Universal acyclic resolutions for arbitrary coefficient groups

by

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Abstract. We prove that for every compactum $X$ and every integer $n \geq 2$ there are a compactum $Z$ of dimension $\leq n + 1$ and a surjective $UV^{n-1}$-map $r : Z \to X$ such that for every abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

1. Introduction. This paper is devoted to proving the following theorem which was announced in [8].

THEOREM 1.1. Let $X$ be a compactum. Then for every integer $n \geq 2$ there are a compactum $Z$ of dimension $\leq n + 1$ and a surjective $UV^{n-1}$-map $r : Z \to X$ having the property that for every abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

The cohomological dimension $\dim_G X$ of $X$ with respect to an abelian group $G$ is the least number $n$ such that $H^{n+1}(X, A; G) = 0$ for every closed subset $A$ of $X$. A space is $G$-acyclic if its reduced Čech cohomology groups modulo $G$ are trivial; a map is $G$-acyclic if every fiber is $G$-acyclic. By the Vietoris–Begle theorem a surjective $G$-acyclic map of compacta cannot raise the cohomological dimension $\dim_G$. A compactum $X$ is approximately $n$-connected if any embedding of $X$ into an ANR has the $UV^n$-property, i.e. for every neighborhood $U$ of $X$ there is a smaller neighborhood $X \subset V \subset U$ such that the inclusion $V \subset U$ induces the zero homomorphism of the homotopy groups in dimensions $\leq n$. An approximately $n$-connected compactum has trivial reduced Čech cohomology groups in dimensions $\leq n$ with respect to any group $G$. A map is called a $UV^n$-map if every fiber is approximately $n$-connected.

Theorem 1.1 generalizes the following results of [6, 7].

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Theorem 1.2 ([6]). Let $G$ be an abelian group and let $X$ be a compactum with $\dim_G X \leq n$, $n \geq 2$. Then there are a compactum $Z$ with $\dim_G Z \leq n$ and $\dim Z \leq n + 1$ and a $G$-acyclic map $r : Z \to X$ from $Z$ onto $X$.

Theorem 1.3 ([7]). Let $X$ be a compactum with $\dim_Z X \leq n \geq 2$. Then there exist a compactum $Z$ with $\dim Z \leq n$ and a cell-like map $r : Z \to X$ from $Z$ onto $X$ such for every integer $k \geq 2$ and every group $G$ such that $\dim_G X \leq k$ we have $\dim_G Z \leq k$.

Theorem 1.2 obviously follows from Theorem 1.1. Theorem 1.3 can be derived from Theorem 1.1 as follows. Recall that a compactum is cell-like if any map from the compactum to a CW-complex is null-homotopic. A map is cell-like if its fibers are cell-like. Let $X$ have $\dim_Z X < 1$ and let $r : Z \to X$ satisfy the conclusions of Theorem 1.1 for $n = \dim_Z X + 1$. Then $\dim_Z Z \leq \dim_Z X \leq n - 1$ and because $Z$ is finite-dimensional we have $\dim Z = \dim_Z Z \leq n - 1$. Since $r$ is $UV^{n-1}$ and $\dim Z \leq n - 1$ we find that $r$ is cell-like. Let $\dim_G X \leq k \geq 2$ for a group $G$. If $k \leq n$ then $\dim_G Z \leq k$ by Theorem 1.1, and if $k > n$ then $\dim_G Z \leq k$ since $\dim Z \leq n - 1$. Thus Theorem 1.1 implies Theorem 1.3.

It was observed in [7] that the restriction $k \geq 2$ in Theorem 1.3 cannot be omitted. Therefore Theorem 1.1 does not hold for $k = 1$.

Let us discuss possible generalizations of Theorem 1.1. One is tempted to reduce the dimension of $Z$ to $n$. This is partially justified by

Theorem 1.4 ([8]). Let $X$ be a compactum. Then for every integer $n \geq 2$ there are a compactum $Z$ of dimension $\leq n$ and a surjective $UV^{n-1}$-map $r : Z \to X$ having the property that for every finitely generated abelian group $G$ and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and $r$ is $G$-acyclic.

However, Theorem 1.4 does not hold for arbitrary groups $G$. Indeed, one can show that a $\mathbb{Q}$-acyclic $UV^1$-map from a compactum of dimension $\leq 2$ must be $\mathbb{Z}$-acyclic (even cell-like). Thus a compactum $X$ with $\dim_Z X = 3$ and $\dim_{\mathbb{Q}} Z = 2$ cannot be the image of a compactum of dimension $\leq 2$ under a $\mathbb{Q}$-acyclic $UV^1$-map.

The situation becomes more complicated if we drop in Theorem 1.1 the requirement that $r$ is $UV^{n-1}$ and consider

Problem 1.5. Given a compactum $X$, an integer $n \geq 2$ and a collection $\mathcal{G}$ of abelian groups such that $\dim_G X \leq n$ for every $G \in \mathcal{G}$, do there exist a compactum $Z$ of dimension $\leq n$ and a $\mathcal{G}$-acyclic surjective map $r : Z \to X$ such that $\dim_G Z \leq \max\{\dim_G X, 2\}$ for every $G \in \mathcal{G}$? (The $\mathcal{G}$-acyclicity means the $G$-acyclicity for every $G \in \mathcal{G}$.)

In general the answer to Problem 1.5 is negative [5]. Indeed, let $X$ be a compactum with $\dim_Z X = 3$, $\dim_{\mathbb{Q}} X = 2$ and $\dim_{\mathbb{Z}_p} X = 2$ for every
prime \( p \) and let \( G = \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z}) \). Clearly \( \dim_G X = 2 \) and the \( G \)-acyclicity implies both the \( \mathbb{Q} \)- and \( (\mathbb{Q}/\mathbb{Z}) \)-acyclicity. Then it follows from the Bockstein sequence generated by

\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
\]

that the \( G \)-acyclicity implies the \( \mathbb{Z} \)-acyclicity and therefore there is no \( G \)-acyclic resolution for \( X \) from a compactum of dimension \( \leq 2 \).

The situation described in the example can be interpreted in terms of Bockstein theory. Let \( G \) be a collection of abelian groups. Denote by \( \sigma(G) \) the union of the Bockstein bases \( \sigma(G) \) of all \( G \in G \). Based on the Bockstein inequalities define the closure \( \overline{\sigma(G)} \) of \( \sigma(G) \) as a collection of abelian groups containing \( \sigma(G) \) and possibly some additional groups determined by:

- \( \mathbb{Z}_p \in \overline{\sigma(G)} \) if \( \mathbb{Z}_p^{\infty} \in \sigma(G) \);
- \( \mathbb{Z}_p^{\infty} \in \overline{\sigma(G)} \) if \( \mathbb{Z}_p \in \sigma(G) \);
- \( \mathbb{Q} \in \overline{\sigma(G)} \) if \( Z_{(p)} \in \sigma(G) \);
- \( \mathbb{Z}_{(p)} \in \overline{\sigma(G)} \) if \( \mathbb{Q}, \mathbb{Z}_p^{\infty} \in \sigma(G) \).

One can show that for compact metric spaces the \( G \)-acyclicity implies the \( \mathbb{Q} \)-acyclicity. This motivates the following

**Conjecture 1.6.** Problem 1.5 can be answered positively if \( \dim_E X \leq n \) for every \( E \in \sigma(G) \).

The key open case of this conjecture seems to be constructing a \( \mathbb{Q} \)-acyclic resolution \( r : Z \to X \) for a compactum \( X \) with \( \dim_{\mathbb{Q}} X \leq n, n \geq 2 \), from a compactum \( Z \) of dimension \( \leq n \).

2. Preliminaries. All groups are assumed to be abelian, and functions between groups are homomorphisms. \( \mathcal{P} \) stands for the set of primes. For a non-empty subset \( \mathcal{A} \) of \( \mathcal{P} \) let

\[
S(\mathcal{A}) = \{p_1^{n_1}p_2^{n_2} \ldots p_k^{n_k} : p_i \in \mathcal{A}, n_i \geq 0\}
\]

be the set of positive integers with prime factors from \( \mathcal{A} \), and for the empty set define \( S(\emptyset) = \{1\} \). Let \( G \) be a group and \( g \in G \). We say that \( g \) is \( \mathcal{A} \)-torsion if there is \( n \in S(\mathcal{A}) \) such that \( ng = 0 \), and \( g \) is \( \mathcal{A} \)-divisible if for every \( n \in S(\mathcal{A}) \) there is \( h \in G \) such that \( nh = g \). \( \text{Tor}_\mathcal{A} G \) is the subgroup of \( \mathcal{A} \)-torsion elements of \( G \). The group \( G \) is \( \mathcal{A} \)-torsion if \( G = \text{Tor}_\mathcal{A} G \); \( G \) is \( \mathcal{A} \)-torsion free if \( \text{Tor}_\mathcal{A} G = 0 \); and \( G \) is \( \mathcal{A} \)-divisible if every element of \( G \) is \( \mathcal{A} \)-divisible.

\( G \) is \( \mathcal{A} \)-local if it is \( (\mathcal{P} \setminus \mathcal{A}) \)-divisible and \( (\mathcal{P} \setminus \mathcal{A}) \)-torsion free. The \( \mathcal{A} \)-localization of \( G \) is the homomorphism \( G \to G \otimes \mathbb{Z}(\mathcal{A}) \) defined by \( g \to g \otimes 1 \), where \( \mathbb{Z}(\mathcal{A}) = \{n/m : n \in \mathbb{Z}, m \in S(\mathcal{P} \setminus \mathcal{A})\} \). \( G \) is \( \mathcal{A} \)-local if and only if the \( \mathcal{A} \)-localization of \( G \) is an isomorphism. A simply connected CW-complex is
A-local if its homotopy groups are A-local. A map between two simply connected CW-complexes is an A-localization if the induced homomorphisms of the homotopy and (reduced integral) homology groups are A-localizations.

The extensional dimension of a compactum $X$ is said not to exceed a CW-complex $K$, written $e$-$\dim X \leq K$, if for every closed subset $A$ of $X$ and every map $f : A \to K$ there is an extension of $f$ over $X$. It is well known that $\dim X \leq n$ is equivalent to $e$-$\dim X \leq \mathbb{S}^n$, and $\dim_G X \leq n$ is equivalent to $e$-$\dim X \leq K(G,n)$, where $K(G,n)$ is an Eilenberg–Mac Lane complex of type $(G,n)$.

A map between CW-complexes is said to be combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain. Let $M$ be a simplicial complex and let $M^{[k]}$ be the $k$-skeleton of $M$ (the union of all simplexes of $M$ of dimension $\leq k$). By a resolution of $M$ we mean a CW-complex $EW(M,k)$ and a combinatorial map $\omega : EW(M,k) \to M$ such that $\omega$ is 1-to-1 over $M^{[k]}$. Let $f : N \to K$ be a map of a subcomplex $N$ of $M$ into a CW-complex $K$. The resolution is said to be suitable for $f$ if $f \circ \omega_{\omega^{-1}(N)}$ extends to a map $f' : EW(M,k) \to K$. We call $f'$ a resolving map for $f$. The resolution is said to be suitable for a compactum $X$ if $e$-$\dim X \leq \omega^{-1}(\Delta)$ for every simplex $\Delta$ of $M$. Note that if $\omega : EW(M,k) \to M$ is a resolution suitable for $X$ then for every map $\phi : X \to M$ there is a map $\psi : X \to EW(M,k)$ such that $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta$ for every simplex $\Delta$ of $M$. We call $\psi$ a combinatorial lifting of $\phi$.

Let $M$ be a finite simplicial complex and let $f : N \to K$ be a cellular map from a subcomplex $N$ of $M$ to a CW-complex $K$ such that $M^{[k]} \subset N$. A standard way of constructing a resolution suitable for $f$ is described in [7]. Such a resolution $\omega : EW(M,k) \to M$ is called the standard resolution of $M$ for $f$ and it has the following properties:

- $EW(M,k)$ is $(k-1)$-connected if so are $M$ and $K$;
- $\omega$ is a surjective map and for every simplex $\Delta$ of $M$, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K$;
- for every subcomplex $T$ of $M$, $\omega|_{\omega^{-1}(T)} : EW(T,k) = \omega^{-1}(T) \to T$ is the standard resolution of $T$ for $f|_{N \cap T} : N \cap T \to K$.

Let $G$ be a group, let $\alpha : L \to M$ be a surjective combinatorial map of a CW-complex $L$ and a finite simplicial complex $M$, and let $n$ be a positive integer such that $\tilde{H}_i(\alpha^{-1}(\Delta);G) = 0$ for every $i < n$ and every simplex $\Delta$ of $M$. One can show by induction on the number of simplexes of $M$ using the Mayer–Vietoris sequence and the Five Lemma that $\alpha_* : \tilde{H}_i(L;G) \to \tilde{H}_i(M;G)$ is an isomorphism for $i < n$. We will refer to this fact as the combinatorial Vietoris–Begle theorem.

**Proposition 2.1** ([2]). Let $G$ be a group and $p \in \mathcal{P}$. The following conditions are equivalent:

...
• $G$ is $p$-divisible;
• $\text{Ext}(\mathbb{Z}_p\infty, G)$ is $p$-divisible;
• $\text{Ext}(\mathbb{Z}_p\infty, G) = 0$.

**Proposition 2.2.** Let $G$ be a group, let $2 \leq k \leq n$ be integers and let $\mathcal{F} \subset \mathcal{P}$ and $p \in \mathcal{P} \setminus \mathcal{F}$. Let $M$ be an $(n - 1)$-connected finite simplicial complex such that $H_n(M)$ is $\mathcal{F}$-torsion, and let $\omega : L = EW(M, k) \to M$ be the standard resolution of $M$ for a cellular map $f : N \to K(G, k)$ from a subcomplex $N$ of $M$ containing $M[k]$. Then $L$ is $(k - 1)$-connected and for every $1 \leq i \leq n - 1$:

(i) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ are $p$-torsion if $G = \mathbb{Z}_p$;

(ii) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ are $p$-torsion and $\pi_k(L)$ is $p$-divisible if $G = \mathbb{Z}_p\infty$;

(iii) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ are $q$-divisible and $\pi_i(L)$ is $q$-torsion free for every $q \in \mathcal{P}$, $q \neq p$ if $G = \mathbb{Z}_{(p)}$;

(iv) $\pi_i(L)$ and $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ are $q$-divisible and $\pi_i(L)$ is $q$-torsion free for every $q \in \mathcal{P}$ if $G = \mathbb{Q}$.

**Proof.** Recall that $\omega$ is a combinatorial surjective map, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K(G, k)$ for every simplex $\Delta$ of $M$, and $L$ is $(k - 1)$-connected because so are $M$ and $K(G, k)$. Since $M$ is $(n - 1)$-connected and $H_n(M)$ is $\mathcal{F}$-torsion we have $H_n(M; \mathbb{Q}) = 0$ and $H_n(M; \mathbb{Z}_q) = 0$, $H_n(M; \mathbb{Z}_{(q)}) = 0$ for $q \in \mathcal{P} \setminus \mathcal{F}$ and $H_n(M; \mathbb{Z}_q\infty) = 0$ for every $q \in \mathcal{P}$.

(i) By the generalized Hurewicz theorem $\tilde{H}_*(K(\mathbb{Z}_p, k))$ is $p$-torsion. Then $\tilde{H}_*(K(\mathbb{Z}_p, k); \mathbb{Q}) = 0$. Hence by the combinatorial Vietoris–Begle theorem $\tilde{H}_i(L; \mathbb{Q}) = 0$ for $i \leq n$ and therefore $\tilde{H}_i(L)$ is torsion for $i \leq n$.

Let $q \in \mathcal{P}$ and $q \neq p$ and $i \leq n - 1$. Note that $\tilde{H}_*(K(\mathbb{Z}_p, k); \mathbb{Z}_{(q)}) = 0$ and hence by the combinatorial Vietoris–Begle theorem $\tilde{H}_i(L; \mathbb{Z}_{(q)}) = 0$. Then $\tilde{H}_i(L) \otimes \mathbb{Z}_{(q)} = 0$. Thus $\tilde{H}_i(L)$ is torsion and $q$-torsion free and hence $\tilde{H}_i(L)$ is $p$-torsion.

Now let $q \in \mathcal{P} \setminus \mathcal{F}$ and $q \neq p$. Recall $H_n(M; \mathbb{Z}_{(q)}) = 0$. Then using the previous argument we conclude that $H_n(L)$ is $q$-torsion free and hence $H_n(L)$ is $(\mathcal{F} \cup \{p\})$-torsion.

By the generalized Hurewicz theorem $\pi_i(L)$ is $p$-torsion for $i \leq n - 1$ and $\pi_n(L)$ is $(\mathcal{F} \cup \{p\})$-torsion. Thus $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ is $p$-torsion and (i) follows.

(ii) The argument used in (i) applies to show that $\pi_i(L)$, $i \leq n - 1$, and $\pi_n(L)/\text{Tor}_\mathcal{F} \pi_n(L)$ are $p$-torsion. Note that $\pi_k(L) = H_k(L)$. We will show that $H_k(L)$ is $p$-divisible and this will imply (ii). Observe that $H_k(K(\mathbb{Z}_p\infty, k); \mathbb{Z}_p) = \mathbb{Z}_p\infty \otimes \mathbb{Z}_p = 0$. Then since $H_k(M; \mathbb{Z}_p) = 0$ the combi-
nal torial Vietoris–Begle theorem implies that $H_k(L; \mathbb{Z}_p) = 0$. Thus $H_k(L) \otimes \mathbb{Z}_p = 0$ and therefore $H_k(L)$ is $p$-divisible.

(iii) Since $\mathbb{Z}_{(p)}$ is $p$-local we deduce that $\tilde{H}_*(K(\mathbb{Z}_{(p)}; k))$ is $p$-local and therefore $\tilde{H}_* (K(\mathbb{Z}_{(p)}; k); \mathbb{Z}_q) = \tilde{H}_* (K(\mathbb{Z}_{(p)}; k); \mathbb{Z}_{q\infty}) = 0$ for every $q \in \mathcal{P}$, $q \neq p$.

Let $q \in \mathcal{P}$, $q \neq p$. Recall that $\tilde{H}_i (M; \mathbb{Z}_{q\infty}) = 0$ for $i \leq n$. Then by the combinatorial Vietoris–Begle theorem $\tilde{H}_i (L; \mathbb{Z}_{q\infty}) = 0$ for $i \leq n$. Hence by the universal coefficient theorem $\tilde{H}_i (L) \otimes \mathbb{Z}_{q\infty} = 0$ and $\tilde{H}_i (L) \otimes \mathbb{Z}_{q\infty} = 0$ for $i \leq n - 1$ and therefore $\tilde{H}_i (L)$ is $q$-torsion free and $q$-divisible for $i \leq n - 1$.

Let $q \in \mathcal{P}$, $q \neq p$ and $q \notin \mathcal{F}$. Recall that $H_n (M; \mathbb{Z}_q) = 0$. By the combinatorial Vietoris–Begle theorem $H_n (L; \mathbb{Z}_q) = 0$. Hence $H_n (L) \otimes \mathbb{Z}_q = 0$ and therefore $H_n (L)$ is $q$-divisible.

Let $q \in \mathcal{P}$, $q \neq p$ and $q \notin \mathcal{F}$. Then $H_n (M; \mathbb{Z}_{q\infty}) = 0$. By the combinatorial Vietoris–Begle theorem $H_n (L; \mathbb{Z}_{q\infty}) = 0$. Hence $H_n (L) \otimes \mathbb{Z}_{q\infty} = 0$ and therefore $H_n (L)/\text{Tor}_q H_n (L)$ is $q$-divisible.

Now using completion and localization theories [1] we will pass to the homotopy groups of $L$.

Let $q \in \mathcal{P}$. Define $\mathcal{A} = \mathcal{P} \setminus \{q\}$. Let $\alpha : L \to L_\alpha$ be an $\mathcal{A}$-localization of $L$. Recall that $\tilde{H}_i (L)$ is $q$-torsion free and $q$-divisible for $i \leq n - 1$. Then $\alpha$ induces an isomorphism of the groups $\tilde{H}_* (L)$ and $\tilde{H}_* (L_\alpha)$ in dimensions $\leq n - 1$. Hence by the Whitehead theorem, $\alpha$ induces an isomorphism of the homotopy groups in dimensions $\leq n - 1$ and therefore $\pi_i (L)$ is $\mathcal{A}$-local (that is, $q$-divisible and $q$-torsion free) for $i \leq n - 1$.

Let $q \in \mathcal{P}$, $q \neq p$ and $q \notin \mathcal{F}$. Let $\beta : L \to L_\beta$ be a $q$-completion of $L$. Then $\beta$ induces an isomorphism of $\tilde{H}_* (L; \mathbb{Z}_q)$ and $\tilde{H}_* (L_\beta; \mathbb{Z}_q)$; since $H_n (L; \mathbb{Z}_q) = 0$ we get $H_n (L; \mathbb{Z}_q) = 0$ and therefore $H_n (L_\beta)$ is $q$-divisible. Now since $\pi_i (L)$ is $q$-divisible and $q$-torsion free we have $\text{Hom} (\mathbb{Z}_{q\infty}, \pi_i (L)) = 0$, $i \leq n - 1$, and by Proposition 2.1, $\text{Ext} (\mathbb{Z}_{q\infty}, \pi_i (L)) = 0$, $i \leq n - 1$. Then the exact sequence

$$0 \to \text{Ext} (\mathbb{Z}_{q\infty}, \pi_i (L)) \to \pi_i (L) \to \text{Hom} (\mathbb{Z}_{q\infty}, \pi_{i-1} (L)) \to 0$$

implies that $L_\beta$ is $(n - 1)$-connected and $\text{Ext} (\mathbb{Z}_{q\infty}, \pi_n (L)) = \pi_n (L_\beta)$. Thus $\pi_n (L_\beta) = H_n (L_\beta)$ and hence $\text{Ext} (\mathbb{Z}_{q\infty}, \pi_n (L))$ is $q$-divisible. Then by Proposition 2.1, $\pi_n (L)$ is $q$-divisible and therefore $\pi_n (L)/\text{Tor}_q \pi_n (L)$ is $q$-divisible.

Now assume that $q \in \mathcal{F}$. Once again let $\mathcal{A} = \mathcal{P} \setminus \{q\}$ and let $\alpha : L \to L_\alpha$ be an $\mathcal{A}$-localization of $L$. Recall that $\tilde{H}_i (L)$ is $q$-torsion free and $q$-divisible for $i \leq n - 1$ and therefore $\alpha$ induces an isomorphism of the groups $\tilde{H}_* (L)$ and $\tilde{H}_* (L_\alpha)$ in dimensions $\leq n - 1$. Note that $H_n (L) \otimes \mathbb{Z} (\mathcal{A}) = H_n (L) \otimes \mathbb{Z} [1/q] = (H_n (L)/\text{Tor}_q H_n (L)) \otimes \mathbb{Z} [1/q]$ and hence since $H_n (L)/\text{Tor}_q H_n (L)$ is $q$-divisible we infer that the $\mathcal{A}$-localization of $H_n (L)$ is an epimorphism. Then by the Whitehead theorem $\alpha$ induces an epimorphism of the $n$th ho-
motopy groups of $L$ and $L_\alpha$ and therefore the $A$-localization of $\pi_n(L)$ is an epimorphism. This happens only if $\pi_n(L)/\Tor_q \pi_n(L)$ is $q$-divisible, and (iii) is proved.

(iv) The proof is similar to the proof of (iii). ■

Let $X$ be a compactum and let $n$ be a positive integer. The Bockstein basis of abelian groups is the collection $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_p^\infty, \mathbb{Z}(p) : p \in \mathcal{P}\}$ of groups. Define the Bockstein basis of $X$ in dimensions $\leq n$ as $\sigma(X,n) = \{E \in \sigma : \dim_E X \leq n\}$. Following [6] define:

$$T(X,n) = \{p \in \mathcal{P} : \mathbb{Z}_p \text{ or } \mathbb{Z}_p^\infty \in \sigma(X,n)\};$$

$$\mathcal{D}(X,n) = \begin{cases} \emptyset & \text{if } \sigma(X,n) \text{ contains only torsion groups,} \\ \mathcal{P} & \text{if } \mathbb{Q} \in \sigma(X,n) \text{ but } \mathbb{Z}(p) \in \sigma(X,n) \text{ for no } p \in \mathcal{P}, \\ \mathcal{P} \setminus \{p \in \mathcal{P} : \mathbb{Z}(p) \in \sigma(X,n)\} & \text{otherwise}; \end{cases}$$

$$\mathcal{F}(X,n) = \mathcal{D}(X,n) \setminus T(X,n).$$

Note that for every group $G$ such that $\dim_G X \leq n$, $G$ is $\mathcal{F}(X,n)$-torsion free.

PROPOSITION 2.3. Let $X$ be a compactum such that $\mathcal{D}(X,n) \neq \emptyset$. Then $\dim_H X \leq n$ for every group $H$ such that $H$ is $\mathcal{D}(X,n)$-divisible and $\mathcal{F}(X,n)$-torsion free.

Proof. Let $G = \bigoplus \{E : E \in \sigma(X,n)\}$. Then $\dim_G X \leq n$. One can easily verify that in the notations of Proposition 2.4 of [6], $\mathcal{D}(G) = \mathcal{D}(X,n)$ and $\mathcal{F}(G) = \mathcal{F}(X,n)$. Then the result follows from Proposition 2.4 of [6]. ■

In the proof of Theorem 1.1 we will also use the following facts.

PROPOSITION 2.4 ([7]). Let $K$ be a simply connected CW-complex such that $K$ has only finitely many non-trivial homotopy groups. Let $X$ be a compactum such that $\dim_{\pi_i(K)} X \leq i$ for $i > 1$. Then $e$-$\dim X \leq K$.

Let $K'$ be a simplicial complex. We say that maps $h : K \to K'$, $g : L \to L'$, $\alpha : L \to K$ and $\alpha' : L' \to K'$ combinatorially commute if $(\alpha' \circ g)(h \circ \alpha)^{-1}(\Delta)) \subset \Delta$ for every simplex $\Delta$ of $K'$. Recall that a map $h' : K \to L'$ is a combinatorial lifting of $h$ to $L'$ if $(\alpha' \circ h')(h^{-1}(\Delta)) \subset \Delta$ for every simplex $\Delta$ of $K'$.

For a simplicial complex $K$ and $a \in K$, $\text{st}(a)$ denotes the union of all the simplexes of $K$ containing $a$.

PROPOSITION 2.5 ([7]). (i) Let a compactum $X$ be represented as the inverse limit $X = \varprojlim K_i$ of finite simplicial complexes $K_i$ with bonding maps $h_j^i : K_j \to K_i$. Fix $i$ and let $\omega : EW(K_i,k) \to K_i$ be a resolution of $K_i$ which is suitable for $X$. Then there is a sufficiently large $j$ such that $h_j^i$ admits a combinatorial lifting to $EW(K_i,k)$. 

(ii) Let \( h : K \rightarrow K', h' : K \rightarrow L' \) and \( \alpha' : L' \rightarrow K' \) be maps of a simplicial complex \( K' \) and CW-complexes \( K \) and \( L' \) such that \( h \) and \( \alpha' \) are combinatorial and \( h' \) is a combinatorial lifting of \( h \). Then there is a cellular approximation of \( h' \) which is also a combinatorial lifting of \( h \).

(iii) Let \( K \) and \( K' \) be simplicial complexes, let maps \( h : K \rightarrow K', g : L \rightarrow L', \alpha : L \rightarrow K \) and \( \alpha' : L' \rightarrow K' \) combinatorially commute and let \( h \) be combinatorial. Then

\[
g(\alpha^{-1}(st(x))) \subset \alpha'^{-1}(st(h(x))) \quad \text{and} \quad h(st((\alpha(z))) \subset st((\alpha' \circ g)(z))
\]

for every \( x \in K \) and \( z \in L \).

3. Proof of Theorem 1.1. Write \( D = D(X, n) \) and \( F = F(X, n) \). Represent \( X \) as the inverse limit \( X = \varprojlim (K_i, h_i) \) of finite simplicial complexes \( K_i \) with combinatorial bonding maps \( h_{i+1} : K_{i+1} \rightarrow K_i \) and the projections \( p_i : X \rightarrow K_i \) such that \( \text{diam}(p_i^{-1}(\Delta)) \leq 1/i \) for every simplex \( \Delta \) of \( K_i \). Following A. Dranishnikov [3, 4] we construct by finite induction CW-complexes \( L_i \) and maps \( g_{i+1} : L_{i+1} \rightarrow L_i, \alpha_i : L_i \rightarrow K_i \) such that:

(a) \( L_i \) is \((n+1)\)-dimensional and obtained from \( K_i^{[n+1]} \) by replacing some \((n+1)\)-simplexes by \((n+1)\)-cells attached to the boundary of the replaced simplexes by a map of degree \( \in S(F) \). Then \( \alpha_i \) is a projection of \( L_i \) taking the new cells to the original ones such that \( \alpha \) is 1-to-1 over \( K_i^{[n]} \). We define a simplicial structure on \( L_i \) for which \( \alpha_i \) is a combinatorial map and refer to this simplicial structure while constructing resolutions of \( L_i \). Note that for \( F = \emptyset \) we do not replace simplexes of \( K_i^{[n+1]} \) at all.

(b) The maps \( h_{i+1}, g_{i+1}, \alpha_{i+1} \) and \( \alpha_i \) combinatorially commute. Recall that this means that \( (\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta \) for every simplex \( \Delta \) of \( K_i \).

We will construct \( L_i \) in such a way that \( Z = \varprojlim(L_i, g_i) \) will admit a map \( r : Z \rightarrow X \) such that \( Z \) and \( r \) satisfy the conclusions of the theorem.

Let \( E \in \sigma \) be such that \( \dim E X \leq k, 2 \leq k \leq n \), and let \( f : N \rightarrow K(E, k) \) be a cellular map from a subcomplex \( N \) of \( L_i \) with \( L_i^k \subset N \). Let \( \omega_L : EW(L_i, k) \rightarrow L_i \) be the standard resolution of \( L_i \) for \( f \). We are going to construct from \( \omega_L \) a resolution \( \omega : EW(K_i, k) \rightarrow K_i \) of \( K_i \) suitable for \( X \). If \( \dim K_i \leq k \) set \( \omega = \alpha_i \circ \omega_L : EW(K_i, k) = EW(L_i, k) \rightarrow K_i \).

If \( \dim K_i > k \) set \( \omega_k = \alpha_i \circ \omega_L : EW_k(K_i, k) = EW(L_i, k) \rightarrow K_i \). We will construct by induction resolutions \( \omega_j : EW_j(K_i, k) \rightarrow K_i, \) \( k+1 \leq j \leq \dim K_i \), such that \( EW_j(K_i, k) \) is a subcomplex of \( EW_{j+1}(K_i, k) \) and \( \omega_{j+1} \) extends \( \omega_j \) for every \( k \leq j < \dim K_i \).

Assume that \( \omega_j : EW_j(K_i, k) \rightarrow K_i, \) \( k \leq j < \dim K_i \), is constructed. For every \((j+1)\)-simplex \( \Delta \) of \( K_i \) consider the subcomplex \( \omega_j^{-1}(\Delta) \) of \( EW_j(K_i, k) \). Enlarge \( \omega_j^{-1}(\Delta) \) by attaching \((n+1)\)-cells in order to kill the
elements of $\text{Tor}_F \pi_n(\omega_j^{-1}(\Delta))$, and attaching cells of dimension $> n + 1$ in order to get a subcomplex with trivial homotopy groups in dimensions $> n$.

Let $EW_{j+1}(K_i, k)$ be $EW_j(K_i, k)$ with all the cells attached for all $(j+1)$-simplexes $\Delta$ of $K_i$ and let $\omega_{j+1} : EW_{j+1}(K_i, k) \to K_i$ be an extension of $\omega_j$ sending the interior points of the attached cells to the interior of the corresponding $\Delta$.

Finally, define $EW(K_i, k) = EW_j(K_i, k)$ and $\omega = \omega_j : EW_j(K_i, k) \to K_i$ for $j = \dim K_i$. Note that since we attach cells only of dimension $> n$, the $n$-skeleton of $EW(K_i, k)$ coincides with the $n$-skeleton of $EW(L_i, k)$.

Let us show that $EW(K_i, k)$ is suitable for $X$. Fix a simplex $\Delta$ of $K_i$. Since $\omega^{-1}(\Delta)$ is contractible if $\dim \Delta \leq k$, assume that $\dim \Delta > k$. Set $T = \alpha_i^{-1}(\Delta)$. Note that it follows from the construction that $T$ is $(n-1)$-connected, $H_n(T)$ is $\mathcal{F}$-torsion, $\omega^{-1}(\Delta)$ is $(k-1)$-connected, $\pi_n(\omega^{-1}(\Delta)) = \pi_n(\omega^1_L(T))/\text{Tor}_F \pi_n(\omega^1_L(T))$, $\pi_j(\omega^{-1}(\Delta)) = 0$ for $j \geq n+1$ and $\pi_j(\omega^{-1}(\Delta)) = \pi_j(\omega^1_L(T))$ for $j \leq n-1$.

Consider the following cases.

**Case 1:** $E = \mathbb{Z}_p$. By (i) of Proposition 2.2, $\pi_j(\omega^1_L(T))$, $j \leq n - 1$, and $\pi_n(\omega^1_L(T))/\text{Tor}_F \pi_n(\omega^1_L(T))$ are $p$-torsion. Then $\pi_j(\omega^{-1}(\Delta))$ is $p$-torsion for $j \leq n$. Therefore $\dim \pi_j(\omega^{-1}(\Delta)) X \leq \dim_{\mathbb{Z}_p} X \leq k$ for $j \geq k$ and hence by Proposition 2.4, $\text{e-dim } X \leq \omega^{-1}(\Delta)$.

**Case 2:** $E = \mathbb{Z}_{p^\infty}$. Then by (ii) of Proposition 2.2, $\pi_j(\omega^1_L(T))$, $j \leq n - 1$, and $\pi_n(\omega^1_L(T))/\text{Tor}_F \pi_n(\omega^1_L(T))$ are $p$-torsion and $\pi_k(\omega^1_L(T))$ is $p$-divisible. Then $\pi_j(\omega^{-1}(\Delta))$ is $p$-torsion for $j \leq n$ and $\pi_k(\omega^{-1}(\Delta))$ is $p$-divisible. Therefore by the Bockstein theorem we have the inequalities $\dim \pi_k(\omega^{-1}(\Delta)) X \leq \dim_{\mathbb{Z}_{p^\infty}} X \leq k$ and $\dim \pi_j(\omega^{-1}(\Delta)) X \leq \dim_{\mathbb{Z}_{p^\infty}} X + 1 \leq k + 1$ for $j \geq k + 1$. Hence $\text{e-dim } X \leq \omega^{-1}(\Delta)$ by Proposition 2.4.

**Case 3:** $E = \mathbb{Z}_{(p)}$. Then by (iii) of Proposition 2.2, $\pi_j(\omega^1_L(T))$, $j \leq n - 1$, is $p$-local and $\pi_n(\omega^1_L(T))/\text{Tor}_F \pi_n(\omega^1_L(T))$ is $q$-divisible for every $q \in \mathcal{P}$, $q \neq p$. Then $\pi_j(\omega^{-1}(\Delta))$, $j \leq n - 1$, is $p$-local and $\pi_n(\omega^{-1}(\Delta))$ is $\mathcal{D}$-divisible and $\mathcal{F}$-torsion free. Therefore $\dim \pi_j(\omega^{-1}(\Delta)) X \leq k$ for $j \leq n - 1$ and by Proposition 2.3, $\dim \pi_n(\omega^{-1}(\Delta)) X \leq n$. Hence $\text{e-dim } X \leq \omega^{-1}(\Delta)$ by Proposition 2.4.

**Case 4:** $E = \mathbb{Q}$. This case is similar to the previous one.

Thus we have shown that $EW(K_i, k)$ is suitable for $X$. Now replacing $K_{i+1}$ by $K_j$ with a sufficiently large $j$ we may assume by Proposition 2.5(i) that there is a combinatorial lifting of $h_{i+1}$ to $h_{i+1}' : K_{i+1} \to EW(K_i, k)$. By Proposition 2.5(ii) we replace $h_{i+1}'$ by its cellular approximation preserving the property of $h_{i+1}'$ of being a combinatorial lifting of $h_{i+1}$. 
Let $\Delta$ be a simplex of $K_i$ and let $\tau : (\alpha_i \circ \omega_L)^{-1}(\Delta) \to \omega^{-1}(\Delta)$ be the inclusion. Note that from the construction it follows that the kernel of the induced homomorphism $\tau_* : \pi_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) \to \pi_n(\omega^{-1}(\Delta))$ is $\mathcal{F}$-torsion. Using this fact and the reasoning in the proof of Theorem 1.2 of [6] one can construct from $K_{i+1}^{[n+1]}$ a CW-complex $L_{i+1}$ by replacing some $(n+1)$-simplexes of $K_{i+1}^{[n+1]}$ by $(n+1)$-cells attached to the boundary of the replaced simplexes by a map of degree $\in S(\mathcal{F})$ such that $h_{i+1}$ restricted to $K_{i+1}^{[n]}$ extends to a map $g_{i+1}^j : L_{i+1} \to EW(L_i, n)$ such that $g_{i+1}^j, \alpha_{i+1}, h_{i+1}$ and $\alpha_i \circ \omega_L$ combinatorially commute, where $\alpha_{i+1}$ is a projection of $L_{i+1}$ into $K_{i+1}$ taking the new cells to the original ones in such a way that $\alpha_{i+1}$ is 1-to-1 over $K_{i+1}^{[n]}$.

Now define $g_{i+1} = \omega_L \circ g_{i+1}^j : L_{i+1} \to L_i$ and finally define a simplicial structure on $L_{i+1}$ for which $\alpha_{i+1}$ is a combinatorial map. It is easy to check that the properties (a) and (b) are satisfied. Since the triangulation of $L_{i+1}$ can be replaced by any of its barycentric subdivisions we may also assume that

$$(c) \quad \text{diam } g_{i+1}^j(\Delta) \leq 1/i \text{ for every simplex } \Delta \text{ in } L_{i+1} \text{ and } j \leq i,$$

where $g_i^j = g_{i+1} \circ g_{i+2} \circ \ldots \circ g_i : L_i \to L_j$.

Define $Z = \lim_{\mathcal{F}}(L_i, g_i)$ and let $r_i : Z \to L_i$ be the projections. Clearly $\dim Z \leq n+1$. To construct $L_{i+1}$ we used an arbitrary map $f : N \to K(E, k)$ such that $E \in \sigma, \dim_E X \leq k, 2 \leq k \leq n$ and $N$ is a subcomplex of $L_i$ containing $L_i^{[k]}$. By a standard reasoning described in detail in the proof of Theorem 1.6 of [7] one can show that choosing $E$ and $f$ in an appropriate way for each $i$ we can achieve that $\dim_E Z \leq k$ for every integer $2 \leq k \leq n$ and every $E \in \sigma$ such that $\dim_{Z^p} X \leq k$. Then by the Bockstein theorem $\dim_G Z \leq k$ for every group $G$ such that $\dim_G X \leq k, 2 \leq k \leq n$.

By Proposition 2.5(iii), properties (a) and (b) imply that for every $x \in X$ and $z \in Z$ the following holds:

$$(d1) \quad g_{i+1}(\alpha_i^{-1}(\text{st}(p_i(x)))) \subset \alpha_i^{-1}(\text{st}(p_i(x))),$$
$$(d2) \quad h_{i+1}(\text{st}(\alpha_i^{-1}(\text{st}(p_i(z)))) \subset \text{st}(\alpha_i^{-1}(\text{st}(p_i(z))))).$$

Define a map $r : Z \to X$ by $r(z) = \bigcap \{p_i^{-1}(\text{st}(\alpha_i^{-1}(\text{st}(p_i(z)))) : i = 1, 2, \ldots \}$. Then (d2) implies that $r$ is indeed well defined and continuous.

Properties (d1) and (d2) also imply that for every $x \in X$,

$$r^{-1}(x) = \lim_{\mathcal{F}}(\alpha_i^{-1}(\text{st}(p_i(x))), g_i|_{\alpha_i^{-1}(\text{st}(p_i(x)))})$$

where the map $g_i|_{\ldots}$ is considered as a map to $\alpha_i^{-1}(\text{st}(p_{i-1}(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, $r$ is onto. Fix $x \in X$ and let us show that $r^{-1}(x)$ satisfies the conclusions of the theorem. Since $T = \alpha_i^{-1}(\text{st}(p_i(x)))$ is $(n-1)$-connected we see that $r^{-1}(x)$ is approximately
(n - 1)-connected as the inverse limit of (n - 1)-connected finite simplicial complexes.

Let a group $G$ be such that $\dim_G X \leq n$. Note that $H_n(T)$ is $\mathcal{F}$-torsion and $G$ is $\mathcal{F}$-torsion free. Then by the universal coefficient theorem $H^n(T; G) = \text{Hom}(H_n(T), G) = 0$. Thus $\tilde{H}^k(r^{-1}(x); G) = 0$ for $k \leq n$ and since $\dim_G Z \leq n$, we have $\tilde{H}^k(r^{-1}(x); G) = 0$ for $k \geq n + 1$. Hence $r$ is $G$-acyclic and this completes the proof. ■

References


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