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Miller spaces and spherical resolvability of finite complexes

by

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Abstract. Let $\mathcal{A}$ be a fixed collection of spaces, and suppose K is a nilpotent space that can be built from spaces in $\mathcal{A}$ by a succession of cofiber sequences. We show that, under mild conditions on the collection $\mathcal{A}$, it is possible to construct K from spaces in $\mathcal{A}$ using, instead, homotopy (inverse) limits and extensions by fibrations. One consequence is that if K is a nilpotent finite complex, then ΩK can be built from finite wedges of spheres using homotopy limits and extensions by fibrations. This is applied to show that if $\mathrm{map}_*(X, S^n)$ is weakly contractible for all sufficiently large n, then $\mathrm{map}_*(X, K)$ is weakly contractible for any nilpotent finite complex K.

Introduction. A *Miller space* is a CW complex X with the property that the space $\mathrm{map}_*(X, K)$ of pointed maps from X to K is weakly contractible for any nilpotent finite complex K (cf. [8, p. 46]). They are named for Haynes Miller, who proved in his landmark paper [16] that if G is a (locally) finite group, then the classifying space BG is a Miller space. In this paper we prove the following simple recognition principle for Miller spaces.

THEOREM (Corollary 11). *Let X be a space and let $N \in \mathbb{N}$. Then the following are equivalent*:

(a) $\mathrm{map}_*(X, K) \sim *$ *for every nilpotent finite complex* K,
(b) $\mathrm{map}_*(X, S^n) \sim *$ *for all* $n \geq N$.

In the stable category, one can define a *Miller spectrum* by the property that the mapping spectrum $F(X, K)$ is contractible for every finite spectrum K. Since cofiber sequences and fiber sequences are the same in the stable category, a finite spectrum K with m cells is the fiber in a fiber sequence $K \to L \to S^n$ in which L has only $m-1$ cells; in the terminology of [6, 14], this means that K is *spherically resolvable with weight* m. An easy induction shows that X is a Miller spectrum if and only if $F(X, S^0) \simeq *$.

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Thus our theorem is the unstable analog of an elementary stable observation. Of course, the proof of the stable version is not available to us because cofiber sequences are not fiber sequences, unstably. To prove our result, it is necessary to determine the extent to which a finite complex can be constructed from spheres in a more general way, i.e., by arbitrary homotopy (inverse) limits [4, Ch. XI] and extensions by fibrations.

We call a class $\overline{\mathcal{R}}$ of spaces a *strong resolving class* if it is closed under weak equivalence, homotopy limits and extensions by fibrations. The principal example is the class $\{K \mid \mathrm{map}_*(X, K) \sim *\}$, for a fixed space X. We derive Corollary 11 from the following property of strong resolving classes.

THEOREM 10 *Let $\overline{\mathcal{R}}$ be a strong resolving class and let $N \in \mathbb{N}$. Then the following are equivalent*:

(a) *$K \in \overline{\mathcal{R}}$ for every nilpotent finite complex K,*

(b) *$\bigvee S^n \in \overline{\mathcal{R}}$ for every finite wedge of spheres with $n \geq N$.*

Another corollary of Theorem 10 is that if K is a nilpotent finite complex, then ΩK is *spherically resolvable* in the following sense: ΩK belongs to $\overline{\mathcal{R}}(\mathcal{S})$, the smallest strong resolving class that contains S^n for each n.

We now summarize the organization of the paper. In Section 1 we recall the notion of cone length *with respect to a collection $\mathcal{A}$ of spaces*. The principal result of the section is an upper bound for the cone length of the suspension of certain homotopy fibers. Section 2 is where we introduce (strong) resolving classes and prove that some special strong resolving classes are closed under certain finite type wedges. Section 3 contains the statement of the main theorem of the paper in its full generality and the derivation from it of the results described in the introduction. The main theorem is finally proved in Section 4. The last section contains a simple proof of a desuspension result that plays a key role in the proof of the main theorem.

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1. Cone length and collections of spaces. The proof of our main theorem proceeds by induction on a certain kind of cone length. In this section we recall the notion of cone length with respect to a collection $\mathcal{A}$ of spaces and derive the basic properties that we will need later.

DEFINITION 1. Let $\mathcal{A}$ be a collection of spaces. The $\mathcal{A}$-*cone length* of a space K, denoted $\mathrm{cl}_{\mathcal{A}}(K)$, is the least integer n for which there are cofiber sequences

$$A_i \to K_i \to K_{i+1} \quad \text{for } 0 \le i < n,$$

with $K_0 \sim *$, $K_n \sim K$ and each $A_i \in \mathcal{A}$. If $K \sim *$ then $\mathrm{cl}_{\mathcal{A}}(K) = 0$; if no such n exists, then $\mathrm{cl}_{\mathcal{A}}(K) = \infty$.

Here, and throughout the paper, $\sim$ denotes weak equivalence. Intuitively, $\mathrm{cl}_{\mathcal{A}}(X)$ is the number of steps it takes to build X from $\mathcal{A}$ using cofibrations [7, 1].

If $\mathcal{A}$ is a collection of spaces, then we denote by $\Sigma\mathcal{A}$ the collection of all suspensions of spaces in $\mathcal{A}$. We will be concerned with the following closure properties of collections $\mathcal{A}$: $\mathcal{A}$ is *closed under suspension* if $\Sigma\mathcal{A} \subseteq \mathcal{A}$; $\mathcal{A}$ is *closed under smash products* if $A \wedge B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$; and $\mathcal{A}$ is *closed under wedges* if $A \vee B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$.

The following result is easily checked, and we omit the proof.

PROPOSITION 2. *Let $\mathcal{A}$ be a collection of spaces and let K be a space. Then*

(a) $\mathrm{cl}_{\Sigma\mathcal{A}}(\Sigma K) \le \mathrm{cl}_{\mathcal{A}}(K)$,

(b) *if $\mathcal{A}$ is closed under suspension*, *then* $\mathrm{cl}_{\mathcal{A}}(\Sigma K) \le \mathrm{cl}_{\mathcal{A}}(K)$,

(c) *if $\mathcal{A}$ is closed under smash products and $A \in \mathcal{A}$*, *then* $\mathrm{cl}_{\mathcal{A}}(A \wedge K) \le \mathrm{cl}_{\mathcal{A}}(K)$,

(d) *if $\mathcal{A}$ is closed under wedges*, *and L is any space*, *then* $\mathrm{cl}_{\mathcal{A}}(K \vee L) \le \max(\mathrm{cl}_{\mathcal{A}}(K), \mathrm{cl}_{\mathcal{A}}(L))$.

A *finite type wedge* of spaces in $\mathcal{A}$ is a space W that is weakly equivalent to $\bigvee A_i$ with each $A_i \in \mathcal{A}$ and such that, for each n, all but finitely many of the A_i are n-connected. If $\mathcal{A}$ is a collection of spaces, then we write $\mathcal{A}^{\vee}$ for the collection of all finite type wedges of spaces in $\mathcal{A}$. Notice that $\mathcal{A}^{\vee}$ is closed under finite type wedges.

REMARK. A space W is a finite type wedge of spaces in $\mathcal{A}$ if it has such a wedge decomposition. This does not mean that W does not have another decomposition which is *not* of finite type. For example, let $\mathcal{A}$ be the two-member collection $\{S^2, \bigvee_{n=1}^{\infty} S^2\}$. Then $W = \bigvee_{n=1}^{\infty} S^2$ is a member of $\mathcal{A}^{\vee}$ since, in fact, $W \in \mathcal{A}$. However, W can also be considered as an infinite (and not of finite type) wedge of S^2 with itself.

Examples. The following examples are of particular interest.

(a) We write $\mathcal{S} = \{S^n \mid n \geq 0\}$. Clearly $\mathcal{S}$ is closed under smash products. Then $\mathrm{cl}_{\mathcal{S}}(K) < \infty$ if and only if K is a connected finite complex.

(b) The collection $\mathcal{S}^{\vee}$ is closed under smash products and finite type wedges. It follows that any finite type wedge K of suspensions of a given connected finite complex has $\mathrm{cl}_{\mathcal{S}^{\vee}}(K) < \infty$.

(c) Let $\mathcal{S}_\infty = \{\bigvee_{\alpha \in I} S^n \mid n \geq 0,\ I \text{ an index set}\}$ denote the collection of all unidimensional wedges of spheres. Then $\mathcal{S}_\infty^{\vee}$ is simply the collection of all wedges of spheres. If K is a wedge of suspensions of a given connected finite-dimensional space, then $\mathrm{cl}_{\mathcal{S}_\infty^{\vee}}(K) < \infty$.

We are now ready to prove two key results. The first is a formal result about collections which follows easily from the Hilton–Milnor theorem.

Proposition 3. *If $\mathcal{A}$ is closed under smash products and suspension, then $\Sigma\Omega(\Sigma^2\mathcal{A}^{\vee}) \subseteq \Sigma^3\mathcal{A}^{\vee}$.*

Proof. Let $A = \bigvee \Sigma^2 A_i \in \Sigma^2\mathcal{A}^{\vee}$. By the Hilton–Milnor theorem [12], ΩA is a finite type product of the form

$$\Omega A \sim \prod \Omega(\Sigma^2 A_{i_1} \wedge \ldots \wedge \Sigma^2 A_{i_k}).$$

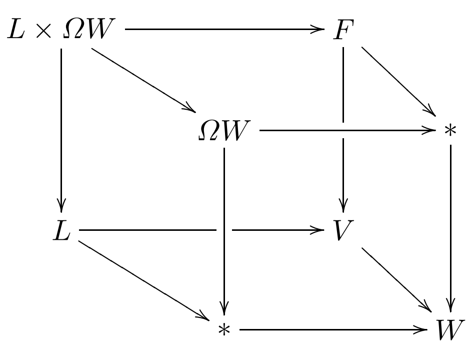
The splitting of the product after one suspension and the James splitting of $\Sigma\Omega\Sigma X$ when X is simply connected imply that $\Sigma\Omega A \in \Sigma^3 A^{\vee}$. ■

The second result of this section gives an upper bound for the cone length of the suspension of the homotopy fiber of certain maps.

Proposition 4. *Let $L \to V \to W$ be a cofiber sequence and let F be the homotopy fiber of $V \to W$. Then*

$$\Sigma F \simeq \Sigma L \vee (L \wedge \Sigma\Omega W).$$

Proof. Convert the maps $L \xrightarrow{*} W$, $V \to W$ and $W \xrightarrow{=} W$ to fibrations. The total spaces and fibers form the commutative diagram



in which the bottom square is a homotopy pushout. Since all the fibrations have the same base (namely W), the fibers of the vertical maps in the cube have the same fiber (namely ΩW). Therefore each vertical face of the cube is a homotopy pullback square and it follows from Mather's cube theorem [15] that the top square is also a homotopy pushout. (Alternatively, one can simply appeal to a result of V. Puppe [17] to conclude that the top square is a homotopy pushout.) Hence, the cofiber ΣF of the map $F \to *$ has the same homotopy type as the cofiber of $L \times \Omega W \to \Omega W$, namely $\Sigma L \vee (L \wedge \Sigma\Omega W)$, as can be seen from the diagram

$$\begin{array}{ccccc} L \times \Omega W & \longrightarrow & \Omega W & \longrightarrow & \Sigma F \\ \big\downarrow & \text{homotopy pushout} & \big\downarrow & & \big\| \\ L & \xrightarrow[*]{} & L * \Omega W & \longrightarrow & \Sigma L \vee (L \wedge \Sigma\Omega W) \end{array}$$

whose rows are cofiber sequences. ∎

COROLLARY 5. *Let $\mathcal{A}$ be a collection of spaces that is closed under smash products and suspension. Let $L \to V \to W$ be a cofiber sequence with $W \in \Sigma^2\mathcal{A}^\vee$ and let F be the homotopy fiber of $V \to W$. Then* $\mathrm{cl}_{\mathcal{A}^\vee}(\Sigma F) \le \mathrm{cl}_{\mathcal{A}^\vee}(L)$.

Proof. By Proposition 4, $\Sigma F \simeq \Sigma L \vee (L \wedge \Sigma\Omega W)$. Now $\Sigma\Omega W \in \Sigma^3\mathcal{A}^\vee$ by Proposition 3, so

$$\mathrm{cl}_{\mathcal{A}^\vee}(\Sigma F) \le \mathrm{cl}_{\mathcal{A}^\vee}(\Sigma L) \le \mathrm{cl}_{\mathcal{A}^\vee}(L)$$

by parts (d), (c) and (b) of Proposition 2, in that order. ∎

2. Resolving classes. The second piece of terminology that we need to state our main theorem is the notion of (strong) resolving classes.

DEFINITION 6. We call a nonempty class $\mathcal{R}$ of spaces a *resolving class* if it is closed under weak equivalences and pointed homotopy limits. It is a *strong resolving class* if it is further closed under extensions by fibrations; i.e., if whenever $F \to E \to B$ is a fiber sequence with $F, B \in \mathcal{R}$, then $E \in \mathcal{R}$.

Resolving classes are dual to closed classes as defined in [5] and [8, p. 45].

Every resolving class $\mathcal{R}$ contains the one-point space $*$ (cf. [8, p. 47]). From this, it follows that if $F \to E \to B$ is a fiber sequence with $E, B \in \mathcal{R}$, then $F \in \mathcal{R}$.

EXAMPLES. (a) The class $\{K \mid K \sim *\}$ is a strong resolving class.

(b) If $\mathcal{R}$ is a (strong) resolving class, then the class $\{K \mid \mathrm{map}_*(X, K) \in \mathcal{R}\}$ is also a (strong) resolving class. In particular, $\{K \mid \mathrm{map}_*(X, K) \sim *\}$ is a strong resolving class.

(c) If $f : A \to B$ is any map then the class of all f-local spaces is a resolving class [8, p. 5]. This includes, for example, the class of all h_*-local spaces, where h_* is a homology theory. If P is a set of primes, then the class of all P-local spaces is a strong resolving class.

If $\mathcal{A}$ is a collection of spaces, then $\mathcal{R}(\mathcal{A})$ denotes the smallest resolving class which contains $\mathcal{A}$, and $\overline{\mathcal{R}}(\mathcal{A})$ denotes the smallest strong resolving class which contains $\mathcal{A}$. We say that a space K is $\mathcal{A}$-*resolvable* if $K \in \overline{\mathcal{R}}(\mathcal{A})$.

The collection $\mathcal{S}$ of all spheres is of particular interest. A space K is *spherically resolvable* if $K \in \overline{\mathcal{R}}(\mathcal{S})$. This concept is related to, but not the same as, the notion of spherical resolvability described in [6, 14].

We now show that if $\overline{\mathcal{R}} = \{K \mid \mathrm{map}_*(X, K) \sim *\}$ for some space X, then $\overline{\mathcal{R}}$ is closed under the operation of taking certain finite type wedges of suspensions of members of $\overline{\mathcal{R}}$. The statement and proof of Proposition 7 in the representative special case $\mathcal{A} = \mathcal{S}$ are due to Bill Dwyer [9].

Proposition 7. *Let $\mathcal{A}$ be a collection of spaces that is closed under suspension and smash products and let $N \geq 2$. Let $\overline{\mathcal{R}} = \{K \mid \mathrm{map}_*(X, K) \sim *\}$ for some space X. If $\Sigma^N \mathcal{A} \subseteq \overline{\mathcal{R}}$, then $\Sigma^N \mathcal{A}^\vee \subseteq \overline{\mathcal{R}}$.*

Proof. Define a relation $<$ on $\Sigma^N \mathcal{A}^\vee$ as follows: $V < W$ if either

(1) the connectivity of V is greater than the connectivity of W, or

(2) the connectivity of V equals the connectivity of W (say both are $(n-1)$-connected) and the minimum number of $(n-1)$-connected summands of V in any wedge decomposition of V is less than the minimum number of $(n-1)$-connected summands of W in any wedge decomposition of W.

Claim. *Suppose that $V \in \Sigma^N \mathcal{A}^\vee$ is $(n-1)$-connected. Then there is a map $f : V \to \Sigma^N A$ with $A \in \mathcal{A}$ such that the homotopy fiber U of f belongs to $\Sigma^N \mathcal{A}^\vee$ and $U < V$.*

Proof of Claim. Write $V \sim V' \vee \Sigma^N A$ with $\Sigma^N A$ $(n-1)$-connected and $V' < V$. Let $f : V \to \Sigma^N A$ be the map which collapses V'. By [12], the homotopy fiber of f is

$$(V' \times \Omega\Sigma^N A)/(* \times \Omega\Sigma^N A) \sim V' \wedge (\Omega\Sigma^N A)_+ \sim V' \vee \Big(V' \wedge \bigvee_{m=0}^{\infty} (\Sigma^{N-1} A)^{\wedge m}\Big) \in \Sigma^N \mathcal{A}^\vee,$$

using the James splitting of $\Sigma\Omega\Sigma Z$, which applies because $\Sigma^N A$ is a simply connected suspension and V' is a suspension. This completes the proof of the Claim.

Now let $V = V_0 \in \Sigma^N \mathcal{A}$. For $n \geq 0$, define V_{n+1} as the fiber of a map $f : V_n \to \Sigma^N A_n$ as in the Claim. The result is a tower of spaces

$$V_0 \leftarrow V_1 \leftarrow \ldots \leftarrow V_n \leftarrow V_{n+1} \leftarrow \ldots$$

with $V_n \in \Sigma^N \mathcal{A}^\vee$ and $V_{n+1} < V_n$ for each n. Since the spaces in this tower become arbitrarily highly connected as n increases, $\mathrm{holim}_n V_n \sim *$.

For each n the fiber sequence $V_n \to V_{n+1} \to \Sigma^N A_n$ gives rise to a fiber sequence

$$\mathrm{map}_*(X, V_{n+1}) \to \mathrm{map}_*(X, V_n) \to \overbrace{\mathrm{map}_*(X, \Sigma^N A_n)}^{*}.$$

It follows by induction that each map $V_n \to V$ induces a weak equivalence $\mathrm{map}_*(X, V_n) \sim \mathrm{map}_*(X, V)$. Finally, we compute

$$\begin{aligned}\mathrm{map}_*(X, V) &\sim \mathrm{holim}_n \, \mathrm{map}_*(X, V) \sim \mathrm{holim}_n \, \mathrm{map}_*(X, V_n) \\ &\sim \mathrm{map}_*(X, \mathrm{holim}_n \, V_n) \sim \mathrm{map}_*(X, *) \sim *. \ \blacksquare\end{aligned}$$

3. The main results. We can now state our main result.

THEOREM 8. *Let $\mathcal{A}$ be a collection of spaces that is closed under suspension and smash products and let $N \in \mathbb{N}$. If K is a nilpotent space with* $\mathrm{cl}_{\mathcal{A}^\vee}(K) = n < \infty$, *then*

(a) $K \in \overline{\mathcal{R}}(\Sigma^N \mathcal{A}^\vee)$,
(b) $\Omega^n K \in \mathcal{R}(\Sigma^N \mathcal{A}^\vee)$.

The proof will be given in Section 4.

The collections $\mathcal{S}$ and $\mathcal{S}_\infty$ satisfy the conditions of Theorem 8. For the collection $\mathcal{S}$, the theorem implies that every nilpotent finite complex is $\mathcal{S}^\vee$-resolvable. Our first corollary shows that the loop space of a nilpotent finite complex is spherically resolvable.

COROLLARY 9. *If K is a nilpotent finite complex, then ΩK is spherically resolvable.*

Proof. Let $\mathcal{R} = \{K \mid \Omega K \in \overline{\mathcal{R}}(\mathcal{S})\}$, which is a strong resolving class. If $\bigvee S^{n_\alpha}$ is a simply connected finite type wedge of spheres, then by the Hilton–Milnor theorem [12], $\Omega \bigvee S^{n_\alpha}$ is a product of loop spaces of spheres, and so is in $\overline{\mathcal{R}}(\mathcal{S})$. Thus $\Sigma^2 \mathcal{S}^\vee \subseteq \mathcal{R}$, and Theorem 8(a) guarantees that every nilpotent finite complex is in $\overline{\mathcal{R}}(\Sigma^2 \mathcal{S}^\vee) \subseteq \mathcal{R}$. ∎

We also obtain the following general version of our recognition principle for Miller spaces.

THEOREM 10. *Let $\mathcal{A}$ be a collection of spaces that is closed under suspension and smash products and let $N \geq 1$. Then the following are equivalent*:

(a) $\mathrm{map}_*(X, \Sigma^N A) \sim *$ *for each* $A \in \mathcal{A}$,
(b) $\mathrm{map}_*(X, K) \sim *$ *for every nilpotent space K with* $\mathrm{cl}_{\mathcal{A}^\vee}(K) < \infty$.

Proof. Let $\mathcal{M}$ be the class of all spaces K such that $\mathrm{map}_*(X, K) \sim *$; we have already seen that $\mathcal{M}$ is a strong resolving class. We have $\Sigma^N \mathcal{A} \subseteq \mathcal{M}$

by assumption, so $\overline{\mathcal{R}}(\Sigma^N\mathcal{A}^\vee) \subseteq \mathcal{M}$ by Proposition 7. By Theorem 8(a), $\overline{\mathcal{R}}(\Sigma^N\mathcal{A}^\vee)$ contains every nilpotent space K with $\mathrm{cl}_{\mathcal{A}^\vee}(K) < \infty$. ■

We obtain the theorem cited in the introduction by specializing to $\mathcal{A} = \mathcal{S}$.

Corollary 11. *Let X be a space and let $N \in \mathbb{N}$. Then the following are equivalent*:

(a) $\mathrm{map}_*(X, K) \sim *$ *for every nilpotent finite complex* K,
(b) $\mathrm{map}_*(X, S^n) \sim *$ *for all* $n \geq N$.

If X is simply connected then Theorem 10 can be strengthened somewhat. If L is a space with a nilpotent covering space K having $\mathrm{cl}_{\mathcal{A}}(K) < \infty$, then it is easy to see that $\mathrm{map}_*(X, L) \sim *$. The same sort of argument proves the following reduction of Miller's theorem.

Corollary 12. *For any $N \in \mathbb{N}$ the following are equivalent*:

(a) $\mathrm{map}_*(B\mathbb{Z}/p, K) \sim *$ *for every finite-dimensional complex* K,
(b) $\mathrm{map}_*(B\mathbb{Z}/p, \bigvee S^n) \sim *$ *for each wedge of n-spheres with* $n \geq N$.

Proof. Let K be a finite-dimensional CW complex and let $\widetilde{K}$ be its (finite-dimensional) universal cover. Applying Theorem 10 in the case $\mathcal{A} = \mathcal{S}_\infty$, we see that $\mathrm{map}_*(B\mathbb{Z}/p, \widetilde{K}) \sim *$. But according to [16, Thm. 10.1], the induced map

$$\pi_*(\mathrm{map}_*(B\mathbb{Z}/p, \widetilde{K})) \to \pi_*(\mathrm{map}_*(B\mathbb{Z}/p, K))$$

is surjective, so $\mathrm{map}_*(B\mathbb{Z}/p, K) \sim *$. ■

Actually, since $B\mathbb{Z}/p$ is a countable CW complex, it suffices to check that $\mathrm{map}_*(B\mathbb{Z}/p, \bigvee S^n) \sim *$ for *countable* wedges of spheres.

4. Proof of Theorem 8. We will make essential use of the following desuspension result.

Proposition 13. *Let $\mathcal{R}$ be a resolving class and let $N \in \mathbb{N}$. If K is a connected nilpotent space and $\bigvee_{i=1}^m \Sigma^N K \in \mathcal{R}$ for each m, then $K \in \mathcal{R}$.*

We defer further discussion of Proposition 13 until Section 5, and proceed directly to the proof of Theorem 8.

Proof of Theorem 8. Since $\mathcal{A}$ is closed under suspension, we may, and do, assume that $N \geq 2$.

We prove assertion (a) by induction on $\mathrm{cl}_{\mathcal{A}^\vee}(K)$. If $\mathrm{cl}_{\mathcal{A}^\vee}(K) = 1$, then $K \in \Sigma\mathcal{A}^\vee$. Therefore each finite wedge $\bigvee_{i=1}^m \Sigma^N K$ is in $\Sigma^N\mathcal{A}^\vee$ and Proposition 13 proves the assertion in the initial case.

Now assume that the result is known for all nilpotent spaces with $\mathcal{A}^\vee$-cone length less than n, and that K is nilpotent with $\mathrm{cl}_{\mathcal{A}^\vee}(K) = n$. By Proposition 13, it is enough to show $\bigvee_{i=1}^m \Sigma^N K \in \overline{\mathcal{R}}(\Sigma^N\mathcal{A}^\vee)$ for each m.

Write $V = \bigvee_{i=1}^m \Sigma^N K$. By parts (a) and (d) of Proposition 2, we have $\mathrm{cl}_{\Sigma^N \mathcal{A}^\vee}(V) \le \mathrm{cl}_{\mathcal{A}^\vee}(K) = n$. Thus V has a $\Sigma^N \mathcal{A}^\vee$-cone decomposition

$$A_i \to V_i \to V_{i+1} \quad \text{for } 0 \le i < n$$

with $A_i \in \Sigma^N \mathcal{A}^\vee$ for each i. Hence, there is a cofiber sequence $L \to V \to W$ of simply connected spaces with $\mathrm{cl}_{\mathcal{A}^\vee}(L) \le \mathrm{cl}_{\Sigma^N \mathcal{A}^\vee}(L) < n$ and $W \in \Sigma^{N+1}\mathcal{A}^\vee$ (specifically, $L = V_{n-1}$ and $W = \Sigma A_{n-1}$).

Let F denote the homotopy fiber of $V \to W$. By Corollary 5, $\mathrm{cl}_{\mathcal{A}^\vee}(\Sigma F) \le \mathrm{cl}_{\mathcal{A}^\vee}(L) < n$. Consequently, $\mathrm{cl}_{\mathcal{A}^\vee}(\bigvee_{i=1}^m \Sigma F) < n$ for each m by Proposition 2(d). By the inductive hypothesis, $\bigvee_{i=1}^m \Sigma F \in \overline{\mathcal{R}}(\Sigma^N \mathcal{A})$ for each m. Since W is 2-connected and V is simply connected, F is at least simply connected, and hence nilpotent. Now Proposition 13 implies that $F \in \overline{\mathcal{R}}(\Sigma^N \mathcal{A})$.

Since W and F are in the strong resolving class $\overline{\mathcal{R}}(\Sigma^N \mathcal{A}^\vee)$, we conclude that $V \in \overline{\mathcal{R}}(\Sigma^N \mathcal{A}^\vee)$. This completes the proof of (a).

The proof of (b) is similar. To prove the inductive step, we write $V = \bigvee_{i=1}^m \Sigma^N K$ and show that $\Omega^n V \in \mathcal{R}(\Sigma^N \mathcal{A})$. As before, we consider the cofiber sequence $L \to V \to W$ with $W \in \Sigma\mathcal{A}$ and the corresponding fiber sequence $F \to V \to W$. This gives us a fiber sequence

$$\Omega^n V \to \Omega^n W \to \Omega^{n-1} F$$

with $\Omega^n W \in \mathcal{R}(\Sigma^N \mathcal{A})$. It now suffices to prove that $\Omega^{n-1} F \in \mathcal{S}$, which follows by induction using Proposition 13. ∎

5. Desuspension in resolving classes. Proposition 13 is an immediate consequence of the following theorem, various incarnations of which can be found in work of Barratt, Hopkins and Bousfield [2, 13, 3]. It has also been studied by Goerss [10].

THEOREM 14. *If K is a nilpotent space, then K is homotopy equivalent to the homotopy limit of a tower*

$$M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow M_{n+1} \leftarrow \ldots$$

of spaces, each of which is a homotopy limit of a natural diagram of spaces of the form $\bigvee_{i=1}^m \Sigma K$.

This has the following consequence in our terminology.

COROLLARY 15. *Let $\mathcal{A}$ be a collection of nilpotent spaces and let $\mathcal{R}$ be a resolving class. If $\mathcal{A}$ is closed under wedges and $\Sigma\mathcal{A} \subseteq \mathcal{R}$, then $\mathcal{A} \subseteq \mathcal{R}$.*

It is difficult to find a satisfactory (that is, complete and fairly elementary) account of this resolution of K by finite wedges $\bigvee \Sigma K$ in the literature. The resolution was first described (for simply connected spaces) in lecture notes by M. Barratt [2]. It is stated (without proof) for nilpotent spaces in Hopkins' paper [13]. In [3, Ex. 4.9], Bousfield uses a deep theorem on the convergence of the homology spectral sequence of a cosimplicial space to show that, for *any* space K, the homotopy limit of the tower is weakly equivalent to the $\mathbb{Z}$-completion $\mathbb{Z}_\infty(K)$. This completion is weakly equivalent to K if K is nilpotent.

In view of Bousfield's work, it is not strictly necessary to provide a proof of Theorem 14. However there is a simple geometric proof of Theorem 14 (suggested by Wojciech Chachólski) under the additional assumption that K is simply connected. We include that proof here.

We begin by recalling Goodwillie's generalization of the classical Blakers–Massey excision theorem. Let $\mathbf{Sp}_*$ denote a reasonable category of pointed topological spaces and let C_n denote the category of subsets of $\{1, \ldots, n\}$ whose morphisms are inclusions. A diagram $\mathcal{X} : C_n \to \mathbf{Sp}_*$ is called a *cubical diagram.* Such a diagram is *strongly cocartesian* if every 2-dimensional face is a homotopy pushout square. Write $f_i : \mathcal{X}(\emptyset) \to \mathcal{X}(\{i\})$; if $\mathcal{X}$ is strongly cocartesian, then $\mathcal{X}$ is completely determined by the maps f_i.

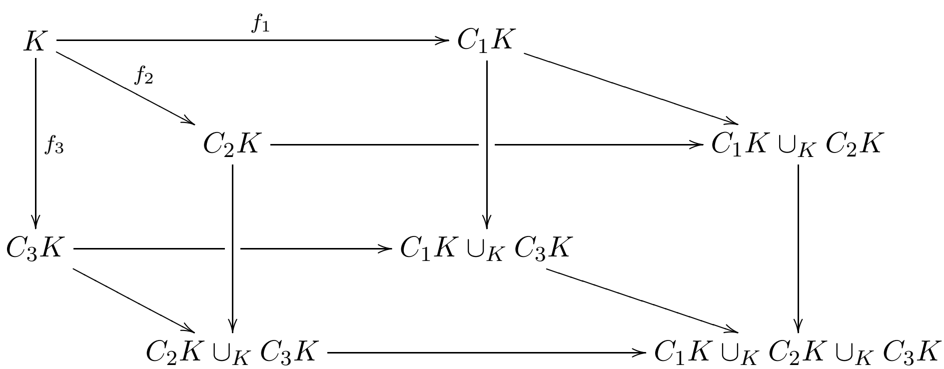
Let $\widetilde{C}_n = \{A \subseteq \{1, \ldots, n\} \mid A \neq \emptyset\}$ and write $\widetilde{\mathcal{X}} = \mathcal{X}|_{\widetilde{C}_n}$.

Theorem [11, Thm. 3.21]. *Let $\mathcal{X}$ be a strongly cocartesion cubical diagram. Let F_i denote the homotopy fiber of the map $f_i : \mathcal{X}(\emptyset) \to \mathcal{X}(\{i\})$ and assume that F_i is c_i-connected. Then the homotopy fiber of the natural map*

$$\mathcal{X}(\emptyset) \to \operatorname{holim} \widetilde{\mathcal{X}}$$

is at least $(\sum_i c_i)$-connected.

Proof of the simply connected case of Theorem 14. Let $\mathcal{X}_n$ denote the n-dimensional strongly cocartesian diagram which is determined by the inclusions $f_i : K \hookrightarrow C_i K$ of K into the cone on K. For the case $n = 3$, this is the diagram



It is easy to see that $\mathcal{X}_n(A)$ is homotopy equivalent to a wedge of suspensions of K if $A \neq \emptyset$ (simply collapse one of the cones to a point). Furthermore, the homotopy fiber of f_i (namely K) is 1-connected for each i by assumption.

Define $M_n = \operatorname{holim} \widetilde{\mathcal{X}}_n$. Using naturality of homotopy limits, we obtain a commutative ladder

$$\begin{array}{ccccccccc} K & = & K & =\cdots= & K & = & K & = & \cdots \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n+1}} & & \\ M_1 & \leftarrow & M_2 & \leftarrow\cdots\leftarrow & M_n & \leftarrow & M_{n+1} & \leftarrow & \cdots \end{array}$$

Goodwillie's theorem implies that the homotopy fiber of the map h_n is at least n-connected. It follows that the fiber of the natural map $h : K \to \operatorname{holim}_n M_n$ is weakly contractible—in other words, h is a weak equivalence. ∎

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