations of the system. Then we apply $\mathcal{E} \circ \mathcal{A}$ to the equations. In certain cases we project the system onto another quotient.

If $f_s(r, 1, c_1, c_2)$ is the homomorphism on $\pi_1(M(\phi(1, \eta)))$ defined before, then the terms appearing in the system are denoted by $A = \tilde{\alpha}w^r$, $r \in \mathbb{Z}$, $F = \tilde{\beta}vw^{1-s}$, $s \in \{0, 1\}$, and $C = w^{c_1}(vw)^{c_2}\tilde{c}_0$.

Because $w = \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a}^{-1}$ and $v = \tilde{a}\tilde{b}$ we have $\kappa(v) = \kappa(\tilde{a}\tilde{b}) = u_1 v_1$, $\kappa(w) = \kappa(\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\tilde{a}^{-1}) = u_1(v_1 u_1^{-1} v_1^{-1} u_1^{-1}) = u_1$ and therefore $\kappa(\tilde{\alpha}\tilde{a}^r) =$

$\kappa(\tilde{\alpha}w^r) = \kappa(A)$, $\kappa(\tilde{\beta}\tilde{a}^s\tilde{b}) = \kappa(\tilde{\beta}vw^{1-s}) = \kappa(F)$ and $\kappa(C) = \kappa(w^{c_1}(vw)^{c_2}\tilde{c}_0)$
$= u_1^{c_1}(u_1v_1u_1)^{c_2}c_{01} = u_1^{c_1}v_1^{c_2}c_{01} = \kappa(\tilde{a}^{c_1}\tilde{b}^{c_2}\tilde{c}_0)$.

PROPOSITION 6.2. *If $A = \tilde{\alpha}w^r$, $F = \tilde{\beta}vw^{1-s}$ where $r \in \mathbb{Z}$ and $s \in \{0,1\}$ then $AFAF^{-1} = 1$. Hence $\mathcal{E} \circ \mathcal{A}(AFAF^{-1}) = 0$.*

Proof. We have

$$AFAF^{-1} = \tilde{\alpha}w^r\tilde{\beta}vw^{1-s}\tilde{\alpha}w^r w^{s-1}v^{-1}\tilde{\beta}^{-1}$$
$$= \tilde{\alpha}w^r\tilde{\alpha}^{-1}\tilde{\alpha}\tilde{\beta}vw^{1-s}\tilde{\beta}^{-1}\tilde{\alpha}^{-1}(\tilde{\alpha}\tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1})\tilde{\beta}w^{r+s-1}v^{-1}\tilde{\beta}^{-1}$$
$$= w^{r+s}vwBv^{-1}w^{-r-s+1} = w^{r+s-1}w^{-r-s+1} = 1.$$

The second part is clear. ∎

Now, let us consider the term $CAC^{-1}A^{-1}$. If we let $a = \frac{-1+\eta}{2}$ then we have

$$CAC^{-1}A^{-1} = w^{c_1}(vw)^{c_2}\tilde{c}_0\tilde{\alpha}w^r\tilde{c}_0^{-1}(vw)^{-c_2}w^{-c_1}w^{-r}\tilde{\alpha}^{-1}$$
$$= w^{c_1}(vw)^{c_2}B^a\tilde{\alpha}v^aw^{\eta r}v^{-a}(vw)^{-c_2}w^{-c_1}w^{-r}\tilde{\alpha}^{-1}$$
$$= (w^{c_1}(vw)^{c_2}B^a(vw)^{-c_2}w^{-c_1})$$
$$\cdot (w^{c_1}(vw)^{c_2}(wvw)^aw^{\eta r}(wvw)^{-a}(wvww)^{-c_2}w^{-r}w^{-c_1}).$$

In Cases I and II we have $\eta = 1$ and therefore $a = 0$. So

$$CAC^{-1}A^{-1} = w^{c_1}[(vw)^{c_2}w^r(wvww)^{-c_2}w^{-r}]w^{-c_1}.$$

PROPOSITION 6.3. *If $\eta = 1$ then $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = c_2$. Hence:*
(1) *In Cases* I.1 *and* II, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = 2k$, $k \in \mathbb{Z}$.
(2) *In Case* I.2, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = 2k+1$, $k \in \mathbb{Z}$.

Proof. Consider $CAC^{-1}A^{-1}$ for $r = 0$, which is $w^{c_1}[(vw)^{c_2}(w^{-1}Bv^{-1})^{c_2}]$ $\cdot w^{-c_1}$. We will show that $\mathcal{E} \circ \mathcal{A}((vw)^{c_2}(w^{-1}Bv^{-1})^{c_2}) = c_2$ for $c_2 \in \mathbb{Z}$, which implies the assertion for Case I.2.

If $c_2 = 1$ then $(vw)^{c_2}(w^{-1}Bv^{-1})^{c_2} = vww^{-1}Bv^{-1} = vBv^{-1}$ and therefore $\mathcal{E} \circ \mathcal{A}(vBv^{-1}) = 1$. Suppose that the assertion is true for $c_2 - 1$. Then

$$(vw)^{c_2}(w^{-1}Bv^{-1})^{c_2} = (vw)^{c_2-1}(vw)(w^{-1}Bv^{-1})(w^{-1}Bv^{-1})^{c_2-1}$$
$$= (vw)^{c_2-1}(vBv^{-1})(w^{-1}Bv^{-1})^{c_2-1}$$
$$= [(vw)^{c_2-1}(vBv^{-1})(vw)^{1-c_2}](vw)^{c_2-1}(w^{-1}Bv^{-1})^{c_2-1}.$$

Therefore

$$\mathcal{E} \circ \mathcal{A}((vw)^{c_2}(w^{-1}Bv^{-1})^{c_2}) = \mathcal{E} \circ \mathcal{A}((vw)^{c_2-1}(vBv^{-1})(vw)^{1-c_2})$$
$$+ \mathcal{E} \circ \mathcal{A}((vw)^{c_2-1}(w^{-1}Bv^{-1})^{c_2-1})$$
$$= 1 + (c_2 - 1) = c_2.$$

The proof for Cases I.1 and II follows from the two lemmas below. ∎

LEMMA 6.4. $\mathcal{E} \circ \mathcal{A}([v^2, w^r]) = 0$ *for any* $r \geq 0$.

Proof. The case $r = 0$ is trivial. The rest of the proof is straightforward induction on r. ∎

LEMMA 6.5. *If* c_2 *is even then* $\mathcal{E} \circ \mathcal{A}((vw)^{c_2} w^r (w^{-1} B v^{-1})^{c_2} w^{-r}) = c_2$.

Proof. The case $c_2 = 0$ is trivial. If $c_2 = 2$ the conclusion follows by straightforward calculation using Lemma 6.4. For the general case we use induction on c_2. ∎

In Cases III and IV we have $\eta = -1$ and therefore $a = -1$. So
$$CAC^{-1}A^{-1} = (w^{c_1}(vw)^{c_2} B^{-1}(vw)^{-c_2} w^{-c_1})$$
$$\cdot (w^{c_1}(vw)^{c_2}(wvw)^{-1} w^{-r}(wvw)(wvww)^{-c_2} w^{-r} w^{-c_1}).$$

PROPOSITION 6.6. *If* $\eta = -1$ *then* $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = c_2 + r - 1$. *Hence*:

(1) *In Case* III.1, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = 2k + r - 1$, $r \geq 0$, $k \in \{0, 1\}$.
(2) *In Case* III.2, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = 0$.
(3) *In Case* IV, $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = r - 1$, $r \in \mathbb{Z}$.

Proof. Consider $CAC^{-1}A^{-1}$ in Case III.2, where $c_2 = 1$, $c_1 = s$ and $r = 0$. So we have
$$CAC^{-1}A^{-1} = (w^s(vw)B^{-1}(vw)^{-1}w^{-s})$$
$$\cdot (w^s(vw)(wvw)^{-1}(wvw)(wvww)^{-1}w^{-s})$$
$$= (w^s(vw)B^{-1}(vw)^{-1}w^{-s})(w^s v B v^{-1} w^{-s}),$$
and therefore $\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = -1 + 1 = 0$.

The proof for Cases III.1 and IV follows from the two lemmas below. ∎

LEMMA 6.7. $\mathcal{E} \circ \mathcal{A}(v^{-1}w^{-r}vw^{-r}) = r$ *and* $\mathcal{E} \circ \mathcal{A}((vw)^{-1}w^{-r}(vw)w^{-r})$ $= r$ *for* $r \in \mathbb{Z}$.

Proof. For the first part assume that $r \geq 0$. The case $r \leq 0$ is similar and left to the reader.

If $r = 0$ then the assertion is trivial. The rest of the proof is induction on r. ∎

LEMMA 6.8. *If* c_2 *is even then*
$$\mathcal{E} \circ \mathcal{A}(CAC^{-1}A^{-1}) = -1 + \mathcal{E} \circ \mathcal{A}((vw)^{-1}w^{-r}vww^{-r}) + c_2.$$

Proof. We have
$$CAC^{-1}A^{-1} = (w^{c_1}(vw)^{c_2} B^{-1}(vw)^{-c_2} w^{-c_1})$$
$$\cdot (w^{c_1}(vw)^{c_2}(wvw)^{-1} w^{-r}(wvw)(wvww)^{-c_2} w^{-r} w^{-c_1}).$$

Using Lemma 6.5 we obtain the result. ∎

Now consider the term $CFC^{-1}FA^{-p}$. We compute $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-p})$ in Cases III and IV. Because we will show that the necessary conditions obtained so far for Cases I and II are also sufficient, we do not need to compute $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-p})$ in Cases I and II.

PROPOSITION 6.9. *In Case* III, *where $p = 0$, we have*
$$\mathcal{E} \circ \mathcal{A}(CFC^{-1}F) = \begin{cases} -2k + s - 1 & \text{in Case } \text{III.1}, \\ -2, & \text{in Case } \text{III.2}. \end{cases}$$

Proof. In Case III.1 we have $F = \tilde{\beta}vw^{1-s}$ with $s \in \{0, 1\}$ and $C = (vw)^{2k}\tilde{c}_0$ with $k \in \{0, 1\}$. So $CFC^{-1}F = (vw)^{2k}v^{-1}w^{1-s}v(vw^{-1})^{-2k}w^{1-s}$.

If $k = 0$ and $s = 0$ then $CFC^{-1}F = v^{-1}wvw = B^{-1}$ and it follows that $\mathcal{E} \circ \mathcal{A}(CFC^{-1}F) = -1$.

If $k = 0$ and $s = 1$ then $CFC^{-1}F = v^{-1}v = 1$ and hence $\mathcal{E} \circ \mathcal{A}(CFC^{-1}F) = 0$.

If $k = 1$ and $s = 0$ then
$$CFC^{-1}F = [(vw)^2 B^{-1}(vw)^{-2}](v^2 B^{-1}v^{-2})(w^{-1}vB^{-1}v^{-1}w)$$
and so $\mathcal{E} \circ \mathcal{A}(CFC^{-1}F) = -3$.

If $k = 1$ and $s = 1$ then $CFC^{-1}F = (v^2 B^{-1}v^{-2})(v^2 wB^{-1}w^{-1}v^{-2})$ and therefore $\mathcal{E} \circ \mathcal{A}(CFC^{-1}F) = -2$.

In Case III.2 we have $F = \tilde{\beta}vw^{1-s}$ and $C = w^s vw\tilde{c}_0$ where $s \in \{0, 1\}$. So $CFC^{-1}F = (w^{s-1}vB^{-1}v^{-1}w^{1-s})(w^{-1}vB^{-1}v^{-1}w)$ and the result follows. ∎

Now we consider Case IV. Then $A = \tilde{\alpha}w^{2r+1}$, $F = \tilde{\beta}vw^{1-s}$ and $C = w^r\tilde{c}_0$ where $r \in \mathbb{Z}$ and $s \in \{0, 1\}$.

PROPOSITION 6.10. *In Case* IV *we have $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-1}) = s - 1$ where $F = \tilde{\beta}vw^{1-s}$, $s \in \{0, 1\}$.*

Proof. We have
$$CFC^{-1}FA^{-1} = w^r\tilde{c}_0\tilde{\beta}vw^{1-s}\tilde{c}_0^{-1}w^{-r}\tilde{\beta}vw^{1-s}w^{-2r-1}\tilde{\alpha}^{-1}$$
$$= w^r\tilde{\alpha}\tilde{\beta}^{-1}v^{-1}w^{-1}v^{-1}w^{s-1}vw^{-r}\tilde{\beta}vw^{-s-2r}\tilde{\alpha}^{-1}.$$
If $s = 1$, then $CFC^{-1}FA^{-1} = 1$ and therefore $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-1}) = 0$. If $s = 0$, then $CFC^{-1}FA^{-1} = w^r B^{-1}w^{-r}$ and so $\mathcal{E} \circ \mathcal{A}(CFC^{-1}FA^{-1}) = -1$. ∎

Now, we are going to study the behavior of $\mathcal{E} \circ \mathcal{A}$ after conjugating the variable $Z = w^m v^n B^k v^{-n}w^{-m}$ by $\tilde{\alpha}, \tilde{\alpha}^{-1}, \tilde{\beta}, \tilde{\beta}^{-1}$ and $\tilde{c}_0$ in each case of $\phi_p(1, \eta)$, where $B = w^{-1}v^{-1}w^{-1}v$.

PROPOSITION 6.11. *If $Z = w^m v^n B^k v^{-n}w^{-m}$ then*

(1) $\mathcal{E} \circ \mathcal{A}(Z) = \mathcal{E} \circ \mathcal{A}(\tilde{\alpha}Z\tilde{\alpha}^{-1}) = \mathcal{E} \circ \mathcal{A}(\tilde{\alpha}^{-1}Z\tilde{\alpha}) = -\mathcal{E} \circ \mathcal{A}(\tilde{\beta}Z\tilde{\beta}^{-1})$
$$= -\mathcal{E} \circ \mathcal{A}(\tilde{\beta}^{-1}Z\tilde{\beta}) = k.$$

(2) *In each case of $\phi_p(1, \eta)$ we have $\mathcal{E} \circ \mathcal{A}(\tilde{c}_0 Z\tilde{c}_0^{-1}) = \eta k$.*

Proof. In fact, since $Z = w^m v^n B^k v^{-n} w^{-m}$ and since $\tilde{\alpha} B \tilde{\alpha}^{-1} = w^{-1} B w$ from the table (4.3), it follows that $\mathcal{E} \circ \mathcal{A}(Z) = k$ and $\mathcal{E} \circ \mathcal{A}(\tilde{\alpha} Z \tilde{\alpha}^{-1}) = k$. Also $\tilde{\beta} B \tilde{\beta}^{-1} = v^{-1} w B^{-1} w^{-1} v$ and so $\mathcal{E} \circ \mathcal{A}(\tilde{\beta} Z \tilde{\beta}^{-1}) = -k$.

Now $\tilde{\alpha}^{-1} B \tilde{\alpha} = w B w^{-1}$ and therefore $\mathcal{E} \circ \mathcal{A}(\tilde{\alpha}^{-1} Z \tilde{\alpha}) = k$. Also $\tilde{\beta}^{-1} B \tilde{\beta} = v w B^{-1} w^{-1} v^{-1}$ and therefore $\mathcal{E} \circ \mathcal{A}(\tilde{\beta}^{-1} Z \tilde{\beta}) = -k$. Finally, according to the table (4.4), $\tilde{c}_0 B \tilde{c}_0^{-1} = B^\eta$ and therefore $\mathcal{E} \circ \mathcal{A}(\tilde{c}_0 Z \tilde{c}_0^{-1}) = \eta k$. ∎

COROLLARY 6.12. *If* $Z = w^m v^n B^k v^{-n} w^{-m}$, $A = \tilde{\alpha} w^r$, $F = \tilde{\beta} v w^{1-s}$ *and* $C = w^{c_1} (vw)^{c_2} \tilde{c}_0$, *then in each case of* $\phi_p(1, \eta)$,
- $\mathcal{E} \circ \mathcal{A}(AZA^{-1}) = \mathcal{E} \circ \mathcal{A}(FA^{-1}ZAF^{-1}) = -\mathcal{E} \circ \mathcal{A}(A^p F^\eta Z F^{-\eta} A^{-p}) = k$,
- $\mathcal{E} \circ \mathcal{A}(CZC^{-1}) = \eta k$.

Proof. This is an immediate consequence of Proposition 6.11. ∎

Now, we will apply $\mathcal{E} \circ \mathcal{A}$ to the second and the third equations of the system (I). By Corollary 6.12 and Proposition 6.2 we do not obtain any information if we apply the homomorphism to the first equation.

Let
$$Z_1 = \prod_i w^{u_i} v^{v_i} B^{t_1^i} v^{-v_i} w^{-u_i}, \quad Z_2 = \prod_i w^{m_i} v^{n_i} B^{t_2^i} v^{-n_i} w^{-m_i},$$
$$Z_3 = \prod_i w^{x_i} v^{y_i} B^{t_3^i} v^{-y_i} w^{-x_i} \quad \text{and} \quad t_j = \sum_i t_j^i$$

where t_j^i is the exponent of B in the ith factor of Z_j.

THEOREM 6.13. *If we apply* $\mathcal{E} \circ \mathcal{A}$ *to the second equation of the system* (I) *we obtain* $c_2 = 0$ *in Cases* I *and* II, *and* $-2t_1 - 1 + c_2 + r = 0$ *in cases* III *and* IV.

Proof. This follows from Corollary 6.12 and Propositions 6.3 and 6.6. ∎

We will see that the conditions given by Theorem 6.13 are also sufficient to solve the problem for Cases I and II.

For the other cases we have

THEOREM 6.14. *In Cases* III.1 *and* IV, *i.e.* $\phi_p(1, -1)$ *where* $p = 0, 1$ *respectively, if we apply* $\mathcal{E} \circ \mathcal{A}$ *to the third equation of the system* (I) *we obtain* $2(t_3 - t_2) - pt_1 - c_2 + s - 1 = 0$.

Proof. This follows from Corollary 6.12 and Propositions 6.9 and 6.10. ∎

In order to solve Case III.1 completely we need some further results. The strategy will now change slightly.

Let us consider Case III.I, which corresponds to $\phi_0(1, -1)$. We first consider the maps $f_1(r, 1, 0, 0)$ with r odd, $r \geq 3$.

Recall that from the system generated by the input data ($A = \alpha w^r, F = \beta v, C = c_0; r, 1, (0, 0)$) the second equation is

$$Z_3(CZ_1C^{-1})\Big(\prod_{i=1}^{r-1} w^i B w^{-i}\Big)(AZ_3^{-1}A^{-1})Z_1^{-1} = 1$$

once $CAC^{-1}A^{-1} = \prod_{i=1}^{r-1} w^i B w^{-i}$.

For Z_1 and Z_3 as above we have

$$\mathcal{A}(CZ_1C^{-1}) = \sum_i (-t_1^i)\bar{w}^{u_i}\bar{v}^{-v_i},$$

$$\mathcal{A}(Z_1^{-1}) = \sum_i (-t_1^i)\bar{w}^{u_i}\bar{v}^{v_i},$$

$$\mathcal{A}(AZ_3^{-1}A^{-1}) = \sum_i (-t_3^i)\bar{w}^{r+x_i-(-1)^{y_i}}\bar{v}^{y_i},$$

$$\mathcal{A}(Z_3) = \sum_i t_3^i \bar{w}^{x_i}\bar{v}^{y_i},$$

$$\mathcal{A}(CAC^{-1}A^{-1}) = \sum_{i=1}^{r-1} 1\bar{w}^i.$$

We claim that the second equation does not have a solution.

PROPOSITION 6.15. *For Case* III.1 *with* $\phi_0(1, -1)$ *and* $f_1(r, 1, 0, 0)$ *with* $r \geq 3$, *the second equation of the system,*

$$Z_3(CZ_1C^{-1})(CAC^{-1}A^{-1})(AZ_3^{-1}A^{-1})Z_1^{-1} = 1,$$

has no solution where $A = \alpha w^r$ *and* $C = c_0$.

Proof. The second equation is

$$Z_3(CZ_1C^{-1})\Big(\prod_{i=1}^{r-1} w^i B w^{-i}\Big)(AZ_3^{-1}A^{-1})Z_1^{-1} = 1$$

and by taking $\mathcal{A}(\text{second equation}) = 0$ we get

$$\mathcal{A}(Z_3) - w^r \mathcal{A}(Z_3)w^{-1} - 2\mathcal{A}(Z_1) + \sum_{i=1}^{r-1} 1\bar{w}^i = 0.$$

Let $H \subset \mathbb{Z}[\pi_1(K)]$ be the subring generated by the homogeneous elements of the form $\bar{w}^i$ for $i \in \mathbb{Z}$. As a group, H is isomorphic to the free abelian group generated by $\bar{w}^i$ with $i \in \mathbb{Z}$. It is easy to see that H is a subring which can be identified with the ring of Laurent polynomials over the integers. If we project the equation above on the group H (the projection maps $\bar{w}^{u_i}\bar{v}^{v_i}$ to 0 if $v_i \neq 0$ and to $\bar{w}^{u_i}$ otherwise), and if by abuse of notation we write Z_i

for the projection of Z_i to H, we get

$$\mathcal{A}(Z_3)(1 - \bar{w}^{r-1}) - 2\mathcal{A}(Z_1) + \sum_{i=1}^{r-1} 1\bar{w}^i = 0,$$

which is equivalent to

$$\mathcal{A}(Z_3)(1 - \bar{w})(1 + \bar{w} + \cdots + \bar{w}^{r-2}) - 2\mathcal{A}(Z_1) + \bar{w}(1 + \bar{w} + \cdots + \bar{w}^{r-2}) = 0.$$

Now let us look at this equation on $H/2H$. Then we obtain

$$(\mathcal{A}(Z_3)(1 - \bar{w}) + \bar{w})(1 + \bar{w} + \cdots + \bar{w}^{r-2}) = 0.$$

This is an equation in the ring $H/2H$ and this ring has no zero divisors by Theorem 1.4 in [KLM88]. Therefore we must have

$$\mathcal{A}(Z_3)(1 - \bar{w}) + \bar{w} = 0, \quad \text{or} \quad \mathcal{A}(Z_3)(\bar{w} - 1) = \bar{w}$$

since $r \geq 3$. But the last equation has no solution (set $\bar{w} = 1$). Hence the result follows. ∎

Now we will consider the maps $f_1(r, 1, 0, 2)$ with r odd, $r \geq 1$. This will complete the analysis of Case III.1.

We have $A = \alpha w^r$, $F = \beta v$ and $C = (vw)^2 c_0$. In order to study the second equation we need the following calculation:

$$CAC^{-1}A^{-1} = [(vw)^2 B(vw)^{-2}](v^2 B^{-2}v^{-2})(vw^{1-r}vBv^{-1}w^{r-1}v^{-1})$$

$$\cdot \prod_{i=0}^{r} vw^{i-r}Bw^{r-i}v^{-1}$$

and therefore

$$\mathcal{A}(CAC^{-1}A^{-1}) = 1\bar{v}^2 - 2\bar{v}^2 + 1\bar{w}^{r-1}\bar{v}^2 + \sum_{i=0}^{r} 1\bar{w}^{r-i}\bar{v}.$$

Moreover,

$$\mathcal{A}(CZ_1C^{-1}) = \sum_i (-t_1^i)\bar{w}^{u_i}\bar{v}^{2-v_i}.$$

We claim:

PROPOSITION 6.16. *For Case* III.1 *with* $\phi_0(1, -1)$ *and* $f_1(r, 1, 0, 2)$ *with* $r \geq 0$, *the second equation of the system*,

$$Z_3(CZ_1C^{-1})(CAC^{-1}A^{-1})(AZ_3^{-1}A^{-1})Z_1^{-1} = 1,$$

has no solution.

Proof. As before we consider $\mathcal{A}$(second equation) $= 0$. The summands $\mathcal{A}(Z_3)$ and $\mathcal{A}(AZ_3^{-1}A^{-1})$ are as stated before Proposition 6.15. The summands $\mathcal{A}(CZ_1C^{-1})$ and $\mathcal{A}(CAC^{-1}A^{-1})$ are calculated above. So we deduce

that

$$\sum_i t_3^i \bar{w}^{x_i} \bar{v}^{y_i} + \sum_i (-t_1^i) \bar{w}^{u_i} \bar{v}^{2-v_i} + (1-\bar{w})(1+\bar{w}+\cdots+\bar{w}^{r-2})(-\bar{v}^2) + \sum_{i=0}^{r} 1\bar{w}^{r-i}\bar{v}$$

$$+ \sum_i (-t_3^i) \bar{w}^{r+x_i-(-1)^{y_i}} \bar{v}^{y_i} + \sum_i (-t_1^i) \bar{w}^{u_i} \bar{v}^{v_i} = 0,$$

where the indices run over the i's such that $v_i = y_i = 1$.

A homogeneous summand of Z_3 and its conjugate which appear in the equation are indexed by elements of the same horizontal line. A homogeneous summand of Z_1 and its conjugate which appear in the equation are indexed by elements of two horizontal lines symmetric to the line $\bar{w} = 1$. In particular, if a summand is indexed by an element which belongs to the line $\bar{w} = 1$ its conjugate is also a summand which is indexed by an element which belongs to the line $\bar{w} = 1$. So we let H be the abelian group generated by the elements $\bar{w}^i \bar{v}$ with $i \in \mathbb{Z}$ and we project the equation on H. Then we obtain the equation

$$(1 - \bar{w}^{r+1})\Big(\sum_i t_3^i \bar{w}^{x_i}\Big) + 2\sum_i (-t_1^i) \bar{w}^{u_i} + \sum_{i=0}^{r} 1\bar{w}^{r-i} = 0,$$

where the last equality follows from the fact that $\mathbb{Z}[\pi_1(K)]$ has no zero divisors (see Theorem 1.4 in [KLM88]).

Now we consider this equation mod $2H$, and using the fact that the group ring $\mathbb{Z}_2[\mathbb{Z}]$ has no zero divisors (Theorem 1.4 in [KLM88]) we obtain

$$\Big(\sum_i t_3^i \bar{w}^{x_i}\Big)(1-\bar{w})\Big(\sum_{i=0}^{r} 1\bar{w}^{r-i}\Big) + \sum_{i=0}^{r} 1\bar{w}^{r-i} = 0.$$

But this implies that

$$\Big(\sum_i t_3^i \bar{w}^{x_i}\Big)(1-\bar{w}) + 1 = 0,$$

which, as we have already seen in the proof of Proposition 6.15, does not have a solution. So the result follows. ∎

6.2. *The classification theorem.* Let us consider first the cases:

I.1. $\phi_0(1,1)$ and $f_s(r,1,0,2k)$, $r \geq 0$, $s \in \{0,1\}$ and $k \in \mathbb{Z}$,
I.2. $\phi_0(1,1)$ and $f_s(0,1,s,2k+1)$, $s \in \{0,1\}$ and $k \in \mathbb{Z}$,
II. $\phi_1(1,1)$ and $f_s(2r+1,1,r,2k)$, $r \geq 0$, $s \in \{0,1\}$ and $k \in \mathbb{Z}$.

PROPOSITION 6.17.

(i) *Cases* I.1 *and* II *have no solution for* $k \neq 0$ *and Case* I.2 *has no solution for all* $k \in \mathbb{Z}$.

(ii) *In Case* I.1 *for* $f_s(r,1,0,0)$ *and* II *for* $f_s(2r+1,1,r,0)$, $r \geq 0$, $s \in \{0,1\}$ *there is a lifting.*

Proof. For (i) we have $c_2 \neq 0$ and the result follows from Theorem 6.13.

For (ii) let us first consider Case I.1. It is sufficient to prove that the system generated by the input data $(A = \tilde{\alpha}w^r, F = \tilde{\beta}vw^{1-s}, C = \tilde{c}_0; 0, 0, (0, 1))$ has trivial solution. In fact, in this case we have

$$AFAF^{-1} = CAC^{-1}A^{-1} = CFC^{-1}F^{-1} = 1.$$

So we have the trivial solution.

Similarly for II, if we take $A = \tilde{\alpha}w^{2r+1}, F = \tilde{\beta}vw^{1-s}$ and $C = w^r \tilde{c}_0$ then the system generated by the data $(A, F, C; 2r + 1, s, (r, 0))$ has trivial solution. ∎

Now, let us consider Case III.1 with $\phi_0(1, -1)$ and $f_s(r, 1, 0, 2k)$, $s \in \{0, 1\}$, $r \geq 0$ and $k \in \{0, 1\}$.

COROLLARY 6.18. *Case* III.1 *has no solution for* r *even or* s *even.*

Proof. The condition from Theorem 6.13 is not satisfied modulo 2 for r even. The condition from Theorem 6.14 is not satisfied modulo 2 for s even. ∎

COROLLARY 6.19. *In Case* III.1 *if* $r \geq 3$ *is odd then the system generated by the input data* $(A = \tilde{\alpha}w^r, F = \tilde{\beta}v, C = \tilde{c}_0; r, 1, (0, 0))$ *has no solution.*

Proof. This follows from Proposition 6.15. ∎

COROLLARY 6.20. *In Case* III.1, *if* $r \geq 1$ *is odd then the system generated by the input data* $(A = \tilde{\alpha}w^r, F = \tilde{\beta}v, C = (vw)^2 \tilde{c}_0; r, 1, (0, 2))$ *has no solution.*

Proof. This follows from Proposition 6.16. ∎

THEOREM 6.21. *In Case* III.1, *if* $s = 1$, $r = 1$ *and* $c_2 = 0$ *then the system generated by the input data* $(A = \tilde{\alpha}w, F = \tilde{\beta}v, C = \tilde{c}_0; 1, 1, (0, 0))$ *has a solution.*

Proof. In fact, using the tables (4.3) and (4.4) we obtain

$$AFAF^{-1} = CAC^{-1}A^{-1} = CFC^{-1}F = 1. \quad \blacksquare$$

Now let us consider Case III.2 with $\phi_0(1, -1)$ and $f_s(0, 1, s, 1)$, $s \in \{0, 1\}$.

THEOREM 6.22. *The system generated by the input data* $(A = \tilde{\alpha}, F = \tilde{\beta}vw^{1-s}, C = w^{s-1}v\tilde{c}_0; 0, s, (s, 1))$ *for* $s \in \{0, 1\}$ *has trivial solution. So there is a lifting for* $f_s(0, 1, s, 1)$ *with* $s \in \{0, 1\}$.

Proof. In fact, we have $AFAF^{-1} = 1$ (see proof of Proposition 6.2). Moreover, using the tables (4.3) and (4.4) we obtain

$$CAC^{-1}A^{-1} = CFC^{-1}F = 1. \quad \blacksquare$$

Now let us consider Case IV with $\phi_1(1, -1)$ and $f_s(2r+1, 1, r, 0)$, $r \in \mathbb{Z}$, $s \in \{0, 1\}$.

PROPOSITION 6.23. *In Case* IV, *if* $r - s = 2l$, $l \in \mathbb{Z}$ *and* $s \in \{0, 1\}$, *then the system generated by the input data* $(A = \tilde{\alpha}w^{2r+1}$, $F = \tilde{\beta}vw^{1-s}$, $C = w^r \tilde{c}_0; 2r+1, s, (r, 0))$ *has no solution.*

Proof. In fact, in Case IV, $p = 1$ and $c_2 = 0$ and therefore the hypotheses make it impossible to solve the system

$$\begin{cases} \mathcal{E} \circ \mathcal{A}(\text{second equation}) \\ \mathcal{E} \circ \mathcal{A}(\text{third equation}) \end{cases} = \begin{cases} -2t_1 + 2r = 0 \\ 2(t_3 - t_2) - t_1 + s - 1 = 0. \end{cases} \blacksquare$$

Now it is necessary to decide whether or not there is a lifting in each of the cases $f_0(2r+1, 1, r, 0)$ with r odd, and $f_1(2r+1, 1, r, 0)$ with r even.

LEMMA 6.24. *In Case* IV, *if* $q = v^{-1}wv$ *then*

$\tilde{\alpha}q\tilde{\alpha}^{-1} = w^{-1}qw$	
$\tilde{\beta}^{-1}q\tilde{\beta} = w^{-1}$	$\tilde{\beta}q\tilde{\beta}^{-1} = v^{-1}q^{-1}v$
$\tilde{\beta}w\tilde{\beta}^{-1} = q^{-1}$	$\tilde{c}_0^{-1}q\tilde{c}_0 = w^{-1}$
$\tilde{c}_0^{-1}w\tilde{c}_0 = q^{-1}$	$\tilde{c}_0 w\tilde{c}_0^{-1} = q^{-1}$

Proof. All the above relations follow immediately from the tables (4.3) and (4.4). $\blacksquare$

PROPOSITION 6.25. *In Case* IV *of Corollary* 5.4 *there is a lifting for* $f_s(2r+1, 1, r, 0)$ *and* $r + s = 2l + 1$, $l \in \mathbb{Z}$ *and* $s \in \{0, 1\}$.

Proof. Consider the system generated by the input data $(A = CFC^{-1}F$, $F = \tilde{\beta}w^{s-1}v$, $C = q^{1-s}w^{l+1-s}q^{s-l-1}\tilde{c}_0; 4l+3-2s, s, (2l+1-s, 0))$, where $q = v^{-1}wv$.

It follows from Lemma 6.24 that $AFAF^{-1} = 1$ and $CAC^{-1}A^{-1} = 1$. Also, since $A = CFC^{-1}F$ we have $CFC^{-1}FA^{-1} = 1$. Therefore the system generated by the input data above has trivial solution and so there is a lifting for $f_s(4l+3-2s, 1, 2l+1-s, 0)$, $l \in \mathbb{Z}$ and $s \in \{0, 1\}$. $\blacksquare$

The considerations above together with Remark 5.1 allow us to state the main result. Let ϕ be a homeomorphism of K, where K denotes the Klein bottle, and let $M(\phi)$ be $K \times [0, 1]$ with $(x, 0)$ with $(\phi(x), 1)$ identified. Then $M(\phi)$ is a fiber bundle over the circle S^1 with fiber K.

THEOREM 6.26. *Let* $f : M(\phi) \to M(\phi)$ *be a fiber-preserving map over* S^1. *If* f *belongs to one of the cases of Theorem* 2.4 *then it can be deformed to a fixed point free map* g *by a fiberwise homotopy over* S^1 *if and only if* f *belongs to the corresponding case in the table below.*

Case I $\phi_0(1,1)$	$f_s(r,1,0,0): \alpha \mapsto \alpha^r, \beta \mapsto \alpha^s\beta, c_0 \mapsto c_0, \ r,s \in \mathbb{Z}$
Case II $\phi_1(1,1)$	$f_s(2r+1,1,r,0): \alpha \mapsto \alpha^{2r+1}, \beta \mapsto \alpha^s\beta, c_0 \mapsto \alpha^r c_0, \ r,s \in \mathbb{Z}$
Case III $\phi_0(1,-1)$	$f_{2s+1}(1,1,0,4k): \alpha \mapsto \alpha, \beta \mapsto \alpha^{2s+1}\beta, c_0 \mapsto \beta^{4k}c_0, \ s,k \in \mathbb{Z}$ $f_{2s+1}(-1,1,0,4k+2): \alpha \mapsto \alpha^{-1}, \beta \mapsto \alpha^{2s+1}\beta, c_0 \mapsto \beta^{4k+2}c_0, \ s,k \in \mathbb{Z}$ $f_s(0,1,s,2k+1): \alpha \mapsto 1, \beta \mapsto \alpha^s\beta, c_0 \mapsto \alpha^s\beta^{2k+1}c_0 \ s,k \in \mathbb{Z}$
Case IV $\phi_1(1,-1)$	$f_{2s}(4l+3,1,2l+1,4k): \alpha \mapsto \alpha^{4l+3}, \beta \mapsto \alpha^{2s}\beta, \ c_0 \mapsto \alpha^{2l+1}\beta^{4k}c_0, \ s,k,l \in \mathbb{Z}$ $f_{2s}(4l+1,1,2l,4k+2): \alpha \mapsto \alpha^{4l+1}, \beta \mapsto \alpha^{2s}\beta, \ c_0 \mapsto \alpha^{2l}\beta^{4k+2}c_0, \ s,k,l \in \mathbb{Z}$ $f_{2s+1}(4l+3,1,2l+1,4k+2): \alpha \mapsto \alpha^{4l+3}, \beta \mapsto \alpha^{2s+1}\beta, c_0 \mapsto \alpha^{2l+1}\beta^{4k+2}c_0, \ s,k \in \mathbb{Z}$ $f_{2s+1}(4l+1,1,2l,4k): \alpha \mapsto \alpha^{4l+1}, \beta \mapsto \alpha^{2s+1}\beta, c_0 \mapsto \alpha^{2l}\beta^{4k}c_0, \ s,k,l \in \mathbb{Z}$

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