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Amenability and Ramsey theory in the metric setting

by

Adriane Kaïchouh (Lyon)

Abstract. Moore [Fund. Math. 220 (2013)] characterizes the amenability of the automorphism groups of countable ultrahomogeneous structures by a Ramsey-type property. We extend this result to the automorphism groups of metric Fraïssé structures, which encompass all Polish groups. As an application, we prove that amenability is a G_δ condition.

Introduction. In recent years, there has been a flurry of activity relating notions linked to amenability of groups on one side, and combinatorial conditions linked to Ramsey theory on the other side. In this paper, we extend a result of Moore [Mo, Theorem 7.1] on the amenability of closed subgroups of S_∞ to general Polish groups. A topological group is said to be *amenable* if every continuous action of the group on a compact Hausdorff space admits an invariant probability measure.

Moore's result is the counterpart of a theorem of Kechris, Pestov and Todorčević [KPT] on extreme amenability. A topological group is said to be *extremely amenable* if every continuous action of the group on a compact Hausdorff space admits a fixed point. In the context of closed subgroups of S_∞, which are exactly the automorphism groups of Fraïssé structures, Kechris, Pestov and Todorčević characterize extreme amenability by a combinatorial property of the associated Fraïssé classes (in the case where its objects are rigid), namely, the Ramsey property. A class $\mathcal{K}$ of structures is said to have the *Ramsey property* if for all structures A and B in $\mathcal{K}$, and all integers k, there is a structure C in $\mathcal{K}$ such that for every coloring of the set of copies of A in C with k colors, there exists a copy of B in C within which all copies of A have the same color.

Thus, extreme amenability, which provides fixed points, corresponds to colorings having a "fixed", meaning monochromatic, set. Amenability, on

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the other hand, provides invariant measures. Since a measure is not far from being a barycenter of point masses, the natural mirror image of the Ramsey property in that setting should be for a coloring to have a "monochromatic convex combination of sets". Indeed, Tsankov (in an unpublished note) and Moore introduced a *convex Ramsey property* and proved that a Fraïssé class has the convex Ramsey property if and only if the automorphism group of its Fraïssé limit is amenable.

Moreover, the Kechris–Pestov–Todorčević result was extended to general Polish groups by Melleray and Tsankov [MT1]. They use the framework of continuous logic [BYBHU] via the observation that every Polish group is the automorphism group of an approximately homogeneous metric structure [Me, Theorem 6], that is, of a metric Fraïssé limit (in the sense of [MT1]; these were built by Ben Yaacov [BY]). They define an *approximate Ramsey property* for classes of metric structures, and then show that a metric Fraïssé class has the approximate Ramsey property if and only if the automorphism group of its Fraïssé limit is extremely amenable.

In this paper, we "close the diagram" by giving a metric version of Moore's result. We replace the classical notion of a coloring with the metric one (from [MT1]) to define a *metric convex Ramsey property*, and we prove the exact analogue of Moore's theorem (Theorem 4.3):

THEOREM 0.1. *Let $\mathcal{K}$ be a metric Fraïssé class,* $\mathbf{K}$ *its Fraïssé limit and G the automorphism group of* $\mathbf{K}$*. Then G is amenable if and only if $\mathcal{K}$ has the metric convex Ramsey property.*

From this result, we deduce some interesting structural consequences about amenability. First, we improve the previously known characterization of amenability mentioned below.

If G is a topological group, all minimal continuous actions of G on compact Hausdorff spaces can be embraced in a single one: the action of G by translation on its *greatest ambit* $S(G)$ (see [P1]). In particular, the topological group G is amenable if and only if the action of G on $S(G)$ admits an invariant Borel probability measure. The greatest ambit of G is none other than the *Samuel compactification*, which is characterized by the property that every right uniformly continuous bounded function on G extends to a continuous function on $S(G)$. Thus, amenability can be characterized as follows.

THEOREM 0.2 (see [P1, Theorem 3.5.12]). *Let G be a topological group. Then the following are equivalent:*

(1) *G is amenable.*
(2) *There is an invariant mean* ([1]) *on the space* $\mathrm{RUCB}(G)$ *of right uniformly continuous bounded functions on G.*

([1]) Positive linear form of norm 1.

(3) *For every positive integer N and all $f_1, \dots, f_N$ in* $\mathrm{RUCB}(G)$*, there exists a mean Λ on* $\mathrm{RUCB}(G)$ *that is invariant on the orbits of $f_1, \dots, f_N$, that is, for every $j \leq N$ and every g in G, one has $\Lambda(g^{-1} \cdot f_j) = \Lambda(f_j)$.*

(4) *For every $\epsilon > 0$, every finite subset F of G, every positive integer N and all $f_1, \dots, f_N$ in* $\mathrm{RUCB}(G)$*, there is a finitely supported probability measure μ on G such that for every $j \leq N$ and every $h \in F$, one has*

$$\Big|\int_G f_j \, d\mu - \int_G f_j \, d(h \cdot \mu)\Big| < \epsilon.$$

The implications (4)⇒(3)⇒(2) follow from the weak*-compactness of the space of means on $\mathrm{RUCB}(G)$ (which is a consequence of the Banach–Alaoglu theorem), while the implication (2)⇒(4) follows from an application of the Riesz representation theorem to the Samuel compactification of G, and the fact that every Borel probability measure on a compact space can be approximated by finitely supported probability measures. Condition (4) is known as *Day's weak*-asymptotic invariance condition*.

In the course of the proof of Theorem 0.1, we provide several reformulations of the metric convex Ramsey property, among which the following (Theorem 5.1).

THEOREM 0.3. *Let G be a Polish group. Then the following are equivalent:*

(1) *G is amenable.*

(2) *For every $\epsilon > 0$, every finite subset F of G and every left uniformly continuous map $f : G \to [0, 1]$, there exist elements $g_1, \dots, g_n$ of G and barycentric coefficients $\lambda_1, \dots, \lambda_n$ such that for all $h, h' \in F$, one has*

$$\Big|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\Big| < \epsilon.$$

(3) *For every $\epsilon > 0$, every finite subset F of G and every $f \in$* $\mathrm{RUCB}(G)$*, there is a finitely supported probability measure μ on G such that for every h in F, one has*

$$\Big|\int_G f \, d\mu - \int_G f \, d(h \cdot \mu)\Big| < \epsilon.$$

It is a strengthening of Day's weak*-asymptotic invariance condition for Polish groups: to check that a Polish group is amenable, it suffices to verify Day's condition for a single function. This result was motivated by a similar result obtained by Moore for discrete groups [Mo, Theorem 2.1]. Besides, the same is true for extreme amenability with *multiplicative* means.

It is interesting that to make this reduction from multiple functions to only one function, we need to express the Polish group as the automorphism group of a metric Fraïssé structure (as in [Me]) and then combine multiple colorings into one coloring, whereas it is unclear how to directly combine finitely many right uniformly continuous functions on the group.

Applying the Riesz representation theorem to the Samuel compactification, as in Theorem 0.2, we obtain the following as a corollary (Corollary 5.2).

Corollary 0.4. *Let G be a Polish group. Then the following are equivalent:*

(1) *G is amenable.*
(2) *For every right uniformly continuous bounded function on G, there exists a mean on* RUCB(G) *such that for all $g \in G$, one has $\Lambda(g \cdot f) = \Lambda(f)$.*

Another advantage of Theorems 0.1 and 0.3 is to express amenability in a finitary way, which allows us to compute its Borel complexity. In [P1], it was shown that extreme amenability is equivalent to a Ramsey-theoretic property called *finite oscillation property*, a slight reformulation of which turns out to be a G_δ condition, as observed by Melleray and Tsankov [MT2]. We prove that amenability is also a G_δ condition (Corollary 5.4).

From this, a Baire category argument leads to the following sufficient condition for a Polish group to be amenable (Corollary 5.6).

Corollary 0.5. *Let G be a Polish group such that for every positive $n \in \mathbb{N}$, the set*

$$F_n = \{(g_1, \dots, g_n) \in G^n : \langle g_1, \dots, g_n\rangle \textit{ is amenable } (\textit{as a subgroup of } G)\}$$

is dense in G^n. Then G is amenable.

This is a slight strengthening of the fact that a Polish group whose finitely generated subgroups are amenable is itself amenable (see [G, Theorem 1.2.7]), and also admits a direct proof (see Remark 5.7).

1. A bit of continuous logic. In this section, we briefly set up the framework of continuous logic and of metric Fraïssé classes.

Definition 1.1.

- A *relational continuous language* $\mathcal{L}$ is a sequence of pairs (n, k), where n is an integer and k a positive real number.
- If $\mathcal{L}$ is a relational continuous language, then an *$\mathcal{L}$-structure* is a complete metric space (M, d) endowed, for every $l = (n, k)$ in $\mathcal{L}$, with an n-ary map $R_l : M^n \to \mathbb{R}$ which is k-Lipschitz for the supremum metric on M^n. The maps R_l are called *predicates*.

Definition 1.2. Let $\mathcal{L}$ be a relational continuous language and M be an $\mathcal{L}$-structure. An *automorphism* of M is an isometry of (M, d) that preserves all the predicates, that is, for every $l = (n, k)$ in $\mathcal{L}$ and every $(x_1, \dots, x_n)$ in M^n, one has

$$R_l(g(x_1), \dots, g(x_n)) = R_l(x_1, \dots, x_n).$$

The set of all automorphisms of M is called the *automorphism group* of M and is denoted by $\mathrm{Aut}(M)$.

We turn $\mathrm{Aut}(M)$ into a topological group by endowing it with the topology of pointwise convergence. If the structure M is separable, then $\mathrm{Aut}(M)$ is a Polish group.

Definition 1.3. Let $\mathcal{L}$ be a relational continuous language, and M and M' two $\mathcal{L}$-structures.

- An *embedding* of M' into M is an isometric map $h : M' \to M$ that preserves all the predicates.
- The structure M is said to be *approximately ultrahomogeneous* if for every positive ϵ, every finite subset A of M and every embedding h of A into M, there exists an automorphism g of M such that for all a in A, one has $d(g(a), h(a)) < \epsilon$.

In model-theoretic terms, a structure is approximately ultrahomogeneous if any two finite tuples having the same quantifier-free type can be sent arbitrarily close to each other by an automorphism of the full structure.

Melleray [Me, Theorem 6] showed that every Polish group can be realized as the automorphism group of a separable approximately ultrahomogeneous metric structure. If M is such a structure, its *age*, which is the class of all its finite substructures, has good amalgamation properties. Classes of finite metric structures that enjoy the same properties are called *metric Fraïssé classes*, a precise definition of which can be found in [BY] or [MT1]. A continuous version of the Fraïssé construction was developed by Ben Yaacov [BY], ensuring that every such class is in fact the age of a unique (up to isomorphism) separable approximately ultrahomogeneous structure, its *Fraïssé limit*. For our purposes, however, we may simply take the following as a definition of a Fraïssé class.

Definition 1.4. Let $\mathcal{L}$ be a countable relational continuous language. A class $\mathcal{K}$ of finite $\mathcal{L}$-structures is said to be a *metric Fraïssé class* if it is the age of a separable approximately ultrahomogeneous $\mathcal{L}$-structure $\mathbf{K}$. In that case, $\mathbf{K}$ is called the *Fraïssé limit* of the class $\mathcal{K}$.

Examples 1.5.

- Every classical Fraïssé class can be seen as a metric one by endowing the structures with the discrete metric.

- The class of finite metric spaces is a metric Fraïssé class: it is the age of the universal Urysohn space $\mathbb{U}$.

REMARK 1.6. We could allow languages to contain function symbols; the reasoning would then adapt to classes of *finitely generated* structures by considering finite generating sets for them. Fraïssé limits then include the infinite-dimensional separable Hilbert space, the measure algebra of $[0,1]$, L^p spaces.

Thus, every Polish group is the automorphism group of a Fraïssé limit. We will use this description of Polish groups to give combinatorial characterizations of amenability.

2. The metric convex Ramsey property. We use the notation of [MT1].

DEFINITION 2.1. Let $\mathcal{L}$ be a relational continuous language, $\mathbf{A}$ and $\mathbf{B}$ two finite $\mathcal{L}$-structures and $\mathbf{M}$ an arbitrary $\mathcal{L}$-structure.

- We denote by ${}^{\mathbf{A}}\mathbf{M}$ the set of all embeddings of $\mathbf{A}$ into $\mathbf{M}$. We endow ${}^{\mathbf{A}}\mathbf{M}$ with the metric $\rho_{\mathbf{A}}$ defined by
$$\rho_{\mathbf{A}}(\alpha,\alpha') = \max_{a\in A} d(\alpha(a),\alpha'(a)).$$
- A *coloring* of ${}^{\mathbf{A}}\mathbf{M}$ is a 1-Lipschitz map from $({}^{\mathbf{A}}\mathbf{M},\rho_{\mathbf{A}})$ to $[0,1]$.
- We denote by $\langle{}^{\mathbf{A}}\mathbf{M}\rangle$ the set of all finitely supported probability measures on ${}^{\mathbf{A}}\mathbf{M}$. We will identify embeddings with their associated Dirac measures.
- If $\kappa : {}^{\mathbf{A}}\mathbf{M} \to [0,1]$ is a coloring, then we extend κ to $\langle{}^{\mathbf{A}}\mathbf{M}\rangle$ linearly: if ν in $\langle{}^{\mathbf{A}}\mathbf{M}\rangle$ is of the form $\nu = \sum_{i=1}^n \lambda_i\delta_{\alpha_i}$, we set
$$\kappa(\nu) = \sum_{i=1}^n \lambda_i\kappa(\alpha_i).$$
- Moreover, we extend composition of embeddings to finitely supported measures bilinearly. Namely, if ν in $\langle{}^{\mathbf{A}}\mathbf{B}\rangle$ and ν' in $\langle{}^{\mathbf{B}}\mathbf{M}\rangle$ are of the form $\nu = \sum_{i=1}^n \lambda_i\delta_{\alpha_i}$ and $\nu' = \sum_{j=1}^m \lambda'_j\delta_{\alpha'_j}$, we define
$$\nu'\circ\nu = \sum_{j=1}^m\sum_{i=1}^n \lambda'_j\lambda_i\delta_{\alpha'_j\circ\alpha_i}.$$
- If ν is a measure in $\langle{}^{\mathbf{B}}\mathbf{M}\rangle$, we denote by $\langle{}^{\mathbf{A}}\mathbf{M}(\nu)\rangle$ the set of all finitely supported measures which factor through ν, and by ${}^{\mathbf{A}}\mathbf{M}(\nu)$ the set of those which factor through ν via an embedding. More precisely, if $\nu\in\langle{}^{\mathbf{B}}\mathbf{M}\rangle$ is of the form $\sum_{i=1}^n \lambda_i\delta_{\beta_i}$, we define
$${}^{\mathbf{A}}\mathbf{M}(\nu) = \{\nu\circ\delta_\alpha : \alpha\in{}^{\mathbf{A}}\mathbf{B}\},\qquad \langle{}^{\mathbf{A}}\mathbf{M}(\nu)\rangle = \{\nu\circ\nu' : \nu'\in\langle{}^{\mathbf{A}}\mathbf{B}\rangle\}.$$

Throughout the paper, $\mathcal{K}$ will be a metric Fraïssé class in a relational continuous language, and $\mathbf{K}$ will be its Fraïssé limit.

DEFINITION 2.2. The class $\mathcal{K}$ is said to have the *metric convex Ramsey property* if for every $\epsilon > 0$ and all structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, there exists a structure $\mathbf{C}$ in $\mathcal{K}$ such that for every coloring $\kappa : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$, there is ν in $\langle {}^{\mathbf{B}}\mathbf{C} \rangle$ such that for all $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\nu)$, one has $|\kappa(\eta) - \kappa(\eta')| < \epsilon$.

INTUITION 2.3. In the classical setting, the Ramsey property states that given two structures A and B, we can find a bigger structure C such that whenever we color the copies of A in C, we can find a copy of B in C wherein every copy of A has the same color. Here, it basically says that we can find a convex combination of copies of B in C wherein every compatible convex combination of copies of A has almost the same color (see Figure 1).

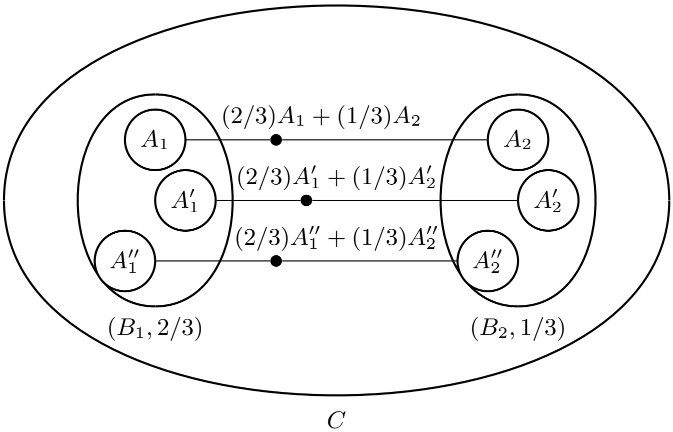

Fig. 1. Black points are barycenters of two corresponding copies of A in B_1 and B_2 with the coefficients $2/3$ and $1/3$. The metric convex Ramsey property says that all these points have almost the same color.

REMARK 2.4. One can replace the assumption $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\nu)$ with the stronger one $\eta, \eta' \in \langle {}^{\mathbf{A}}\mathbf{C}(\nu) \rangle$ in the above definition, as is done in [Mo]. Indeed, the property is preserved under convex combinations.

The following proposition states that the metric convex Ramsey property allows us to stabilize any finite number of colorings at once.

PROPOSITION 2.5. *The following are equivalent:*

(1) *The class $\mathcal{K}$ has the metric convex Ramsey property.*
(2) *For every $\epsilon > 0$, all positive integers $N \in \mathbb{N}$ and all structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, there exists a structure $\mathbf{C}$ in $\mathcal{K}$ such that for all colorings $\kappa_1, \ldots, \kappa_N : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$, there is μ in $\langle {}^{\mathbf{B}}\mathbf{C} \rangle$ such that for all j in $\{1, \ldots, N\}$ and all η, η' in ${}^{\mathbf{A}}\mathbf{C}(\mu)$, one has $|\kappa_j(\eta) - \kappa_j(\eta')| < \epsilon$.*

Remark. Condition (2) above is equivalent to the metric convex Ramsey property for colorings into $[0,1]^N$, where $[0,1]^N$ is endowed with the supremum metric. It follows that the metric convex Ramsey property is equivalent to the same property for colorings taking values in any convex compact metric space.

Proof of Proposition 2.5. The second condition clearly implies the first. For simplicity, we prove the other implication for $N = 2$; the same argument carries over to arbitrary N. Let $\mathbf{A}$ and $\mathbf{B}$ be structures in $\mathcal{K}$, and $\epsilon > 0$. We apply the metric convex Ramsey property twice consecutively.

We find a structure $\mathbf{C}_1$ in $\mathcal{K}$ witnessing the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{B}$ and ϵ, that is, if $\kappa : {}^{\mathbf{A}}\mathbf{C}_1 \to [0,1]$ is a coloring, then there exists $\nu \in \langle {}^{\mathbf{B}}\mathbf{C}_1 \rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{B}$, we have $|\kappa(\nu \circ \delta_\alpha) - \kappa(\nu\delta_{\alpha'})| < \epsilon$. Then we find a structure $\mathbf{C}$ in $\mathcal{K}$ witnessing the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{C}_1$ and ϵ, that is, if $\kappa : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$ is a coloring, then there exists $\nu \in \langle {}^{\mathbf{C}_1}\mathbf{C} \rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\kappa(\nu \circ \delta_\alpha) - \kappa(\nu \circ \delta_{\alpha'})| < \epsilon$.

We now show that $\mathbf{C}$ has the desired property. To this end, let $\kappa_1, \kappa_2 : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$ be two colorings. By definition of the structure of $\mathbf{C}$, there exists $\nu \in \langle {}^{\mathbf{C}_1}\mathbf{C} \rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\kappa_1(\nu \circ \delta_\alpha) - \kappa_1(\nu \circ \delta_{\alpha'})| < \epsilon$.

We then lift the second coloring κ_2 to $\tilde{\kappa}_2 : {}^{\mathbf{A}}\mathbf{C}_1 \to [0,1]$ by setting $\tilde{\kappa}_2(\alpha) = \kappa_2(\nu \circ \delta_\alpha)$. This process corresponds to the classical going color-blind argument: here, instead of forgetting one color, we forget all embeddings that are not channelled through $\mathbf{C}_1$ via ν. The map $\tilde{\kappa}_2$ we obtain is again a coloring. Therefore, we may apply our assumption on $\mathbf{C}_1$ to $\tilde{\kappa}_2$: there exists ν_1 in $\langle {}^{\mathbf{B}}\mathbf{C}_1 \rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\tilde{\kappa}_2(\nu_1 \circ \delta_\alpha) - \tilde{\kappa}_2(\nu_1 \circ \delta_{\alpha'})| < \epsilon$.

Then $\mu = \nu \circ \nu_1$ is as desired. Indeed, let $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\mu)$. There exist $\alpha, \alpha' \in {}^{\mathbf{A}}\mathbf{C}_1$ such that $\eta = \mu \circ \delta_\alpha$ and $\eta' = \mu \circ \delta_{\alpha'}$. Then

$$\begin{aligned} |\kappa_2(\eta) - \kappa_2(\eta')| &= |\kappa_2(\mu \circ \delta_\alpha) - \kappa_2(\mu \circ \delta_{\alpha'})| \\ &= |\kappa_2(\nu \circ \nu_1 \circ \delta_\alpha) - \kappa_2(\nu \circ \nu_1 \circ \delta_{\alpha'})| \\ &= |\tilde{\kappa}_2(\nu_1 \circ \delta_\alpha) - \tilde{\kappa}_2(\nu_1 \circ \delta_{\alpha'})| < \epsilon. \end{aligned}$$

Moreover, whenever $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\mu)$, they are in ${}^{\mathbf{A}}\mathbf{C}(\nu)$ too, hence the assumption on ν yields $|\kappa_1(\eta) - \kappa_1(\eta')| < \epsilon$. ■

Remark 2.6. For the sake of simplicity, we state the results for only one coloring at a time; the previous proposition will imply that we can do the same with any finite number of colorings.

We now give an infinitary reformulation of the metric convex Ramsey property, which is what will be used in the proof of Theorem 4.3 in showing that amenability implies the metric convex Ramsey property.

Proposition 2.7. *The following are equivalent:*

(1) *The class $\mathcal{K}$ has the metric convex Ramsey property.*
(2) *For every $\epsilon > 0$, all structures* $\mathbf{A}$ *and* $\mathbf{B}$ *in $\mathcal{K}$ and all colorings $\kappa : {}^{\mathbf{A}}\mathbf{K} \to [0,1]$, there exists ν in $\langle {}^{\mathbf{B}}\mathbf{K}\rangle$ such that for all η, η' in ${}^{\mathbf{A}}\mathbf{K}(\nu)$, one has $|\kappa(\eta) - \kappa(\eta')| < \epsilon$.*

Proof. (1)⇒(2). Fix $\epsilon > 0$, two structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, and let $\mathbf{C} \in \mathcal{K}$ witness the metric convex Ramsey property for A, B and ϵ. We may assume that $\mathbf{C}$ is a substructure of $\mathbf{K}$. Now every coloring of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a coloring of ${}^{\mathbf{A}}\mathbf{C}$, so if ν is the measure given by $\mathbf{C}$ for a coloring κ, then ν has the desired property.

(2)⇒(1). We use a standard compactness argument. Suppose that $\mathcal{K}$ does not have the metric convex Ramsey property. We can then find structures $\mathbf{A}$, $\mathbf{B}$ in $\mathcal{K}$ and $\epsilon > 0$ such that for every $\mathbf{C} \in \mathcal{K}$, there exists a *bad* coloring $\kappa_{\mathbf{C}}$ of ${}^{\mathbf{A}}\mathbf{C}$, that is, for all $\nu \in \langle {}^{\mathbf{B}}\mathbf{C}\rangle$, the oscillation of $\kappa_{\mathbf{C}}$ on ${}^{\mathbf{A}}\mathbf{C}(\nu)$ is greater than ϵ.

We fix an ultrafilter $\mathcal{U}$ on the collection of finite subsets of $\mathbf{K}$ such that for every finite $D \subseteq \mathbf{K}$, the set $\{E \subseteq \mathbf{K} \text{ finite} : D \subseteq E\}$ belongs to $\mathcal{U}$. We consider the map $\kappa = \lim\limits_{\mathcal{U}} \kappa_{\mathbf{C}}$ on ${}^{\mathbf{A}}\mathbf{K}$ defined by

$$\kappa(\alpha) = t \ \Leftrightarrow\ \forall r > 0, \{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \kappa_{\mathbf{C}}(\alpha) \in [t - r, t + r]\} \in \mathcal{U}.$$

Note that the assumption on $\mathcal{U}$ implies that for all $\alpha \in {}^{\mathbf{A}}\mathbf{K}$, the set $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \alpha(A) \subseteq C\}$ is in $\mathcal{U}$, so $\kappa_{\mathbf{C}}(\alpha)$ is defined $\mathcal{U}$-everywhere and the above definition makes sense. Moreover, since all the $\kappa_{\mathbf{C}}$ are 1-Lipschitz, so is κ, and is thus a coloring of ${}^{\mathbf{A}}\mathbf{K}$. We prove that κ fails property (2).

Let $\nu \in \langle {}^{\mathbf{B}}\mathbf{K}\rangle$ and write $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\beta_i}$ with the β_i's in ${}^{\mathbf{B}}\mathbf{K}$. Then, for all $i \in \{1, \dots, n\}$, the sets $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \beta_i(B) \subseteq \mathbf{C}\}$ belong to $\mathcal{U}$, and so does their intersection U_ν. Furthermore, the set ${}^{\mathbf{A}}\mathbf{K}(\nu)$, which is the same as ${}^{\mathbf{A}}\mathbf{C}(\nu)$ for any $\mathbf{C}$ in U_ν, is finite—note that this is not true of $\langle {}^{\mathbf{A}}\mathbf{K}(\nu)\rangle$ (so choosing the definition of Remark 2.4 for the Ramsey property would require an additional appeal to the compactness of $\langle {}^{\mathbf{A}}\mathbf{K}(\nu)\rangle$). For every $\mathbf{C}$ in U_ν, there exist η, η' in ${}^{\mathbf{A}}\mathbf{C}(\nu)$ such that $|\kappa_{\mathbf{C}}(\eta) - \kappa_{\mathbf{C}}(\eta')| \geq \epsilon$. So there exist η, η' in ${}^{\mathbf{A}}\mathbf{K}(\nu)$ such that the set $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : |\kappa_{\mathbf{C}}(\eta) - \kappa_{\mathbf{C}}(\eta')| \geq \epsilon\}$ belongs to $\mathcal{U}$. By definition of κ, this implies that $|\kappa(\eta) - \kappa(\eta')| \geq \epsilon$, which shows that (2) fails for ν. As ν was arbitrary, this completes the proof. ∎

3. The metric convex Ramsey property for the automorphism group. Let G be the automorphism group of $\mathbf{K}$.

In this section, we reformulate the metric convex Ramsey property in terms of properties of G.

Definition 3.1. Let $\mathbf{A}$ be a finite substructure of $\mathbf{K}$. We define a pseudometric $d_{\mathbf{A}}$ on G by

$$d_{\mathbf{A}}(g,h) = \max_{a\in A} d(g(a),h(a)).$$

We will denote by $(G,d_{\mathbf{A}})$ the induced metric quotient space.

Remark 3.2. The pseudometrics $d_{\mathbf{A}}$, for finite substructures $\mathbf{A}$ of $\mathbf{K}$, generate the topology on G, and hence also the left uniformity. For an introduction to uniformities, see for example [P1].

The pseudometric $d_{\mathbf{A}}$ is the counterpart of the metric $\rho_{\mathbf{A}}$ on ${}^{\mathbf{A}}\mathbf{K}$ on the side of the group. More specifically, as pointed out in [MT1, Lemma 3.8], the restriction map $\Phi_{\mathbf{A}} : (G,d_{\mathbf{A}}) \to ({}^{\mathbf{A}}\mathbf{K},\rho_{\mathbf{A}})$ defined by $g \mapsto g{\restriction_A}$ is distance-preserving, and its image $\Phi_{\mathbf{A}}(G)$ is dense in ${}^{\mathbf{A}}\mathbf{K}$. As a consequence, every 1-Lipschitz map $f : (G,d_{\mathbf{A}}) \to [0,1]$ extends uniquely, via $\Phi_{\mathbf{A}}$, to a coloring κ_f of ${}^{\mathbf{A}}\mathbf{K}$, while every coloring κ of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a 1-Lipschitz map $f_\kappa : (G,d_{\mathbf{A}}) \to [0,1]$.

Proposition 3.3. *The following are equivalent:*

1. *The class $\mathcal{K}$ has the metric convex Ramsey property.*
2. *For every $\epsilon > 0$, every finite substructure $\mathbf{A}$ of $\mathbf{K}$, every finite subset F of G and every 1-Lipschitz map $f : (G,d_{\mathbf{A}}) \to [0,1]$, there exist elements $g_1,\dots,g_n$ of G and barycentric coefficients $\lambda_1,\dots,\lambda_n$ such that for all h,h' in F, one has*
$$\Big|\sum_{i=1}^n \lambda_i f(g_i h) - \sum_{i=1}^n \lambda_i f(g_i h')\Big| < \epsilon.$$
3. *For every $\epsilon > 0$, every finite subset F of G and every left uniformly continuous map $f : G \to [0,1]$, there exist elements $g_1,\dots,g_n$ of G and barycentric coefficients $\lambda_1,\dots,\lambda_n$ such that for all h,h' in F, one has*
$$\Big|\sum_{i=1}^n \lambda_i f(g_i h) - \sum_{i=1}^n \lambda_i f(g_i h')\Big| < \epsilon.$$

Remark 3.4. The finite subset F of G in condition (2) is the counterpart of the structure $\mathbf{B}$ in the Ramsey property: by approximate ultrahomogeneity of the limit $\mathbf{K}$, it corresponds, up to a certain error, to the set of all embeddings of $\mathbf{A}$ into $\mathbf{B}$.

Proof of Proposition 3.3. (1)⇒(2). We set $\mathbf{B} = \mathbf{A} \cup \bigcup_{h\in F} h(\mathbf{A})$. Let κ_f be the unique coloring of ${}^{\mathbf{A}}\mathbf{K}$ that extends f. We then apply Proposition 2.7 to $\mathbf{A}$, $\mathbf{B}$, ϵ and κ_f: there is ν in $\langle {}^{\mathbf{B}}\mathbf{K}\rangle$ such that for all α,α' in ${}^{\mathbf{A}}\mathbf{B}(\nu)$, we have $|\kappa_f(\nu\circ\delta_\alpha) - \kappa_f(\nu\circ\delta_{\alpha'})| < \epsilon$.

Write $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\beta_i}$ with the β_i's in ${}^{\mathbf{B}}\mathbf{K}$. Since the structure $\mathbf{K}$ is a Fraïssé limit, it is approximately ultrahomogeneous. This implies that for each i in $\{1, \dots, n\}$, there exists an element g_i of its automorphism group G such that $\rho_{\mathbf{B}}(g_i, \beta_i) < \epsilon$. It is straightforward to check, using the triangle inequality and the 1-Lipschitzness of the coloring κ_f, that these g_i's and λ_i's have the desired property.

(2)⇒(3). We approximate uniformly continuous functions by Lipschitz ones. More precisely, let $f : G \to [0,1]$ be left uniformly continuous and let $\epsilon > 0$. There exists an entourage V in the left uniformity $\mathcal{U}_L(G)$ on G such that for all x, y in G, if $(x, y) \in V$, then $|f(x) - f(y)| < \epsilon$. Moreover, Remark 3.2 implies that there exist a finite substructure $\mathbf{A}$ of $\mathbf{K}$ and $r > 0$ such that for all x, y in G, if $d_{\mathbf{A}}(x, y) < r$, then $(x, y) \in V$.

Now, for a positive integer k, we can define a map $f_k : (G, d_{\mathbf{A}}) \to [0,1]$ by

$$f_k(x) = \inf_{y \in G} (f(y) + k d_{\mathbf{A}}(x, y)).$$

It is k-Lipschitz as the infimum of k-Lipschitz functions. Note also that f_k is smaller than f.

Take k large enough that $3/k < r$, and let x be any element of G. By definition of f_k, there exists an element y of G such that $f(y) + k d_{\mathbf{A}}(x, y) \le f_k(x) + \epsilon$. Since both f and f_k are bounded by 1, this implies that for small enough ϵ, we have $d_{\mathbf{A}}(x, y) \le 3/k < r$. Thus, the left uniform continuity of f gives $|f(x) - f(y)| < \epsilon$. But then

$$\begin{aligned} |f(x) - f_k(x)| = f(x) - f_k(x) &\le f(x) - f(y) - k d_{\mathbf{A}}(x, y) + \epsilon \\ &\le f(x) - f(y) + \epsilon < 2\epsilon. \end{aligned}$$

We have therefore obtained a good uniform approximation of f by a Lipschitz function.

We then apply (3) to f_k/k, which is 1-Lipschitz, and to ϵ/k: for every finite subset F of G, there exist elements $g_1, \dots, g_n$ of G and barycentric coefficients $\lambda_1, \dots, \lambda_n$ such that for all $h, h' \in F$, we have

$$\Big| \sum_{i=1}^{n} \lambda_i \frac{1}{k} f_k(g_i h) - \sum_{i=1}^{n} \lambda_i \frac{1}{k} f_k(g_i h') \Big| < \frac{\epsilon}{k},$$

hence

$$\Big| \sum_{i=1}^{n} \lambda_i f_k(g_i h) - \sum_{i=1}^{n} \lambda_i f_k(g_i h') \Big| < \epsilon.$$

Then, for all $h, h' \in F$, the triangle inequality gives

$$\Big| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \Big| < 3\epsilon.$$

(3)⇒(1). Let $\mathbf{A}$ and $\mathbf{B}$ be two structures in $\mathcal{K}$, let $\epsilon > 0$ and let $\kappa : {}^{\mathbf{A}}\mathbf{K} \to [0,1]$ be a coloring. Since $\mathcal{K}$ is approximately ultrahomogeneous, for every α in ${}^{\mathbf{A}}\mathbf{B}$, we may choose h_α in G such that $\rho_{\mathbf{A}}(h_\alpha, \alpha) < \epsilon$. Let F be the (finite) set of all such h_α's.

Now consider the restriction f_κ of the coloring κ to $(G, d_{\mathbf{A}})$. It is left uniformly continuous from G to $[0,1]$. We apply condition (3) to f_κ, F and ϵ: there exist elements $g_1, \ldots, g_n$ of G and barycentric coefficients $\lambda_1, \ldots, \lambda_n$ such that for all $h_\alpha, h_{\alpha'}$ in F, one has

$$\Big|\sum_{i=1}^{n} \lambda_i f_\kappa(g_i h_\alpha) - \sum_{i=1}^{n} \lambda_i f_\kappa(g_i h_{\alpha'})\Big| < \epsilon.$$

Set $\nu = \sum_{i=1}^{n} \lambda_i \delta_{g_i} \in \langle {}^{\mathbf{B}}\mathbf{K} \rangle$. Using the triangle inequality and the Lipschitzness of κ, it is now straightforward to check that ν witnesses the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{B}$ and 3ϵ. ∎

Notice that conditions (3) and (4) do not depend on the Fraïssé class, but only on its automorphism group.

By Remark 2.6, the metric convex Ramsey property is equivalent to condition (3) for any finite number of colorings at once. It is that condition which will imply amenability in Theorem 4.3.

Moreover, if G is endowed with a compatible left-invariant metric, Lipschitz functions are uniformly dense in left uniformly continuous bounded ones (the proof is similar to that of (2)⇒(3) above), so we can replace left uniformly continuous maps by 1-Lipschitz maps in (3) to obtain the following.

COROLLARY 3.5. *Let d be any compatible left-invariant metric on G. Then the following are equivalent:*

- *The class $\mathcal{K}$ has the metric convex Ramsey property.*
- *For every $\epsilon > 0$, every finite subset F of G and every 1-Lipschitz map $f : (G, d) \to [0,1]$, there exist elements $g_1, \ldots, g_n$ of G and barycentric coefficients $\lambda_1, \ldots, \lambda_n$ such that for all h, h' in F, one has*

$$\Big|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\Big| < \epsilon.$$

4. A criterion for amenability. Given a compact space X, we denote by $P(X)$ the set of all Borel probability measures on X. It is a subset of the dual space of continuous maps on X. Indeed, if μ is in $P(X)$ and f is a continuous function on X, we set $\mu(f) = \int_X f\, d\mu$. Moreover, if we endow $P(X)$ with the induced weak* topology, it is compact.

If G is a group that acts on X, then one can define an action of G on $P(X)$ by

$$(g \cdot \mu)(f) = \int_X f(g^{-1} \cdot x)\, d\mu(x).$$

Definition 4.1. A topological group G is said to be *amenable* if every continuous action of G on a compact Hausdorff space X admits a measure in $P(X)$ which is invariant under the action of G.

Although amenability is not preserved under subgroups (not even closed subgroups), it is preserved when taking dense subgroups.

Proposition 4.2. *A subgroup of a topological group is amenable* (*with respect to the induced topology*) *if and only if such is its closure.*

Proof. Let H be a dense subgroup of G. It is straightforward to show that that every continuous action of H on a compact Hausdorff space extends to a continuous action of G. Thus, if G is amenable, then so is H. ∎

We are now ready to prove the main theorem.

Theorem 4.3. *Let $\mathcal{K}$ be a metric Fraïssé class, $\mathbf{K}$ its Fraïssé limit and G the automorphism group of $\mathbf{K}$. Then the following are equivalent:*

(1) *The topological group G is amenable.*
(2) *The class $\mathcal{K}$ has the metric convex Ramsey property.*

Proof. (1)⇒(2). Suppose G is amenable. Let $\mathbf{A}, \mathbf{B}$ be structures in the class $\mathcal{K}$, let $\epsilon > 0$ and let $\kappa_0 : {}^{\mathbf{A}}\mathbf{K} \to [0,1]$ be a coloring. We show that there exists $\nu \in \langle {}^{\mathbf{B}}\mathbf{K} \rangle$ such that for all $\alpha, \alpha' \in {}^{\mathbf{A}}\mathbf{B}$, we have $|\kappa_0(\nu \circ \delta_\alpha) - \kappa_0(\nu \circ \delta_{\alpha'})| < \epsilon$, which will imply the metric convex Ramsey property (by Proposition 2.7). We adapt Moore's proof [Mo, (6)⇒(1) in Theorem 7.1] to the metric setting.

The group G acts on the compact Hausdorff space $[0,1]^{{}^{\mathbf{A}}\mathbf{K}}$ by $g \cdot \kappa(\alpha) = \kappa(g^{-1} \circ \alpha)$. Denote by Y the orbit of the coloring κ_0 under this action, and by X its closure, which is compact Hausdorff. Note that all the functions in X are colorings as well. We consider the restriction of the action to X: it is continuous. Thus, since G is amenable, there is an invariant probability measure μ on X.

The map $\alpha \mapsto \int_X \kappa(\alpha)\, d\mu(\kappa)$ is constant on ${}^{\mathbf{A}}\mathbf{K}$. Indeed, the invariance of μ implies that it is constant on every orbit of the action of G on ${}^{\mathbf{A}}\mathbf{K}$. But, by the approximate ultrahomogeneity of $\mathbf{K}$, every such orbit is dense in ${}^{\mathbf{A}}\mathbf{K}$, so our map is constant on the whole of ${}^{\mathbf{A}}\mathbf{K}$ because it is continuous (even 1-Lipschitz).

Let r denote this constant value.

Further, Y being dense in X, the collection of finitely supported probability measures on Y is dense in $P(X)$. In particular, there exist barycentric coefficients $\lambda_1, \ldots, \lambda_n$ and elements $g_1, \ldots, g_n$ of G such that for all α in ${}^{\mathbf{A}}\mathbf{B}$, we have $|\sum_{i=1}^n \lambda_i \kappa_0(g_i^{-1} \circ \alpha) - r| < \epsilon$.

Finally, we may assume that $\mathbf{B}$ is a substructure of $\mathbf{K}$, and set $\beta_i = g_i^{-1} {\restriction} \mathbf{B}$ for i in $\{1, \ldots, n\}$, and $\nu = \sum_{i=1}^n \lambda_i \delta_{\beta_i} \in \langle {}^{\mathbf{B}}\mathbf{K} \rangle$. Then ν is as desired.

Indeed, if α, α' are in ${}^{\mathbf{A}}\mathbf{B}$, and thus in ${}^{\mathbf{A}}\mathbf{K}$, then

$$\begin{aligned}|\kappa_0(\nu\circ\delta_\alpha)-\kappa_0(\nu\circ\delta_{\alpha'})| &= \Big|\sum_{i=1}^n \lambda_i\kappa_0(\beta_i\circ\alpha)-\sum_{i=1}^n \lambda_i\kappa_0(\beta_i\circ\alpha')\Big| \\ &\le \Big|\sum_{i=1}^n \lambda_i\kappa_0(\beta_i\circ\alpha)-r\Big|+\Big|r-\sum_{i=1}^n \lambda_i\kappa_0(\beta_i\circ\alpha')\Big| \\ &= \Big|\sum_{i=1}^n \lambda_i\kappa_0(g_i^{-1}\circ\alpha)-r\Big|+\Big|r-\sum_{i=1}^n \lambda_i\kappa_0(g_i^{-1}\circ\alpha')\Big| \\ &< 2\epsilon.\end{aligned}$$

(2)⇒(1). Conversely, suppose that $\mathcal{K}$ has the metric convex Ramsey property, and let G act continuously on a compact Hausdorff space X. We show that X admits an invariant probability measure. Since $P(X)$ is compact, it suffices to show that if $f_1,\dots,f_N : X \to [0,1]$ are uniformly continuous with respect to the unique (see [P1, Exercise 1.1.3]) uniformity on X, $\epsilon > 0$ and F is a finite subset of G, then there exists μ in $P(X)$ such that for all j in $\{1,\dots,N\}$ and all h in F, $|h\cdot\mu(f_j)-\mu(f_j)|<\epsilon$.

Fix x in X. For j in $\{1,\dots,N\}$, we lift f_j to a map $\tilde{f}_j : G \to [0,1]$ by setting $\tilde{f}_j(g) = f_j(g^{-1}\cdot x)$. Since the action of G on X is continuous and X is compact, for all x in X the map $g\mapsto g^{-1}\cdot x$ is left uniformly continuous (see [P1, Lemma 2.1.5]). It follows that the map $\tilde{f}_j$ is left uniformly continuous.

We then apply Proposition 3.3 to $F\cup\{1\}$, ϵ and $\tilde{f}_1,\dots,\tilde{f}_N$ to obtain barycentric coefficients $\lambda_1,\dots,\lambda_n$ and elements $g_1,\dots,g_n$ of G such that for all j in $\{1,\dots,N\}$ and all h in F (and $h'=1$), we have

$$\Big|\sum_{i=1}^n \lambda_i\tilde{f}_j(g_ih)-\sum_{i=1}^n \lambda_i\tilde{f}_j(g_i)\Big|<\epsilon.$$

Then $\mu=\sum_{i=1}^n \lambda_i\delta_{g_i^{-1}\cdot x}$ is as desired. Indeed, let $j\in\{1,\dots,N\}$ and $h\in F$. We have

$$\mu(f_j)=\sum_{i=1}^n \lambda_i f_j(g_i^{-1}\cdot x)=\sum_{i=1}^n \lambda_i\tilde{f}_j(g_i)$$

and

$$h\cdot\mu(f_j)=\sum_{i=1}^n \lambda_i(h\cdot f_j)(g_i^{-1}\cdot x)=\sum_{i=1}^n \lambda_i f_j(h^{-1}g_i^{-1}\cdot x)=\sum_{i=1}^n \lambda_i\tilde{f}_j(g_ih),$$

so finally

$$|h\cdot\mu(f_j)-\mu(f_j)|=\Big|\sum_{i=1}^n \lambda_i\tilde{f}_j(g_ih)-\sum_{i=1}^n \lambda_i\tilde{f}_j(g_i)\Big|<\epsilon,$$

which completes the proof. ■

Example 4.4. Let $\mathcal{K}$ be the class of finite sets with no additional structure. The Fraïssé limit of $\mathcal{K}$ is the countable set $\mathbb{N}$. It is well known that its automorphism group, S_∞, is amenable, as the union of the finite (hence amenable) symmetric groups is dense in S_∞ (see e.g. [BdlHV, Proposition G.2.2.(iii)]), but not extremely amenable [P2, Theorem 6.5]. In fact, the class of finite sets has the classical Ramsey property (by the Ramsey theorem [R]), but since finite sets are not rigid (every permutation is an automorphism), the Kechris–Pestov–Todorčević result does not apply. However, we can still use this classical Ramsey property to recover the amenability of S_∞: we circumvent the problem of non-rigidity by averaging the colors of all permutations of the small structure to obtain the convex Ramsey property.

More precisely, let $\mathbf{A}$ be a finite set in $\mathcal{K}$. An embedding of $\mathbf{A}$ into $\mathbb{N}$ is given by its image, which is a copy of $\mathbf{A}$ in $\mathbb{N}$, together with an automorphism of $\mathbf{A}$, sending $\mathbf{A}$ to its copy $\mathbf{A}'$. Let now $\mathbf{B}$ be another structure in $\mathcal{K}$, $\kappa : {}^{\mathbf{A}}\mathbb{N} \to [0,1]$ be a coloring and $\epsilon > 0$. Without loss of generality, we may assume that the coloring κ takes its values in a finite set $\{1, \dots, k\}$ for a large enough k.

For each automorphism σ of $\mathbf{A}$, consider the coloring κ_σ of the set $\binom{\mathbb{N}}{\mathbf{A}}$ of copies of $\mathbf{A}$ in $\mathbb{N}$ defined as follows. For each copy $\mathbf{A}'$ of $\mathbf{A}$ in the Fraïssé limit, $\kappa_\sigma(\mathbf{A}')$ is the color that κ gives to the embedding defined by $\mathbf{A}'$ and σ. Apply the Ramsey property to each coloring κ_σ to get a copy $\mathbf{B}_\sigma$ of $\mathbf{B}$ in $\mathbb{N}$ such that κ_σ is constant on the set $\binom{\mathbf{B}_\sigma}{\mathbf{A}}$ of all copies of $\mathbf{A}$ in $\mathbf{B}_\sigma$. Then the isobarycenter of these structures $\mathbf{B}_\sigma$ is the desired measure.

We do not know if this technique generalizes to other non-rigid classes.

5. Structural consequences. As a consequence of Theorem 4.3, Proposition 3.3 and the fact that every Polish group is the automorphism group of some metric Fraïssé structure [Me, Theorem 6], we obtain the following intrinsic characterization of amenability (and its reformulation in terms of finitely supported measures).

Theorem 5.1. *Let G be a Polish group. Then the following are equivalent:*

(1) *G is amenable.*
(2) *For every $\epsilon > 0$, every finite subset F of G and every left uniformly continuous map $f : G \to [0,1]$, there exist elements $g_1, \dots, g_n$ of G and barycentric coefficients $\lambda_1, \dots, \lambda_n$ such that for all $h, h' \in F$, one has*
$$\Big|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\Big| < \epsilon.$$

(3) *For every $\epsilon > 0$, every finite subset F of G and every $f \in \mathrm{RUCB}(G)$ there is a finitely supported probability measure μ on G such that for every h in F, one has $|\mu(f) - (h \cdot \mu)(f)| < \epsilon$.*

The equivalence of (2) and (3) follows from the fact that inversion exchanges left and right uniformly continuous functions.

We recognize Day's weak*-asymptotic invariance condition with only one function from $\mathrm{RUCB}(G)$ needed to check the amenability of G.

Corollary 5.2. *Let G be a Polish group. Then the following are equivalent:*

(1) *G is amenable.*
(2) *For every right uniformly continuous bounded function on G, there exists a mean on $\mathrm{RUCB}(G)$ such that for all $g \in G$, one has $\Lambda(g \cdot f) = \Lambda(f)$.*

Proof. (1)⇒(2). If G is amenable, then the action of G on its Samuel compactification $S(G)$ admits an invariant Borel probability measure μ. The integral against μ gives rise to an invariant mean on the space of all continuous functions on $S(G)$. But continuous functions on the Samuel compactification of G are exactly right uniformly continuous bounded ones, hence condition (2) is satisfied.

(2)⇒(1). Since $\mathrm{RUCB}(G)$ is exactly the space of all continuous functions on the Samuel compactification $S(G)$, we can apply the Riesz representation theorem: for each f in $\mathrm{RUCB}(G)$, there exists a Borel probability measure on $S(G)$ such that for all g in G, we have $\mu(g \cdot f) = \mu(f)$.

But since G is dense in $S(G)$, every Borel probability measure on $S(G)$ can be approximated by finitely supported measures on G. Thus, for every $\epsilon > 0$, every finite subset F of G and every $f \in \mathrm{RUCB}(G)$, there is a finitely supported probability measure μ on G such that for every h in F, one has $|\mu(f) - (h \cdot \mu)(f)| < \epsilon$. Theorem 5.1 then shows that G is amenable. ∎

Similarly, Corollary 3.5 gives the Lipschitz counterpart of Theorem 5.1.

Theorem 5.3. *Let G be a Polish group and d a left-invariant metric on G which induces the topology. Then the following are equivalent:*

(1) *The topological group G is amenable.*
(2) *For every $\epsilon > 0$, every finite subset F of G and every 1-Lipschitz map $f : (G, d) \to [0, 1]$, there exist elements $g_1, \ldots, g_n$ of G and barycentric coefficients $\lambda_1, \ldots, \lambda_n$ such that for all $h, h' \in F$, one has*

$$\Big|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\Big| < \epsilon.$$

It follows that amenability is a G_δ condition in the following sense (see [MT2, Theorem 3.1]).

COROLLARY 5.4. *Let Γ be a countable group and G a Polish group. Then the set of representations of Γ in G whose image is an amenable subgroup of G is G_δ in the space of representations of Γ in G, endowed with the topology of pointwise convergence.*

Proof. Let π be a homomorphism from Γ to G and let d be a compatible left-invariant metric on G. By Proposition 4.2, the image $\pi(\Gamma)$ is amenable if and only if such is its closure, and its closure is Polish (as a closed subset of a Polish space). Then, by Theorem 5.3, $\overline{\pi(\Gamma)}$ is amenable if and only if for every $\epsilon > 0$, every finite subset F of $\overline{\pi(\Gamma)}$ and every 1-Lipschitz function $f : (\overline{\pi(\Gamma)}, d) \to [0,1]$, there exist elements $g_1, \ldots, g_n$ of $\overline{\pi(\Gamma)}$ and barycentric coefficients $\lambda_1, \ldots, \lambda_n$ such that for all h, h' in F, one has

$$\Big|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\Big| < \epsilon.$$

Using the same compactness argument as in Proposition 2.7, one can show that the condition is equivalent to the following:

$\forall \epsilon > 0$, $\forall F \subseteq \overline{\pi(\Gamma)}$ finite, $\exists K \subseteq \overline{\pi(\Gamma)}$ finite,
$\forall f : (KF, d) \to [0,1]$ 1-Lipschitz, $\exists k_1, \ldots, k_n \in K$, $\exists \lambda_1, \ldots, \lambda_n$, $\forall h, h' \in F$,

$$\Big|\sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h')\Big| < \epsilon.$$

It is easily seen that this is again equivalent to the following:

$$\forall \epsilon > 0, \forall F \subseteq \Gamma \text{ finite}, \exists K \subseteq \Gamma \text{ finite},$$

$$(*)\quad \begin{cases} \forall f : KF \to [0,1] \text{ if } \forall \gamma, \gamma' \in KF,\ |f(\gamma) - f(\gamma')| \le d(\pi(\gamma), \pi(\gamma')), \\ \text{then} \\ \exists k_1, \ldots, k_n \in K,\ \exists \lambda_1, \ldots, \lambda_n,\ \forall h, h' \in F, \\ \qquad \Big|\sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h')\Big| < \epsilon. \end{cases}$$

We now prove that, if ϵ, F and K are fixed, the set of representations π satisfying condition $(*)$ above is open, which will imply that the condition is indeed G_δ. We prove that its complement is closed. To that end, take a sequence (π_k) of representations in the complement that converges to some representation π. Let $f_k : KF \to [0,1]$ witness that π_k is in the complement. Since KF is finite, the maps from KF to $[0,1]$ form a compact set, so we may assume that (f_k) converges to some f. Since being Lipschitz is a closed condition, for all γ, γ' in KF, we have $|f(\gamma) - f(\gamma')| \le d(\pi(\gamma), \pi(\gamma'))$.

By the choice of f_k, for all $k_1, \dots, k_n$ in K and all $\lambda_1, \dots, \lambda_n$, there exist h_k, h'_k in F such that

$$\Big|\sum_{i=1}^{n} \lambda_i f_k(k_i h_k) - \sum_{i=1}^{n} \lambda_i f_k(k_i h'_k)\Big| \geq \epsilon.$$

Since F is finite, we may again assume that there are h and h' in F such that for all k, we have $h_k = h$ and $h'_k = h'$. We then take the limit of the above inequality to get

$$\Big|\sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h')\Big| \geq \epsilon,$$

which implies that π does not satisfy condition $(*)$ either, and thus completes the proof. ∎

Remark 5.5. The same argument works if, instead of condition (2) of Theorem 5.3, we use a version of Day's weak*-asymptotic invariance condition with Lipschitz maps. Thus, Corollary 5.4 holds more generally for all topological groups.

This yields the following criterion for amenability, which can however be obtained without the use of Ramsey theory.

Corollary 5.6. *Let G be a Polish group such that for every positive n in $\mathbb{N}$, the set*

$$F_n = \{(g_1, \dots, g_n) \in G^n : \langle g_1, \dots, g_n \rangle \text{ is amenable}\}$$

is dense in G^n. Then G is amenable.

Proof. We use a Baire category argument. By the above corollary applied to the free group $\mathbb{F}_n$ on n generators (identifying $\mathrm{Hom}(\mathbb{F}_n, G)$ with G^n), for all n, the set F_n is dense G_δ in G^n. By the Baire category theorem, the set

$$F = \{(g_k) \in G^{\mathbb{N}} : \forall n, (g_1, \dots, g_n) \in F_n\}$$

is dense and G_δ too. Further, the set of sequences which are dense in G is also dense and G_δ. Then the Baire category theorem gives a sequence (g_k) in their intersection. Thus, the group generated by the g_k's is dense and amenable, hence so is G. ∎

Remark 5.7. The criterion of Corollary 5.6 can also be proven directly using the following compactness argument. Let G act continuously on a compact Hausdorff space X. For every finite subset F of G and every entourage V in the uniformity on $P(X)$, we approximate the elements of F by a tuple in some F_n to find a measure $\mu_{F,V}$ in $P(X)$ which is V-invariant under every element of F. Since $P(X)$ is compact, the net $\{\mu_{F,V}\}$ admits a limit point, which is invariant under the action of G.

The same argument works with extreme amenability as well, and it allows one to slightly simplify the arguments of [MT2]: to show that the groups $\mathrm{Iso}(\mathbb{U})$, $U(H)$ and $\mathrm{Aut}(X,\mu)$ are extremely amenable, Melleray and Tsankov use their Theorem 7.1 along with the facts that extreme amenability is a G_δ property, and that Polish groups are generically $\aleph_0$-generated. This is not necessary, as the core of their proof is basically the above criterion: in each case, they prove that the set of tuples which generate a subgroup that is contained in an extremely amenable group (some $L^0(U(m))$, as it happens) is dense.

6. Concluding remarks. One would expect the characterization of Theorem 4.3 to yield new examples of amenable groups, or at least simpler proofs of the amenability of known groups. However, proving the convex Ramsey property for a concrete Fraïssé class is quite technical and difficult.

Maybe our characterization can be used the other way around, that is, to find new Ramsey-type results. There is also hope that the criterion of Corollary 5.6 may lead to (new) examples of amenable groups.

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Adriane Kaïchouh
Institut Camille Jordan
Université Lyon 1
69622 Villeurbanne, France
E-mail: kaichouh@math.univ-lyon1.fr

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