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Universal acyclic resolutions for arbitrary coefficient groups

by

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Abstract. We prove that for every compactum X and every integer $n \geq 2$ there are a compactum Z of dimension $\leq n+1$ and a surjective UV^{n-1}-map $r : Z \to X$ such that for every abelian group G and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and r is G-acyclic.

1. Introduction. This paper is devoted to proving the following theorem which was announced in [8].

THEOREM 1.1. *Let X be a compactum. Then for every integer $n \geq 2$ there are a compactum Z of dimension $\leq n+1$ and a surjective UV^{n-1}-map $r : Z \to X$ having the property that for every abelian group G and every integer $k \geq 2$ such that $\dim_G X \leq k \leq n$ we have $\dim_G Z \leq k$ and r is G-acyclic.*

The *cohomological dimension* $\dim_G X$ of X with respect to an abelian group G is the least number n such that $\check{H}^{n+1}(X, A; G) = 0$ for every closed subset A of X. A space is *G-acyclic* if its reduced Čech cohomology groups modulo G are trivial; a map is *G-acyclic* if every fiber is G-acyclic. By the Vietoris–Begle theorem a surjective G-acyclic map of compacta cannot raise the cohomological dimension $\dim_G$. A compactum X is *approximately n-connected* if any embedding of X into an ANR has the *UV^n-property*, i.e. for every neighborhood U of X there is a smaller neighborhood $X \subset V \subset U$ such that the inclusion $V \subset U$ induces the zero homomorphism of the homotopy groups in dimensions $\leq n$. An approximately n-connected compactum has trivial reduced Čech cohomology groups in dimensions $\leq n$ with respect to any group G. A map is called a *UV^n-map* if every fiber is approximately n-connected.

Theorem 1.1 generalizes the following results of [6, 7].

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Theorem 1.2 ([6]). *Let G be an abelian group and let X be a compactum with $\dim_G X \le n$, $n \ge 2$. Then there are a compactum Z with $\dim_G Z \le n$ and $\dim Z \le n+1$ and a G-acyclic map $r : Z \to X$ from Z onto X.*

Theorem 1.3 ([7]). *Let X be a compactum with $\dim_{\mathbb{Z}} X \le n \ge 2$. Then there exist a compactum Z with $\dim Z \le n$ and a cell-like map $r : Z \to X$ from Z onto X such for every integer $k \ge 2$ and every group G such that $\dim_G X \le k$ we have $\dim_G Z \le k$.*

Theorem 1.2 obviously follows from Theorem 1.1. Theorem 1.3 can be derived from Theorem 1.1 as follows. Recall that a compactum is *cell-like* if any map from the compactum to a CW-complex is null-homotopic. A map is *cell-like* if its fibers are cell-like. Let X have $\dim_{\mathbb{Z}} X < \infty$ and let $r : Z \to X$ satisfy the conclusions of Theorem 1.1 for $n = \dim_{\mathbb{Z}} X + 1$. Then $\dim_{\mathbb{Z}} Z \le \dim_{\mathbb{Z}} X \le n-1$ and because Z is finite-dimensional we have $\dim Z = \dim_{\mathbb{Z}} Z \le n-1$. Since r is UV^{n-1} and $\dim Z \le n-1$ we find that r is cell-like. Let $\dim_G X \le k \ge 2$ for a group G. If $k \le n$ then $\dim_G Z \le k$ by Theorem 1.1, and if $k > n$ then $\dim_G Z \le k$ since $\dim Z \le n-1$. Thus Theorem 1.1 implies Theorem 1.3.

It was observed in [7] that the restriction $k \ge 2$ in Theorem 1.3 cannot be omitted. Therefore Theorem 1.1 does not hold for $k = 1$.

Let us discuss possible generalizations of Theorem 1.1. One is tempted to reduce the dimension of Z to n. This is partially justified by

Theorem 1.4 ([8]). *Let X be a compactum. Then for every integer $n \ge 2$ there are a compactum Z of dimension $\le n$ and a surjective UV^{n-1}-map $r : Z \to X$ having the property that for every finitely generated abelian group G and every integer $k \ge 2$ such that $\dim_G X \le k \le n$ we have $\dim_G Z \le k$ and r is G-acyclic.*

However, Theorem 1.4 does not hold for arbitrary groups G. Indeed, one can show that a $\mathbb{Q}$-acyclic UV^1-map from a compactum of dimension ≤ 2 must be $\mathbb{Z}$-acyclic (even cell-like). Thus a compactum X with $\dim_{\mathbb{Z}} X = 3$ and $\dim_{\mathbb{Q}} Z = 2$ cannot be the image of a compactum of dimension ≤ 2 under a $\mathbb{Q}$-acyclic UV^1-map.

The situation becomes more complicated if we drop in Theorem 1.1 the requirement that r is UV^{n-1} and consider

Problem 1.5. *Given a compactum X, an integer $n \ge 2$ and a collection $\mathcal{G}$ of abelian groups such that $\dim_G X \le n$ for every $G \in \mathcal{G}$, do there exist a compactum Z of dimension $\le n$ and a $\mathcal{G}$-acyclic surjective map $r : Z \to X$ such that $\dim_G Z \le \max\{\dim_G X, 2\}$ for every $G \in \mathcal{G}$? (The $\mathcal{G}$-acyclicity means the G-acyclicity for every $G \in \mathcal{G}$.)*

In general the answer to Problem 1.5 is negative [5]. Indeed, let X be a compactum with $\dim_{\mathbb{Z}} X = 3$, $\dim_{\mathbb{Q}} X = 2$ and $\dim_{\mathbb{Z}_p} X = 2$ for every prime p and let $G = \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z})$. Clearly $\dim_G X = 2$ and the G-acyclicity implies both the $\mathbb{Q}$- and $(\mathbb{Q}/\mathbb{Z})$-acyclicity. Then it follows from the Bockstein sequence generated by

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

that the G-acyclicity implies the $\mathbb{Z}$-acyclicity and therefore there is no G-acyclic resolution for X from a compactum of dimension ≤ 2.

The situation described in the example can be interpreted in terms of Bockstein theory. Let $\mathcal{G}$ be a collection of abelian groups. Denote by $\sigma(\mathcal{G})$ the union of the Bockstein bases $\sigma(G)$ of all $G \in \mathcal{G}$. Based on the Bockstein inequalities define the closure $\overline{\sigma(\mathcal{G})}$ of $\sigma(\mathcal{G})$ as a collection of abelian groups containing $\sigma(\mathcal{G})$ and possibly some additional groups determined by:

- $\mathbb{Z}_p \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Z}_{p^\infty} \in \overline{\sigma(\mathcal{G})}$;
- $\mathbb{Z}_{p^\infty} \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Z}_p \in \overline{\sigma(\mathcal{G})}$;
- $\mathbb{Q} \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Z}_{(p)} \in \overline{\sigma(\mathcal{G})}$;
- $\mathbb{Z}_{(p)} \in \overline{\sigma(\mathcal{G})}$ if $\mathbb{Q}, \mathbb{Z}_{p^\infty} \in \overline{\sigma(\mathcal{G})}$.

One can show that for compact metric spaces the $\mathcal{G}$-acyclicity implies the $\overline{\sigma(\mathcal{G})}$-acyclicity. This motivates the following

CONJECTURE 1.6. *Problem* 1.5 *can be answered positively if* $\dim_E X \leq n$ *for every* $E \in \overline{\sigma(\mathcal{G})}$.

The key open case of this conjecture seems to be constructing a $\mathbb{Q}$-acyclic resolution $r : Z \to X$ for a compactum X with $\dim_{\mathbb{Q}} X \leq n$, $n \geq 2$, from a compactum Z of dimension $\leq n$.

2. Preliminaries. All groups are assumed to be abelian, and functions between groups are homomorphisms. $\mathcal{P}$ stands for the set of primes. For a non-empty subset $\mathcal{A}$ of $\mathcal{P}$ let

$$S(\mathcal{A}) = \{p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} : p_i \in \mathcal{A},\ n_i \geq 0\}$$

be the set of positive integers with prime factors from $\mathcal{A}$, and for the empty set define $S(\emptyset) = \{1\}$. Let G be a group and $g \in G$. We say that g is $\mathcal{A}$-*torsion* if there is $n \in S(\mathcal{A})$ such that $ng = 0$, and g is $\mathcal{A}$-*divisible* if for every $n \in S(\mathcal{A})$ there is $h \in G$ such that $nh = g$. $\operatorname{Tor}_{\mathcal{A}} G$ is the subgroup of $\mathcal{A}$-torsion elements of G. The group G is $\mathcal{A}$-*torsion* if $G = \operatorname{Tor}_{\mathcal{A}} G$; G is $\mathcal{A}$-*torsion free* if $\operatorname{Tor}_{\mathcal{A}} G = 0$; and G is $\mathcal{A}$-*divisible* if every element of G is $\mathcal{A}$-divisible.

G is $\mathcal{A}$-*local* if it is $(\mathcal{P} \setminus \mathcal{A})$-divisible and $(\mathcal{P} \setminus \mathcal{A})$-torsion free. The $\mathcal{A}$-*localization* of G is the homomorphism $G \to G \otimes \mathbb{Z}_{(\mathcal{A})}$ defined by $g \mapsto g \otimes 1$, where $\mathbb{Z}_{(\mathcal{A})} = \{n/m : n \in \mathbb{Z}, m \in S(\mathcal{P} \setminus \mathcal{A})\}$. G is $\mathcal{A}$-local if and only if the $\mathcal{A}$-localization of G is an isomorphism. A simply connected CW-complex is

$\mathcal{A}$-*local* if its homotopy groups are $\mathcal{A}$-local. A map between two simply connected CW-complexes is an $\mathcal{A}$-*localization* if the induced homomorphisms of the homotopy and (reduced integral) homology groups are $\mathcal{A}$-localizations.

The *extensional dimension* of a compactum X is said not to exceed a CW-complex K, written $\text{e-dim}\, X \le K$, if for every closed subset A of X and every map $f : A \to K$ there is an extension of f over X. It is well known that $\dim X \le n$ is equivalent to $\text{e-dim}\, X \le \mathbb{S}^n$, and $\dim_G X \le n$ is equivalent to $\text{e-dim}\, X \le K(G,n)$, where $K(G,n)$ is an Eilenberg–Mac Lane complex of type (G,n).

A map between CW-complexes is said to be *combinatorial* if the preimage of every subcomplex of the range is a subcomplex of the domain. Let M be a simplicial complex and let $M^{[k]}$ be the k-skeleton of M (= the union of all simplexes of M of dimension $\le k$). By a *resolution* of M we mean a CW-complex $EW(M,k)$ and a combinatorial map $\omega : EW(M,k) \to M$ such that ω is 1-to-1 over $M^{[k]}$. Let $f : N \to K$ be a map of a subcomplex N of M into a CW-complex K. The resolution is said to be *suitable for* f if $f \circ \omega|_{\omega^{-1}(N)}$ extends to a map $f' : EW(M,k) \to K$. We call f' a *resolving map* for f. The resolution is said to be *suitable for a compactum* X if $\text{e-dim}\, X \le \omega^{-1}(\Delta)$ for every simplex Δ of M. Note that if $\omega : EW(M,k) \to M$ is a resolution suitable for X then for every map $\phi : X \to M$ there is a map $\psi : X \to EW(M,k)$ such that $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of M. We call ψ a *combinatorial lifting* of ϕ.

Let M be a finite simplicial complex and let $f : N \to K$ be a cellular map from a subcomplex N of M to a CW-complex K such that $M^{[k]} \subset N$. A standard way of constructing a resolution suitable for f is described in [7]. Such a resolution $\omega : EW(M,k) \to M$ is called the *standard resolution* of M for f and it has the following properties:

- $EW(M,k)$ is $(k-1)$-connected if so are M and K;
- ω is a surjective map and for every simplex Δ of M, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to K;
- for every subcomplex T of M, $\omega|_{\omega^{-1}(T)} : EW(T,k) = \omega^{-1}(T) \to T$ is the standard resolution of T for $f|_{N\cap T} : N \cap T \to K$.

Let G be a group, let $\alpha : L \to M$ be a surjective combinatorial map of a CW-complex L and a finite simplicial complex M, and let n be a positive integer such that $\widetilde{H}_i(\alpha^{-1}(\Delta); G) = 0$ for every $i < n$ and every simplex Δ of M. One can show by induction on the number of simplexes of M using the Mayer–Vietoris sequence and the Five Lemma that $\alpha_* : \widetilde{H}_i(L;G) \to \widetilde{H}_i(M;G)$ is an isomorphism for $i < n$. We will refer to this fact as the *combinatorial Vietoris–Begle theorem*.

PROPOSITION 2.1 ([2]). *Let G be a group and $p \in \mathcal{P}$. The following conditions are equivalent*:

- *G is p-divisible*;
- $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, G)$ *is p-divisible*;
- $\operatorname{Ext}(\mathbb{Z}_{p^\infty}, G) = 0$.

PROPOSITION 2.2. *Let G be a group, let $2 \le k \le n$ be integers and let $\mathcal{F} \subset \mathcal{P}$ and $p \in \mathcal{P} \setminus \mathcal{F}$. Let M be an $(n-1)$-connected finite simplicial complex such that $H_n(M)$ is $\mathcal{F}$-torsion, and let $\omega : L = EW(M,k) \to M$ be the standard resolution of M for a cellular map $f : N \to K(G,k)$ from a subcomplex N of M containing $M^{[k]}$. Then L is $(k-1)$-connected and for every $1 \le i \le n-1$:*

(i) *$\pi_i(L)$ and $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ are p-torsion if $G = \mathbb{Z}_p$;*

(ii) *$\pi_i(L)$ and $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ are p-torsion and $\pi_k(L)$ is p-divisible if $G = \mathbb{Z}_{p^\infty}$;*

(iii) *$\pi_i(L)$ and $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ are q-divisible and $\pi_i(L)$ is q-torsion free for every $q \in \mathcal{P}$, $q \ne p$ if $G = \mathbb{Z}_{(p)}$;*

(iv) *$\pi_i(L)$ and $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ are q-divisible and $\pi_i(L)$ is q-torsion free for every $q \in \mathcal{P}$ if $G = \mathbb{Q}$.*

Proof. Recall that ω is a combinatorial surjective map, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K(G,k)$ for every simplex Δ of M, and L is $(k-1)$-connected because so are M and $K(G,k)$. Since M is $(n-1)$-connected and $H_n(M)$ is $\mathcal{F}$-torsion we have $H_n(M;\mathbb{Q}) = 0$ and $H_n(M;\mathbb{Z}_q) = 0$, $H_n(M;\mathbb{Z}_{(q)}) = 0$ for $q \in \mathcal{P} \setminus \mathcal{F}$ and $H_n(M;\mathbb{Z}_{q^\infty}) = 0$ for every $q \in \mathcal{P}$.

(i) By the generalized Hurewicz theorem $\widetilde{H}_*(K(\mathbb{Z}_p,k))$ is p-torsion. Then $\widetilde{H}_*(K(\mathbb{Z}_p,k);\mathbb{Q}) = 0$. Hence by the combinatorial Vietoris–Begle theorem $\widetilde{H}_i(L;\mathbb{Q}) = 0$ for $i \le n$ and therefore $\widetilde{H}_i(L)$ is torsion for $i \le n$.

Let $q \in \mathcal{P}$ and $q \ne p$ and $i \le n-1$. Note that $\widetilde{H}_*(K(\mathbb{Z}_p,k);\mathbb{Z}_{(q)}) = 0$ and hence by the combinatorial Vietoris–Begle theorem $\widetilde{H}_i(L;\mathbb{Z}_{(q)}) = 0$. Then $\widetilde{H}_i(L) \otimes \mathbb{Z}_{(q)} = 0$. Thus $\widetilde{H}_i(L)$ is torsion and q-torsion free and hence $\widetilde{H}_i(L)$ is p-torsion.

Now let $q \in \mathcal{P} \setminus \mathcal{F}$ and $q \ne p$. Recall $H_n(M;\mathbb{Z}_{(q)}) = 0$. Then using the previous argument we conclude that $H_n(L)$ is q-torsion free and hence $H_n(L)$ is $(\mathcal{F} \cup \{p\})$-torsion.

By the generalized Hurewicz theorem $\pi_i(L)$ is p-torsion for $i \le n-1$ and $\pi_n(L)$ is $(\mathcal{F} \cup \{p\})$-torsion. Thus $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ is p-torsion and (i) follows.

(ii) The argument used in (i) applies to show that $\pi_i(L)$, $i \le n-1$, and $\pi_n(L)/\operatorname{Tor}_{\mathcal{F}} \pi_n(L)$ are p-torsion. Note that $\pi_k(L) = H_k(L)$. We will show that $H_k(L)$ is p-divisible and this will imply (ii). Observe that $H_k(K(\mathbb{Z}_{p^\infty},k);\mathbb{Z}_p) = \mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_p = 0$. Then since $H_k(M;\mathbb{Z}_p) = 0$ the combinatorial Vietoris–Begle theorem implies that $H_k(L;\mathbb{Z}_p)=0$. Thus $H_k(L)\otimes\mathbb{Z}_p=0$ and therefore $H_k(L)$ is p-divisible.

(iii) Since $\mathbb{Z}_{(p)}$ is p-local we deduce that $\widetilde{H}_*(K(\mathbb{Z}_{(p)},k))$ is p-local and therefore $\widetilde{H}_*(K(\mathbb{Z}_{(p)},k);\mathbb{Z}_q)=\widetilde{H}_*(K(\mathbb{Z}_{(p)},k);\mathbb{Z}_{q^\infty})=0$ for every $q\in\mathcal{P}$, $q\neq p$.

Let $q\in\mathcal{P}$, $q\neq p$. Recall that $\widetilde{H}_i(M;\mathbb{Z}_{q^\infty})=0$ for $i\le n$. Then by the combinatorial Vietoris–Begle theorem $\widetilde{H}_i(L;\mathbb{Z}_{q^\infty})=0$ for $i\le n$. Hence by the universal coefficient theorem $\widetilde{H}_i(L)*\mathbb{Z}_{q^\infty}=0$ and $\widetilde{H}_i(L)\otimes\mathbb{Z}_{q^\infty}=0$ for $i\le n-1$ and therefore $\widetilde{H}_i(L)$ is q-torsion free and q-divisible for $i\le n-1$.

Let $q\in\mathcal{P}$, $q\neq p$ and $q\notin\mathcal{F}$. Recall that $H_n(M;\mathbb{Z}_q)=0$. By the combinatorial Vietoris–Begle theorem $H_n(L;\mathbb{Z}_q)=0$. Hence $H_n(L)\otimes\mathbb{Z}_q=0$ and therefore $H_n(L)$ is q-divisible.

Let $q\in\mathcal{P}$, $q\neq p$ and $q\in\mathcal{F}$. Then $H_n(M;\mathbb{Z}_{q^\infty})=0$. By the combinatorial Vietoris–Begle theorem $H_n(L;\mathbb{Z}_{q^\infty})=0$. Hence $H_n(L)\otimes\mathbb{Z}_{q^\infty}=0$ and therefore $H_n(L)/\operatorname{Tor}_q H_n(L)$ is q-divisible.

Now using completion and localization theories [1] we will pass to the homotopy groups of L.

Let $q\in\mathcal{P}$. Define $\mathcal{A}=\mathcal{P}\setminus\{q\}$. Let $\alpha: L\to L_\alpha$ be an $\mathcal{A}$-localization of L. Recall that $\widetilde{H}_i(L)$ is q-torsion free and q-divisible for $i\le n-1$. Then α induces an isomorphism of the groups $\widetilde{H}_*(L)$ and $\widetilde{H}_*(L_\alpha)$ in dimensions $\le n-1$. Hence by the Whitehead theorem, α induces an isomorphism of the homotopy groups in dimensions $\le n-1$ and therefore $\pi_i(L)$ is $\mathcal{A}$-local (that is, q-divisible and q-torsion free) for $i\le n-1$.

Let $q\in\mathcal{P}$, $q\neq p$ and $q\notin\mathcal{F}$. Let $\beta: L\to L_\beta$ be a q-completion of L. Then β induces an isomorphism of $\widetilde{H}_*(L;\mathbb{Z}_q)$ and $\widetilde{H}_*(L_\beta;\mathbb{Z}_q)$; since $H_n(L;\mathbb{Z}_q)=0$ we get $H_n(L_\beta;\mathbb{Z}_q)=0$ and therefore $H_n(L_\beta)$ is q-divisible. Now since $\pi_i(L)$ is q-divisible and q-torsion free we have $\operatorname{Hom}(\mathbb{Z}_{q^\infty},\pi_i(L))=0$, $i\le n-1$, and by Proposition 2.1, $\operatorname{Ext}(\mathbb{Z}_{q^\infty},\pi_i(L))=0$, $i\le n-1$. Then the exact sequence

$$0\to\operatorname{Ext}(\mathbb{Z}_{q^\infty},\pi_i(L))\to\pi_i(L_\beta)\to\operatorname{Hom}(\mathbb{Z}_{q^\infty},\pi_{i-1}(L))\to 0$$

implies that L_β is $(n-1)$-connected and $\operatorname{Ext}(\mathbb{Z}_{q^\infty},\pi_n(L))=\pi_n(L_\beta)$. Thus $\pi_n(L_\beta)=H_n(L_\beta)$ and hence $\operatorname{Ext}(\mathbb{Z}_{q^\infty},\pi_n(L))$ is q-divisible. Then by Proposition 2.1, $\pi_n(L)$ is q-divisible and therefore $\pi_n(L)/\operatorname{Tor}_q\pi_n(L)$ is q-divisible.

Now assume that $q\in\mathcal{F}$. Once again let $\mathcal{A}=\mathcal{P}\setminus\{q\}$ and let $\alpha: L\to L_\alpha$ be an $\mathcal{A}$-localization of L. Recall that $\widetilde{H}_i(L)$ is q-torsion free and q-divisible for $i\le n-1$ and therefore α induces an isomorphism of the groups $\widetilde{H}_*(L)$ and $\widetilde{H}_*(L_\alpha)$ in dimensions $\le n-1$. Note that $H_n(L)\otimes\mathbb{Z}_{(\mathcal{A})}=H_n(L)\otimes\mathbb{Z}[1/q]=(H_n(L)/\operatorname{Tor}_q H_n(L))\otimes\mathbb{Z}[1/q]$ and hence since $H_n(L)/\operatorname{Tor}_q H_n(L)$ is q-divisible we infer that the $\mathcal{A}$-localization of $H_n(L)$ is an epimorphism. Then by the Whitehead theorem α induces an epimorphism of the nth homotopy groups of L and L_α and therefore the $\mathcal{A}$-localization of $\pi_n(L)$ is an epimorphism. This happens only if $\pi_n(L)/\operatorname{Tor}_q \pi_n(L)$ is q-divisible, and (iii) is proved.

(iv) The proof is similar to the proof of (iii). ∎

Let X be a compactum and let n be a positive integer. The *Bockstein basis of abelian groups* is the collection $\sigma = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}, \mathbb{Z}_{(p)} : p \in \mathcal{P}\}$ of groups. Define the *Bockstein basis of X in dimensions $\le n$* as $\sigma(X,n) = \{E \in \sigma : \dim_E X \le n\}$. Following [6] define:

$$\mathcal{T}(X,n) = \{p \in \mathcal{P} : \mathbb{Z}_p \text{ or } \mathbb{Z}_{p^\infty} \in \sigma(X,n)\};$$

$$\mathcal{D}(X,n) = \begin{cases} \emptyset & \text{if } \sigma(X,n) \text{ contains only torsion groups,} \\ \mathcal{P} & \text{if } \mathbb{Q} \in \sigma(X,n) \text{ but } \mathbb{Z}_{(p)} \in \sigma(X,n) \text{ for no } p \in \mathcal{P}, \\ \mathcal{P} \setminus \{p \in \mathcal{P} : \mathbb{Z}_{(p)} \in \sigma(X,n)\} & \text{otherwise;} \end{cases}$$

$$\mathcal{F}(X,n) = \mathcal{D}(X,n) \setminus \mathcal{T}(X,n).$$

Note that for every group G such that $\dim_G X \le n$, G is $\mathcal{F}(X,n)$-torsion free.

PROPOSITION 2.3. *Let X be a compactum such that $\mathcal{D}(X,n) \ne \emptyset$. Then $\dim_H X \le n$ for every group H such that H is $\mathcal{D}(X,n)$-divisible and $\mathcal{F}(X,n)$-torsion free.*

Proof. Let $G = \bigoplus\{E : E \in \sigma(X,n)\}$. Then $\dim_G X \le n$. One can easily verify that in the notations of Proposition 2.4 of [6], $\mathcal{D}(G) = \mathcal{D}(X,n)$ and $\mathcal{F}(G) = \mathcal{F}(X,n)$. Then the result follows from Proposition 2.4 of [6]. ∎

In the proof of Theorem 1.1 we will also use the following facts.

PROPOSITION 2.4 ([7]). *Let K be a simply connected CW-complex such that K has only finitely many non-trivial homotopy groups. Let X be a compactum such that $\dim_{\pi_i(K)} X \le i$ for $i > 1$. Then* $\operatorname{e-dim} X \le K$.

Let K' be a simplicial complex. We say that maps $h : K \to K'$, $g : L \to L'$, $\alpha : L \to K$ and $\alpha' : L' \to K'$ *combinatorially commute* if $(\alpha' \circ g)((h \circ \alpha)^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of K'. Recall that a map $h' : K \to L'$ is a *combinatorial lifting* of h to L' if $(\alpha' \circ h')(h^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of K'.

For a simplicial complex K and $a \in K$, $\operatorname{st}(a)$ denotes the union of all the simplexes of K containing a.

PROPOSITION 2.5 ([7]). (i) *Let a compactum X be represented as the inverse limit $X = \varprojlim K_i$ of finite simplicial complexes K_i with bonding maps $h^i_j : K_j \to K_i$. Fix i and let $\omega : EW(K_i,k) \to K_i$ be a resolution of K_i which is suitable for X. Then there is a sufficiently large j such that h^i_j admits a combinatorial lifting to $EW(K_i,k)$.*

(ii) *Let* $h : K \to K'$, $h' : K \to L'$ *and* $\alpha' : L' \to K'$ *be maps of a simplicial complex* K' *and CW-complexes* K *and* L' *such that* h *and* α' *are combinatorial and* h' *is a combinatorial lifting of* h. *Then there is a cellular approximation of* h' *which is also a combinatorial lifting of* h.

(iii) *Let* K *and* K' *be simplicial complexes*, *let maps* $h : K \to K'$, $g : L \to L'$, $\alpha : L \to K$ *and* $\alpha' : L' \to K'$ *combinatorially commute and let* h *be combinatorial. Then*

$$g(\alpha^{-1}(\mathrm{st}(x))) \subset \alpha'^{-1}(\mathrm{st}(h(x))) \quad \textit{and} \quad h(\mathrm{st}((\alpha(z))) \subset \mathrm{st}((\alpha' \circ g)(z))$$

for every $x \in K$ *and* $z \in L$.

3. Proof of Theorem 1.1. Write $\mathcal{D} = \mathcal{D}(X, n)$ and $\mathcal{F} = \mathcal{F}(X, n)$. Represent X as the inverse limit $X = \varprojlim(K_i, h_i)$ of finite simplicial complexes K_i with combinatorial bonding maps $h_{i+1} : K_{i+1} \to K_i$ and the projections $p_i : X \to K_i$ such that $\mathrm{diam}(p_i^{-1}(\Delta)) \leq 1/i$ for every simplex Δ of K_i. Following A. Dranishnikov [3, 4] we construct by finite induction CW-complexes L_i and maps $g_{i+1} : L_{i+1} \to L_i$, $\alpha_i : L_i \to K_i$ such that:

(a) L_i is $(n+1)$-dimensional and obtained from $K_i^{[n+1]}$ by replacing some $(n+1)$-simplexes by $(n+1)$-cells attached to the boundary of the replaced simplexes by a map of degree $\in S(\mathcal{F})$. Then α_i is a projection of L_i taking the new cells to the original ones such that α_i is 1-to-1 over $K_i^{[n]}$. We define a simplicial structure on L_i for which α_i is a combinatorial map and refer to this simplicial structure while constructing resolutions of L_i. Note that for $\mathcal{F} = \emptyset$ we do not replace simplexes of $K_i^{[n+1]}$ at all.

(b) The maps h_{i+1}, g_{i+1}, α_{i+1} and α_i combinatorially commute. Recall that this means that $(\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta$ for every simplex Δ of K_i.

We will construct L_i in such a way that $Z = \varprojlim(L_i, g_i)$ will admit a map $r : Z \to X$ such that Z and r satisfy the conclusions of the theorem.

Let $E \in \sigma$ be such that $\dim_E X \leq k$, $2 \leq k \leq n$, and let $f : N \to K(E, k)$ be a cellular map from a subcomplex N of L_i with $L_i^{[k]} \subset N$. Let $\omega_L : EW(L_i, k) \to L_i$ be the standard resolution of L_i for f. We are going to construct from ω_L a resolution $\omega : EW(K_i, k) \to K_i$ of K_i suitable for X. If $\dim K_i \leq k$ set $\omega = \alpha_i \circ \omega_L : EW(K_i, k) = EW(L_i, k) \to K_i$.

If $\dim K_i > k$ set $\omega_k = \alpha_i \circ \omega_L : EW_k(K_i, k) = EW(L_i, k) \to K_i$. We will construct by induction resolutions $\omega_j : EW_j(K_i, k) \to K_i$, $k + 1 \leq j \leq \dim K_i$, such that $EW_j(K_i, k)$ is a subcomplex of $EW_{j+1}(K_i, k)$ and ω_{j+1} extends ω_j for every $k \leq j < \dim K_i$.

Assume that $\omega_j : EW_j(K_i, k) \to K_i$, $k \leq j < \dim K_i$, is constructed. For every $(j + 1)$-simplex Δ of K_i consider the subcomplex ${\omega_j}^{-1}(\Delta)$ of $EW_j(K_i, k)$. Enlarge ${\omega_j}^{-1}(\Delta)$ by attaching $(n + 1)$-cells in order to kill the elements of $\operatorname{Tor}_{\mathcal{F}} \pi_n({\omega_j}^{-1}(\Delta))$, and attaching cells of dimension $> n+1$ in order to get a subcomplex with trivial homotopy groups in dimensions $> n$. Let $EW_{j+1}(K_i, k)$ be $EW_j(K_i, k)$ with all the cells attached for all $(j+1)$-simplexes Δ of K_i and let $\omega_{j+1} : EW_{j+1}(K_i, k) \to K_i$ be an extension of ω_j sending the interior points of the attached cells to the interior of the corresponding Δ.

Finally, define $EW(K_i, k) = EW_j(K_i, k)$ and $\omega = \omega_j : EW_j(K_i, k) \to K_i$ for $j = \dim K_i$. Note that since we attach cells only of dimension $> n$, the n-skeleton of $EW(K_i, k)$ coincides with the n-skeleton of $EW(L_i, k)$.

Let us show that $EW(K_i, k)$ is suitable for X. Fix a simplex Δ of K_i. Since $\omega^{-1}(\Delta)$ is contractible if $\dim \Delta \le k$, assume that $\dim \Delta > k$. Set $T = \alpha_i^{-1}(\Delta)$. Note that it follows from the construction that T is $(n-1)$-connected, $H_n(T)$ is $\mathcal{F}$-torsion, $\omega^{-1}(\Delta)$ is $(k-1)$-connected, $\pi_n(\omega^{-1}(\Delta)) = \pi_n(\omega_L^{-1}(T))/\operatorname{Tor}_{\mathcal{F}} \pi_n(\omega_L^{-1}(T))$, $\pi_j(\omega^{-1}(\Delta)) = 0$ for $j \ge n+1$ and $\pi_j(\omega^{-1}(\Delta)) = \pi_j(\omega_L^{-1}(T))$ for $j \le n-1$.

Consider the following cases.

Case 1: $E = \mathbb{Z}_p$. By (i) of Proposition 2.2, $\pi_j(\omega_L^{-1}(T))$, $j \le n-1$, and $\pi_n(\omega_L^{-1}(T))/\operatorname{Tor}_{\mathcal{F}} \pi_n(\omega_L^{-1}(T))$ are p-torsion. Then $\pi_j(\omega^{-1}(\Delta))$ is p-torsion for $j \le n$. Therefore $\dim_{\pi_j(\omega^{-1}(\Delta))} X \le \dim_{\mathbb{Z}_p} X \le k$ for $j \ge k$ and hence by Proposition 2.4, $\operatorname{e-dim} X \le \omega^{-1}(\Delta)$.

Case 2: $E = \mathbb{Z}_{p^\infty}$. Then by (ii) of Proposition 2.2, $\pi_j(\omega_L^{-1}(T))$, $j \le n-1$, and $\pi_n(\omega_L^{-1}(T))/\operatorname{Tor}_{\mathcal{F}} \pi_n(\omega_L^{-1}(T))$ are p-torsion and $\pi_k(\omega_L^{-1}(T))$ is p-divisible. Then $\pi_j(\omega^{-1}(\Delta))$ is p-torsion for $j \le n$ and $\pi_k(\omega^{-1}(\Delta))$ is p-divisible. Therefore by the Bockstein theorem we have the inequalities $\dim_{\pi_k(\omega^{-1}(\Delta))} X \le \dim_{\mathbb{Z}_{p^\infty}} X \le k$ and $\dim_{\pi_j(\omega^{-1}(\Delta))} X \le \dim_{\mathbb{Z}_{p^\infty}} X + 1 \le k+1$ for $j \ge k+1$. Hence $\operatorname{e-dim} X \le \omega^{-1}(\Delta)$ by Proposition 2.4.

Case 3: $E = \mathbb{Z}_{(p)}$. Then by (iii) of Proposition 2.2, $\pi_j(\omega_L^{-1}(T))$, $j \le n-1$, is p-local and $\pi_n(\omega_L^{-1}(T))/\operatorname{Tor}_{\mathcal{F}} \pi_n(\omega_L^{-1}(T))$ is q-divisible for every $q \in \mathcal{P}$, $q \ne p$. Then $\pi_j(\omega^{-1}(\Delta))$, $j \le n-1$, is p-local and $\pi_n(\omega^{-1}(\Delta))$ is $\mathcal{D}$-divisible and $\mathcal{F}$-torsion free. Therefore $\dim_{\pi_j(\omega^{-1}(\Delta))} X \le k$ for $j \le n-1$ and by Proposition 2.3, $\dim_{\pi_n(\omega^{-1}(\Delta))} X \le n$. Hence $\operatorname{e-dim} X \le \omega^{-1}(\Delta)$ by Proposition 2.4.

Case 4: $E = \mathbb{Q}$. This case is similar to the previous one.

Thus we have shown that $EW(K_i, k)$ is suitable for X. Now replacing K_{i+1} by K_j with a sufficiently large j we may assume by Proposition 2.5(i) that there is a combinatorial lifting of h_{i+1} to $h'_{i+1} : K_{i+1} \to EW(K_i, k)$. By Proposition 2.5(ii) we replace h'_{i+1} by its cellular approximation preserving the property of h'_{i+1} of being a combinatorial lifting of h_{i+1}.

Let Δ be a simplex of K_i and let $\tau : (\alpha_i \circ \omega_L)^{-1}(\Delta) \to \omega^{-1}(\Delta)$ be the inclusion. Note that from the construction it follows that the kernel of the induced homomorphism $\tau_* : \pi_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) \to \pi_n(\omega^{-1}(\Delta))$ is $\mathcal{F}$-torsion. Using this fact and the reasoning in the proof of Theorem 1.2 of [6] one can construct from $K_{i+1}^{[n+1]}$ a CW-complex L_{i+1} by replacing some $(n+1)$-simplexes of $K_{i+1}^{[n+1]}$ by $(n+1)$-cells attached to the boundary of the replaced simplexes by a map of degree $\in S(\mathcal{F})$ such that h'_{i+1} restricted to $K_{i+1}^{[n]}$ extends to a map $g'_{i+1} : L_{i+1} \to EW(L_i, n)$ such that g'_{i+1}, α_{i+1}, h_{i+1} and $\alpha_i \circ \omega_L$ combinatorially commute, where α_{i+1} is a projection of L_{i+1} into K_{i+1} taking the new cells to the original ones in such a way that α_{i+1} is 1-to-1 over $K_{i+1}^{[n]}$.

Now define $g_{i+1} = \omega_L \circ g'_{i+1} : L_{i+1} \to L_i$ and finally define a simplicial structure on L_{i+1} for which α_{i+1} is a combinatorial map. It is easy to check that the properties (a) and (b) are satisfied. Since the triangulation of L_{i+1} can be replaced by any of its barycentric subdivisions we may also assume that

(c) $\operatorname{diam} g_{i+1}^{j}(\Delta) \le 1/i$ for every simplex Δ in L_{i+1} and $j \le i$,

where $g_i^j = g_{j+1} \circ g_{j+2} \circ \ldots \circ g_i : L_i \to L_j$.

Define $Z = \varprojlim(L_i, g_i)$ and let $r_i : Z \to L_i$ be the projections. Clearly $\dim Z \le n+1$. To construct L_{i+1} we used an arbitrary map $f : N \to K(E, k)$ such that $E \in \sigma$, $\dim_E X \le k$, $2 \le k \le n$ and N is a subcomplex of L_i containing $L_i^{[k]}$. By a standard reasoning described in detail in the proof of Theorem 1.6 of [7] one can show that choosing E and f in an appropriate way for each i we can achieve that $\dim_E Z \le k$ for every integer $2 \le k \le n$ and every $E \in \sigma$ such that $\dim_{\mathbb{Z}_p} X \le k$. Then by the Bockstein theorem $\dim_G Z \le k$ for every group G such that $\dim_G X \le k$, $2 \le k \le n$.

By Proposition 2.5(iii), properties (a) and (b) imply that for every $x \in X$ and $z \in Z$ the following holds:

(d1) $g_{i+1}(\alpha_{i+1}^{-1}(\mathrm{st}(p_{i+1}(x)))) \subset \alpha_i^{-1}(\mathrm{st}(p_i(x)))$,
(d2) $h_{i+1}(\mathrm{st}((\alpha_{i+1} \circ r_{i+1})(z))) \subset \mathrm{st}((\alpha_i \circ r_i)(z))$.

Define a map $r : Z \to X$ by $r(z) = \bigcap\{p_i^{-1}(\mathrm{st}((\alpha_i \circ r_i)(z))) : i = 1, 2, \ldots\}$. Then (d2) implies that r is indeed well defined and continuous.

Properties (d1) and (d2) also imply that for every $x \in X$,

$$r^{-1}(x) = \varprojlim(\alpha_i^{-1}(\mathrm{st}(p_i(x))), g_i|_{\alpha_i^{-1}(\mathrm{st}(p_i(x)))})$$

where the map $g_i|_{\ldots}$ is considered as a map to $\alpha_{i-1}^{-1}(\mathrm{st}(p_{i-1}(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, r is onto. Fix $x \in X$ and let us show that $r^{-1}(x)$ satisfies the conclusions of the theorem. Since $T = \alpha_i^{-1}(\mathrm{st}(p_i(x)))$ is $(n-1)$-connected we see that $r^{-1}(x)$ is approximately

$(n-1)$-connected as the inverse limit of $(n-1)$-connected finite simplicial complexes.

Let a group G be such that $\dim_G X \leq n$. Note that $H_n(T)$ is $\mathcal{F}$-torsion and G is $\mathcal{F}$-torsion free. Then by the universal coefficient theorem $H^n(T;G) = \mathrm{Hom}(H_n(T), G) = 0$. Thus $\tilde{\check{H}}^k(r^{-1}(x);G) = 0$ for $k \leq n$ and since $\dim_G Z \leq n$, we have $\tilde{\check{H}}^k(r^{-1}(x);G) = 0$ for $k \geq n+1$. Hence r is G-acyclic and this completes the proof. ∎

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