

Pesin theory and equilibrium measures on the interval

by

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Abstract. We use Pesin theory to study possible equilibrium measures for a broad class of piecewise monotone maps of the interval and a broad class of potentials.

1. Introduction. Our goal is to study possible equilibrium measures for rather general piecewise monotone maps of the interval, possibly with discontinuities. In [7], we developed Pesin theory for interval transformations with unbounded derivative and studied properties of measures absolutely continuous with respect to Lebesgue measure. Here we use the theory to investigate measures absolutely continuous with respect to some conformal measures, mirroring work we did in the complex setting (see [6]).

In general, equilibrium measures are probability measures which encode dynamical information for large sets of points (those seen by the measure) with some property, and they maximise (or minimise, depending on the definition) the *free energy* with respect to the corresponding potentials. Measures of maximal entropy and absolutely continuous invariant measures are important examples of equilibrium measures. Existence and uniqueness of equilibrium measures have long been of interest (see [1–5, 9–11, 18–20], for example). We show in this paper that equilibrium measures must often be of a certain form. In some cases this allows one to show uniqueness [14]. The main result is Theorem 6.

The techniques and proofs rely heavily on those of [7, 6]. Each of those articles extended work on Pesin theory of F. Ledrappier [12, 13]. We refer to [7, 6] for several proofs, so this paper remains brief.

We consider maps f defined on a finite union of open intervals. This does not preclude gaps where f is undefined and is thus quite general. Our results also apply to smooth multimodal maps, for example, as interesting measures tend not to live on the critical orbits, and removing critical points

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from the domain will give cusp maps (see the definition below). We allow unbounded potentials (from the logarithm of the derivative) and also unbounded derivatives, providing an alternate approach to that of [4]. Using the natural extension and looking at measures with positive Lyapunov exponent lets us control distortion along almost all backward branches.

DEFINITION 1. Let I be a non-degenerate compact interval, and let $I_1, \dots, I_p$ be a finite collection of pairwise disjoint open subintervals of I . Following [7], a map $f : \bigcup_{j=1}^p I_j \rightarrow I$ is a *weak piecewise monotone cusp map* (with constants $C, \epsilon > 0$) if on each I_j :

- $f : I_j \rightarrow f(I_j)$ is a diffeomorphism;
- for all $x, x' \in I_j$ such that $|Df(x)|, |Df(x')| \leq 2$,
 $|Df(x) - Df(x')| \leq C|x - x'|^\epsilon$;
- for all $x, x' \in I_j$ such that $|Df(x)|, |Df(x')| \geq 2^{-1}$,

$$\left| \frac{1}{Df(x)} - \frac{1}{Df(x')} \right| \leq C|x - x'|^\epsilon.$$

We say f is a *piecewise monotone cusp map* if, in addition, on each I_j and for each $x' \in \partial I_j$,

$$(1) \quad \lim_{x \rightarrow x'} Df(x) \text{ equals } 0 \text{ or } \pm \infty.$$

Thus the difference between being *weak* or not is the derivative condition at the boundary of the domain of definition. Note that we assume that f is defined on a finite collection of intervals. In [7], we started off allowing f to be defined on a countable collection of intervals, but for some results subsequently assumed that f has at most a finite number of discontinuities (on top of being piecewise monotone). In this paper, the additional hypothesis appears in the definition.

We shall restrict our attention to Borel-measurable sets and Borel measures, without further mention.

Let $\phi : I \rightarrow \mathbb{R}$ be a Hölder continuous function and let $t \in \mathbb{R}$. Consider the relation

$$(2) \quad m(f(A)) = \int_A e^\phi |Df|^t dm.$$

DEFINITION 2. Let $X', X \subset \mathbb{R}$, and let $f : X' \rightarrow X$ be diffeomorphic on each connected component of X' . We say a measure m is (ϕ, t) -conformal for f if (2) holds for every set A on which f is injective.

We emphasise that we do not require the conformal measure to be finite, though we will require that on some open subinterval for our main result. There are indications this could be useful in applications when it is sometimes hard to construct a well-behaved finite conformal measure.

We denote by h_μ the entropy of an invariant probability measure μ . Given a *potential* function $\psi : X \rightarrow \mathbb{R}$, we can define the *pressure*

$$P(\psi, f) := \sup \left\{ h_\mu + \int_X \psi d\mu \right\}$$

where the supremum is taken over all invariant probability measures μ . Any measure μ realising this supremum is called an *equilibrium state* or *equilibrium measure* (for the potential ψ). An equilibrium state μ for ψ is also an equilibrium state for $\psi + C$ for every constant C , and in particular for the potential $\psi_0 := \psi - P(\psi, f)$. Clearly $P(\psi_0, f) = 0$, so

$$h_\mu = \int -\psi_0 d\mu$$

for the equilibrium state μ . Conversely, if $P(\psi_0, f) = 0$ and $h_\mu = \int -\psi_0 d\mu$, then μ is an equilibrium state. Often one can show that there is a (ϕ, t) -conformal measure when $P(-\phi - t \log |Df|, f) = 0$. If that is the case, then a measure satisfying the equivalent conditions in the main theorem is an equilibrium state.

DEFINITION 3. Let f be a cusp map with a (ϕ, t) -conformal measure m . Let U be an open interval. We call G an *expanding induced Markov map* for (f, m) if there is a countable collection of pairwise disjoint intervals $U_i \subset U$ such that:

- $m(U) > 0$;
- $m(U \setminus \bigcup_i U_i) = 0$;
- for each i , there exists n_i such that $f_{|U_i}^{n_i} = G_{|U_i} : U_i \rightarrow U$ is a diffeomorphism;
- there exist $C_0, \delta > 0$ such that, for each i and for $j = 1, \dots, n_i$,

$$(3) \quad |Df^j(x)| > C_0 e^{\delta j}$$

for all $x \in f^{n_i-j}(U_i)$;

- there exists $C_1 > 0$ such that on each U_i , G has distortion bounded by C_1 and $|DG| > 2$.

If additionally $\sum_i n_i m(U_i) < \infty$, then we say that G has *integrable return time*.

LEMMA 4. Let G be as per Definition 3, and suppose G has integrable return time. Let ν (cf. Lemma 22) be the unique absolutely continuous invariant probability measure for G . Set

$$\mu' := \sum_i \sum_{j=0}^{n_i-1} f_*^j \nu.$$

Then $\mu := \mu'/\mu'(I)$ is an ergodic, absolutely continuous f -invariant probability measure.

Proof. Standard. ■

DEFINITION 5. The measure μ of Lemma 4 is said to be *generated* by G .

THEOREM 6. Let $f : \bigcup_{j=1}^p I_j \rightarrow I$ be a weak piecewise monotone cusp map with a (ϕ, t) -conformal measure m . Let μ be an ergodic invariant probability measure with positive entropy and positive finite Lyapunov exponent χ_μ . Suppose that $\text{Supp}(\mu) \subset \text{Supp}(m)$ and that there is an open interval W with $\mu(W) > 0$ and $m(W) < \infty$. Then

- (a) $h_\mu \leq t\chi_\mu + \int \phi d\mu$;
- (b) if $P(-\phi - t \log |Df|, f) > 0$, then there is no ergodic equilibrium state, with positive entropy and positive finite Lyapunov exponent, whose support is included in the support of m and intersects W .

The following conditions are equivalent:

- (i) $\mu \ll m$;
- (ii) $\text{HD}(\mu) = t + \int \phi d\mu / \chi_\mu$;
- (iii) $h_\mu = t\chi_\mu + \int \phi d\mu$;
- (iv) the density of μ with respect to m exists and is bounded from below by a positive constant on an open interval of positive measure;
- (v) the measures μ and m are not mutually singular;
- (vi) there is an expanding induced Markov map for (f, m) with integrable return time which generates μ .

Should the equivalent conditions hold then $P(-\phi - t \log |Df|, f) \geq 0$, with equality if and only if μ is an equilibrium state.

If one considers piecewise monotone cusp maps rather than weak piecewise monotone cusp maps, then the references to positive entropy can be dropped.

Proof. (a) is shown in Proposition 20, and (b) follows from (a). That (i) implies (iii) is Lemma 17. That (ii) is equivalent to (iii) follows from Proposition 9. Lemmas 21 and 23 show that (iii) implies (iv) and (vi), while by definition, (vi) implies (i). Also by definition, (i) and (iv) each imply (v).

It now suffices to prove, by contrapositive, that (v) implies (i). Suppose that μ is not absolutely continuous, so there exists a set A with $\mu(A) > 0 = m(A)$. By conformality, $m(f^n(A)) = 0$ for every $n \geq 0$. Let $X := \bigcup_{n \geq 0} f^n(A)$. Then $m(X) = 0$ while, by ergodicity, $\mu(X) = 1$, so μ and m are mutually singular. ■

Under some sort of transitivity assumptions, perhaps on the support of m , (iv) will imply that μ is the *unique* absolutely continuous (with respect to m) invariant probability measure.

While f does not have critical points, conformal measures may be supported on points which get mapped outside the domain of definition of f . For example, given a unimodal map with a conformal measure supported on the critical point and its backward orbit, the measure will still be conformal for the corresponding cusp map (with the critical point removed from the domain of definition).

Supposing that $\text{Supp}(m) \supset \text{Supp}(\mu)$ is reasonable. For transitive maps, conformal measures usually have support equal to the entire space. For non-transitive maps, they will often still have support some large completely invariant set, for example the complement of a basin of attraction or some such.

Another way of looking at the first statement of the theorem is that conformal measures can only exist for certain combinations of ϕ and t , while existence of a conformal measure bounds the free energies of invariant probability measures.

2. Referring to prior work. As stated in the introduction, this paper is largely based on two preceding works [7, 6]; we cite many results to keep this paper brief and encourage the reader to be especially familiar with [7]. In this paper we deal with weak piecewise monotone cusp maps and measures with positive entropy, or with (non-weak) piecewise monotone cusp maps, where the difference between weak and non-weak is the derivative hypothesis (1). Somewhat unsatisfactorily (for which the author has himself to blame), only the principal results of [7] are stated in both settings, while for our current purposes we need some intermediate results which were only stated under the hypothesis (1).

However, as stated in Section 1.2 of [7], results in Sections 5 through 9 of [7] only depend on the hypothesis (1) via [7, Theorem 4.1], while [7, Theorem 1.6] says that the conclusions of [7, Theorem 4.1] hold with the positive entropy hypothesis in place of (1). Thus results in Sections 5 through 9 of [7] also hold for weak piecewise monotone cusp maps provided μ has positive entropy. We shall cite [7, Lemma 5.4] (Lemma 8 here) and [7, Proposition 5.7] (in the proof of Proposition 11) without further referring to hypotheses.

3. Proof. Given a map f , let $\mathcal{M}(f)$ denote the set of ergodic, f -invariant probability measures.

As per the statement of Theorem 6, let f be a weak piecewise monotone cusp map and let $\mu \in \mathcal{M}(f)$ have positive entropy and positive Lyapunov exponent χ_μ , or let f be piecewise monotone cusp map and let $\mu \in \mathcal{M}(f)$ have positive Lyapunov exponent χ_μ .

We define the natural extension as per [13]. Let

$$Y := \{y = (y_0 y_1 \dots) : f(y_{i+1}) = y_i \in I\}.$$

Define $F^{-1} : Y \rightarrow Y$ by $F^{-1}((y_0 y_1 \dots)) := (y_1 y_2 \dots)$. Then F^{-1} is invertible with inverse $F : F^{-1}(Y) \rightarrow Y$. The projection $\Pi : Y \rightarrow I$ is defined by $\Pi : y = (y_0 y_1 \dots) \mapsto y_0$. Then $\Pi \circ F = f \circ \Pi$. There exists a unique F -invariant measure $\bar{\mu}$ such that $\Pi_* \bar{\mu} = \mu$; moreover $\bar{\mu} \in \mathcal{M}(F)$ and $\bar{\mu} \in \mathcal{M}(F^{-1})$ (see [16]).

We call the triplet $(Y, F, \bar{\mu})$ the *natural extension* of (f, μ) (it is also called the *Rohlin* (or *Rokhlin*) *extension* or the *canonical extension*).

Let us remark that invariant probability measures give no mass to the sets of points x for which there is an $n > 0$ such that $f^n(x)$ is not defined, nor do they give mass to the set of x for which there exists an $n > 0$ and *no solution* x' to $f^n(x') = x$. Thus, $F^n(y)$ is defined for all $n \in \mathbb{Z}$ for $\bar{\mu}$ -almost every $y \in Y$.

We have the following unstable manifold theorem: around almost every point in the natural extension, one can pull back an interval along the corresponding branch as far as one wants with bounded distortion and exponential shrinking (and without meeting boundary points or discontinuities). It holds for weak piecewise monotone cusp maps under the positive entropy hypothesis, and for piecewise monotone cusp maps without the entropy hypothesis.

THEOREM 7 ([7, Theorem 1.6 (if weak) and Theorem 4.1 (otherwise)]). *There exists a measurable function α on Y , $0 < \alpha < 1/2$ almost everywhere, such that for $\bar{\mu}$ -almost every $y \in Y$ there exists a set $V_y \subset Y$ with the following properties:*

- $y \in V_y$ and $\Pi V_y = B(\Pi y, \alpha(y))$;
- for each $n > 0$, $f^n : \Pi F^{-n} V_y \rightarrow \Pi V_y$ is a diffeomorphism (in particular it is onto);
- for all $y' \in V_y$,

$$(4) \quad \sum_{i=1}^{\infty} |\log |Df(\Pi F^{-i} y')| - \log |Df(\Pi F^{-i} y)|| < \log 2;$$

- for each $\eta > 0$ there exists a measurable function ρ on Y , $0 < \rho(y) < \infty$ almost everywhere, such that

$$\rho(y)^{-1} e^{n(\chi_\mu - \eta)} < |Df^n(\Pi F^{-n} y)| < \rho(y) e^{n(\chi_\mu + \eta)}.$$

In particular, $|\Pi F^{-n} V_y| \leq 2\rho(y) e^{-n(\chi_\mu - \eta)}$.

With a non-trivial amount of work, one can then prove the following lemma (not subsequently used in this paper, but worth restating) and proposition:

LEMMA 8 ([7, Lemma 5.4]). *Given any interval V of positive μ -measure, there is a $j > 0$ such that $\mu(\bigcup_{k=0}^j f^k(V)) = 1$.*

PROPOSITION 9 ([7, Proposition 6.2 and Theorem 1.7]). $\text{HD}(\mu) = h_\mu/\chi_\mu$.

DEFINITION 10 ([8]). An open interval U is *regularly returning* if $f^n(\partial U) \cap U = \emptyset$ for all $n > 0$. This is also called a *nice interval* in the literature.

If A is a connected component of $f^{-n}(U)$ and B is a connected component of $f^{-m}(U)$ with $m \geq n$, it is easy to check that either $A \cap B = \emptyset$ or $B \subset A$, so inverse images of regularly returning intervals are either *nested* or *disjoint*. Indeed, suppose $x \in \partial A \cap B$. Then $f^n(x) \in \partial U$ (since f may be discontinuous, one uses the fact that $x \in B$ to infer that f^n is defined on a neighbourhood of x), but $f^m(x) \in U$, a contradiction.

PROPOSITION 11. *Almost every point with respect to μ is contained inside arbitrarily small regularly returning intervals.*

Proof. First consider the case that μ is atomic, supported on the (necessarily repelling) orbit of a periodic point p of minimal period k , say. Let S denote the set of points not in the orbit of p and with period divisible by k . Then $f(S) = S$ and S is relatively closed in the domain of definition of f . For each $n \geq 1$, let U_n be the maximal interval containing p on which f^n is defined and for which $f^{n-1}(U_n) \cap S = \emptyset$. One can check that U_n is open, f^n is diffeomorphic on U_n and $U_n \supset U_{n+1}$. If f^n is not defined at a point x , then f^{n-j} is not defined at $f^j(x)$ for $j \geq 0$, so $f^j(x) \notin U_n$ for all $j \geq 0$. Similarly, if $f^{n-1}(x) \in S$, then $f^{n-1+j}(x) \in S$, so $f^j(x) \notin U_n$ for all $j \geq 0$. In particular, U_n is regularly returning.

However, $\bigcap_{n \geq 1} U_n$ need not necessarily equal $\{p\}$. Rather, we claim, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and all n sufficiently large, $B(p, \varepsilon) \cap U_n$ is regularly returning. First, choose ε_0 small enough that $B(p, \varepsilon_0) \subset U_k$ and small enough that, for each $\varepsilon \in (0, \varepsilon_0)$, $|f^k(p \pm \varepsilon) - p| > \varepsilon$. Since U_n is a decreasing sequence of sets, for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$,

$$(U_n \setminus U_{n+1}) \cap \partial B(p, \varepsilon) = \emptyset.$$

Now fix $\varepsilon \in (0, \varepsilon_0)$, let $n > N(\varepsilon)$ and set $B_n := B(p, \varepsilon) \cap U_n$. For $x \in \partial B_n$, either $x \in \partial U_n$, in which case $f^j(x) \notin B_n$ for all j since U_n is regularly returning, or $x \in U_j$ for all $j \geq 1$. In this latter case, for all $j \geq 1$, f^j is diffeomorphic on (p, x) and $f^j((p, x)) \cap S = \emptyset$. As p is repelling, each point $p' \neq p$ in the orbit of p is separated from p either by a point from S or a point where f is not defined; we deduce that for $a \geq 1$ and $j = 1, \dots, k-1$,

$$f^{ak+j}((p, x)) \cap U_1 = \emptyset,$$

so $f^{ak+j}(x) \notin B_n$. Finally, by choice of ε_0 , we have $f^{ak}(x) \notin B(p, \varepsilon) \supset B_n$

for each $a \geq 1$. Thus $f^j(x) \notin B_n$ for each $j \geq 1$. Therefore B_n is regularly returning, completing the proof when μ is atomic.

When μ is non-atomic, we cite [7, Proposition 5.7]. ■

Recall that W (of Theorem 6) is an open interval with $\mu(W) > 0$ and $m(W) < \infty$. Let α, ρ and V_y be given by Theorem 7, for $\eta = \chi_\mu/2$ say.

PROPOSITION 12. *There exist a regularly returning interval $U \subset W$ contained in some branch I_j of f , a constant $K > 0$ and a set $A \subset \Pi^{-1}U$ with the following properties:*

- $\bar{\mu}(A) > 0$;
- for $y \in A$, there is a $y' \in A$ such that $y \in V_{y'}$, $\rho(y') < K$, $\Pi V_{y'} \supset U$ and $U_y := V_y \cap \Pi^{-1}U \subset A$.

Proof. Without loss of generality (considering some $W \cap I_j$ with positive measure), we can assume that W is contained in some branch I_j of f . Choose $\alpha_0, K > 0$ such that there is a positive measure set $A_1 \subset Y$ of points y for which $\alpha(y) > \alpha_0$, $\rho(y) < K$ and $\Pi y \in W$. By Proposition 11, there is a collection of regularly returning intervals, with each interval contained in W and of length at most α_0 , such that the collection covers a full-measure subset of ΠA_1 . Since $\mathbb{R}$ is Lindelöf, there is a countable subcover. Therefore there is at least one regularly returning interval $U \subset W$ with $|U| \leq \alpha_0$ for which the set

$$A_2 := A_1 \cap \Pi^{-1}U$$

has positive measure. For $y \in A_2$ we have $\rho(y) < K$ and $\alpha(y) > \alpha_0 \geq |U|$, so the set V_y given by Theorem 7 satisfies $\Pi V_y \supset U$. Finally, set

$$A := \bigcup_{y \in A_2} V_y \cap \Pi^{-1}U.$$

Since $A \supset A_2$, we have $\bar{\mu}(A) > 0$. ■

Thus we can fix a regularly returning interval U , a corresponding set $A \subset Y$ and sets U_y as per the proposition. For each $y \in A$ and $n \geq 0$, f^n maps the exponentially small (in n) interval $\Pi f^{-n}U_y$ diffeomorphically and with distortion bounded by 2 onto U . Let $\mathcal{P}$ denote the partition $\{I_1, \dots, I_p\} \vee \{I \setminus U, U\}$. Denote by $\mathcal{P}_n(x)$ the element of $\bigvee_{i=0}^n f^{-i}\mathcal{P}$ containing x . Almost every x is the projection of a point y which enters A infinitely often, at times n_j say. Thus x is contained in an interval mapped diffeomorphically by f^{n_j} onto U , so the interval must be $\mathcal{P}_{n_j}(x)$. It follows that $\mathcal{P}_n(x)$ shrinks to the point x as $n \rightarrow \infty$ and we have the following (see also [7, Proposition 6.1]):

LEMMA 13. *$\mathcal{P}$ is a generating partition.*

Mean conditional entropy $H(\cdot|\cdot)$ of one measurable partition with respect to another is defined in [17, §5]. Since $\mathcal{P}$ is generating, the finite partition

$\Pi^{-1}\mathcal{P}$ of Y is generating. Setting

$$\zeta := \bigvee_{i=0}^{\infty} F^i \Pi^{-1}\mathcal{P},$$

we trivially have $H(F^{-1}\zeta|\zeta) = H(F^{-1}\Pi^{-1}\mathcal{P}|\zeta)$. By [17, §9.4, §7.1] (with $T = F^{-1}$), we deduce that

$$(5) \quad H(F^{-1}\zeta|\zeta) = h_{\mu}.$$

Now we introduce another partition of Y . For $y \in Y$, let $e(y)$ denote the first entry time of y to A , defined almost everywhere. Set

$$\xi(y) := F^{-e(y)}U_{F^{e(y)}y}.$$

Since U is regularly returning, $\xi(y)$ and $\xi(y')$ are nested or disjoint for each pair $y, y' \in Y$. Hence, for all $y' \in \xi(y)$ we have $e(y) = e(y')$ and $\xi(y) = \xi(y')$. Therefore we can define a measurable partition ξ of Y where the element of ξ containing y is just $\xi(y)$. Note that we have chosen a definition of ξ corresponding to that of [6] rather than [7].

We remark that $\xi(y) = F^{-1}(\xi(Fy))$ unless $y \in A$, in which case we have $\xi(y) \supset F^{-1}(\xi(Fy))$. The projection Π is injective on each element of ξ and thus on each element of $F\xi$, and the images are open intervals. It follows that $F^{-1}\xi$ is a *countable* refinement of ξ , that is, each element of ξ contains at most a countable number of elements of $F^{-n}\xi$. Therefore, for each $n \in \mathbb{N}$, $F^{-n}\xi$ is a countable refinement of ξ .

Associated to a measurable partition η of Y is a canonical system of (conditional) probability measures known as the *Rokhlin decomposition* of $\bar{\mu}$ with respect to η , denoted p_{η} . For each y , $p_{\eta}(y, \cdot)$ is a probability measure (coinciding with $p(y', \cdot)$ for every $y' \in \eta(y)$) on the element $\eta(y)$ of η containing y , and for all $X \subset Y$,

$$p_{\eta}(y, X) = p_{\eta}(y, X \cap \eta(y)) \quad \text{and} \quad \bar{\mu}(X) = \int_Y p_{\eta}(y, X) d\bar{\mu}.$$

If η' is a refinement of η , then $H(\eta'|\eta) = \int -\log p_{\eta}(y, \eta'(y)) d\bar{\mu}$ by definition. We refer to [17, §1.7, §5] for the general theory.

LEMMA 14. *Let $i \in \mathbb{Z}$, and suppose $F^{-i}\eta$ is a countable refinement of η . If $B \subset \eta(F^i y)$, then*

$$(6) \quad p_{\eta}(F^i y, B) = p_{\eta}(y, F^{-i}B) / p_{\eta}(y, F^{-i}(\eta(F^i y))).$$

Proof. First let us show that almost everywhere,

$$(7) \quad \theta(y) := p_{\eta}(y, F^{-i}(\eta(F^i y))) > 0.$$

Indeed, let Z be a measurable set on which $\theta(\cdot) = 0$ and let $y \in Z$. Since $F^{-i}\eta$ is a countable refinement, there is a sequence $y_1, y_2, \dots$ in Z for which

$Z \cap \eta(y) \subset \bigcup_{j \geq 1} F^{-i}(\eta(F^i y_j))$. Thus $p_\eta(y, Z) \leq \sum_{j \geq 1} \theta(y_j) = 0$, and so $p_\eta(y, Z) = 0$ for all $y \in Z$. Therefore $p_\eta(y, Z) = 0$ for all $y \in Y$, so Z has measure zero, showing (7).

Define $p_{\eta,i}(y, X) := p_\eta(F^i y, F^i X)$. Then

$$\int_Y p_{\eta,i}(y, X) = \int_Y p_\eta(F^i y, F^i X) = \bar{\mu}(F^i X) = \bar{\mu}(X).$$

For y' in the same element of $F^{-i}\eta$ as y , we have $\eta(F^i y) = \eta(F^i y')$ and it follows that $p_{\eta,i}(y, \cdot) = p_{\eta,i}(y', \cdot)$. Thus $p_{\eta,i}$ is the Rokhlin decomposition of $\bar{\mu}$ with respect to the partition $F^{-i}\eta$. As $F^{-i}\eta$ is a refinement of η , $p_{\eta,i}$ is also a decomposition of the system of measures p_η (see the *transitivity property* in [17, §1.7]). Hence, on $F^{-i}(\eta(F^i y))$,

$$p_{\eta,i}(y, \cdot)\theta(y) = p_\eta(y, \cdot).$$

Thanks to our definition of $p_{\eta,i}$, we obtain

$$p_\eta(F^i y, B)\theta(y) = p_\eta(y, F^{-i}B),$$

which, once rearranged, is the desired equality. ■

PROPOSITION 15. *For each $n \geq 1$, the entropy of μ satisfies $nh_\mu = H(F^{-n}\xi|\xi)$.*

Proof. See also the proof of [6, Proposition 30]. We have

$$\begin{aligned} H(F^{-n-1}\xi|\xi) &= \int_Y -\log p_\xi(y, F^{-n-1}(\xi(F^{n+1}y))) d\bar{\mu} \\ &= \int_Y -\log p_\xi(y, F^{-n}(\xi(F^n y))) d\bar{\mu} + \int_Y -\log \frac{p_\xi(y, F^{-n-1}(\xi(F^{n+1}y)))}{p_\xi(y, F^{-n}(\xi(F^n y)))} d\bar{\mu} \\ &= H(F^{-n}\xi|\xi) + \int_Y -\log p_\xi(F^n y, F^{-1}(\xi(F^{n+1}y))) d\bar{\mu} \\ &= H(F^{-n}\xi|\xi) + H(F^{-1}\xi|\xi), \end{aligned}$$

having used (6) to pass to the third line, and invariance of $\bar{\mu}$ to pass to the last. Hence $H(F^{-n}\xi|\xi) = nH(F^{-1}\xi|\xi)$, so it suffices to prove the proposition when $n = 1$.

Let p_ζ and p_ξ be the Rokhlin decompositions of $\bar{\mu}$ corresponding to ζ and ξ . From the definition of ξ , if $y \notin A$ then $\xi(y) = F^{-1}(\xi(Fy))$, so $p_\xi(y, F^{-1}(\xi(Fy))) = 1$. Hence

$$\begin{aligned} (8) \quad H(F^{-1}\xi|\xi) &= \int_Y -\log p_\xi(y, F^{-1}(\xi(Fy))) d\bar{\mu} \\ &= \int_A -\log p_\xi(y, F^{-1}(\xi(Fy))) d\bar{\mu}. \end{aligned}$$

Let $r(y)$ denote the first return time of the point y to A . Then on A we have $F^{-1}(\xi(Fy)) = F^{-r(y)}(\xi(F^{r(y)}y))$ and $\xi(y) = \zeta(y) = U_y$. Thus

$$(9) \quad p_\zeta(y, F^{-r(y)}(\zeta(F^{r(y)}y))) = p_\xi(y, F^{-1}(\xi(Fy))).$$

Repeated application of (6) gives

$$(10) \quad \prod_{i=0}^{r(y)-1} p_\zeta(F^i y, F^{-1}(\zeta(F^{i+1}y))) = \prod_{i=0}^{r(y)-1} \frac{p_\zeta(y, F^{-i-1}(\zeta(F^{i+1}y)))}{p_\zeta(y, F^{-i}(\zeta(F^i y)))} \\ = p_\zeta(y, F^{-r(y)}(\zeta(F^{r(y)}y))).$$

Passing from sum of logarithms to logarithm of product in the following, we deduce from (5), (10), (9) and (8) that

$$h_\mu = H(F^{-1}\zeta|\zeta) = \int_Y -\log p_\zeta(y, F^{-1}(\zeta(Fy))) d\bar{\mu} \\ = \int_A -\sum_{i=0}^{r(y)-1} \log p_\zeta(F^i y, F^{-1}(\zeta(F^{i+1}y))) d\bar{\mu} \\ = \int_A -\log p_\xi(y, F^{-1}(\xi(y))) d\bar{\mu} = H(F^{-1}\xi|\xi),$$

completing the proof. ■

Let $\psi := \phi + t \log |Df|$. Let

$$(S_{-n}\psi)(y) := \sum_{i=1}^n \psi \circ \Pi \circ F^{-i}(y).$$

The choice of the partition ξ renders the following lemma relatively simple.

LEMMA 16. *There are uniform (independent of n, y and $y' \in \xi(y)$) upper and lower bounds on $(S_{-n}\psi)(y') - (S_{-n}\psi)(y)$, and the limit*

$$\lim_{n \rightarrow \infty} ((S_{-n}\psi)(y') - (S_{-n}\psi)(y))$$

exists.

Proof. See the proof of [6, Lemma 31]. Exponential decrease of preimages gives a bound on the Hölder part, and Theorem 7 takes care of the derivative part. ■

Define $\Phi(y, \cdot) : \xi(y) \rightarrow \mathbb{R}$ by

$$\Phi(y, y') := \lim_{n \rightarrow \infty} e^{(S_{-n}\psi)(y') - (S_{-n}\psi)(y)},$$

so $\Phi(y, \cdot)$ is uniformly bounded away from zero and infinity, by Lemma 16.

LEMMA 17. *Suppose μ is absolutely continuous with respect to m . Then $h_\mu = t\chi_\mu + \int \phi d\mu$.*

Proof. Recall that almost every $x \in I$ is the projection of a point y that enters A infinitely often, at times n_j , say. Then $\mathcal{P}_{n_j}(x)$ is an interval containing x mapped by f^{n_j} with distortion bounded by 2 onto U . By Lemma 16, ergodicity and conformality, it follows that

$$-\frac{1}{n_j} \log m(\mathcal{P}_{n_j}(x))$$

converges to $t\chi_\mu + \int \phi d\mu$. Meanwhile, the Shannon–McMillan–Breiman Theorem says that

$$-\frac{1}{n_j} \log \mu(\mathcal{P}_{n_j}(x))$$

converges to h_μ almost everywhere. Set $\gamma := h_\mu - t\chi_\mu - \int \phi d\mu$ and suppose $\gamma \neq 0$. Then for some set $X \subset W$ with $\mu(X) = \mu(W) > 0$, for each $x \in X$ there are arbitrarily large n for which the sets $\mathcal{P}_n(x) \subset W$ are intervals satisfying

$$(11) \quad \left| -\frac{1}{n} \log \mu(\mathcal{P}_n(x)) + \frac{1}{n} \log m(\mathcal{P}_n(x)) - \gamma \right| < \left| \frac{\gamma}{2} \right|.$$

Of course, as partition elements, these intervals have the nested or disjoint property. For any large N there is therefore a disjoint cover of X by such intervals (of the form $\mathcal{P}_n(x) \subset W$ and satisfying (11)) with $n \geq N$. Supposing $\gamma > 0$, we deduce that

$$\mu(X) \leq e^{-N\gamma/2} m(W) < m(W) < \infty,$$

recalling that $m(W) < \infty$ by hypothesis. Letting N tend to infinity we deduce $\mu(X) = 0$, a contradiction. On the other hand, if $\gamma < 0$ then similarly

$$\mu(W) \geq e^{-N\gamma/2} m(X) \geq m(X).$$

Letting $N \rightarrow \infty$ we derive a contradiction as μ is finite, while $m(X) > 0$ by absolute continuity. ■

Note that we actually proved something extra in deriving a contradiction from $\gamma > 0$, namely that $h_\mu \leq t\chi_\mu + \int \phi d\mu$, which we shall reprove shortly. Also, in the proof, note that the regularly returning property ensured that our partition elements $\mathcal{P}_{n_j}(x)$ were intervals.

Let $p := p_\xi$ denote the Rokhlin decomposition of $\bar{\mu}$ with respect to ξ . Writing ξ_{-n} for $F^{-n}\xi$ for clarity and recalling Proposition 15, we get

$$(12) \quad nh_\mu = H(F^{-n}\xi|\xi) = - \int_Y \log p(y, \xi_{-n}(y)) d\bar{\mu}.$$

We note that $p(y, \xi_{-n}(y)) > 0$ almost everywhere. Let

$$(13) \quad q(y, dz) := \frac{\Phi(y, z) dm_{\xi(y)}(z)}{\int_{\xi(y)} \Phi(y, y') dm_{\xi(y)}(y')},$$

where $m_{\xi(y)}$ is the pullback by $\Pi|_{\xi(y)}$ of the conformal measure m restricted to $\Pi\xi(y)$. Because $\text{Supp}(m) \supset \text{Supp}(\mu)$ and elements of ξ_{-n} project onto open intervals, we see that $q(y, \xi_{-n}(y)) > 0$ $\bar{\mu}$ -almost everywhere. However, $q(\cdot, \xi_{-n}(\cdot))$ may be positive on sets (of zero $\bar{\mu}$ -measure) where $p(\cdot, \xi_{-n}(\cdot))$ is not. The function q is our best guess as to how p would look, were μ absolutely continuous, informed by the change of variables formula and the notion that on elements of ξ_{-n} for large n , the densities should be almost constant.

LEMMA 18.

$$(14) \quad - \int \log q(y, \xi_{-n}(y)) d\bar{\mu} = n \left(t\chi_\mu + \int \phi d\mu \right).$$

Proof. This is shown in the proof of [6, Proposition 32]. ■

Comparing q and p will, using equations (12) and (14), allow us to relate h_μ and χ_μ . Define $\bar{\nu}$ on measurable subsets of Y by

$$\bar{\nu}(B) = \int_Y q(y, B) d\bar{\mu}(y).$$

Denote by Y_n the quotient space $Y|_{\xi_{-n}}$, and by $\bar{\nu}_n$ and $\bar{\mu}_n$ the corresponding push-forwards of $\bar{\nu}$ and $\bar{\mu}$ under the quotient map. For each point $V_n \in Y_n$, we define $q_n(V_n) := q(y, V)$ and $p_n(V_n) := p(y, V)$ for any $y \in V \subset Y$, where V is the element of ξ_{-n} which projects to V_n . By (7), $p_n > 0$ almost everywhere with respect to $\bar{\mu}_n$.

LEMMA 19. *Almost everywhere with respect to $\bar{\mu}_n$,*

$$q_n/p_n = d\bar{\nu}_n/d\bar{\mu}_n.$$

Proof. Let $\kappa, c, \varepsilon > 0$. Let $H_n \subset Y_n$ be a set on which $p_n > \kappa > 0$ and $|q_n - cp_n| < \varepsilon$. Let H be the corresponding subset of Y . Furthermore, given $y \in Y$ let $H_n^y \subset H_n$ denote the projection of $H \cap \xi(y)$ in Y_n or, equivalently, the set H_n intersected with the projection of $\xi(y)$. Then for $y \in H$,

$$(15) \quad 1 \geq p(y, H) = \sum_{V \in H_n^y} p_n(V) > \kappa \# H_n^y.$$

In particular, $\# H_n^y < 1/\kappa$, so

$$(16) \quad |q(y, H) - cp(y, H)| < \frac{1}{\kappa} \sup_{H_n} |q_n - cp_n| \leq \varepsilon/\kappa.$$

Let $H^* := \{y : \xi(y) \cap H \neq \emptyset\}$. On $Y \setminus H^*$, $q(y, H) = p(y, H) = 0$, while on H^* , $p(y, H^*) = 1 \leq p(y, H)/\kappa$, the inequality holding by (15). Integrating the latter gives

$$(17) \quad \bar{\mu}(H^*) \leq \bar{\mu}(H)/\kappa.$$

Writing $\theta := \int_Y (q(y, H) - cp(y, H)) d\bar{\mu}$, we derive from (16) and (17) that

$|\theta| \leq \varepsilon \bar{\mu}(H)/\kappa^2$. Thus

$$\begin{aligned} \int_{H_n} \frac{d\bar{\nu}_n}{d\bar{\mu}_n} d\bar{\mu}_n &= \bar{\nu}_n(H_n) = \int_Y q(y, H) d\bar{\mu} = \theta + \int_Y cp(y, H) d\bar{\mu} \\ &= \theta + \int_H c d\bar{\mu} = \theta + \int_{H_n} \frac{q_n}{p_n} d\bar{\mu}_n + \int_{H_n} \left(c - \frac{q_n}{p_n} \right) d\bar{\mu}_n. \end{aligned}$$

Therefore

$$\int_{H_n} \frac{d\bar{\nu}_n}{d\bar{\mu}_n} d\bar{\mu}_n = \int_{H_n} \frac{q_n}{p_n} d\bar{\mu}_n + \varepsilon_*$$

for some $|\varepsilon_*| \leq \bar{\mu}_n(H_n)2\varepsilon/\kappa^2$. Noting the lack of dependence on c , we deduce that for any set $J_n \subset Y_n$ with $p_n > \kappa > 0$ on J_n ,

$$\int_{J_n} \left| \frac{d\bar{\nu}_n}{d\bar{\mu}_n} - \frac{q_n}{p_n} \right| d\bar{\mu}_n \leq \frac{2\varepsilon}{\kappa^2}.$$

Letting ε and then κ go to zero yields the lemma. ■

REMARK. In the preceding lemma, we do not at all claim absolute continuity of $\bar{\nu}_n$. It may clarify matters to give an example where $\bar{\nu}_n$ is not absolutely continuous with respect to $\bar{\mu}_n$. Suppose $f : x \mapsto 2x \pmod{1}$. Let m be Lebesgue measure, a $(0, 1)$ -conformal measure. Take for U the interval $(0, 1/2)$, and for A the set $\Pi^{-1}U$, so for $y \in A$, $\Pi\xi(y) = (0, 1/2)$. Then $\Phi \equiv 1$ on each $\xi(y)$. By (13),

$$q(y, \xi(y) \cap \Pi^{-1}(0, 1/4)) = 1/2.$$

If $\mu((0, 1/4)) = 0$ and $\mu((1/4, 1/2)) = c > 0$, say, it follows that

$$\bar{\nu}_n(\Pi^{-1}(0, 1/4)) = c/2.$$

Thus for each $n \geq 1$, $\bar{\nu}_n(\Pi^{-1}(0, 1/4)) > 0$ while its $\bar{\mu}_n$ -measure is 0.

PROPOSITION 20. *We have $h_\mu \leq t\chi_\mu + \int \phi d\mu$. If equality holds, then $q = p$ and μ is absolutely continuous with respect to m .*

Proof. By equations (12) and (14),

$$n \left(h_\mu - t\chi_\mu - \int \phi d\mu \right) = \int_{Y_n} \log \frac{q_n}{p_n} d\bar{\mu}_n \leq \log \int_{Y_n} \frac{q_n}{p_n} d\bar{\mu}_n,$$

the latter by concavity of logarithm. But by Lemma 19, the latter expression is bounded above by $\log \bar{\nu}_n(Y_n) = 0$ (it may be negative if $\bar{\nu}_n$ is not absolutely continuous with respect to $\bar{\mu}_n$). Thus,

$$h_\mu \leq t\chi_\mu + \int \phi d\mu.$$

Equality can only hold if $q_n = p_n$ almost everywhere. If this holds for all n , then $q = p$ almost everywhere, so μ is absolutely continuous. ■

LEMMA 21. *Suppose $h_\mu = t\chi_\mu + \int \phi d\mu$. Then the density of μ with respect to m is bounded away from zero on the interval U .*

Proof. We recall that A was defined subsequent to the proof of Proposition 12. Let $\bar{\mu}_A$ denote the restriction of $\bar{\mu}$ to A . Since the function Φ defined after the proof of Lemma 16 is uniformly bounded away from zero (and infinity) on A and since (13) is an expression for the Rokhlin decomposition q of $\bar{\mu}$, the density of $\Pi_*\bar{\mu}_A$ is bounded away from zero on U . For each $C \subset U$, we have $\Pi_*\bar{\mu}_A(C) \leq \bar{\mu}(\Pi^{-1}C) = \mu(C)$, hence the density of μ is bounded from below away from zero on U . ■

LEMMA 22. *Let G be an expanding induced Markov map for (f, m) with range $V \subset W$. Then there is an ergodic absolutely continuous G -invariant probability measure ν with density uniformly bounded away from zero and infinity m -almost everywhere on V .*

Proof. See [6, Lemma 35] for the proof, which is a little more involved than the standard Folklore Lemma. ■

LEMMA 23. *Suppose $h_\mu = t\chi_\mu + \int \phi d\mu$. There exists an expanding induced Markov map for (f, m) , with range U and integrable return time, which generates μ .*

Proof. See also [6, Proposition 34]. Let $N > 0$ be large enough that for all $n \geq N$ and all $y \in A$ we have $|Df^n(\Pi F^{-n}y)| > 2$, and for each $y \in A$ let $s(y)$ denote the N th return time of y to A . In particular $s(y) \geq N$ almost everywhere on A , and by a simple extension of Kac's Theorem [15, Theorem 2.4.6],

$$\int_A s(y) d\bar{\mu} = N.$$

Then $\mathcal{Q} := \{\Pi F^{-s(y)}(\xi(F^{s(y)})) : y \in A\}$ is a collection of nested or disjoint open intervals which each get mapped by an iterate of f diffeomorphically onto U . Moreover, the set $X := \bigcup_{J \in \mathcal{Q}} J$ has $\mu(X) = \mu(U)$. By the nested or disjoint property, the maximal (under inclusion) elements of $\mathcal{Q}$ form a countable partition $\{U_i\}_{i \in \mathbb{N}}$ of X . Let f^{n_i} be the corresponding iterates.

Defining $G : \bigcup U_i \rightarrow U$ by $G|_{U_i} := f^{n_i}_{|U_i}$, we obtain an expanding induced Markov map for (f, m) :

By construction, the topological conditions of Definition 3 are verified, as is the lower bound on the derivative, $|DG| > 2$. Since $\mu(X) = \mu(U)$, Lemma 21 implies that $m(X) = m(U)$, so $m(U \setminus \bigcup_i U_i) = 0$.

The distortion bound of 2 ensues from (4).

Let us check condition (3): By choice of A there is a $K > 0$ and for each $y \in A$ there is a $y' \in U_y$ such that $\rho(y') < K$ (see Proposition 12 and, subsequent to its proof, the definition of A). Then by the definitions of

ρ and U_i and by bounded distortion, $|Df^j| \geq e^{j\chi_\mu/2}/2K$ on $f^{n_i-j}(U_i)$ for $j = 1, \dots, n_i$.

Finally, let us check that G has integrable return time: Let $\varepsilon > 0$ be a lower bound for the density of μ on U given by Lemma 21. We remark that for each i , we have $n_i = \inf\{s(y) : y \in A \cap \Pi^{-1}U_i\}$ since the U_i were maximal under inclusion. Thus, writing $n(x) := n_i$ if $x \in U_i$, we get

$$\begin{aligned} \sum_i n_i m(U_i) &= \int_U n(x) dm \leq \frac{1}{\varepsilon} \int_U n(x) d\Pi_* \bar{\mu}_A = \frac{1}{\varepsilon} \int_A n(\Pi y) d\bar{\mu}_A \\ &\leq \frac{1}{\varepsilon} \int_A s(y) d\bar{\mu}_A = \frac{1}{\varepsilon} \int_A s(y) d\bar{\mu} = \frac{1}{\varepsilon} N < \infty, \end{aligned}$$

as required, concluding the proof. ■

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