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The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds

by

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Abstract. Using fiberings, we determine the cup-length and the Lyusternik–Shnirel'man category for some infinite families of real flag manifolds $O(n_1 + \ldots + n_q)/O(n_1) \times \ldots \times O(n_q)$, $q \geq 3$. We also give, or describe ways to obtain, interesting estimates for the cup-length of any $O(n_1 + \ldots + n_q)/O(n_1) \times \ldots \times O(n_q)$, $q \geq 3$. To present another approach (combining well with the "method of fiberings"), we generalize to the real flag manifolds Stong's approach used for calculations in the $\mathbb{Z}_2$-cohomology algebra of the Grassmann manifolds.

1. Introduction. For fixed positive integers $n_1, \ldots, n_q$ ($q \geq 2$), a *flag of type* $(n_1, \ldots, n_q)$ (see e.g. [7]) is defined to be a q-tuple $(S_1, \ldots, S_q)$ of mutually orthogonal subspaces in $\mathbb{R}^n$, where $n = n_1 + \ldots + n_q$ and $\dim(S_i) = n_i$. The set $F(n_1, \ldots, n_q)$ of all the flags of type $(n_1, \ldots, n_q)$ may obviously be identified with a quotient space of the orthogonal group, $O(n)/O(n_1) \times \ldots \times O(n_q)$. This identification makes $F(n_1, \ldots, n_q)$ into a closed manifold of dimension $\delta(n_1, \ldots, n_q) := \sum_{1 \leq i < j \leq q} n_i n_j$ (in some cases, we shall just write δ, when the sequence $(n_1, \ldots, n_q)$ is clear from the context). In particular, $F(n_1, n_2)$ is the Grassmann manifold of all n_1-dimensional vector subspaces in $\mathbb{R}^{n_1+n_2}$.

Over the flag manifold $F(n_1, \ldots, n_q)$, one has q canonical vector bundles, $\gamma_1, \ldots, \gamma_q$, with $\dim(\gamma_i) = n_i$; the fiber of γ_i over $(S_1, \ldots, S_q) \in F(n_1, \ldots, n_q)$ may be identified with S_i. The Whitney sum $\bigoplus_{i=1}^{q} \gamma_i$ is the trivial n-dimensional vector bundle ε^n.

With any closed positive-dimensional manifold M one can associate a homotopy invariant called the ($\mathbb{Z}_2$-)*cup-length* of M (briefly: $\mathrm{cup}(M)$), that is, the maximum c such that there are cohomology classes $a_1, \ldots, a_c \in H^*(M; \mathbb{Z}_2)$, all of positive dimensions, such that their cup-product $a_1 \cup \ldots \cup a_c$ is nonzero. The number $\mathrm{cup}(M)$ is well known to provide a lower bound for another very interesting, but not easily calculable, homotopy invariant: the

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Lyusternik–Shnirel'man category of M (briefly, $\mathrm{cat}(M)$). Recall (cf. [6]) that $\mathrm{cat}(M)$ is the minimum number of open subsets of M covering M, each of which is contractible in M; one has $1+\dim(M) \geq \mathrm{cat}(M) \geq 1+\mathrm{cup}(M)$. For the flag manifolds, the cup-length and the Lyusternik–Shnirel'man category are in general unknown.

In this paper, using suitable fiberings, we explicitly determine the cup-length for some infinite families of flag manifolds $F(n_1,\ldots,n_q)$, $q \geq 3$. Using the same approach, we also present some estimates, or describe ways to obtain estimates, for $\mathrm{cup}(F(n_1,\ldots,n_q))$ in general. At the end of the paper, we adapt to the flag manifolds the approach which Robert Stong used in [14] for calculations in the $\mathbb{Z}_2$-cohomology algebra of Grassmann manifolds (he succeeded in actually giving explicit formulae for the cup-length of certain infinite families of Grassmann manifolds).

To obtain an approximation to the value of $\mathrm{cup}(F(n_1,\ldots,n_q))$, one's first idea certainly is to use the well known fact that the first cohomology group $H^1(F(n_1,\ldots,n_q);\mathbb{Z}_2)$ is always nontrivial; more precisely, it is the direct sum of $q-1$ copies of $\mathbb{Z}_2$. Then for any nonvanishing $x \in H^1(F(n_1,\ldots,n_q);\mathbb{Z}_2)$ one can ask what is the height of x (denoted $\mathrm{height}(x)$) or, in other words, what is the maximum p such that $x^p \neq 0$. Of course, if one finds the answer, then one has the corresponding lower bound for $\mathrm{cup}(F(n_1,\ldots,n_q))$; at the other extreme, one obviously has

$$\mathrm{cup}(F(n_1,\ldots,n_q)) \leq \delta(n_1,\ldots,n_q).$$

For all Grassmann manifolds $F(n_1,n_2)$, the height of the first Stiefel–Whitney class $w_1(\gamma_1)=w_1(\gamma_2)$ is known thanks to Stong [14]. More generally, in the $\mathbb{Z}_2$-cohomology $H^*(F(n_1,\ldots,n_q);\mathbb{Z}_2)$ one always has nontrivial Stiefel–Whitney classes $w_1(\gamma_1),\ldots,w_1(\gamma_{q-1})$. Some $\mathbb{Z}_2$-linear combinations (for instance $w_1(\gamma_1)+\ldots+w_1(\gamma_{q-1})=w_1(\gamma_q)$) will also be nonzero. In particular, when $F(n_1,\ldots,n_q)$ is nonorientable, then $w_1(F(n_1,\ldots,n_q))$ (which is defined to be the first Stiefel–Whitney class of the tangent bundle of $F(n_1,\ldots,n_q)$ and is a $\mathbb{Z}_2$-linear combination of $w_1(\gamma_1),\ldots,w_1(\gamma_{q-1})$; cf. [7]) is nonzero. When trying to find an estimate for $\mathrm{cup}(F(n_1,\ldots,n_q))$, one may start by asking what is the height of any of the above-mentioned first Stiefel–Whitney classes.

In 2000, Ilori and Ajayi [5] calculated the height of $w_1(F(n_1,\ldots,n_q))$ for some of those flag manifolds $F(n_1,\ldots,n_q)$ which are nonorientable. More recently, the second named author has shown in [10] that a slight modification of their approach is enough for obtaining a complete result, that is, for calculating the height of the first Stiefel–Whitney class of *any* nonorientable real flag manifold. In Section 2, we show that it also is possible to calculate $\mathrm{height}(w_1(\gamma_i))$, $i=1,\ldots,q$. The numbers $\mathrm{height}(w_1(\gamma_i))$ are sometimes bigger than $\mathrm{height}(w_1(F(n_1,\ldots,n_q)))$ in the case of nonorientable flag manifolds $F(n_1,\dots,n_q)$. Another advantage is that the numbers height$(w_1(\gamma_i))$ are *always* nontrivial, even when $w_1(F(n_1,\dots,n_q)) = 0$ (in other words, when $F(n_1,\dots,n_q)$ is orientable), giving also in that case interesting lower bounds for $\mathrm{cup}(F(n_1,\dots,n_q))$.

In Section 3, we present another way to obtain lower bounds for the cup-length of $F(n_1,\dots,n_q)$, based on a lemma of Horanská and Korbaš [4] (they attribute the lemma to R. Stong), applied to natural fiberings of $F(n_1,\dots,n_q)$ with $q \geq 3$. We also find some infinite families of flag manifolds $F(n_1,\dots,n_q)$ $(q \geq 3)$ with the cup-length equal to $\delta(n_1,\dots,n_q)$. In addition, we derive a necessary condition for $\mathrm{cup}(F(n_1,\dots,n_q)) = \delta(n_1,\dots,n_q)$, we derive a nontrivial upper bound for $\mathrm{cup}(F(n_1,\dots,n_q))$ when the necessary condition is not satisfied, and we determine the number $\mathrm{cup}(F(1,2,n_3))$ for all $n_3 \geq 3$. Finally, we generalize to $F(n_1,\dots,n_q)$ Stong's approach from [14], and show how this can be used to calculate $\mathrm{cup}(F(n_1,\dots,n_q))$.

In what follows, all cohomology groups will be understood to have coefficients in $\mathbb{Z}_2$.

2. Heights of the first Stiefel–Whitney classes

2.1. *On* height$(w_1(F(n_1,\dots,n_q)))$. Let $w_i(\gamma_j) \in H^i(F(n_1,\dots,n_q))$ be the ith Stiefel–Whitney characteristic class of the canonical vector bundle γ_j over $F(n_1,\dots,n_q)$. Recall that, by Borel [2, Theorem 11.1], the cohomology ring $H^*(F(n_1,\dots,n_q))$ can be represented as a quotient ring of the polynomial ring

$$\mathbb{Z}_2[w_1(\gamma_1),\dots,w_{n_1}(\gamma_1),\dots,w_1(\gamma_q),\dots,w_{n_q}(\gamma_q)]$$

by the ideal given by the identity

$$\prod_{j=1}^{q}(1 + w_1(\gamma_j) + \ldots + w_{n_j}(\gamma_j)) = 1.$$

Of course, the identity comes from the fact that $\bigoplus_{i=1}^q \gamma_i \cong \varepsilon^n$.

In [5], a partial result on the height of $w_1(F(n_1,\dots,n_q))$ for *nonorientable* flag manifolds $F(n_1,\dots,n_q)$ has been derived. Recently, J. Lörinc [10] proved the following complete result.

Theorem 2.1.1 (Lörinc [10]). *Let $F(n_1,\dots,n_q)$ for $q \geq 2$ be any nonorientable real flag manifold*; *hence not all of $n_1,\dots,n_q$ have the same parity. Letting p be the sum of all even numbers among $n_1,\dots,n_q$, put $k = \min\{p, n-p\}$. If s is the uniquely determined integer such that $2^s < n \leq 2^{s+1}$, then*

$$\operatorname{height}(w_1(F(n_1,\dots,n_q))) = \begin{cases} n-1 & \textit{if } k=1,\\ 2^{s+1}-2 & \textit{if } k=2 \textit{ or}\\ & \textit{if } k=3 \textit{ and } n=2^s+1,\\ 2^{s+1}-1 & \textit{otherwise.}\end{cases}$$

For any orientable flag manifold, its first Stiefel–Whitney class vanishes, hence it makes no sense to define its height.

2.2. *Heights of the first Stiefel–Whitney classes of the canonical vector bundles.* Regardless of orientability of the flag manifold $F(n_1,\dots,n_q)$, the heights of $w_1(\gamma_i)$ $(i=1,\dots,q)$ are always of interest, because they *always* provide a nontrivial lower bound for $\operatorname{cup}(F(n_1,\dots,n_q))$.

The height of $w_1(\gamma_1) = w_1(\gamma_2) \in H^*(F(p,n-p))$ is known due to Stong [14]: If s is such that $2^s < n \le 2^{s+1}$ and $k=\min\{p,n-p\}$, then

$$\operatorname{height}(w_1(\gamma_1)) = \begin{cases} n-1 & \text{if } k=1,\\ 2^{s+1}-2 & \text{if } k=2 \text{ or}\\ & \text{if } k=3 \text{ and } n=2^s+1,\\ 2^{s+1}-1 & \text{otherwise.}\end{cases}$$

We observe that now for any $F(n_1,\dots,n_q)$ the numbers $\operatorname{height}(w_1(\gamma_i))$ $(i=1,\dots,q)$ can readily be computed. One just needs to use a suitable fibering of the manifold $F(n_1,\dots,n_q)$ over a Grassmann manifold. For instance, for $i=1$, one uses the fiber bundle

$$\begin{array}{ccc} F(n_2,\dots,n_q) & \hookrightarrow & F(n_1,\dots,n_q)\\ & & \downarrow\\ & & F(n_1,n_2+\dots+n_q)\end{array} \tag{1}$$

More generally (see e.g. [13, 7.4]), one has the fiber bundle with the bundle projection defined by sending the flag $(S_1,\dots,S_q)\in F(n_1,\dots,n_q)$ to the flag $(S_1,\dots,S_t,S_{t+1}\oplus\dots\oplus S_q)\in F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)$, for a fixed t (in (1), we have $t=1$). For any of these fiber bundles the Leray–Hirsch theorem applies. Indeed, if $i: F(n_{t+1},\dots,n_q)\hookrightarrow F(n_1,\dots,n_q)$ is the fiber inclusion, then the pullbacks $i^*(\gamma_{t+1}),\dots,i^*(\gamma_q)$ can obviously be identified with the canonical vector bundles over $F(n_{t+1},\dots,n_q)$. Keeping in mind the description of the cohomology ring $H^*(F(n_1,\dots,n_q))$ and the well known properties of the Stiefel–Whitney classes, we see that the map i induces an epimorphism in $\mathbb{Z}_2$-cohomology. This fact will be used repeatedly in what follows.

As a consequence of the Leray–Hirsch theorem, the $\mathbb{Z}_2$-cohomology homomorphism induced by the bundle projection considered in (1) is a monomorphism. Hence the height of $w_1(\gamma_1)\in H^*(F(n_1,\dots,n_q))$ is the same as the height of $w_1(\gamma_1) \in H^*(F(n_1, n_2 + \ldots + n_q))$. In general, if we define (as always) $n = n_1 + \ldots + n_q$, if s is such that $2^s < n \le 2^{s+1}$, and if $k_i = \min\{n_i, n - n_i\}$, then for γ_i over $F(n_1, \ldots, n_q)$,

$$\operatorname{height}(w_1(\gamma_i)) = \begin{cases} n-1 & \text{if } k_i = 1, \\ 2^{s+1}-2 & \text{if } k_i = 2 \text{ or} \\ & \text{if } k_i = 3 \text{ and } n = 2^s + 1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

For any $F(n_1, \ldots, n_q)$ nonorientable, we know (cf. 2.1.1) the value of the height of $w_1(F(n_1, \ldots, n_q))$. But the following example shows that, in infinitely many cases, the height of the first Stiefel–Whitney class of some of the canonical vector bundles may even exceed $\operatorname{height}(w_1(F(n_1, \ldots, n_q)))$.

Example 2.2.1. Let $q \ge 3$, and let $t_2, \ldots, t_q$ be integers such that $2 \le t_2 \le \ldots \le t_q$ and $2^s < 1 + 2t_2 + \ldots + 2t_q < 2^{s+1}$, for a suitable integer s. Then $\operatorname{height}(w_1(F(1, 2t_2, \ldots, 2t_q))) = 2t_2 + \ldots + 2t_q$, while $\operatorname{height}(w_1(\gamma_2)) = 2^{s+1} - 1$. Of course, $\operatorname{height}(w_1(\gamma_2)) > \operatorname{height}(w_1(F(1, 2t_2, \ldots, 2t_q)))$.

3. On the cup-length of flag manifolds

3.1. *When is* $\operatorname{cup}(F(n_1, \ldots, n_q)) = \delta(n_1, \ldots, n_q)$? We first derive a partial answer to the question of for which flag manifolds their cup-length and dimension coincide.

For the Grassmann manifolds, Berstein proved the following.

Lemma 3.1.1 (Berstein [1]). *For $F(k, n-k)$ one has $\operatorname{cup}(F(k, n-k)) = \delta(k, n-k)$ only if $k = 1$, or if $k = 2$ and n is one plus a power of 2.*

To obtain a similar (although not so complete) piece of information for the flag manifolds $F(n_1, \ldots, n_q)$ with $q \ge 3$, we shall use the following lemma from Horanská and Korbaš [4].

Lemma 3.1.2. *Let $p : E \to B$ be a smooth fiber bundle with connected base B and connected fiber F. Suppose that the fiber inclusion induces an epimorphism in $\mathbb{Z}_2$-cohomology. Then* $\operatorname{cup}(E) \ge \operatorname{cup}(F) + \operatorname{cup}(B)$.

For $a = 1$ or 2, we will abbreviate

$$a^{\cdots k} = \underbrace{a, \ldots, a}_{k}.$$

We are now able to prove the following.

Theorem 3.1.3. (a) *For any $m \ge 1$, $k \ge 1$, one has*

$$\operatorname{cup}(F(1^{\cdots k}, m)) = \delta(1^{\cdots k}, m),$$
$$\operatorname{cat}(F(1^{\cdots k}, m)) = 1 + \delta(1^{\cdots k}, m).$$

(b) *Let $m \geq 2$, $d > 0$ and $j > 0$ be integers. Taking t to be the integer such that $2^t \leq m < 2^{t+1}$, suppose that $j \geq 2^{t+d} - m - 2d + 1$. Then*

$$\mathrm{cup}(F(1^{\dots j}, 2^{\dots d}, m)) = \delta(1^{\dots j}, 2^{\dots d}, m)$$

and, as an obvious consequence,

$$\mathrm{cat}(F(1^{\dots j}, 2^{\dots d}, m)) = 1 + \delta(1^{\dots j}, 2^{\dots d}, m).$$

REMARK (a). If the condition $j \geq 2^{t+d} - m - 2d + 1$ is not satisfied, then it need not be true that $\mathrm{cup}(F(1^{\dots j}, 2^{\dots d}, m)) = \delta(1^{\dots j}, 2^{\dots d}, m)$; this will be seen in 3.2.4. At the same time, we do not know whether or not in general $j < 2^{t+d} - m - 2d + 1$ implies that $\mathrm{cup}(F(1^{\dots j}, 2^{\dots d}, m)) < \delta(1^{\dots j}, 2^{\dots d}, m)$. In this context, see also Remark (c) after 3.2.2.

Proof of Theorem 3.1.3. (a) It is enough to prove the result on cup-length. We have the fiber bundle

$$\begin{array}{ccc} F(1,m) & \hookrightarrow & F(1^{\dots k+1}, m) \\ & & \downarrow \\ & & F(1^{\dots k}, m+1). \end{array}$$

Using Lemma 3.1.2 and induction on k (for $F(1,m)$, the claim is obviously true), one immediately obtains

$$\delta(1^{\dots k+1}, m) \geq \mathrm{cup}(F(1^{\dots k+1}, m)) \geq \delta(1^{\dots k+1}, m).$$

This proves part (a).

(b) Again, we shall prove the result on cup-length. We proceed by induction on d. For $d = 1$, the conditions are $j > 0$, $j \geq 2^{t+1} - m - 1$. If $j - 2^{t+1} + m + 1 = 0$, we have a sequence of fiber bundles

$$\begin{array}{ccc} F(1,m) & \hookrightarrow & F(1^{\dots j}, 2, m) \\ & & \downarrow \\ F(1,m+1) & \hookrightarrow & F(1^{\dots j-1}, 2, m+1) \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ F(1, 2^{t+1}-2) & \hookrightarrow & F(1, 2, 2^{t+1}-2) \\ & & \downarrow \\ & & F(2, 2^{t+1}-1) \end{array}$$

Then 3.1.2 and 3.1.1 imply (in the right-hand “tower”, we go from the penultimate to the first space) that $\mathrm{cup}(F(1,2,2^{t+1}-2)) = \delta(1,2,2^{t+1}-2)$ etc., and $\mathrm{cup}(F(1^{\dots j}, 2, m)) = \delta(1^{\dots j}, 2, m)$.

If $j - 2^{t+1} + m + 1 > 0$ we again form a sequence of fiber bundles

$$\begin{array}{ccc}
F(1,m) & \hookrightarrow & F(1^{\cdots j},2,m) \\
 & & \downarrow \\
F(1,m+1) & \hookrightarrow & F(1^{\cdots j-1},2,m+1) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
F(1,2^{t+1}-1) & \hookrightarrow & F(1^{\cdots j-2^{t+1}+1+m},2,2^{t+1}-1) \\
 & & \downarrow \\
 & & F(1^{\cdots j-2^{t+1}+1+m},2^{t+1}+1)
\end{array}$$

Then 3.1.1, 3.1.2, and 3.1.3(a) imply that

$$\operatorname{cup}(F(1^{\cdots j-2^{t+1}+1+m},2,2^{t+1}-1)) = \delta(1^{\cdots j-2^{t+1}+1+m},2,2^{t+1}-1)$$

etc., and $\operatorname{cup}(F(1^{\cdots j},2,m)) = \delta(1^{\cdots j},2,m)$. We have verified the claim for $d = 1$.

Now suppose that $d \geq 2$ and that the claim is true for $d-1$. To deal with the flag manifold $F(1^{\cdots j},2^{\cdots d},m)$, we also in this case construct a sequence of obvious fiber bundles

$$\begin{array}{ccc}
F(1,m) & \hookrightarrow & F(1^{\cdots j},2^{\cdots d},m) \\
 & & \downarrow \\
F(1,m+1) & \hookrightarrow & F(1^{\cdots j-1},2^{\cdots d},m+1) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
F(2,2^{t+1}-1)) & \hookrightarrow & F(1^{\cdots j-2^{t+1}+1+m},2^{\cdots d},2^{t+1}-1) \\
 & & \downarrow \\
 & & F(1^{\cdots j-2^{t+1}+1+m},2^{\cdots d-1},2^{t+1}+1)
\end{array}$$

with fibers covered by Lemma 3.1.1. Note that the number $j-2^{t+1}+m+1$ is now positive.

The last base space in the right-hand tower satisfies the assumptions of Theorem 3.1.3(b). That means that the induction hypothesis applies. Using Lemma 3.1.2, one readily deduces that the cup-length of

$$F(1^{\cdots j-2^{t+1}+1+m},2^{\cdots d},2^{t+1}-1)$$

is the same as its dimension. Then applying Lemma 3.1.2 to the penultimate fiber bundle in the above sequence, we also find that for its total space the cup-length and dimension coincide. After a finite number of repetitions of this step we conclude that $\delta(F(1^{\cdots j}, 2^{\cdots d}, m)) = \mathrm{cup}(F(1^{\cdots j}, 2^{\cdots d}, m))$, as needed. This completes the proof of Theorem 3.1.3.

3.2. *On* $\mathrm{cup}(F(n_1,\dots,n_q)) < \delta(n_1,\dots,n_q)$. Usually, Lemma 3.1.2 (applied to suitable fiber bundles) can be used to obtain lower bounds for $\mathrm{cup}(F(n_1,\dots,n_q))$ higher than those given merely by the heights of the first Stiefel–Whitney classes.

EXAMPLE 3.2.1. For $\mathrm{cup}(F(1,2,3,4,5,6))$, we have 31 as a lower bound, given e.g. by $\mathrm{height}(w_1(F(1,2,3,4,5,6)))$. A better result can be derived from the following sequence of obvious fiber bundles:

$$\begin{array}{ccc}
F(5,6) & \hookrightarrow & F(1,2,3,4,5,6) \\
 & & \downarrow \\
F(4,11) & \hookrightarrow & F(1,2,3,4,11) \\
 & & \downarrow \\
F(3,15) & \hookrightarrow & F(1,2,3,15) \\
 & & \downarrow \\
F(2,18) & \hookrightarrow & F(1,2,18) \\
 & & \downarrow \\
 & & F(1,20)
\end{array}$$

Indeed, using Stong's results [14] we calculate that

$$\mathrm{cup}(F(2,18)) = 33, \quad \mathrm{cup}(F(3,15)) = 38, \quad \mathrm{cup}(F(4,11)) = 27.$$

For $F(5,6)$, we have $\mathrm{height}(w_1(\gamma_1)) = 15$; by Poincaré duality, there is a cohomology class $b \in H^{15}(F(5,6))$ such that $w_1^{15} \cup b \neq 0$. Since the cohomology ring $H^*(F(5,6))$ is generated by $w_i(\gamma_1)$, $i = 1,2,3,4,5$, and we have $H^{30}(F(5,6)) \cong \mathbb{Z}_2$, it is clear that $\mathrm{cup}(F(5,6)) \geq 18$. Then we apply Lemma 3.1.2 to the last fiber bundle and obtain $\mathrm{cup}(F(1,2,18)) \geq 53$ (in 3.2.4, we shall see that $\mathrm{cup}(F(1,2,18)) = 53$). Hence from the penultimate fiber bundle we obtain $\mathrm{cup}(F(1,2,3,15)) \geq 91$ etc. Eventually, we arrive at $\mathrm{cup}(F(1,2,3,4,5,6)) \geq 136$. Since $\delta(1,2,3,4,5,6) = 175$, we have $136 \leq \mathrm{cup}(F(1,2,3,4,5,6)) \leq 175$.

It is natural to try to find (if possible) some upper bound smaller than that given by the dimension. We first derive the following result on upper bounds, and then come back to the above example.

PROPOSITION 3.2.2. *Let* $h(i) := \mathrm{height}(w_1(\gamma_i))$, $i = 1,\dots,q-1$, *and let* $S := h(1) + \dots + h(q-1)$. *If* $S < \delta(n_1,\dots,n_q)$, *then*

$$\mathrm{cup}(F(n_1,\dots,n_q)) \leq S + \left[\frac{\delta(n_1,\dots,n_q) - S}{2}\right] < \delta(n_1,\dots,n_q).$$

In particular, a necessary condition for $\mathrm{cup}(F(n_1,\ldots,n_q)) = \delta(n_1,\ldots,n_q)$ *is* $S \geq \delta(n_1,\ldots,n_q)$.

Proof. We know the structure of the cohomology ring $H^*(F(n_1,\ldots,n_q))$. The fact that the top nonzero cohomology group,

$$H^{\delta(n_1,\ldots,n_q)}(F(n_1,\ldots,n_q)),$$

is $\mathbb{Z}_2$ implies (cf. Horanská and Korbaš [4, p. 26]) that any cup-product of maximum length in $H^*(F(n_1,\ldots,n_q))$ can be expressed as a *monomial* in the Stiefel–Whitney classes of the canonical vector bundles $\gamma_1,\ldots,\gamma_{q-1}$.

Suppose now that $S < \delta(n_1,\ldots,n_q)$ and put $k = \delta(n_1,\ldots,n_q) - S$. Then no *candidates* for nonzero cup-products are longer than

$$w_1^{h(1)}(\gamma_1)\ldots w_1^{h(q-1)}(\gamma_{q-1})w_2^{k/2}(\gamma_j)$$

if k is even, or

$$w_1^{h(1)}(\gamma_1)\ldots w_1^{h(q-1)}(\gamma_{q-1})w_2^{(k-3)/2}(\gamma_j)w_3(\gamma_t)$$

if k is odd, where $j,t \in \{1,\ldots,q-1\}$. Hence then

$$\mathrm{cup}(F(n_1,\ldots,n_q)) \leq S + \left[\frac{\delta(n_1,\ldots,n_q) - S}{2}\right],$$

and the proof is complete.

Remark (b). In general, if we just know that $S \geq \delta(n_1,\ldots,n_q)$, we cannot claim that $\mathrm{cup}(F(n_1,\ldots,n_q)) = \delta(n_1,\ldots,n_q)$. Indeed, for the latter we would need $w_1^{s_1}(\gamma_1)\ldots w_1^{s_{q-1}}(\gamma_{q-1}) \neq 0$ for some $s_i \leq h(i)$ $(i = 1,\ldots$ $\ldots,q-1)$ such that $s_1 + \ldots + s_{q-1} = \delta(n_1,\ldots,n_q)$. A procedure for this will be described in 3.3.

Remark (c). If the condition $j \geq 2^{t+d} - m - 2d + 1$ of Theorem 3.1.3 is not satisfied, let k be the maximum nonnegative integer such that $j < 2^{t+d-k} - m - 2d + 1$. Using 3.2.2, one can verify that if $\binom{j}{2} < 2d^2 + 2md - d\cdot 2^{t+d-k}$, then

$$\mathrm{cup}(F(1^{\cdots j}, 2^{\cdots d}, m)) < \delta(1^{\cdots j}, 2^{\cdots d}, m);$$

one has $S < \delta$ in these cases.

We do not know what is $\mathrm{cat}(F(1^{\cdots j}, 2^{\cdots d}, m))$ if $S < \delta$. We are just able to prove, using Korbaš's [7, 1.1] and Koschorke's [9, 3.10], that if λ_i is the orientation bundle of the canonical vector bundle γ_i over $F(1^{\cdots j}, 2^{\cdots d}, m)$ with $S < \delta$, then the δ-multiple, $\delta\lambda_1 \otimes \ldots \otimes \lambda_{j+d}$, of the line bundle $\lambda_1 \otimes \ldots \otimes \lambda_{j+d}$ over $F(1^{\cdots j}, 2^{\cdots d}, m)$ has a nowhere vanishing cross-section. As a consequence of Korbaš and Szűcs's [8, 1.1], the latter is a geometric necessary condition for

$$\mathrm{cat}(F(1^{\cdots j}, 2^{\cdots d}, m)) \leq \delta.$$

We believe that the following may be a reasonable general conjecture.

Conjecture. *For the flag manifolds* $F(n_1,\dots,n_q)$, *let* λ_i *be the orientation bundle of the canonical vector bundle* γ_i. *Then* $\operatorname{cat}(F(n_1,\dots,n_q)) \le \delta(n_1,\dots,n_q))$ *if and only if the vector bundle* $\delta\lambda_1 \otimes \dots \otimes \lambda_q$ *has a nowhere vanishing cross-section.* (We observe that, in view of Korbaš and Szűcs's [8, 1.1], "only" the if-part of our conjecture remains to be proved (or disproved).)

Using Proposition 3.2.2, we now improve the upper bound given in Example 3.2.1.

Example 3.2.3. Proposition 3.2.2 implies that $\operatorname{cup}(F(5,6)) \le 22$; using results of [14], one readily verifies that $w_1^{14}(\gamma_1)w_2^8(\gamma_1)$ is nonzero, hence we have $\operatorname{cup}(F(5,6)) = 22$. In addition, again applying 3.2.2, we now obtain $\delta(1,2,3,4,5,6) - S = 175 - 140 = 35$, hence

$$140 \le \operatorname{cup}(F(1,2,3,4,5,6)) \le 157.$$

It turns out that (at least in some cases) the upper bounds given by 3.2.2 are very good: for instance, they allow us to compute the exact value of $\operatorname{cup}(F(1,2,n_3))$ for any $n_3 \ge 3$.

Proposition 3.2.4. *For any integer* $n_3 \ge 3$, *let* s *be the only integer such that* $2^s \le n_3 < 2^{s+1}$. *Then*

$$\operatorname{cup}(F(1,2,n_3)) = \begin{cases} 3n_3 + 2 \ (= \delta(1,2,n_3)) & \text{if } n_3 = 2^{s+1} - 1 \text{ or} \\ & \text{if } n_3 = 2^{s+1} - 2, \\ 2^s + 2n_3 + 1 & \text{otherwise.} \end{cases}$$

As a consequence, $\operatorname{cat}(F(1,2,n_3)) = 1 + \delta(1,2,n_3)$ *if* $n_3 = 2^{s+1} - 1$ *or if* $n_3 = 2^{s+1} - 2$, *and* $\operatorname{cat}(F(1,2,n_3)) \ge 2^s + 2n_3 + 2$ *if* $2^s \le n_3 \le 2^{s+1} - 3$.

Proof. We just prove the result on the cup-length; the result on the category is then obvious. If $n_3 = 2^{s+1} - 1$ or $n_3 = 2^{s+1} - 2$, then $F(1,2,n_3)$ satisfies the assumptions of Theorem 3.1.3(b), hence $\operatorname{cup}(F(1,2,n_3)) = \delta(1,2,n_3)$ in these cases. In the remaining cases we have $2^s \le n_3 \le 2^{s+1} - 3$. To obtain a lower bound for $\operatorname{cup}(F(1,2,n_3))$, we apply Lemma 3.1.2 to the fiber bundle

$$\begin{array}{ccc} F(2,n_3) & \hookrightarrow & F(1,2,n_3) \\ & & \downarrow \\ & & F(1,2+n_3) \end{array}$$

One has $\operatorname{cup}(F(1,2+n_3)) = 2 + n_3$. In addition, as a consequence of Hiller's results [3] (see Remark (e) below), we have $\operatorname{cup}(F(2,n_3)) = n_3 + 2^s - 1$ if $2^s \le n_3 \le 2^{s+1} - 3$. Therefore $\operatorname{cup}(F(1,2,n_3)) \ge 2 + n_3 + n_3 + 2^s - 1 = 2n_3 + 2^s + 1$ if $2^s \le n_3 \le 2^{s+1} - 3$.

To obtain upper bounds, we shall use Proposition 3.2.2. One has $S = h(1) + h(2) = 2 + n_3 + 2^{s+1} - 2 = n_3 + 2^{s+1}$ if $2^s \le n_3 \le 2^{s+1} - 3$.

Hence (by 3.2.2) we obtain

$$\mathrm{cup}(F(1,2,n_3)) \le n_3 + 2^{s+1} + \frac{3n_3 + 2 - n_3 - 2^{s+1}}{2} = 2n_3 + 2^s + 1$$

if $2^s \le n_3 \le 2^{s+1} - 3$. We see that our upper and lower bounds for $\mathrm{cup}(F(1,2,n_3))$ coincide, which finishes the proof.

REMARK (d). One readily verifies that the manifold $F(1,2,n_3)$ considered in 3.2.4 has its cup-length equal to its dimension precisely when the condition $j \ge 2^{t+d} - m - 2d + 1$ from Theorem 3.1.3(b) is satisfied.

REMARK (e). Due to Hiller (cf. [3] or [4]), it is known that if $n \ge 4$, $2^s < n \le 2^{s+1}$, then $w_1^{2^{s+1}-2}(\gamma_1) w_2^{n-2^s-1}(\gamma_1) \in H^{2(n-2)}(F(2,n-2))$ realizes the cup-length of $F(2,n-2)$. Of course, for $F(1,n-1)$, the cohomology class $w_1^{n-1}(\gamma_1)$ realizes its cup-length. Then the method of proof of Theorem 3.1.3 together with the Leray–Hirsch theorem (see the proof of Lemma 3.1.2 in [4]) enables one to write down a monomial in the cohomology of any of those flag manifolds covered by 3.1.3 which realizes its cup-length. A similar observation applies to the manifolds $F(1,2,n_3)$ of Proposition 3.2.4.

REMARK (f). For the flag manifold $F(n_1,\ldots,n_q)$ one has its universal covering (consisting of oriented flags)

$$\widetilde{F}(n_1,\ldots,n_q) = SO(n_1 + \ldots + n_q)/SO(n_1) \times \ldots \times SO(n_q).$$

Then clearly (cf. [6] if needed) $\mathrm{cat}(\widetilde{F}(n_1,\ldots,n_q)) \le \mathrm{cat}(F(n_1,\ldots,n_q))$, and therefore any lower bound for $\mathrm{cat}(\widetilde{F}(n_1,\ldots,n_q))$ is also a lower bound for $\mathrm{cat}(F(n_1,\ldots,n_q))$. If $SO(n_1) \times \ldots \times SO(n_q)$ is a maximal torus in the group $SO(n_1 + \ldots + n_q)$, then the Lyusternik–Shnirel'man category $\mathrm{cat}(\widetilde{F}(n_1,\ldots,n_q))$ can be calculated. Namely, from [12, Theorem 2], one knows that

$$\mathrm{cat}(G/T) = \tfrac{1}{2}(\dim(G) - \mathrm{rank}(G)) + 1$$

if G is a compact connected Lie group and T is a maximal torus of G. Applying this, we obtain

$$\mathrm{cat}(\widetilde{F}(1,2^{\cdots n})) = n^2 + 1, \tag{2}$$

$$\mathrm{cat}(\widetilde{F}(2^{\cdots n})) = n^2 - n + 1. \tag{3}$$

We note that the same result can be obtained in another way: the lower bound implied by the ($\mathbb{Z}$-cohomology) cup-length of $\widetilde{F}(1,2^{\cdots n})$ or $\widetilde{F}(2^{\cdots n})$ (these cup-lengths can readily be found using the results of T. Watanabe [15]) coincides with the Grossman upper bound (see e.g. James [6, 5.1]).

As a consequence of (2) and (3), we have

$$\operatorname{cat}(F(1, 2^{\cdots n})) \geq n^2 + 1, \tag{4}$$

$$\operatorname{cat}(F(2^{\cdots n})) \geq n^2 - n + 1. \tag{5}$$

It is possible to show that the lower bounds given in (4) and (5) are in fact worse than the lower bounds which we can derive from suitable fiberings, using Lemma 3.1.2 (we illustrated such a procedure in 3.2.1).

3.3. *An "easy" way to find* $\operatorname{cup}(F(n_1, \ldots, n_q))$. We adapt to the case of real flag manifolds the approach which R. Stong [14] used for the special case of the Grassmann manifolds.

For the flag manifold $F(1^{\cdots m})$, we define $e_i := w_1(\gamma_i)$. From Borel's description (cf. 2.1) of the cohomology algebra $H^*(F(1^{\cdots m}))$, one can derive (or find in [14]) that the nonzero monomials in the top dimension, hence in $H^{\binom{m}{2}}(F(1^{\cdots m}))$, are precisely those of the form $e_{\sigma(1)}^{m-1} \ldots e_{\sigma(i)}^{m-i} \ldots e_{\sigma(m)}^{0}$ for any permutation σ of the set $\{1, \ldots, m\}$, hence the monomials with no repeated exponents.

For the flag manifold $F(n_1, \ldots, n_q)$, we put $\nu_j := n_1 + \ldots + n_j$ for $j = 0, \ldots, q$ (in particular, $\nu_0 = 0$ and $\nu_q = n$). The map

$$p : F(1^{\cdots n_1}, \ldots, 1^{\cdots n_q}) \to F(n_1, \ldots, n_q),$$
$$p(S_1, \ldots, S_{n_1}, \ldots, S_{\nu_{q-1}+1}, \ldots, S_n) = (S_1 \oplus \ldots \oplus S_{n_1}, \ldots, S_{\nu_{q-1}+1} \oplus \ldots \oplus S_n),$$

is obviously the projection of the corresponding fiber bundle with fiber

$$F(1^{\cdots n_1}) \times \ldots \times F(1^{\cdots n_q}).$$

The Leray–Hirsch theorem now applies; therefore the induced homomorphism $p^* : H^*(F(n_1, \ldots, n_q)) \to H^*(F(1^{\cdots n_1}, \ldots, 1^{\cdots n_q}))$ is injective and the cohomology algebra $H^*(F(1^{\cdots n_1}, \ldots, 1^{\cdots n_q}))$ is a free module over $H^*(F(n_1, \ldots, n_q))$, via p^*, with the obvious basis. In particular, since the top class of the basis is

$$e_1^{n_1-1} e_2^{n_1-2} \ldots e_{\nu_1-1} e_{\nu_1+1}^{n_2-1} e_{\nu_1+2}^{n_2-2} \ldots e_{\nu_2-1} \ldots e_{\nu_{q-1}+1}^{n_q-1} e_{\nu_{q-1}+2}^{n_q-2} \ldots e_{\nu_q-1},$$

we have the following generalization of Stong's [14, Observation, p. 106].

Observation. The value of $u \in H^{\mathrm{top}}(F(n_1, \ldots, n_q))$ on the fundamental class of $F(n_1, \ldots, n_q)$ (briefly: the value of u) is the same as the value of

$$p^*(u) \cdot e_1^{n_1-1} e_2^{n_1-2} \ldots e_{\nu_1-1} e_{\nu_1+1}^{n_2-1} e_{\nu_1+2}^{n_2-2} \ldots e_{\nu_2-1} \ldots e_{\nu_{q-1}+1}^{n_q-1} e_{\nu_{q-1}+2}^{n_q-2} \ldots e_{\nu_q-1}$$

on the fundamental class of $F(1^{\cdots n_1}, \ldots, 1^{\cdots n_q})$ (note that the latter is, in theory, always easily calculable, because we know when a monomial in $e_1, \ldots, e_n$ in $H^{\binom{n}{2}}(F(1^{\cdots n_1}, \ldots, 1^{\cdots n_q}))$ vanishes and when it is nonzero).

In the use of the Observation, it is important to keep in mind (see Borel's description of $H^*(F(n_1,\dots,n_q))$ in 2.1) that

• the class u can always be expressed in terms of the Stiefel–Whitney classes of just $q-1$ of the canonical vector bundles, e.g. $\gamma_1,\dots,\gamma_{q-1}$;

• the pull-back $p^*(\gamma_i)$ splits as the Whitney sum of line bundles, $p^*(\gamma_i) = \gamma_{\nu_{i-1}+1}\oplus\gamma_{\nu_{i-1}+2}\oplus\ldots\oplus\gamma_{\nu_i}$, for $i=1,\dots,q$;

• as a consequence, $p^*(w_j(\gamma_i))$ (where $j=1,\dots,n_i$, $i=1,\dots,q$) is the jth elementary symmetric function in $e_{\nu_{i-1}+1}, e_{\nu_{i-1}+2},\dots,e_{\nu_i}$.

The inclusion $a:\mathbb{R}^{n-1}\hookrightarrow\mathbb{R}^n$, $a(x_1,\dots,x_{n-1}) = (x_1,\dots,x_{n-1},0)$, induces inclusions

$$a_i : F(n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_q)\hookrightarrow F(n_1,\dots,n_{i-1},n_i,n_{i+1},\dots,n_q)$$

$(i=1,\dots,q)$ such that $a_i^*(\gamma_t)=\gamma_t$ if $t\neq i$ and $a_i^*(\gamma_i)=\gamma_i\oplus\varepsilon^1$ (as a consequence, the classes $a_i^*(w(\gamma_t))$ and $w(\gamma_t)$ are the same for $t=1,\dots,q$). Now we have the following generalization of Stong's [14, Lemma 1, p. 107]; it is useful in applications.

Lemma 3.3.1. *If $x\in H^{\delta(n_1,\dots,n_i,\dots,n_q)-n+n_i}(F(n_1,\dots,n_i,\dots,n_q))$, then the value of $a_i^*(x)$ in $H^{\mathrm{top}}(F(n_1,\dots,n_i-1,\dots,n_q))$ is the same as the value of*

$$x\cdot w_{n_1}(\gamma_1)\ldots w_{n_{i-1}}(\gamma_{i-1})\cdot w_{n_{i+1}}(\gamma_{i+1})\ldots w_{n_q}(\gamma_q)$$

in $H^{\delta(n_1,\dots,n_i,\dots,n_q)}(F(n_1,\dots,n_i,\dots,n_q))$.

Proof. Since, for fixed $(n_1,\dots,n_q)$ and any permutation σ of the set $\{1,\dots,q\}$, the manifolds $F(n_1,\dots,n_q)$ and $F(n_{\sigma(1)},\dots,n_{\sigma(q)})$ are diffeomorphic, it is enough to prove the lemma for one value of i. So we shall take $i=q$ in the rest of the proof. The value of $a_q^*(x)$ (cf. the Observation) is

$$\begin{aligned}\varDelta_{n-1} := p^*(a_q^*(x))\cdot e_1^{n_1-1}e_2^{n_1-2}&\ldots e_{\nu_1-1}\\ &\times e_{\nu_1+1}^{n_2-1}e_{\nu_1+2}^{n_2-2}\ldots e_{\nu_1+n_2-1}\ldots e_{\nu_{q-2}+1}^{n_{q-1}-1}\\ &\times e_{\nu_{q-2}+2}^{n_{q-1}-2}\ldots e_{\nu_{q-2}+n_{q-1}-1}e_{\nu_{q-1}+1}^{n_q-2}e_{\nu_{q-1}+2}^{n_q-3}\ldots e_{n-2}.\end{aligned}$$

Since $a_q^*(x)$ can be expressed using just the Stiefel–Whitney classes of the vector bundles $\gamma_2,\dots,\gamma_q$, the term $p^*(a_q^*(x))$ can and will be understood as a function symmetric in each of the following sets of variables:

$$\{e_{\nu_1+1},\dots,e_{\nu_1+n_2}=e_{\nu_2}\},\dots,\{e_{\nu_{q-1}+1},\dots,e_{\nu_{q-1}+n_q-1}=e_{n-1}\}.$$

At the same time, the value of $x\cdot w_{n_1}(\gamma_1)w_{n_2}(\gamma_2)\ldots w_{n_{q-1}}(\gamma_{q-1})$ is

$$\begin{aligned}\varDelta_n := p^*(x)\cdot e_1e_2\ldots e_{\nu_1}e_{\nu_1+1}&\ldots e_{\nu_1+n_2}\ldots e_{\nu_{q-2}+1}\ldots e_{\nu_{q-2}+n_{q-1}}\\ \times e_1^{n_1-1}&\ldots e_{\nu_1-1}e_{\nu_1+1}^{n_2-1}\ldots e_{\nu_1+n_2-1}\ldots e_{\nu_{q-2}+1}^{n_{q-1}-1}\ldots e_{\nu_{q-2}+n_{q-1}-1}\\ \times e_{\nu_{q-1}+1}^{n_q-1}&e_{\nu_{q-1}+2}^{n_q-2}\ldots e_{\nu_{q-1}+n_q-1},\end{aligned}$$

that is,

$$p^*(x)\cdot e_1^{n_1}e_2^{n_1-1}\dots e_{\nu_1}e_{\nu_1+1}^{n_2}\dots e_{\nu_1+n_2}\dots e_{\nu_{q-2}+1}^{n_{q-1}}\dots e_{\nu_{q-2}+n_{q-1}} \times e_{\nu_{q-1}+1}^{n_q-1}e_{\nu_{q-1}+2}^{n_q-2}\dots e_{\nu_{q-1}+n_q-1}.$$

The term $p^*(x)$ is understood here as a function symmetric in each of the following sets of variables:

$$\{e_{\nu_1+1},\dots,e_{\nu_1+n_2}=e_{\nu_2}\},\dots,\{e_{\nu_{q-1}+1},\dots,e_{\nu_{q-1}+n_q}=e_n\}.$$

The factor $e_1e_2\dots e_{n-1}$, coming from

$$e_1^{n_1}e_2^{n_1-1}\dots e_{\nu_1}e_{\nu_1+1}^{n_2}\dots e_{\nu_1+n_2}\dots e_{\nu_{q-2}+1}^{n_{q-1}}\dots e_{\nu_{q-2}+n_{q-1}} \times e_{\nu_{q-1}+1}^{n_q-1}e_{\nu_{q-1}+2}^{n_q-2}\dots e_{n-1},$$

annihilates all those monomials in the expansion of $p^*(x)$ which contain e_n. Therefore Δ_n corresponds to

$$\overline{p}^*(a_q^*(x))\cdot e_1^{n_1}e_2^{n_1-1}\dots e_{\nu_1}e_{\nu_1+1}^{n_2}\dots e_{\nu_1+n_2}\dots e_{\nu_{q-2}+1}^{n_{q-1}}\dots e_{\nu_{q-2}+n_{q-1}} \times e_{\nu_{q-1}+1}^{n_q-1}e_{\nu_{q-1}+2}^{n_q-2}\dots e_{n-1},$$

where $\overline{p}^*(a_q^*(x))$ is "the same" as $p^*(a_q^*(x))$, except that it lies in the cohomology of $F(1^{\cdots n})$. That means that we have a 1-1 correspondence between the monomials in the expansion of Δ_{n-1}, and the monomials in the expansion of Δ_n, with each degree being raised by 1. The monomials with (no) repeated exponents in Δ_n are in 1-1 correspondence with the monomials with (no) repeated exponents in Δ_{n-1}. Hence (cf. the Observation) the value of $a_i^*(x)$ is (non)zero precisely when the value of $x\cdot w_{n_1}(\gamma_1)\dots w_{n_{q-1}}(\gamma_{q-1})$ is (non)zero. This finishes the proof.

Now we describe how to calculate the cup-length of any real flag manifold, using what we said above (in particular, the Observation).

PROCEDURE 3.3.2. For any $F(n_1,\dots,n_q)$,

(1) one calculates the numbers $h(1),\dots,h(q-1)$ (see 3.2.2);

(2) one constructs all possible (always finitely many) monomials in

$$H^{\delta(n_1,\dots,n_q)}(F(n_1,\dots,n_q))$$

of the form $w_1^{s_1}(\gamma_1)\dots w_1^{s_{q-1}}(\gamma_{q-1})$ times a monomial in the Stiefel–Whitney classes higher than the first, where $s_i\le h(i)$ $(i=1,\dots,q-1)$;

(3) using the Observation, one "easily" decides, about each of the monomials constructed in (2), whether or not it vanishes;

(4) one finds the maximum length of the nonzero monomials from (3), which is then $\mathrm{cup}(F(n_1,\dots,n_q))$.

Of course, one could try to calculate $\mathrm{cup}(F(n_1,\dots,n_q))$ using just the explicit description of the algebra $H^*(F(n_1,\dots,n_q))$ (see 2.1); this approach may be called a frontal attack. But calculations in $H^*(F(n_1,\dots,n_q))$ turn out to be extremely difficult. It is the main purpose of Procedure 3.3.2 to make the calculation of $\mathrm{cup}(F(n_1,\dots,n_q))$ more manageable, even if not really easy; because of our doubts about the easiness, we put quotation marks round the word easy in the title of the present section and in 3.3.2(3).

The realization of Procedure 3.3.2 can sometimes be facilitated, for instance, by the use of previously known facts or by suitable ad hoc ideas. We illustrate this with the following.

EXAMPLE 3.3.3. To calculate $\mathrm{cup}(F(1,2,2,2))$, first observe that $S = h(1)+h(2)+h(3) = 18 = \delta(1,2,2,2)$. Now we use the Observation to decide whether or not $w_1^6(\gamma_1)w_1^6(\gamma_2)w_1^6(\gamma_3)$ vanishes. For this we calculate, in the cohomology of $F(1,1,1,1,1,1,1)$,

$$p^*(w_1^6(\gamma_1)w_1^6(\gamma_2)w_1^6(\gamma_3))\cdot e_2e_4e_6,$$

hence

$$e_1^6(e_2+e_3)^6(e_4+e_5)^6\cdot e_2e_4e_6.$$

One readily calculates that the latter is zero; therefore $\mathrm{cup}(F(1,2,2,2)) < 18$.

Further one can use Lemma 3.3.1. We have the inclusion

$$a_2 : F(1,1,2,2) \hookrightarrow F(1,2,2,2)$$

and we know, from Theorem 3.1.3, that $\mathrm{cup}(F(1,1,2,2)) = \delta(1,1,2,2) = 13$. Using the proof of 3.1.3(b) and the Leray–Hirsch theorem, one sees that the cup-length of $F(1,1,2,2)$ can be realized by $w_1^2(\gamma_1)w_1^5(\gamma_2)w_1^6(\gamma_3)$.

From 3.3.1, it follows that

$$w_1^2(\gamma_1)w_1^5(\gamma_2)w_1^6(\gamma_3)w_1(\gamma_1)w_2(\gamma_3)w_2(\gamma_4)$$

is nonzero in the top cohomology group of $F(1,2,2,2)$. Expressing $w_2(\gamma_4)$ in terms of the Stiefel–Whitney classes of γ_1, γ_2, and γ_3 (using the fact that the sum of all the canonical bundles is trivial), we deduce that

$$\begin{aligned} &w_1^5(\gamma_1)w_1^5(\gamma_2)w_1^6(\gamma_3)w_2(\gamma_3) + w_1^4(\gamma_1)w_1^6(\gamma_2)w_1^6(\gamma_3)w_2(\gamma_3)\\ &\quad + w_1^3(\gamma_1)w_1^5(\gamma_2)w_1^6(\gamma_3)w_2^2(\gamma_3) + w_1^3(\gamma_1)w_1^5(\gamma_2)w_1^6(\gamma_3)w_2(\gamma_2)w_2(\gamma_3)\end{aligned}$$

is nonzero. Of course, this already implies that $\mathrm{cup}(F(1,2,2,2)) \ge 16$. Now we start testing the monomials. If we take

$$w_1^4(\gamma_1)w_1^6(\gamma_2)w_1^6(\gamma_3)w_2(\gamma_3)$$

and calculate its value using the Observation, we see that $e_1^4(e_2+e_3)^6 \times (e_4+e_5)^6e_4e_5e_2e_4e_6 \neq 0$, and therefore

$$\mathrm{cup}(F(1,2,2,2)) = 17, \qquad 18 \le \mathrm{cat}(F(1,2,2,2)) \le 19.$$

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References

[1] I. Berstein, *On the Lusternik–Schnirelmann category of Grassmannians*, Math. Proc. Cambridge Philos. Soc. 79 (1976), 129–134.

[2] A. Borel, *La cohomologie mod 2 de certains espaces homogènes*, Comment. Math. Helv. 27 (1953), 165–197.

[3] H. L. Hiller, *On the cohomology of real grassmanians*, Trans. Amer. Math. Soc. 257 (1980), 521–533.

[4] L'. Horanská and J. Korbaš, *On cup products in some manifolds*, Bull. Belg. Math. Soc. Simon Stevin 7 (2000), 21–28.

[5] S. A. Ilori and D. O. Ajayi, *The height of the first Stiefel–Whitney class of the real flag manifolds*, Indian J. Pure Appl. Math. 36 (2000), 621–624.

[6] I. M. James, *On category, in the sense of Lusternik–Schnirelmann*, Topology 17 (1978), 331–348.

[7] J. Korbaš, *Vector fields on real flag manifolds*, Ann. Global Anal. Geom. 3 (1985), 173–184.

[8] J. Korbaš and A. Szűcs, *The Lyusternik–Shnirel'man category*, *vector bundles*, *and immersions of manifolds*, Manuscripta Math. 95 (1998), 289–294.

[9] U. Koschorke, *Vector Fields and Other Vector Bundle Morphisms—A Singularity Approach*, Lecture Notes in Math. 847, Springer, Berlin, 1981.

[10] J. Lörinc, *The height of the first Stiefel–Whitney class of any nonorientable real flag manifold*, Math. Slovaca 53 (2003), 91–95.

[11] J. Milnor and J. Stasheff, *Characteristic Classes*, Ann. of Math. Stud. 76, Princeton Univ. Press, Princeton, 1974.

[12] W. Singhof, *On the Lusternik–Schnirelmann category of Lie groups*, Math. Z. 145 (1975), 111–116.

[13] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton, 1951.

[14] R. E. Stong, *Cup products in Grassmannians*, Topology Appl. 13 (1982), 103–113.

[15] T. Watanabe, *On cup products in the cohomology of a flag manifold*, *Part I*, preprint, Osaka Univ., September 1999.

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